Projective metric realizations of cone-manifolds with singularities along 2-bridge knots and links

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Abstract. Using projective metric geometry we develop a technique to describe homogeneous 3-dimensional metrics on cone-manifolds generated by two rotations. In particular, for some cone-manifolds with singularities along 2-bridge knots and links we give explicit descriptions of all possible geometries ($\mathbb{S}^3$, $\widetilde{SL_2(\mathbb{R})}$, and Nil).

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1. Introduction

In the present paper we develop a projective metric approach from [12] to realize homogeneous 3-geometries, which are also referred as *Thurston geometries*:

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \widetilde{SL_2(\mathbb{R})},\ Nil,\ Sol. \tag{1.1}$$

The projective models of Thurston geometries suggested in [12] are summarized in Table 1 below. Here we demonstrate existence of geometric structures on cone-manifolds generated by two rotations. In particular, for some cone-manifolds with singularities along 2-bridge knots and links we give explicit descriptions of all possible geometries ($\mathbb{S}^3$, $\widetilde{SL_2(\mathbb{R})}$, and Nil).

We recall that 2-bridge knots and links $K(p,q)$ are parameterized by coprime integers p and q, $0 < q < p$ (here we don't take care about orientations of components and non-equivalence to a mirror image). We only indicate those discussions would make our paper too long. For basic properties of 2-bridge knots and links we refer

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to [2]. By $\mathcal{C}_{p/q}$ we denote the cone-manifolds whose underlying space is the topological 3-sphere S^3 and whose singular set is $K(p,q)$ (see [1, 6]) for definition of a cone-manifold). If the singular set is a knot (for p odd) we have one cone angle (say, α, and we write $\mathcal{C}_{p/q}(\alpha)$), and if the singular set is a 2-component link (for p even) we have two (different in a general case) cone angles (say, α and β, and we write $\mathcal{C}_{p/q}(\alpha,\beta)$). Remark that we don't need to think about ordering of components, because they are equivalent under a symmetry of S^3. We assume that the cone angles belong to the open interval $(0;2\pi)$.

Since 2-bridge links (in particular, knots) $K(p,q)$ are most studied links in knot theory, cone-manifolds $\mathcal{C}_{p/q}$ have already been intensively investigated by many authors. In particular, spherical geometry structures on cone-manifolds $\mathcal{C}_{p/1}$ were described in [17] and [7]: $\mathcal{C}_{p/1}(\alpha)$ and $\mathcal{C}_{p/1}(\alpha,\alpha)$ are spherical if $\alpha \in (\pi - \frac{2\pi}{p}, \pi + \frac{2\pi}{p})$. The explicit description of spherical geometry on $\mathcal{C}_{3/1}(\alpha)$ was done in [3]. Here we demonstrate our unified method for complete description of some cone-manifolds from the family $\mathcal{C}_{p/1}$, especially with $\mathbb{S}^3$, $\widetilde{SL_2}(\mathbb{R})$, and Nil metric structures, since cone-manifolds $\mathcal{C}_{p/q}$ can be obtained from fundamental polyhedra by two identifying rotations.

Indeed, the combinatorial construction of fundamental polyhedra, which will be referred as $P_{p/q}$, for orbifolds $\mathcal{C}_{p/q}$ was suggested by Minkus [10]. For various applications of Minkus construction see [15] and [16]. Later it was discovered by Mednykh and Rasskazov [8, 9] that this topological construction can be successfully realized in hyperbolic, spherical and Euclidean spaces. Further development of this fruitful idea in the spaces of constant curvature was done in [3, 7, 19, 21]. From [5] the construction of polyhedra $P_{p/q}$ became known as *butterfly polyhedra*.

General schemata for $P_{p/q}$ and polyhedra $P_{2/1}$, $P_{3/1}$, $P_{4/1}$ are presented in Figures 1, 4, 6, 10. The polyhedron $P_{p/q}$ has four (non-planar) faces. For a given geometric realization of $P_{p/q}$, the cone-manifold $\mathcal{C}_{p/q}$ will be obtained by pairwise identifications of its faces by two rotational isometries (denoted by ϕ and ψ) with one axis passing through points A_0, A_1, and another axis passing through points A_2, A_3, as indicated in all above mentioned figures.

Starting with the combinatorial description of $P_{p/q}$, we will realize the polyhedron metrically in the real projective-spherical space $\mathcal{PS}^3(\mathbb{R})$ with the coordinate simplex A_0, A_1, A_2, A_3 (in the case $p/q = p/1$ vertices A_1 and A_3 are adjacent in $P_{p/1}$). Then, using projective models of Thurston geometries, we will provide the metric descriptions of geometric structures on some of $\mathcal{C}_{p/q}(\alpha,\beta)$ depending on p, q, and cone angles α and β.

According to Minkus construction, for the metrical realization, the dihedral angles at edges (A_0A_1) and (A_2A_3) ("essential angles" at axes) must be the same as cone angles of the cone-manifold, i.e. α for one axis and β for another axis (with $\beta = \alpha$ in the case of a knot). In further considerations we will use notation $P_{p/q}(\alpha,\beta)$ for Minkus polyhedron with essential angles equal to α and β.

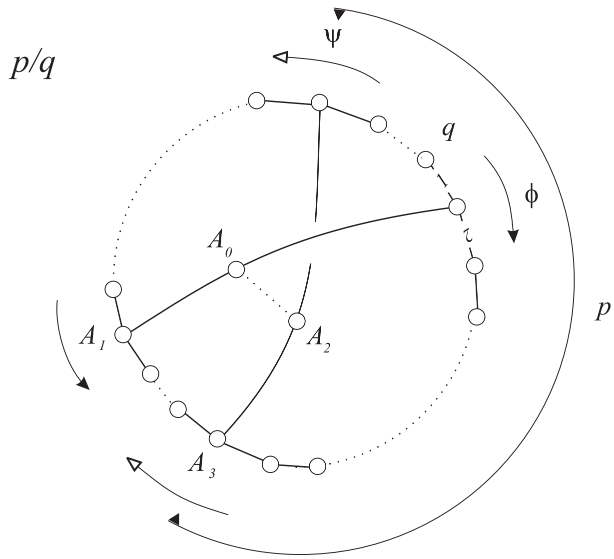

FIGURE 1. The polyhedron $P_{p/q}$

The paper is organized as follows. In Section 2 we will recall some basic facts on the projective spherical space $\mathcal{PS}^3(\mathbb{R})$ in which we are going to realize $P = P_{p/q}(\alpha,\beta)$. In Section 3 we will discuss the invariant polarity, in general. All possible geometric structures ($\mathbb{S}^3$, $\widetilde{SL_2(\mathbb{R})}$, and Nil), arising on cone-manifolds $\mathcal{C}_{2/1}(\alpha,\beta)$, $\mathcal{C}_{3/1}(\alpha)$, and $\mathcal{C}_{4/1}(\alpha,\beta)$ will be proceeded by a unified strategy in Sections 4, 5, and 6, respectively, and summarized in Theorems 4.1, 5.9, 6.5.

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2. Projective models of 3-geometries

2.1. Projective-spherical space

The *real projective-spherical space* $\mathcal{PS}^3(\mathbb{R})$ is represented by the "half subspace" incidence structure of the real vector 4-space $\mathbf{V}^4 = \mathbf{V}^4(\mathbb{R})$ and its dual space $\boldsymbol{V}_4 = \boldsymbol{V}_4(\mathbb{R})$ [12].

Each vector $\mathbf{x} \in \mathbf{V}^4 \setminus \mathbf{0}$, where $\mathbf{0} = (0,0,0,0)$ is the null-vector, determines the ray $(\mathbf{x})$ consisting of vectors $c \cdot \mathbf{x}$, $c \in \mathbb{R}_+$. We suppose that two rays are equivalent, $(\mathbf{x}) \sim (\mathbf{y})$ if the defining vectors are positively proportional, i.e. $\mathbf{y} = c \cdot \mathbf{x}$ for some $c \in \mathbb{R}_+$. Each vector $\mathbf{x} \in \mathbf{V}^4 \setminus \mathbf{0}$ defines a point $X(\mathbf{x}) \in \mathcal{PS}^3$ (presented by the ray $(\mathbf{x})$). Obviously, if vectors $\mathbf{x}, \mathbf{y} \in \mathbf{V}^4 \setminus \mathbf{0}$ are positively proportional then they define the same point $X(\mathbf{x}) = Y(\mathbf{y}) \in \mathcal{PS}^3$.

Analogously, if linear forms $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_4 \setminus \mathbf{0}$ are positively proportional, $\boldsymbol{v} = \boldsymbol{u} \cdot \frac{1}{c}$, $c \in \mathbb{R}_+$, then corresponding "form rays" are equivalent, $(\boldsymbol{u}) \sim (\boldsymbol{v})$, and they define the same (oriented) plane $u(\boldsymbol{u}) = v(\boldsymbol{v}) \in \mathcal{PS}^3$.

For $\mathbf{x} \in \mathbf{V}^4$ and $\boldsymbol{u} \in \boldsymbol{V}_4$ we denote by $(\mathbf{x}\boldsymbol{u}) \in \mathbb{R}$ the result of substituting coordinates of $\mathbf{x}$ into the form $\boldsymbol{u}$. Obviously, if $\mathbf{y} = c \cdot \mathbf{x}$ and $\boldsymbol{v} = \boldsymbol{u} \cdot \frac{1}{c}$ then $(\mathbf{x}\boldsymbol{u}) = (\mathbf{y}\boldsymbol{v})$. Cases $(\mathbf{x}\boldsymbol{u}) > 0$, $= 0$, or < 0 correspond to situations when a point $X(\mathbf{x})$ lies in the half space, on the plane, or in the opposite half space of $u(\boldsymbol{u})$, respectively.

Let $A_0(\mathbf{a}_0)$, $A_1(\mathbf{a}_1)$, $A_2(\mathbf{a}_2)$, $A_3(\mathbf{a}_3)$, where $\mathbf{a}_i \in \mathbf{V}^4 \setminus \mathbf{0}$, be four points $\mathcal{PS}^3$ in a general position, forming a *projective coordinate simplex* $A_0A_1A_2A_3$, and $A(\mathbf{a})$, with $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \in \mathbf{V}^4 \setminus \mathbf{0}$, be the (normalizing) unit point. Since coordinates of rays are determined up to positive proportionality, we have some freedom in choosing of $\mathbf{a}$ (and so, $\mathbf{a}_i$), that will be utilized in further computations.

Analogously, the "*plane simplex*" $b^0b^1b^2b^3$, such that

$$b^0(\boldsymbol{b}^0) = A_1A_2A_3, \qquad b^1(\boldsymbol{b}^1) = A_2A_3A_0,$$
$$b^2(\boldsymbol{b}^2) = A_3A_0A_1, \qquad b^3(\boldsymbol{b}^3) = A_0A_1A_2,$$

with the (normalizing) unit form $b(\boldsymbol{b} = \boldsymbol{b}^0 + \boldsymbol{b}^1 + \boldsymbol{b}^2 + \boldsymbol{b}^3)$, is fitted to the simplex vertices A_0, A_1, A_2, A_3 by choosing $\boldsymbol{b}^j \in \boldsymbol{V}_4 \setminus \mathbf{0}$ such that $(\mathbf{a}_i\boldsymbol{b}^j) = \delta_i^j$ is the Kronecker symbol.

A *projective collineation* λ of the projective spherical space $\mathcal{PS}^3$ is described by a pair of regular linear transformations $\mathbf{V}^4 \to \mathbf{V}^4$ (point $\mapsto$ point) and $\boldsymbol{V}_4 \to \boldsymbol{V}_4$ (plane $\mapsto$ plane) that preserves point-plane incidence. This will be assured if the value $(\mathbf{x}\boldsymbol{u})$ is preserved, where $\mathbf{x} \in \mathbf{V}^4$ and $\boldsymbol{u} \in \boldsymbol{V}_4$. It is known (e.g. by [12]) that a projective collineation λ can be given by $\lambda = \lambda(\Lambda, \Lambda^{-1})$, an inverse matrix pair Λ and Λ^{-1} for points and planes, respectively, preserving their incidences by $(\mathbf{x}\boldsymbol{u}) = (\mathbf{x}\Lambda, \Lambda^{-1}\boldsymbol{u})$. Obviously, for $c \in \mathbb{R}_+$ pairs (Λ, Λ^{-1}) and $(\Lambda \cdot c, \frac{1}{c} \cdot \Lambda^{-1})$ describe the same collineation λ. So, we have a freedom in choosing Λ up to positive proportionality.

Remark 2.1. Thus, we have two kinds of "projective freedom" up to positive real numbers: changing the basis vectors and the corresponding basis forms (so unit point and unit form) $c \cdot \mathbf{a}_i \to \mathbf{a}_i$ and $\boldsymbol{b}^i \cdot \frac{1}{c} \to \boldsymbol{b}^i$, $i = 0, 1, 2, 3$; and changing matrices, representing the collineation $(\Lambda, \Lambda^{-1}) \mapsto (\Lambda \cdot c, \frac{1}{c} \cdot \Lambda^{-1})$.

2.2. The case of skew axes.

Let us assume that Minkus polyhedron $P = P_{p/1}(\alpha, \beta)$ is realized in projective-spherical space $\mathcal{PS}^3$. Here and below we will refer to notations in Figures 1, 4, 6, 10. Let ϕ and ψ be rotational transformations about essential edges on positive angles α and β, respectively, identifying the faces of P. Suppose that axes of ϕ (passing through $A_0(\mathbf{a}_0)$ and $A_1(\mathbf{a}_1)$) and ψ (passing through $A_2(\mathbf{a}_2)$ and $A_3(\mathbf{a}_3)$)

are skew. Thus, simplex $A_0A_1A_2A_3$ is a projective coordinate simplex. and vectors $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ can be taken as a basis in $\mathbf{V}^4$.

Let us (first formally) correspond to transformations ϕ and ψ the matrices $\Phi = (f_j^i)_{j,i=0,1,2,3}$ and $\Psi = (g_j^i)_{j,i=0,1,2,3}$ as follows:

$$\phi \sim \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f_2^0 & f_2^1 & \cos\alpha & \sin\alpha \\ f_3^0 & f_3^1 & -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ F & M_\alpha \end{pmatrix} \tag{2.1}$$

and

$$\psi \sim \Psi = \begin{pmatrix} \cos\beta & \sin\beta & g_0^2 & g_0^3 \\ -\sin\beta & \cos\beta & g_1^2 & g_1^3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} M_\beta & G \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}. \tag{2.2}$$

where $\mathbf{1}_2$, $\mathbf{0}_2$, M_α, M_β, F, and G are 2×2 blocks. In the forthcoming calculations the given 2×2 block presentations of matrices Φ and Ψ will be useful.

Thinking of ϕ as a projective collineation $\phi = (\Phi, \Phi^{-1})$, we see from (2.1) that the ϕ-images of the basis vectors are

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f_2^0 & f_2^1 & \cos\alpha & \sin\alpha \\ f_3^0 & f_3^1 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ F & M_\alpha \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.3}$$

(by row-column multiplication), and with the inverse in 2×2 block matrix form

$$\phi^{-1}(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3) = (\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3) \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ -M_\alpha^T F & M_\alpha^T \end{pmatrix} \tag{2.4}$$

holds, where $M_\alpha^T = M_{-\alpha}$ is the transpose that is also the inverse of M_α.

By (2.3) all points of axis A_0A_1 are fixed under the projective collineation ϕ. Furthermore, planes of the pencil (b^2, b^3), incident to A_0A_1, will be "rotated through angle α", while the planes incident to points $(f_2^0, f_2^1, \cos\alpha - 1, \sin\alpha)$ and $(f_3^0, f_3^1, -\sin\alpha, \cos\alpha - 1)$ will be invariant. This is because collineation ϕ has the doubled eigenvalue 1 and of conjugate complex eigenvalues $(\cos\alpha + i\sin\alpha)$ and $(\cos\alpha - i\sin\alpha)$, as it follows from the theory of real collineations through complex extension.

Note that α is not a metric angle yet, but later on it will be realized as an angle in a suitable projective-spherical metric space. E.g. if $\alpha = \frac{2\pi}{n}$ then the order of the rotational transformation ϕ will be the integer $n \geqslant 2$, as usual for orbifolds with singular set of angular neighbourhood to $\frac{2\pi}{n}$.

It is known that the fundamental group of $S^3 \setminus K(p,1)$ is generated by elements ϕ, ψ with one *relation*:

$$\underbrace{\phi\psi\phi\dots}_{p} = \underbrace{\psi\phi\psi\dots}_{p}. \tag{2.5}$$

To obtain from P a cone-manifold $\mathcal{C}(p/1)$ with singularity along $K(p,1)$ we will require that rotational transformations ϕ and ψ satisfy this relation. This will admit to determine certain parameters for Φ and Ψ in (2.1) and (2.2).

2.3. The case of intersecting axes

As we shall see later, forthcoming *Nil*-realizations (and possible affine, so Euclidean realizations) correspond to intersecting rotational axes for ϕ and ψ. This also includes the situation when axes are parallel (and so, are intersecting in an infinite point). In this case the above suggested simplex $A_0A_1A_2A_3$, with A_0, A_1 on the axis of ϕ and A_2, A_3 on the axis of ψ, can not be a projective coordinate simplex, since these points are not linearly independent.

It is adequate now to consider a projective coordinate simplex $A_0A_1A_2A_3$, where $A_0(\mathbf{a}_0)$ and $A_1(\mathbf{a}_1)$ be still points on the axis of ϕ, but $A_2(\mathbf{a}_2)$ and $A_3(\mathbf{a}_3)$ no more points on the axis of ψ. More exactly, transformations ϕ and ψ are presented by matrices $\Phi = (f^i_j)_{j,i=0,1,2,3}$ and $\Psi = (g^i_j)_{j,i=0,1,2,3}$ as follows:

$$\phi \sim \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f^0_2 & f^1_2 & \cos\alpha & \sin\alpha \\ f^0_3 & f^1_3 & -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ F & M_\alpha \end{pmatrix}, \tag{2.6}$$

$$\psi \sim \Psi = \begin{pmatrix} 1 & 0 & \sin\beta & 1-\cos\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 & G \\ \mathbf{0}_2 & M_\beta \end{pmatrix}. \tag{2.7}$$

The submatrix F in (2.6) is taken the same as in (2.1) for shortening discussions, and the submatrix G in (2.7) is different from that in (2.2).

Thus A_0A_1 is the axis for rotation ϕ, where the plane pencil $(b^2(\boldsymbol{b}^2), b^3(\boldsymbol{b}^3))$ will be rotated by α (see Fig. 2).

Collineation ψ will be a rotation through angle β (one can see that now e.g. by eigenvalues). It is clear from (2.7) that the rotation axis $A_{03}A_1$ with $A_{03}(\mathbf{a}_0+\mathbf{a}_3)$ will be point-wise fixed by ψ since $(c,d,0,c)$, $(c,d) \neq (0,0)$, is fixed by choosing submatrix G of Ψ. A_1 is a common point of the above axes. The plane $b^0(\boldsymbol{b}_0)$ will be chosen in Sections 5 and 6, as a consequence there, the common invariant (ideal) plane, i. e. $f^0_2 = 0 = f^0_3$. Now $b^1(\boldsymbol{b}^1) = A_0A_2A_3$ is an invariant plane for ψ.

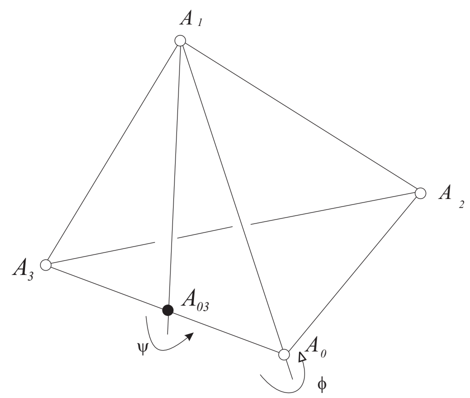

FIGURE 2

2.4. Models for Nil-geometry

It is well known that for $\pi_1(S^3 \setminus K(p,1))$, given by the presentation (2.5), the element

$$(\phi\psi)^p = (\psi\phi)^p, \quad \text{if} \quad p \quad \text{is odd} \quad (\text{here } K(p,1) \text{ is a knot}),$$
$$(\phi\psi)^{p/2} = (\psi\phi)^{p/2}, \quad \text{if} \quad p \quad \text{is even} \quad (\text{here } K(p,1) \text{ is a link}),$$

is central in the group generated by ϕ and ψ. If the metric realizations of rotational transformations ϕ and ψ are such that the center is not trivial, then the construction leads to a compact space. But trivial center may also occur in $\mathbb{S}^3$ realization.

Polyhedron P may have *bent faces* at the end of the construction. We preview in advance a sketch in Fig. 3 on Nil–realization of $\mathcal{C}_{3/1}(\frac{\pi}{3})$. This is fibred over the Euclidean plane $\mathbb{E}^2$, with plane triangle group $T(2,3,6)$, see Section 5. The edge frame refers to Fig. 3. The edges $1, 2, \dots, 6$ provide the relation $\phi\ \psi\ \phi\ \psi^{-1}\ \phi^{-1}\ \psi^{-1} = 1$ by the *Poincaré cycle* of edges equivalent under transformations. Obviously, this relation is a particular case of (2.5) for $p = 3$. This relation implies now $f_2^0 = 0 = f_3^0$ as we shall see in Sections 5 and 6. The faces for identifications $\phi : 163 \mapsto 254$ to $A_0A_{03}^{\phi\psi}$ and $\psi : 125 \mapsto 634$ to axis $A_{03}A_0^{\psi\phi}$ have to be formed carefully for a *solid*. E.g. by choosing a point U on the "*screw axis*" of $\phi\psi$ (see Fig. 3 for imagination) the star-shape triangles $UA_0A_{03}^{\phi\psi}$, $U1$, $U6$, $U3$ and their ϕ-images $U^\phi A_0A_{03}^{\phi\psi}$, $U^\phi 2$, $U^\phi 5$, $U^\phi 4$ can be formed. Here e.g. $U1 = UA_0A_{03}$ denotes also a triangle with the endpoints of segment 1. Then we choose another point V "a bit under" U^ϕ on the screw axis of $\psi\phi$ and take the star-shape triangles $VA_{03}A_0^{\psi\phi}$, $V1$, $V2$, $V5$ and their ψ-images $V^\psi A_{03}A_0^{\psi\phi}$, $V^\psi 6$, $V^\psi 3$, $V^\psi 4$. We can achieve that V^ψ will be "over" U on the screw axis of $\psi\phi$ if $U^\phi V$ smaller than the screw component of $\psi\phi$ that will be computed later from the *center translation* $\phi\psi\phi\psi\phi\psi$. Such a construction of fundamental polyhedron P will be similar also to other cases after having fixed

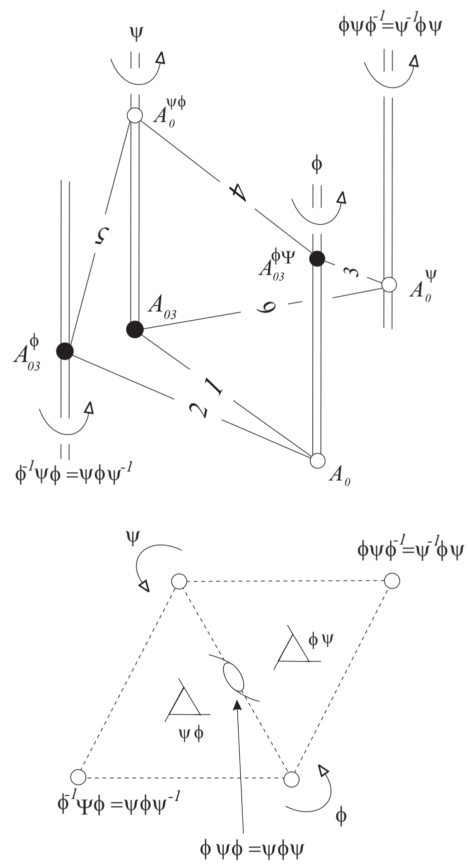

FIGURE 3

the generating rotations in a corresponding projective-spherical model of Thurston geometries.

2.5. Projective models for Thurston geometries

We shall look for a plane $\mapsto$ point polarity

$$(\,)_*: \boldsymbol{V}_4 \to \mathbf{V}^4, \qquad \boldsymbol{b}^i \mapsto \boldsymbol{b}^i_* = b^{ij}\mathbf{a}_j$$

with symmetric matrix of coefficients $B = (b^{ij})$, or *scalar product of forms* defined linearly by

$$\langle\ ,\ \rangle: \boldsymbol{V}_4 \times \boldsymbol{V}_4 \to \mathbb{R}, \quad \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = (\boldsymbol{b}^i_*\, \boldsymbol{b}^j) = (b^{ir}\mathbf{a}_k \boldsymbol{b}^j) = b^{ik}\delta^j_k = b^{ij} \tag{2.8}$$

(with Einstein's sum convention), so that Φ and Ψ from (2.1), (2.2) and (2.6), (2.7), respectively, shall preserve the polarity and the scalar product (up to some projective freedom, see details in Section 3).

The recognition of geometries will be based on the following result of Molnár: the signature of the scalar product and some other information (see [12] and Table 1 from therein) uniquely determine the realizing geometry.

Table 1.

Space $\mathbb{X}$	Signature of polarity $()_*$ or scalar product	Domain of proper points of $\mathbb{X}$ in $\mathcal{PS}^3$	The group $G = \mathrm{Isom}\mathbb{X}$ as a special transformation group of $\mathcal{PS}^3$
$\mathbb{S}^3$	$(+++ +)$	$\mathcal{PS}^3$	Coll. $\mathcal{PS}^3$ preserving $()_*$
$\mathbb{H}^3$	$(-+++)$	$\{(\mathbf{x}) \in \mathcal{P}^3 \;:\; (\mathbf{x},\mathbf{x}) < 0\}$	Coll. $\mathcal{P}^3$ preserving $()_*$
$\widetilde{SL_2(\mathbb{R})}$	$(--++)$ with skew line fibering	Universal covering of $\mathcal{H} = \{[\mathbf{x}] \in \mathcal{PS}^3 : (\mathbf{x},\mathbf{x}) < 0\}$ by fibering transformations	Coll. $\mathcal{PS}^3$ preserving $()_*$ and fibers
$\mathbb{E}^3$	$(0+++)$	$\mathcal{A}^3 = \mathcal{P}^3 \setminus \{\omega^\infty\}$, where $\omega^\infty = (\boldsymbol{b}^0)$, $\boldsymbol{b}^0_* = \mathbf{0}$	Coll. $\mathcal{PS}^3$ preserving $()_*$ generated by plane reflections
$\mathbb{S}^2 \times \mathbb{R}$	$(0+++)$ with $\mathbf{0}$-line bundle fibering	$\mathcal{A}^3 \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is fixed origin	G is generated by plane reflections and sphere inversions, leaving invariant the $\mathbf{0}$-concentric 2-spheres of $()_*$
$\mathbb{H}^2 \times \mathbb{R}$	$(0-++)$ with $\mathbf{0}$-line bundle fibering	$\mathcal{C}^+ = \{X \in \mathcal{A}^3 : \langle\overrightarrow{\mathbf{0}X}, \overrightarrow{\mathbf{0}X}\rangle < 0, \text{half cone}\}$ by fibering	G is generated by plane reflections and hyperboloid inversions, leaving invariant the $\mathbf{0}$-concentric half-hyperboloids in the half-cone $\mathcal{C}^+$ by $()_*$
Sol	$(0-++)$ with parallel plane fibering	$\mathcal{A}^3$ with parallel plane fibering	Coll. $\mathcal{P}^3$ preserving $()_*$ and the parallel plane fibering
Nil	$(000+)$ with parallel line bundle fibering	$\mathcal{A}^3$ with a distinguished parallel plane pencil along each line	Quadr. maps of $\mathcal{P}^3$ conjugate to coll. preserv. $()_*$ and the fibering

3. The invariant polarity

In this section we consider the generators $\phi(\Phi, \Phi^{-1})$ and $\psi(\Psi, \Psi^{-1})$ defined by pairs of formulae (2.1), (2.2) (if axes of ϕ and ψ are skew) and (2.6), (2.7) (if axes of ϕ and ψ are intersecting), respectively, and look for a (non-trivial) plane $\mapsto$ point polarity or a scalar product (2.8), invariant under ϕ and ψ. This step is a crucial technical point of our approach (see also [13] and [14] for similar technique).

Suppose that a scalar product is determined by a non-zero symmetric matrix $B = (b^{ij})$ with the following block form:

$$B = \begin{pmatrix} B^{00} & B^{02} \\ B^{20} & B^{22} \end{pmatrix}. \tag{3.1}$$

Hence ${B^{00}}^T = B^{00}$, ${B^{02}}^T = B^{20}$, ${B^{22}}^T = B^{22}$.

The invariance condition for generator ϕ will be considered as

$$\Phi^T B \Phi = (\pm) B, \tag{3.2}$$

because ${\Phi^{-1}}^T B\Phi^{-1} = (\pm)B$ leads to equivalent condition. The similar condition will be applied for ψ as well.

For the invariant quadratic form $\xi_i b^{ij}\xi_j$ we denote $\zeta_0^T = (\xi_0, \xi_1)$, $\zeta_2^T = (\xi_2, \xi_3)$ and consider the block form

$$(\zeta_0^T, \zeta_2^T)\begin{pmatrix} B^{00} & B^{02} \\ B^{20} & B^{22}\end{pmatrix}\begin{pmatrix}\zeta_0 \\ \zeta_2\end{pmatrix}$$

that will be treated in concrete cases later on.

3.1. The case of skew axes

Let us consider transformations $\phi(\Phi)$ and $\psi(\Psi)$ given by (2.1) and (2.2). For $\phi(\Phi)$ given by (2.1) using (3.1) and (3.2) we get:

$$\begin{pmatrix} \begin{matrix} B^{00} + F^T B^{20} + \\ + B^{02}F + F^T B^{22} F \end{matrix} & B^{02}M_\alpha + F^T B^{22} M_\alpha \\ M_\alpha^T B^{20} + M_\alpha^T B^{22} F & M_\alpha^T B^{22} M_\alpha \end{pmatrix} = \pm \begin{pmatrix} B^{00} & B^{02} \\ B^{20} & B^{22}\end{pmatrix}. \tag{3.3}$$

Analogously, for $\psi(\Psi)$ given by (2.2) we get:

$$\begin{pmatrix} M_\beta^T B^{00} M_\beta & M_\beta^T B^{00} G + M_\beta^T B^{02} \\ G^T B^{00} M_\beta + B^{20} M_\beta & \begin{matrix} G^T B^{00} G + B^{20} G + \\ + G^T B^{02} + B^{22}\end{matrix} \end{pmatrix} = \pm \begin{pmatrix} B^{00} & B^{02} \\ B^{20} & B^{22}\end{pmatrix}. \tag{3.4}$$

Obviously, formulae (3.3) and (3.4) give equations for their 2×2 blocks. In further computations the equations for blocks of (3.3) will be indexed as $(3.3_{[00]})$, $(3.3_{[02]})$, $(3.3_{[20]})$, and $(3.3_{[22]})$ (and analogously for the equation (3.4)).

In general, the signs $(+)$ or $(-)$ in (3.3) and (3.4) can be chosen independently. But as we shall see below, only choosing $(+)$ in both formulae will lead to adequate solution for $B = (b^{ij})$.

Let us start with formula (3.3) and consider all possible cases.

Case 1: The sign $(+)$ in formula (3.3).

Subcase 1.1: $\alpha = \pi \pmod{2\pi}$. In this subcase $(3.3_{[22]})$ allows arbitrary matrix $B^{22} = \begin{pmatrix} b^{22} & b^{23} \\ b^{32} & b^{33}\end{pmatrix}$. Then $(3.3_{[02]})$, as well as $(3.3_{[20]})$, implies $B^{02} = \frac{1}{2}F^T B^{22}\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$. Hence $(3.3_{[00]})$ automatically holds for arbitrary B^{00}. Therefore

$$B = \begin{pmatrix} B^{00} & -\frac{1}{2}F^T B^{22} \\ (-\frac{1}{2}F^T B^{22})^T & B^{22}\end{pmatrix}.$$

Subcase 1.2: $\alpha \neq \pi \pmod{2\pi}$. By $(3.3_{[22]})$, B^{22} can be chosen in the form $B^{22} = \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix} = b\mathbf{1}_2$, for some $b > 0$ (we may assume). From $(3.3_{[02]})$, as well as from $(3.3_{[20]})$, we get

$$B^{02} = F^T B^{22} M_\alpha(\mathbf{1}_2 - M_\alpha)^{-1} = \frac{b}{2\sin\frac{\alpha}{2}} F^T M_{(\frac{\pi}{2}+\frac{\alpha}{2})}.$$

Indeed, since $\det(\mathbf{1}_2 - M_\alpha) = \det \begin{pmatrix} 1-\cos\alpha & -\sin\alpha \\ \sin\alpha & 1-\cos\alpha \end{pmatrix} = 2(1-\cos\alpha) = 4\sin^2\frac{\alpha}{2}$ we have

$$(\mathbf{1}_2 - M_\alpha)^{-1} = \frac{1}{2\sin\frac{\alpha}{2}} \begin{pmatrix} \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \end{pmatrix} =$$
$$= \frac{1}{2\sin\frac{\alpha}{2}} \begin{pmatrix} \cos(\frac{\pi}{2}-\frac{\alpha}{2}) & \sin(\frac{\pi}{2}-\frac{\alpha}{2}) \\ -\sin(\frac{\pi}{2}-\frac{\alpha}{2}) & \cos(\frac{\pi}{2}-\frac{\alpha}{2}) \end{pmatrix} = \frac{1}{2\sin\frac{\alpha}{2}} M_{(\frac{\pi}{2}-\frac{\alpha}{2})}.$$

Obviously, with B^{02} as above, $(3.3_{[00]})$ holds for arbitrary B^{00}. Hence

$$B = \begin{pmatrix} B^{00} & \frac{b}{2\sin\frac{\alpha}{2}} F^T M_{(\frac{\pi}{2}+\frac{\alpha}{2})} \\ (\frac{b}{2\sin\frac{\alpha}{2}} F^T M_{(\frac{\pi}{2}+\frac{\alpha}{2})})^T & b\mathbf{1}_2 \end{pmatrix}. \tag{3.5}$$

Case 2: The sign $(-)$ in (3.3).

Subcase 2.1: $\alpha = \pm\frac{\pi}{2} \pmod{2\pi}$. In this subcase $(3.3_{[22]})$ serves $B^{22} = \begin{pmatrix} b^{22} & b^{23} \\ b^{23} & -b^{22} \end{pmatrix}$, and $b^{33} = -b^{22}$ implies an *isotropic plane* $u(\boldsymbol{u})$, with $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$, through the axis A_0A_1, non adequate for models of 3-geometries (1.1) by [12].

Subcase 2.2: $\alpha \neq \pm\frac{\pi}{2} \pmod{2\pi}$. By $(3.3_{[22]})$ we get $B^{22} = 0$. Since $b^{22} = b^{33} = 0$ means incident polar-pole pair, i.e. isotropic planes through A_0A_1 again, this case is not adequate for compact realizations of 3-geometries (1.1).

Summarizing discussions of subcases **2.1** and **2.2** we get

Lemma 3.1. *The sign (-) in (3.3) doesn't lead to adequate collineation ϕ and to compact realization of polyhedron P in Thurston geometries.*

We analogously discuss (3.4) for angle β.

Case 3: The sign $(+)$ in formula (3.4).

Subcase 3.1: $\beta = \pi \pmod{2\pi}$. Then B^{00} is arbitrary, and

$$B^{02} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} B^{00}G = -\frac{1}{2}B^{00}G.$$

Then $(3.4_{[22]})$ holds, B^{22} is arbitrary. Therefore

$$B = \begin{pmatrix} B^{00} & -\frac{1}{2}B^{00}G \\ (-\frac{1}{2}B^{00}G)^T & B^{22} \end{pmatrix}.$$

Subcase 3.2: $\beta \neq \pi \pmod{2\pi}$. By $(3.4_{[00]})$, B^{00} can be chosen in the form $B^{00} = \begin{pmatrix} \overline{b} & 0 \\ 0 & \overline{b} \end{pmatrix} = \overline{b}\mathbf{1}_2$ for some $\overline{b} > 0$ (we may assume, again; different signs of b and $\overline{b}$ would lead to non-adequate incident polar-pole pair in $\widetilde{SL_2(\mathbb{R})}$ (Table 1), as discussed later on, negative signs for both are equivalent to positive ones). From $(3.4_{[02]})$, as well as from $(3.4_{[20]})$, we get

$$B^{02} = -(\mathbf{1}_2 - M_\beta)^{-1}B^{00}G = \frac{\overline{b}}{2\sin\frac{\beta}{2}} M_{(-\frac{\pi}{2}-\frac{\beta}{2})}G,$$

similarly as in Subcase 1.2. Now with arbitrary B^{22} we get

$$B = \begin{pmatrix} \bar{b}\mathbf{1}_2 & \frac{\bar{b}}{2\sin\frac{\beta}{2}} M_{(-\frac{\pi}{2}-\frac{\beta}{2})} G \\ \left(\frac{\bar{b}}{2\sin\frac{\beta}{2}} M_{(-\frac{\pi}{2}-\frac{\beta}{2})} G\right)^T & B^{22} \end{pmatrix}. \tag{3.6}$$

Case 4: The sign $(-)$ in formula (3.4).

Subcase 4.1: $\beta = (\pm)\frac{\pi}{2} \pmod{2\pi}$. By $(3.5_{[00]})$ we get $B^{00} = \begin{pmatrix} b^{00} & b^{01} \\ b^{01} & -b^{00} \end{pmatrix}$. Since $b^{11} = -b^{00}$ implies an isotropic plane through the axis A_2A_3, this case is non adequate for realizations of 3-geometries (1.1).

Subcase 4.2: $\beta \neq \pm\frac{\pi}{2} \pmod{2\pi}$. From $(3.4_{[00]})$ we get $B^{00} = 0$. Again, $b^{00} = b^{11} = 0$ means incident polar-pole pair, we do not obtain compact realization of 3-geometries (1.1).

Summarizing discussions of subcases **4.1** and **4.2** we get

Lemma 3.2. *The sign (-) in (*3.4*) doesn't lead to adequate collineation ψ and to compact realization of polyhedron P in Thurston geometries.*

3.2. The case of intersecting axes

Let us consider transformations $\phi(\Phi)$ and $\psi(\Psi)$ given by (2.6) and (2.7), and find the invariant polarity (b^{ij}) in the block form, as in (3.1) and (3.2).

The block equation for $\phi(\Phi)$ will take a form as in (3.3). Thus, arguments from subcases **1.1**, **1.2**, **2.1**, and **2.2** of the preceding subsection can be applied.

The block equation for $\Psi(\psi)$ will take a form

$$\begin{pmatrix} B^{00} & B^{00}G + B^{02}M_\beta \\ G^T B^{00} M_\beta^T B^{20} & \begin{matrix} G^T B^{00} G + M_\beta^T B^{20} G + \\ + G^T B^{02} M_\beta + M_\beta^T B^{22} M_\beta \end{matrix} \end{pmatrix} = \begin{pmatrix} \pm B^{00} & \pm B^{02} \\ \pm B^{20} & \pm B^{22} \end{pmatrix} \tag{3.7}$$

with G as in (2.7).

Let us discuss cases of the sign $(+)$ and $(-)$ in the right side of (3.7).

Case 5: Sign $(+)$ in formula (3.7). Then B^{00} is arbitrary from $(3.7_{[00]})$, and we go to the following subcases.

Subcase 5.1: $\beta \neq \pi \pmod{2\pi}$. Firstly we observe that

$$B^{02} = B^{00}G(\mathbf{1}_2 - M_\beta)^{-1}$$
$$= B^{00}G\frac{1}{2\sin\frac{\beta}{2}} \begin{pmatrix} \cos(\frac{\pi}{2}-\frac{\beta}{2}) & \sin(\frac{\pi}{2}-\frac{\beta}{2}) \\ -\sin(\frac{\pi}{2}-\frac{\beta}{2}) & \cos(\frac{\pi}{2}-\frac{\beta}{2}) \end{pmatrix} = \frac{1}{2\sin\frac{\beta}{2}} B^{00} G M_{(\frac{\pi}{2}-\frac{\beta}{2})}$$

follows from $(3.7_{[02]})$, as well as from $(3.7_{[20]})$. The equation $(3.7_{[22]})$ gives

$$G^T B^{00} G + G^T B^{00} G \frac{1}{2\sin\frac{\beta}{2}} M_{(\frac{\pi}{2}+\frac{\beta}{2})} + \frac{1}{2\sin\frac{\beta}{2}} M_{(-\frac{\pi}{2}-\frac{\beta}{2})} G^T B^{00} G = B^{22} - M_\beta^T B^{22} M_\beta.$$

Since $G = \begin{pmatrix} \sin\beta & 1-\cos\beta \\ 0 & 0 \end{pmatrix} = 2\sin\frac{\beta}{2}\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ 0 & 0 \end{pmatrix}$ we get $B^{02} = B^{00}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{00} \\ 0 & b^{10} \end{pmatrix}$ and $b^{33} = b^{00} + b^{22}$, $b^{23} = b^{32} = 0$. Therefore

$$B = \begin{pmatrix} b^{00} & b^{01} & 0 & b^{00} \\ b^{10} & b^{11} & 0 & b^{10} \\ 0 & 0 & b^{22} & 0 \\ b^{00} & b^{10} & 0 & b^{00}+b^{22} \end{pmatrix}. \tag{3.8}$$

Subcase 5.2: $\beta = \pi \pmod{2\pi}$. From (3.7) we get $B^{02} = \frac{1}{2}B^{00}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{00} \\ 0 & b^{10} \end{pmatrix}$, $B^{20} = \begin{pmatrix} 0 & 0 \\ b^{00} & b^{10} \end{pmatrix}$, $b^{00} = 0$, and $B^{22} = B^{22\,T} = \begin{pmatrix} b^{22} & b^{23} \\ b^{32} & b^{33} \end{pmatrix}$. Hence

$$B = \begin{pmatrix} 0 & b^{01} & 0 & 0 \\ b^{10} & b^{11} & 0 & b^{10} \\ 0 & 0 & b^{22} & b^{23} \\ 0 & b^{10} & b^{32} & b^{33} \end{pmatrix}. \tag{3.9}$$

Case 6: Sign $(-)$ in formula (3.7). Then we get $B^{00} = 0$. The equality $b^{11} = 0$ means incident polar-pole pair non-adequate for compact realization (Fig. 2).

Thus we have got

Lemma 3.3. *The sign (-) in (3.7) doesn't lead to adequate collineation ψ and to compact realization of polyhedron P in Thurston geometries.*

4. The Hopf link $K(2,1)$

To investigate geometric structures on $\mathcal{C}_{2/1}(\alpha,\beta)$ with singularities along the Hopf link $K(2,1)$, we recall, that according to Minkus construction, the polyhedron $P_{2/1}(\alpha,\beta)$ looks as in Fig. 4, and transformations ψ and ψ satisfies the relation $\phi\psi = \psi\phi$.

Theorem 4.1. *For any $\alpha, \beta \in (0, 2\pi)$ the polyhedron $P_{2/1}(\alpha,\beta)$ with essential angles α and β can be realized as a compact polyhedron in spherical space $\mathbb{S}^3$ with ϕ and ψ acting by isometries.*

Proof. In the case of skew axes transformations $\phi(\Phi)$ and $\psi(\Psi)$ are given by (2.1) and (2.2). Then the relation $\phi\psi = \psi\phi$ has the following matrix form:

$$\begin{pmatrix} M_\beta & G \\ FM_\beta & FG + M_\alpha \end{pmatrix} = \begin{pmatrix} M_\beta + GF & GM_\alpha \\ F & M_\alpha \end{pmatrix}.$$

Equations $F(\mathbf{1}_2 - M_\beta) = 0$ and $G(\mathbf{1}_2 - M_\alpha) = 0$ imply $F = 0$ and $G = 0$, since $\alpha \neq 0$ and $\beta \neq 0 \pmod{2\pi}$ by the assumption.

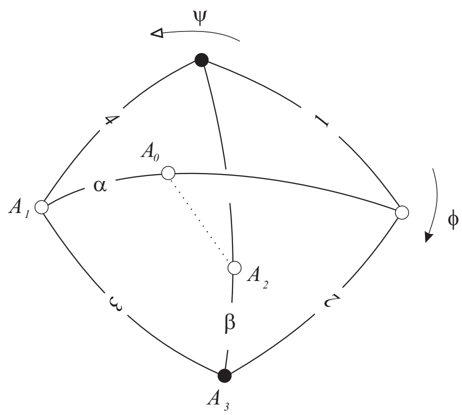

FIGURE 4. The polyhedron $P_{2/1}$

If $\alpha = \pi$ and $\beta = \pi \,(\mathrm{mod}\, 2\pi)$ then from subcases **1.1** and **3.1** we conclude that the polarity invariant under Φ and Ψ is given by

$$B = \begin{pmatrix} B^{00} & \mathbf{0}_2 \\ \mathbf{0}_2 & B^{22} \end{pmatrix} = \begin{pmatrix} b^{00} & b^{01} & 0 & 0 \\ b^{10} & b^{11} & 0 & 0 \\ 0 & 0 & b^{22} & b^{23} \\ 0 & 0 & b^{32} & b^{33} \end{pmatrix}. \tag{4.1}$$

The signature $(+ + ++)$ will lead to the spherical geometry. As one can see, other signatures in B^{00} or in B^{22} yield incident polar-pole pair, i. e. an isotropic plane through the rotation axis A_2A_3 or A_0A_1, respectively, and so, geometrization is not possible for these signatures. Namely, signature $(- - ++)$ would lead to $\widetilde{SL_2(\mathbb{R})}$ geometry in Table 1 with its hyperboloid model (see Fig. 8). One of two our axes would be inside of the hyperboloid (proper axis), and the other – outside (improper axis), that gives a contradiction.

If $\alpha \neq \pi$ and $\beta \neq \pi \,(\mathrm{mod}\, 2\pi)$, then from subcases **1.2** and **3.2** we conclude that the polarity invariant under Φ and Ψ is given by

$$B = \begin{pmatrix} \bar{b}\mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & b\mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} b^{00} & 0 & 0 & 0 \\ 0 & b^{00} & 0 & 0 \\ 0 & 0 & b^{22} & 0 \\ 0 & 0 & 0 & b^{22} \end{pmatrix} \tag{4.2}$$

for some $\bar{b} = b^{00} > 0$ and $b = b^{22} > 0$ again.

E.g. $\alpha \neq \pi$, $\beta = \pi$ would lead to

$$B = \begin{pmatrix} b^{00} & b^{01} & 0 & 0 \\ b^{10} & b^{11} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

as a combination of (4.1) and (4.2). Again, by Table 1, signature $(+ + ++)$ is suitable only.

In each (++++) case we can change basis by linear combinations so that we get $B = \mathbf{1}_4$.

In the case of intersecting axes transformations $\phi(\Phi)$ and $\psi(\Psi)$ are given by (2.6) and (2.7). Then the relation $\phi\psi = \psi\phi$ has the following matrix form:

$$\begin{pmatrix} \mathbf{1}_2 & G \\ F & FG + M_\alpha M_\beta \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 + GF & GM_\alpha \\ M_\beta F & M_\beta M_\alpha \end{pmatrix}.$$

Hence $G = 0$ or $\alpha = 0\,(\mathrm{mod} 2\pi)$, and by (2.7) we get $\beta = 0$ or $\alpha = 0$. This gives the contradiction with non-triviality of α and β. □

By (2.1) and (2.2) one can find coordinates of vertices of the polyhedron.

Corollary 4.2. *Four vertices of $P_{2/1}(\alpha, \beta)$ are the following:*

$$A_1(0,1,0,0,),\ A_3(0,0,0,1),\ A_3^{\phi^{-1}}(0,0,\sin\alpha,\cos\alpha),\ A_1^{\psi^{-1}}(\sin\beta,\cos\beta,0,0).$$

The angles at edges A_1A_3, $A_3A_1^{\psi^{-1}}$, $A_1^{\psi^{-1}}A_3^{\phi^{-1}}$, $A_3^{\phi^{-1}}A_1$ are right angles. The essential lengths are $A_1A_1^{\psi^{-1}} = \beta$, $A_3A_3^{\phi^{-1}} = \alpha$, $A_1A_3 = \frac{\pi}{2}$ so as the other three edges.

This prescribes the positions of A_0 and A_2 as well. We can calculate any metrical data of $P_{\alpha,\beta}(2/1)$ by standard formulae of the spherical geometry. E.g. the length of the singular line $A_1A_1^{\psi^{-1}}$ with cone angle α will be β by

$$\cos A_1A_1^{\psi^{-1}} = \frac{\langle \mathbf{a}_1, \mathbf{a}_1\Psi^{-1}\rangle}{|\mathbf{a}_1||\mathbf{a}_1\Psi^{-1}|} = \cos\beta$$

by some additional arguments.

5. The trefoil knot $K(3,1)$

5.1. Geometries in the case of skew axes

It is known that the trefoil knot $K(3,1)$ and its mirror image are not equivalent up to orientation preserving homeomorphisms of S^3. Thus, we distinguish the right-hand trefoil knot and the left-hand trefoil knot. The right-hand trefoil knot is presented in Fig. 5. According to Minkus construction, polyhedron $P_{3/1}(\alpha)$ looks as in Fig. 6 and transformations ϕ and ψ, defined by (2.1) and (2.2) satisfy the relation $\phi\psi\phi = \psi\phi\psi$. This relation has the following matrix form:

$$\begin{pmatrix} M_\beta + GF & GM_\alpha \\ FM_\beta + FGF + M_\alpha F & FGM_\alpha + M_\alpha M_\alpha \end{pmatrix} = \begin{pmatrix} M_\beta M_\beta + GFM_\beta & M_\beta G + GFG + GM_\alpha \\ FM_\alpha & FG + M_\alpha \end{pmatrix}, \tag{5.1}$$

i.e. $GF = -M_\beta$ and $FG = -M_\alpha$ lead to $M_\alpha F = FM_\beta$ and $M_\beta G = GM_\alpha$. Thus, either

1: $\beta = \alpha \pmod{2\pi}$, $\mu + \nu = \alpha + \pi \pmod{2\pi}$, $F = f \cdot \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} = f \cdot M_\mu$, $G = g \cdot \begin{pmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{pmatrix} = g \cdot M_\nu$, $f \cdot g = 1$ $(f, g > 0)$ in the case of right-hand trefoil knot, or

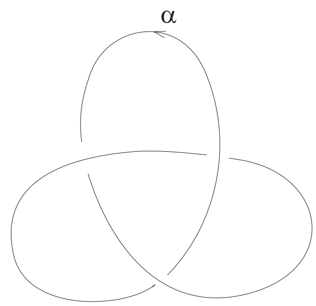

FIGURE 5. The right-hand trefoil knot

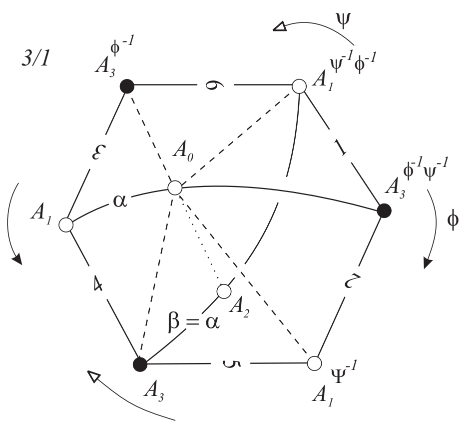

FIGURE 6. The polyhedron $P_{3/1}$

2: $\beta = -\alpha \pmod{2\pi}$, $\nu - \mu = \alpha + \pi \pmod{2\pi}$, $F = f \cdot \begin{pmatrix} \cos\mu & \sin\mu \\ \sin\mu & -\cos\mu \end{pmatrix} = f \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot M_\mu$, $G = g \cdot \begin{pmatrix} \cos\nu & \sin\nu \\ \sin\nu & -\cos\nu \end{pmatrix} = g \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot M_\nu$, $f \cdot g = 1$ $(f, g > 0)$ in the case of left-hand trefoil knot occur.

By changing some of basis vectors as $\mathbf{a}_{0'} := f \cdot \mathbf{a}_0$, $\boldsymbol{b}^{0'} := \boldsymbol{b}^0 \cdot \frac{1}{f}$; and $\mathbf{a}_{1'} := f \cdot \mathbf{a}_1$, $\boldsymbol{b}^{1'} := \boldsymbol{b}^1 \cdot \frac{1}{f}$, we achieve $f = g = 1$ in both cases. After that, to simplify notations we change indices back: $0' \mapsto 0$ and $1' \mapsto 1$.

Remark 5.1. We will not discuss the case **2** of the left-hand trefoil knot since it can be done completely analogously to the case **1**.

Lemma 5.2. *Projective collineations $\phi = \phi(\Phi, \Phi^{-1})$ and $\psi = \psi(\Psi, \Psi^{-1})$, presented by (2.1) and (2.2), with $\beta = \alpha \in (0, 2\pi)$, that identify faces of $P_{3/1}$, preserve scalar products* *with following signatures:*

$$(++++) \quad \textit{iff} \quad \frac{\pi}{3} < \alpha < \frac{5\pi}{3};$$
$$(--++) \quad \textit{iff} \quad 0 < \alpha < \frac{\pi}{3},\ \frac{5\pi}{3} < \alpha < 2\pi;$$
$$(0\ 0\ ++) \quad \textit{iff} \quad \alpha = \frac{\pi}{3} \ \textit{or} \ \alpha = \frac{5\pi}{3}.$$

Proof. By Lemma 3.1 and Lemma 3.2 we consider only sign $(+)$ in (3.2). By Remark 5.1 we consider only case $\beta = \alpha$ in Minkus construction.

(i). Assume that $\alpha = \pi$. Then transformations ϕ and ψ are involutive halfturns. Moreover, $\mu + \nu = 0 \pmod{2\pi}$ implies that for

$$\Phi = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ M_\mu & M_\alpha \end{pmatrix} \qquad \text{and} \qquad \Psi = \begin{pmatrix} M_\alpha & M_\nu \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}$$

the following relations hold:

$$F^T = F^{-1} = M_\mu^T = M_{(2\pi-\mu)} = M_\nu = G.$$

It is easy to see that

$$\Phi\Psi\Phi = \Psi\Phi\Psi = \begin{pmatrix} \mathbf{0}_2 & M_{(\pi-\mu)} \\ M_{(\pi+\mu)} & \mathbf{0}_2 \end{pmatrix};$$

and $(\Phi\Psi)^3 = (\Psi\Phi)^3 = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}$ is trivial. We see that $\phi\psi\phi = \psi\phi\psi$ is an involutive halfturn with axis pair

$$(x^0, x^1, x^2, x^3), \quad \text{where } x^2 = -x^0 \cdot \cos\mu - x^1 \cdot \sin\mu, \quad x^3 = x^0 \cdot \sin\mu - x^1 \cdot \cos\mu,$$
$$(y_0, y_1, y_2, y_3), \quad \text{where } y_2 = -\cos\mu \cdot y_0 - \sin\mu \cdot y_1, \quad y_3 = \sin\mu \cdot y_0 - \cos\mu \cdot y_1$$

for points and for planes, respectively. Then

$$\Phi\Psi = \begin{pmatrix} -1 & 0 & \cos\mu & -\sin\mu \\ 0 & -1 & \sin\mu & \cos\mu \\ -\cos\mu & -\sin\mu & 0 & 0 \\ \sin\mu & -\cos\mu & 0 & 0 \end{pmatrix};$$

and $(\psi\phi) = (\phi\psi)^{-1} = (\phi\psi)^2$ are "3_1 screw motions" (that is a translation along a line with a rotation on $2\pi/3$ about it) permuting the previous halfturns in a cyclic way. E.g. $(\phi\psi)^{-1}\phi(\phi\psi) = \psi\phi\psi = \phi\psi\phi$ and $(\phi\psi)^{-1}(\phi\psi\phi)(\phi\psi) = \psi$, $(\phi\psi)^{-1}\psi(\phi\psi) = \psi\phi\psi\phi\psi = \phi$.

The realization with compact fundamental domain is possible in the spherical space $\mathbb{S}^3$ not in the usual sense, e.g. a lens realization comes (see Fig. 7.a) with lens angle $\frac{\pi}{3}$, just as a special case of the next subcase for extension to $\alpha = \beta = \pi$.

Now let us look for the invariant polarity B. By subcases 1.1 and 3.1 in Section 3 submatrices of B are related by

$$\frac{1}{2} F^T B^{22} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = B^{02} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} B^{00} G, \qquad B^{20^T} = B^{02},$$

we get

$$\frac{1}{2} M_{(\pi-\mu)} B^{22} = B^{02} = \frac{1}{2} B^{00} M_{(\pi-\mu)}, \qquad B^{20^T} = B^{02},$$

with $M_{(\pi-\mu)} = \begin{pmatrix} \cos(\pi-\mu) & \sin(\pi-\mu) \\ -\sin(\pi-\mu) & \cos(\pi-\mu) \end{pmatrix}$. These yield the invariant quadratic form $\xi_i b^{ij} \xi_j$ for linear forms $\boldsymbol{b}^i \xi$ of the dual space $V_4(\mathbb{R})$, i.e. for planes of $\mathcal{PS}^3(\mathbb{R})$ in block matrices, where $\eta_0^T = (\xi_0, \xi_1)$ and $\eta_2^T = (\xi_2, \xi_3)$:

$$(\eta_0^T, \eta_2^T) \begin{pmatrix} B^{00} & B^{02} \\ B^{20} & B^{22} \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_2 \end{pmatrix} = \eta_0^T B^{00} \eta_0 + \eta_2^T B^{20} \eta_0 + \eta_0^T B^{02} \eta_2 + \eta_2^T B^{22} \eta_2 =$$
$$= \left[\eta_0^T + \frac{1}{2} \eta_2^T M_{(\pi+\mu)} \right] B^{00} \left[\eta_0 + \frac{1}{2} M_{(\pi-\mu)} \eta_2 \right] + \frac{3}{4} \eta_2^T M_{(\pi+\mu)} B^{00} M_{(\pi-\mu)} \eta_2.$$

Thus, if B^{00} has signature (σ_1, σ_2), then B has signature $(\sigma_1, \sigma_1, \sigma_2, \sigma_2)$. Since possible signatures for B^{00} are $(++)$, $(0\ +)$ and $(-+)$, possible signatures for B are $(++ ++), (0\ 0\ ++)$ and $(--++)$. As well as in Section 4, to exclude isotropic plane, only signature $(++++)$ is possible.

Remark that $\alpha = \pi$ implies that $\mathcal{C}_{3/1}(\pi)$ is an orbifold, and its orbifold group $\langle \phi, \psi \mid \phi^2 = \psi^2 = 1,\ \phi\psi\phi = \psi\phi\psi \rangle$ is dihedral group of order 6. Hence, only signature $(++++)$ leads to compact fundamental domain as desired.

(ii). Assume that $\alpha \neq \pi$. Then $\mu + \nu = \alpha + \pi \pmod{2\pi}$. By subcases 1.2 and 3.2 in Section 3, i.e. by formulas (3.5) and (3.6), submatrices of B satisfy the following relations:

$$\frac{b}{2\sin\frac{\alpha}{2}} F^T M_{(\frac{\pi}{2}+\frac{\alpha}{2})} = B^{02} = \frac{\overline{b}}{2\sin\frac{\alpha}{2}} M_{(-\frac{\pi}{2}-\frac{\alpha}{2})} G,$$
$$B^{20^T} = B^{02}, \qquad B^{00} = \overline{b} \cdot \mathbf{1}_2, \qquad B^{22} = b \cdot \mathbf{1}_2.$$

Since $F = M_\mu$ (so $F^T = M_{(-\mu)}$) and $G = M_\nu$ we get

$$\frac{b}{2\sin\frac{\alpha}{2}} M_{(\frac{\alpha}{2}+\frac{\pi}{2}-\mu)} = \frac{\overline{b}}{2\sin\frac{\alpha}{2}} M_{(-\frac{\alpha}{2}-\frac{\pi}{2}+\nu)}.$$

Using $\mu + \nu = \alpha + \pi \pmod{2\pi}$, from this relation we get $\overline{b} = b$. Recalling that $B^{02} = \frac{b}{2\sin\frac{\alpha}{2}} M_{(\frac{\alpha}{2}+\frac{\pi}{2}-\mu)} = \frac{b}{2\sin\frac{\alpha}{2}} \begin{pmatrix} -\sin(\frac{\alpha}{2}-\mu) & \cos(\frac{\alpha}{2}-\mu) \\ -\cos(\frac{\alpha}{2}-\mu) & -\sin(\frac{\alpha}{2}-\mu) \end{pmatrix}$, we obtain

$$B = \begin{pmatrix} b & 0 & \frac{-b\sin(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & \frac{b\cos(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} \\ 0 & b & \frac{-b\cos(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & \frac{-b\sin(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} \\ \frac{-b\sin(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & \frac{-b\cos(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & b & 0 \\ \frac{b\cos(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & \frac{-b\sin(\frac{\alpha}{2}-\mu)}{2\sin\frac{\alpha}{2}} & 0 & b \end{pmatrix} \tag{5.2}$$

with some projective freedom, of course. Namely, $b = 1$ can be chosen. The polyhedron P and so the coordinate simplex are flexible by the angle μ (and ν). E.g. $\mu = \nu = \frac{\alpha}{2} - \frac{\pi}{2} \pmod{2\pi}$ leads to the usual angle metric convention for simplex planes in formula (5.6) and Figures 7 and 8 below. Then we have

$$B = (b^{ij}) = \begin{pmatrix} 1 & 0 & \frac{-1}{2\sin\frac{\alpha}{2}} & 0 \\ 0 & 1 & 0 & \frac{-1}{2\sin\frac{\alpha}{2}} \\ \frac{-1}{2\sin\frac{\alpha}{2}} & 0 & 1 & 0 \\ 0 & \frac{-1}{2\sin\frac{\alpha}{2}} & 0 & 1 \end{pmatrix} \tag{5.3}$$

with $\det(b^{ij}) = \left[1 - \left(\frac{1}{2\sin\frac{\alpha}{2}}\right)^2\right]^2 = \left(\frac{\frac{1}{2}-\cos\alpha}{1-\cos\alpha}\right)^2$ and the quadratic form for a variable plane defined by form $\boldsymbol{b}^i\xi_i$

$$\xi_i b^{ij}\xi_j = \left(\xi_0 - \frac{1}{2\sin\frac{\alpha}{2}}\xi_2\right)^2 + \left(\xi_1 - \frac{1}{2\sin\frac{\alpha}{2}}\xi_3\right)^2 + \frac{\frac{1}{2}-\cos\alpha}{1-\cos\alpha}(\xi_2\xi_2 + \xi_3\xi_3)$$

with signature

$$\begin{aligned}
&\mathbf{ii_1}: \ (++++) \quad \text{iff} \quad -1 < \cos\alpha < \frac{1}{2},\ \frac{\pi}{3} < \alpha < \pi,\ \pi < \alpha < \frac{5\pi}{3},\\
&\mathbf{ii_2}: \ (--++) \quad \text{iff} \quad \cos\alpha > \frac{1}{2},\ 0 < \alpha < \frac{\pi}{3},\ \frac{5\pi}{3} < \alpha < 2\pi;\\
&\mathbf{ii_3}: \ (0\ 0\ ++) \quad \text{iff} \quad \alpha = \frac{\pi}{3} \text{ or } \alpha = \frac{5\pi}{3}.
\end{aligned}$$

□

5.2. Computations for skew axes

Remark 5.3. Using the agreement $\mu = \nu = \frac{\alpha}{2} - \frac{\pi}{2} \pmod{2\pi}$, we get the following matrices for transformations:

$$\Phi = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ M_{(\frac{\alpha}{2}-\frac{\pi}{2})} & M_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\frac{\alpha}{2} & -\cos\frac{\alpha}{2} & \cos\alpha & \sin\alpha \\ \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} & -\sin\alpha & \cos\alpha \end{pmatrix},$$

$$\Psi = \begin{pmatrix} M_\alpha & M_{(\frac{\alpha}{2}-\frac{\pi}{2})} \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & \sin\frac{\alpha}{2} & -\cos\frac{\alpha}{2} \\ -\sin\alpha & \cos\alpha & \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Phi^{-1} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ M^T_{(\frac{\pi}{2}-\frac{\alpha}{2})} & M^T_\alpha \end{pmatrix}, \qquad \Psi^{-1} = \begin{pmatrix} M^T_\alpha & M^T_{(\frac{\pi}{2}-\frac{\alpha}{2})} \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} \tag{5.4}$$

$$\Phi\Psi = \begin{pmatrix} M_\alpha & M_{(\frac{\alpha}{2}-\frac{\pi}{2})} \\ M_{(\frac{3\alpha}{2}-\frac{\pi}{2})} & \mathbf{0}_2 \end{pmatrix}, \qquad \Psi\Phi = \begin{pmatrix} \mathbf{0}_2 & M_{(\frac{3\alpha}{2}-\frac{\pi}{2})} \\ M_{(\frac{\alpha}{2}-\frac{\pi}{2})} & M_\alpha \end{pmatrix},$$

$$\Phi^{-1}\Psi^{-1} = \begin{pmatrix} M_{(-\alpha)} & M_{(\frac{\pi}{2}-\frac{\alpha}{2})} \\ M_{(\frac{\pi}{2}-\frac{3\alpha}{2})} & \mathbf{0}_2 \end{pmatrix}, \quad \Psi^{-1}\Phi^{-1} = \begin{pmatrix} \mathbf{0}_2 & M_{(\frac{\pi}{2}-\frac{3\alpha}{2})} \\ M_{(\frac{\pi}{2}-\frac{\alpha}{2})} & M_{(-\alpha)} \end{pmatrix},$$

$$\Phi\Psi\Phi = \Psi\Phi\Psi = \begin{pmatrix} \mathbf{0}_2 & M_{(\frac{3\alpha}{2}-\frac{\pi}{2})} \\ M_{(\frac{3\alpha}{2}-\frac{\pi}{2})} & \mathbf{0}_2 \end{pmatrix},$$

and

$$(\Phi\Psi)^3 = \begin{pmatrix} M_{(3\alpha-\pi)} & \mathbf{0}_2 \\ \mathbf{0}_2 & M_{(3\alpha-\pi)} \end{pmatrix}.$$

While $B = (b^{ij})$ is "responsible" for the metric of planes (angles and distances), its inverse (if exists) $a_{ij} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle = (b^{ij})^{-1}$ determines the distances of points ([12], [11]).

Of course, we can compute these data from (5.2), in more general, as well. We have obtained

$$(a_{ij}) = (b^{ij})^{-1} = \frac{1-\cos\alpha}{\frac{1}{2}-\cos\alpha}\begin{pmatrix} 1 & 0 & \frac{1}{2\sin\frac{\alpha}{2}} & 0 \\ 0 & 1 & 0 & \frac{1}{2\sin\frac{\alpha}{2}} \\ \frac{1}{2\sin\frac{\alpha}{2}} & 0 & 1 & 0 \\ 0 & \frac{1}{2\sin\frac{\alpha}{2}} & 0 & 1 \end{pmatrix} \tag{5.5}$$

if $\frac{1}{2} - \cos\alpha \neq 0$.

Lemma 5.4. *If axes of ϕ and ψ are skew, then generalized dihedral angles of coordinate simplex $A_0A_1A_2A_3$ for $P_{3/1}(\alpha)$ are given by*

$$\beta^{01} = \beta^{03} = \beta^{12} = \beta^{23} = \frac{\pi}{2}, \quad \cos\beta^{02} = \cos\beta^{13} = \frac{1}{2\sin\frac{\alpha}{2}},$$

where β^{ij} is angle between faces b^i and b^j, that is dihedral angle at edge (A_k, A_ℓ), where $\{k,\ell\} = \{0,1,2,3\} \setminus \{i,j\}$. At signature $(--++)$ the angle measure can be extended by complex cos *function (see Subsection 5.3 and* [11]*, e.g.).*

Proof. The angle β^{02} between planes b^0 and b^2 at simplex edge A_1A_3 (see Fig. 6) can be obtained by (the complementary angle of their poles (normals) $\boldsymbol{b}_*^0 = b^{00}\mathbf{a}_0 + b^{02}\mathbf{a}_2$ and $\boldsymbol{b}_*^2 = b^{20}\mathbf{a}_0 + b^{22}\mathbf{a}_2$:

$$\cos\beta^{02} = \frac{-b^{02}}{\sqrt{b^{00}b^{22}}} = \frac{1}{2\sin\frac{\alpha}{2}}, \tag{5.6}$$

that is less than 1 if $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$ as in case of signature $(++++)$.

Computations for other angles are completely analogous. □

Lemma 5.5. *If axes of ϕ and ψ are skew, then vertices of polyhedron $P_{3/1}(\alpha)$ are the following:*

$$\begin{gathered} A_1(0,1,0,0), \;\; A_3(0,0,0,1), \\ A_3^{\phi^{-1}}\left(-\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, \sin\alpha, \cos\alpha\right), \;\; A_1^{\psi^{-1}}\left(\sin\alpha, \cos\alpha, -\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\right), \\ A_3^{\phi^{-1}\psi^{-1}}\left(-\cos\frac{3\alpha}{2}, \sin\frac{3\alpha}{2}, 0, 0\right), \;\; A_1^{\psi^{-1}\phi^{-1}}\left(0, 0, -\cos\frac{3\alpha}{2}, \sin\frac{3\alpha}{2}\right). \end{gathered} \tag{5.7}$$

Proof. By direct calculations using formulas according to Remark 5.3. □

Lemma 5.6. *Lengths of the six equal edges $A_1A_3 = A_3A_1^{\psi^{-1}} = \cdots = A_3^{\phi^{-1}}A_1$ of $P_{3/1}(\alpha)$, in the spherical case, equals to ρ, with $\cos\rho = \frac{1}{2\sin\frac{\alpha}{2}}$. The length measure is also extended (formally) by complex* cos *function* [11] *to signature $(--++)$.*

Proof. Positive distances can be computed by using (5.5):

$$\cos A_0A_2 = \frac{a_{02}}{\sqrt{a_{00}a_{22}}} = \frac{1}{2\sin\frac{\alpha}{2}}, \qquad \cos A_1A_3 = \frac{a_{13}}{\sqrt{a_{11}a_{33}}} = \frac{1}{2\sin\frac{\alpha}{2}},$$

that is less than 1 in the case of signature $(++++)$. As one can see from Fig. 6, by (5.7), $\cos A_1A_3^{\phi^{-1}\psi^{-1}} = \sin\frac{3\alpha}{2} = \cos(\frac{3\alpha}{2} - \frac{\pi}{2})$, hence

$$A_1A_3^{\phi^{-1}\psi^{-1}} = \frac{3\alpha}{2} - \frac{\pi}{2}.$$

The length of the whole singular set is

$$\ell := A_1 A_3^{\phi^{-1}\psi^{-1}} + A_3 A_1^{\psi^{-1}\phi^{-1}} = 3\alpha - \pi.$$

Since $\ell = 3\alpha - \pi$ holds for any $\alpha \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, pairs $(\alpha, \ell) = (\frac{4\pi}{3}, 3\pi)$; $(\alpha, \ell) = (\frac{6\pi}{4}, 4\pi - \frac{\pi}{2})$; and $(\alpha, \ell) = (\frac{8\pi}{5}, 4\pi - \frac{\pi}{5})$ are also realizable by polyhedron P with vertices in (5.7). Let us imagine $\mathbb{S}^3$ as $\mathbb{E}^3 \cup \{\infty\}$ in conformal (circle geometric) interpretation in a symbolic picture Fig. 7. Then the edges $1, 2, \dots, 6$ represent a segment class of angular neighbourhood 2π, indeed.

To see this, we consider the case $\alpha = \pi$, first symbolically in Fig. 7.a. The lens here is of angle $\frac{\pi}{3}$ by (5.6). Thus the six equivalent segments yield angle $6 \cdot \frac{\pi}{3} = 2\pi$. Then, as we have seen, the angular distances are $A_3^{\phi^{-1}\psi^{-1}} A_1 = A_1^{\psi^{-1}\phi^{-1}} A_3 = \pi$. E.g. $A_3^{\phi^{-1}\psi^{-1}}(\cos(\pi - \frac{3\alpha}{2}), \sin(\pi - \frac{3\alpha}{2}), 0, 0)$, $A_0(\cos(0), \sin(0), 0, 0)$, $A_1(\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}), 0, 0)$ show in (5.7), how to introduce the "angle coordinate" (fibre coordinate [12]) $\pi - \frac{3\alpha}{2}$ for $A_3^{\phi^{-1}\psi^{-1}}$, increasing from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ if α decreases from π to $\frac{\pi}{3}$, and decreasing $-\frac{\pi}{2}$ to $-\frac{3\pi}{2} \equiv \frac{\pi}{2} \pmod{2\pi}$ if α varies from π to $\frac{5\pi}{3}$ (see Fig. 7.c). □

Let us introduce the midpoint N of arc $A_1 A_3^{\phi^{-1}\psi^{-1}}$ (as north pole, see Fig. 7.b), and similarly the midpoint S of arc $A_3 A_1^{\psi^{-1}\phi^{-1}}$ (south pole). Then the star shape triangles, with plane coordinates (forms expressed by $\boldsymbol{b}^i$-s), bound a nice symmetric solid as fundamental domain P in the following:

Lemma 5.7. *If axes of ϕ and ψ are skew, then north pole, south pole, and triangular faces of polyhedron $P_{3/1}(\alpha)$ are given by the following formulas:*

$$N\Big(\cos\frac{3(\pi-\alpha)}{4}, \sin\frac{3(\pi-\alpha)}{4}, 0, 0\Big); \quad S\Big(0, 0, \cos\frac{3(\pi-\alpha)}{4}, \sin\frac{3(\pi-\alpha)}{4}\Big);$$

$$SA_3A_1(1,0,0,0)^T \sim \boldsymbol{b}^0, \quad NA_1A_3(0,0,1,0)^T \sim \boldsymbol{b}^2;$$

$$SA_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}\psi^{-1}} \Big(\sin\frac{3\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, \cos\frac{3\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, 0, 0\Big)^T;$$

$$NA_3^{\phi^{-1}\psi^{-1}} A_1^{\psi^{-1}\phi^{-1}} \Big(0, 0, \sin\frac{3\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, \cos\frac{3\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}\Big)^T;$$

$$SA_1A_3^{\phi^{-1}} \Big(\cos\frac{\pi-\alpha}{4}, 0, \cos\frac{\alpha}{2}\sin\frac{3(\pi-\alpha)}{4}, -\cos\frac{\alpha}{2}\cos\frac{3(\pi-\alpha)}{4}\Big)^T;$$

$$NA_3^{\phi^{-1}} A_1 \Big(0, 0, -\cos\alpha\cos\frac{3(\pi-\alpha)}{4}, \sin\alpha\cos\frac{3(\pi-\alpha)}{4}\Big)^T;$$

$$SA_3^{\phi^{-1}} A_1^{\psi^{-1}\phi^{-1}} \Big(-\sin\frac{\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, -\cos\frac{\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, 0, 0\Big)^T;$$

$$NA_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}} \Big(-\cos\frac{\alpha}{2}\sin\frac{3(\pi-\alpha)}{4}, \cos\alpha\cos\frac{3(\pi-\alpha)}{4}, -\sin\frac{3\alpha}{2}\sin\frac{\pi+\alpha}{4}, -\cos\frac{3\alpha}{2}\sin\frac{\pi+\alpha}{4}\Big)^T;$$

$$SA_3^{\phi^{-1}\psi^{-1}} A_1^{\psi^{-1}} \Big(-\sin\frac{3\alpha}{2}\sin\frac{\pi+\alpha}{4}, -\cos(3\alpha)\sin\frac{\pi+\alpha}{4}, -\cos\frac{\alpha}{2}\sin\frac{3(\pi-\alpha)}{4}, \cos\frac{\alpha}{2}\cos\frac{3(\pi-\alpha)}{4}\Big)^T;$$

$$NA_1^{\psi^{-1}} A_3^{\phi^{-1}\psi^{-1}} \Big(0, 0, -\sin\frac{\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}, -\cos\frac{\alpha}{2}\sin\frac{3(\pi+\alpha)}{4}\Big)^T;$$

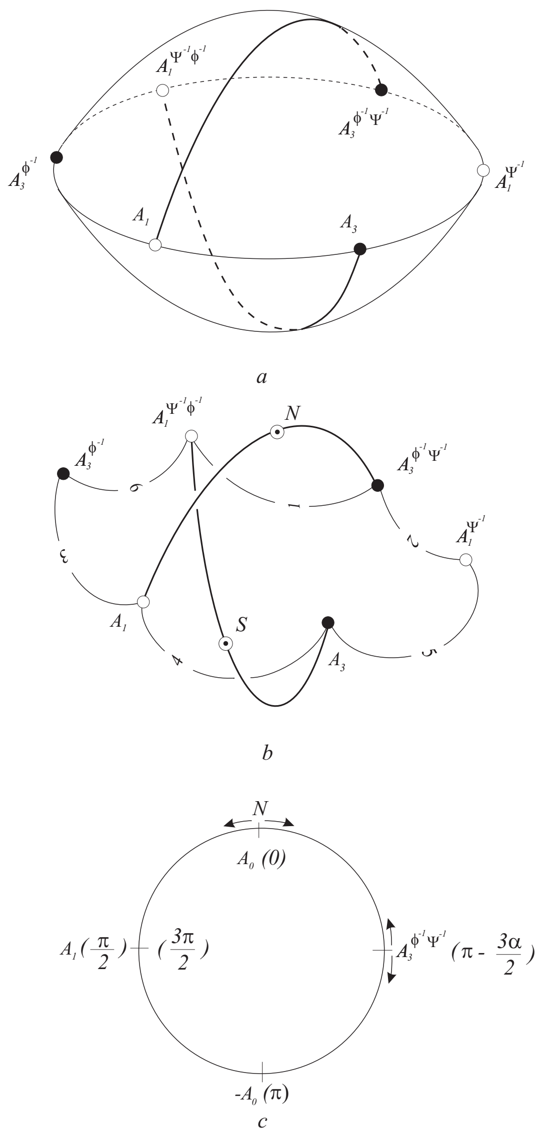

FIGURE 7. a, b, c

$$SA_1^{\psi^{-1}}A_3\Big(-\cos\alpha\cos\frac{3(\pi-\alpha)}{4},-\sin\alpha\cos\frac{3(\pi-\alpha)}{4},0,0\Big)^T;$$
$$NA_3A_1^{\psi^{-1}}\Big(\cos\frac{\alpha}{2}\sin\frac{3(\pi-\alpha)}{4},-\cos\frac{\alpha}{2}\cos\frac{3(\pi-\alpha)}{4},\cos\frac{(\pi-\alpha)}{4},0\Big)^T.$$

Now we are able to compute dihedral angles of $P_{3/1}$ by the scalar product matrix (b^{ij}) in (5.3).

Lemma 5.8. *The essential dihedral angles of $P_{3/1}(\alpha)$ are (also in complex extended form) as follows:*

$$\cos(\angle A_3A_1) = \cos(\angle A_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}\psi^{-1}}) = \frac{1}{2\sin\frac{\alpha}{2}};$$

the other four dihedral angles are equal, e.g.

$$\cos(\angle A_1A_3^{\phi^{-1}}) = \frac{\cos\frac{3(\pi-\alpha)}{4}}{\sqrt{(1+\sin\frac{\alpha}{2})2\sin\frac{\alpha}{2}(2\sin\frac{\alpha}{2}-1)}}.$$

Proof. The calculations of dihedral angles of polyhedron $P_{3/1}(\alpha)$ can be done as follows.

First of all, just to illustrate our technique, we will check the known result that dihedral angle of $P_{3/1}$ at the axis of ϕ, passing through A_0A_1, is α. Indeed, we observe that the simplex plane $b^2 = A_0A_1A_3$ intersects with $A_0A_1A_3^{\phi^{-1}}$, that is the image of b^2 under ϕ^{-1} (given by Remark 5.3). Since $\Phi\boldsymbol{b}^2 = \boldsymbol{b}^2\cos\alpha - \boldsymbol{b}^3\sin\alpha$, in the sense of (2.4) for $\phi^{-1}(\Phi^{-1},\Phi)$, we get

$$\begin{aligned}\cos(\boldsymbol{b}^2,\Phi\boldsymbol{b}^2) &= \frac{\langle\boldsymbol{b}^2,\Phi\boldsymbol{b}^2\rangle}{\sqrt{\langle\boldsymbol{b}^2,\boldsymbol{b}^2\rangle\langle\Phi\boldsymbol{b}^2,\Phi\boldsymbol{b}^2\rangle}} = \langle\boldsymbol{b}^2,\Phi\boldsymbol{b}^2\rangle = (\boldsymbol{b}^2_*\Phi\boldsymbol{b}^2)\\ &= (b^{2i}\mathbf{a}_i,\Phi\boldsymbol{b}^2) = \Big(\frac{1}{2\sin\frac{\alpha}{2}}\mathbf{a}_0 + 1\cdot\mathbf{a}_2, \boldsymbol{b}^2\cos\alpha - \boldsymbol{b}^3\sin\alpha\Big) = \cos\alpha,\end{aligned}$$

as expected. Of course, dihedral angle of $P_{3/1}(\alpha)$ at the axis of ψ, passing through A_2A_3, is α too.

As we already demonstrated in Lemma 5.4, $\cos\angle A_3A_1 = \frac{1}{2\sin\frac{\alpha}{2}}$.

Other dihedral angles can be found from scalar products (defined by matrix B in (2.2)) of forms of incident faces given in Lemma 5.7. Since $\angle A_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}\psi^{-1}}$ is the angle between planes $NA_3^{\phi^{-1}\psi^{-1}} A_1^{\psi^{-1}\phi^{-1}}$ and $SA_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}\psi^{-1}}$, we get

$$\cos\angle A_1^{\psi^{-1}\phi^{-1}} A_3^{\phi^{-1}\psi^{-1}} = \frac{1}{2\sin\frac{\alpha}{2}} \frac{\sin^2\frac{3(\pi+\alpha)}{4}\cdot\big(\sin^2\frac{3\alpha}{2}+\cos^2\frac{3\alpha}{2}\big)}{\sin^2\frac{3(\pi+\alpha)}{4}} = \frac{1}{2\sin\frac{\alpha}{2}}.$$

The other four angles are equal, as e.g. the computations by Lemma 5.7 show. Since $\angle A_1A_3^{\phi^{-1}}$ is the angle between planes $NA_3^{\phi^{-1}}A_1$ and $SA_1A_3^{\phi^{-1}}$, we get

$$\begin{aligned}\cos\angle A_1A_3^{\phi^{-1}} = \cos\frac{\alpha}{2}\Bigg[&\Big(\sin\frac{3(\pi-\alpha)}{4}\cos\alpha + \cos\frac{3(\pi-\alpha)}{4}\sin\alpha\Big) - \\ &-\frac{1}{2\sin\frac{\alpha}{2}}\cos\alpha\cos\frac{\pi-\alpha}{4}\Bigg]\cdot\\ \cdot\Big(\cos^2\frac{\alpha}{2} + \cos^2\frac{\pi-\alpha}{4} - \frac{2}{2\sin\frac{\alpha}{2}}&\cos\frac{\pi-\alpha}{4}\sin\frac{3(\pi-\alpha)}{4}\cos\frac{\alpha}{2}\Big)^{-1/2} =\end{aligned}$$

$$= \frac{\sin\alpha\sin\frac{\pi-\alpha}{4} - \cos\alpha\cos\frac{\pi-\alpha}{4}}{\sqrt{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} + 4\sin^2\frac{\alpha}{2}\cos^2\frac{\pi-\alpha}{4} - \sin\alpha\left[\sin(\pi-\alpha)+\sin\frac{\pi-\alpha}{2}\right]}}$$

$$= \frac{\cos\frac{3(\pi-\alpha)}{4}}{\sqrt{(1-\cos\alpha)\left(1+\sin\frac{\alpha}{2}\right) - \sin\alpha\cos\frac{\alpha}{2}}},$$

and so on, as in Lemma 5.6. Obviously, if $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$ then $-\frac{\pi}{2} < \frac{3(\pi-\alpha)}{4} < \frac{\pi}{2}$.

Direct computations give

$$\cos\angle A_3 A_1^{\psi^{-1}} = \cos\angle A_1^{\psi^{-1}} A_3^{\phi^{-1}\psi^{-1}} = \cos\angle A_3^{\phi^{-1}} A_1^{\psi^{-1}\phi^{-1}} = \cos\angle A_1 A_3^{\phi^{-1}}$$

and $2\angle A_1 A_3^{\phi^{-1}} = \pi - \angle A_3 A_1$. Thus, the sum of the six mentioned dihedral angles is equal to 2π.

Remark, as particular cases, that $\alpha = \frac{\pi}{2}$ leads to $\angle A_3 A_1 = \frac{\pi}{4}$ and $\angle A_1 A_3^{\phi^{-1}} = \frac{3\pi}{8}$, as well as $\alpha = \frac{3\pi}{2}$ leads to these angles too. □

The center of $\pi_1(S^3 \setminus K(3,1))$ is generated by $\tau := (\phi\psi)^3 = (\psi\phi)^3$. By Remark 5.3 this τ has the following matrix presentation depending on α:

$$\mathcal{T}_\alpha := (\Phi\Psi)^3 = (\Psi\Phi)^3 = \begin{pmatrix} M_{3\alpha-\pi} & \mathbf{0}_2 \\ \mathbf{0}_2 & M_{3\alpha-\pi} \end{pmatrix}.$$

This is trivial if $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{5\pi}{3}$. By Lemma 5.2 for these angles the signature is $(0\,0\,{+}{+})$. By (5.7) we get that for these angles $A_1 = A_3^{\phi^{-1}\psi^{-1}}$, $A_3 = A_1^{\psi^{-1}\phi^{-1}}$, $A_3^{\phi^{-1}} = -A_1^{\psi^{-1}}$, hence polyhedron $P_{3/1}$ degenerates.

Note that for spherical orbifolds we get the following cases: $\mathcal{T}_\pi$ is trivial;

$$\mathcal{T}_{\frac{2\pi}{3}} = \begin{pmatrix} M_\pi & \mathbf{0}_2 \\ \mathbf{0}_2 & M_\pi \end{pmatrix} = -\mathbf{1}_4;\ \mathcal{T}_{\frac{\pi}{2}} = \begin{pmatrix} M_{\frac{\pi}{2}} & \mathbf{0}_2 \\ \mathbf{0}_2 & M_{\frac{\pi}{2}} \end{pmatrix};\mathcal{T}_{\frac{2\pi}{5}} = \begin{pmatrix} M_{\frac{\pi}{5}} & \mathbf{0}_2 \\ \mathbf{0}_2 & M_{\frac{\pi}{5}} \end{pmatrix}$$

.

5.3. Computations for $\widetilde{SL_2(\mathbb{R})}$

By Lemma 5.2 the case $0 < \alpha < \frac{\pi}{3}$ gives signature $(-\,-\,{+}{+})$. According to Table 1 it leads to $\widetilde{SL_2(\mathbb{R})}$ geometry by its one-sheet hyperboloid solid model in projective-spherically extended $\mathbb{E}^3$ [12] (see also Section 6), roughly sketched in Fig. 8. Unfortunately, the case of $\widetilde{SL_2(\mathbb{R})}$ geometry is more difficult to imagine, since analogies with the spherical case are not always valid.

At each inner point (e.g. at A_0 in Fig. 8.a) we can form the asymptotic cone. Lines pointing into interior of this cone coincide with planes of positive form squares and their angle can usually be measured. E.g. for the angle between coordinate planes b^2 and b^3, incident with A_0A_1, we get $\beta^{23} = \frac{\pi}{2}$ as in the spherical cases. But $b^{02} = \frac{-1}{2\sin\frac{\alpha}{2}} < -1$, if $0 < \alpha < \frac{\pi}{3}$, and so

$$\cos\beta^{02} = \frac{-b^{02}}{\sqrt{b^{00}b^{22}}} = \frac{1}{2\sin\frac{\alpha}{2}} > 1,$$

provide complex angle $\beta^{02} = x$ and distance xi of poles by $\cos x = \cosh(xi)$ (see e.g. [11] for more details). Here we only remark, that the cross ratio (p, q, u, v) of a plane pencil in the complex extension defines the projective measure of (p, q) by $\frac{1}{2i} \log(p, q, u, v)$, where u, v are the isotropic elements (e.g. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$) in the pencil of (p, q), of course, with some discussions.

The distance metric is defined analogously to (5.5), but now

$$\cosh A_0A_2 = \frac{-a_{02}}{\sqrt{a_{00}a_{22}}} = \frac{1}{2\sin\frac{\alpha}{2}} > 1$$

is taken, again through complex extension. Formula (5.4), Lemma 5.4, and Lemma 5.7 hold as well. The length of the singular set is given by

$$\ell := A_1A_3^{\phi^{-1}\psi^{-1}} + A_3A_1^{\psi^{-1}\phi^{-1}} = |3\alpha - \pi|$$

also with orbifold cases realized for $\alpha = \frac{2\pi}{7}, \frac{2\pi}{8}, \dots$. The equal distances of $A_1A_3 = A_0A_2 = A_3^{\phi^{-1}\psi^{-1}} A_1^{\psi^{-1}}$, etc. can be obtained by

$$\cosh A_1A_3 = \frac{-a_{13}}{\sqrt{a_{11}a_{33}}} = \frac{1}{2\sin\frac{\alpha}{2}} = \frac{-a_{02}}{\sqrt{a_{00}a_{22}}} = \cosh A_0A_2 > 1.$$

Comparing with Fig. 7, we see that the angular (fibre) coordinate $\pi - \frac{3}{2}\alpha$ of $A_3^{\phi^{-1}\psi^{-1}}$ in Fig. 7.c is just $\frac{\pi}{2}$ of A_1 if $\alpha = \frac{\pi}{3}$, i.e. we get degenerate case, But, for $\frac{\pi}{3} > \alpha > 0$, $A_3^{\phi^{-1}\psi^{-1}}(\cos(\pi - \frac{3}{2}\alpha), \sin(\pi - \frac{3}{2}\alpha), 0, 0)$ moves on line A_0A_1 over A_1 up to $(-1, 0, 0, 0)$ in the projective sphere, i.e. up to the opposite point of A_0 (see Fig. 8 for imagination in the sense of universal cover). The same holds for $A_1^{\psi^{-1}\phi^{-1}}$ on the line A_2A_3 over A_3 up to the opposite of A_2. Thus the coordinates of P in (5.7) remain valid together with coordinates of "poles" N and S in Lemma 5.7, moreover with the plane coordinates there. The hyperboloid has the equation by (5.5) as follows for points $(\eta^i \mathbf{a}_i)$

$$\begin{aligned} 0 = \eta^i a_{ij} \eta^j = \frac{1 - \cos\alpha}{\frac{1}{2} - \cos\alpha} \Bigg[\left(\eta^0 + \frac{1}{2\sin\frac{\alpha}{2}} \eta^2 \right)^2 + \\ + \left(\eta^1 + \frac{1}{2\sin\frac{\alpha}{2}} \eta^3 \right)^2 + \frac{\frac{1}{2} - \cos\alpha}{1 - \cos\alpha} \left(\eta^2\eta^2 + \eta^3\eta^3 \right) \Bigg]. \end{aligned} \tag{5.8}$$

E.g. we get negative values for all $\langle \mathbf{a}_i, \mathbf{a}_i \rangle$ $(i = 0, 1, 2, 3)$ and for all vertices and poles N and S of P.

To visualize our model P in projective-spherically extended Euclidean space $\mathbb{E}^3$ (in the sense of Fig. 8, but imagine it also in the "back side" Fig. 8.b), we can introduce new coordinate simplex by (5.8) as follows:

$$(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}) = (\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3) \begin{pmatrix} \frac{\sin\frac{\alpha}{2}}{\sqrt{\frac{1}{4} - \sin^2\frac{\alpha}{2}}} & 0 & 0 & 0 \\ 0 & \frac{\sin\frac{\alpha}{2}}{\sqrt{\frac{1}{4} - \sin^2\frac{\alpha}{2}}} & 0 & 0 \\ \frac{1}{2\sqrt{\frac{1}{4} - \sin^2\frac{\alpha}{2}}} & 0 & 1 & 0 \\ 0 & \frac{1}{2\sqrt{\frac{1}{4} - \sin^2\frac{\alpha}{2}}} & 0 & 1 \end{pmatrix}$$

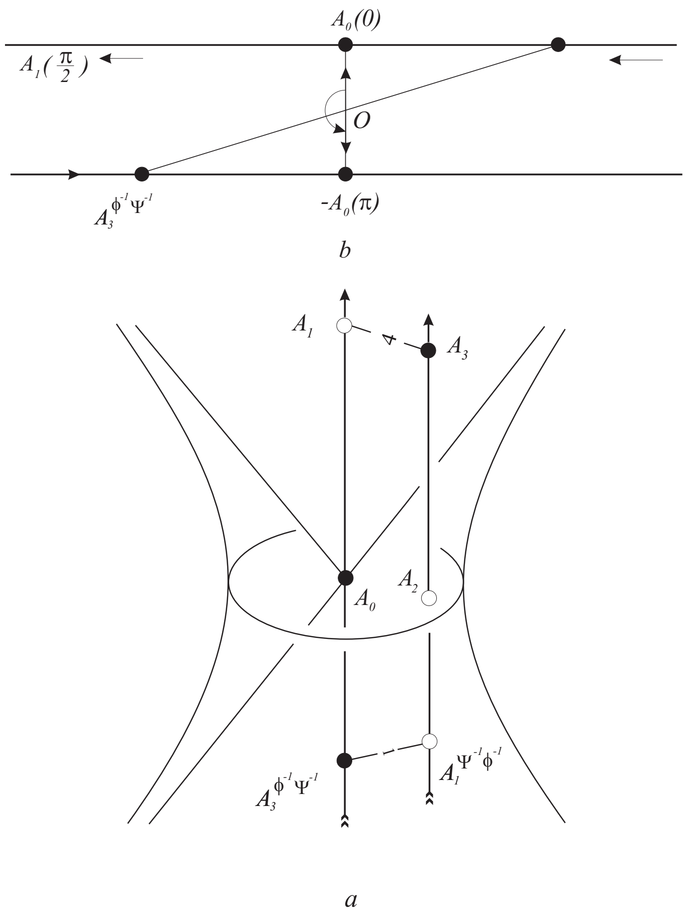

FIGURE 8. Hyperboloid model for $\widetilde{SL_2(\mathbb{R})}$ for $0 < \alpha < \frac{\pi}{3}$

for new basis forms and, by the inverse, we get the new simplex vertices:

$$\begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\frac{1}{4}-\sin^2\frac{\alpha}{2}}}{\sin\frac{\alpha}{2}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\frac{1}{4}-\sin^2\frac{\alpha}{2}}}{\sin\frac{\alpha}{2}} & 0 & 0 \\ -\frac{1}{2\sin\frac{\alpha}{2}} & 0 & 1 & 0 \\ 0 & -\frac{1}{2\sin\frac{\alpha}{2}} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}.$$

By conjugacy we could translate the formulas in (5.4) and (5.7) to get pictures by Fig. 8. However, our original coordinate simplex is more informative in the sense of projective-spherical geometry [12] for $\widetilde{SL_2(\mathbb{R})}$ with transformation matrices given in Remark 5.3

Thus, we have a similar picture as in Fig. 3, now with 2 regular triangles of angle α, a 2_1-screw axis between the triangles and 3_1-screw axes in the middle of 2 regular triangles, however, these are in the hyperbolic base plane $\mathbb{H}^2$ of the fibre space $\widetilde{SL_2(\mathbb{R})}$. The construction of P is also possible by this analogy, but the screw axes cause difficulties.

We have extreme problem in visualizing the cases $\frac{5\pi}{3} < \alpha < 2\pi$. Our formulas by angular (fibre) coordinates (Fig. 8) may help us. For $\alpha = \frac{5\pi}{3}$ we would have $A_3^{\phi^{-1}\psi^{-1}}(\pi - \frac{5\pi}{2}) \sim A_1(-\frac{3\pi}{2} \equiv \frac{\pi}{2})$ as a limit from the $\mathbb{S}^3$–cases. After this degenerate case $\alpha = \frac{5\pi}{3}$, $A_3^{\phi^{-1}\psi^{-1}}(\pi - \frac{3\pi}{2})$ moves further up to $A_0(\pi - 3\pi \equiv 0)$. Thus *an extrapolation* in the universal cover $\widetilde{SL_2(\mathbb{R})}$, just by formulas (5.7) and Lemma 5.7 [12] *is possible.* Then the length of the complete singular line is $l = 3\alpha - \pi$, i.e. it varies in the open interval $(4\pi, 5\pi)$. The fundamental polyhedron P will vary also by α in $\widetilde{SL_2(\mathbb{R})}$.

The topic is related to the tetrahedron tilings (e.g. in [14], Section 4) initiated by I.K. Zhuk [22] from other aspect. Namely, a Zhuk's tetrahedron fibre orbifold is two-fold covered by $\mathcal{C}_{3/1}(\alpha)$, where $\alpha = \frac{2\pi}{k},\ 3 \le k \in \mathbb{N}$.

5.4. Nil structure and other discussions

For the case of signature $(0\ 0\ \ +\,+)$ from Lemma 5.2, arising for $\alpha = \frac{\pi}{3}$ and $\alpha = \frac{5\pi}{3}$, we change our model. As mentioned at formulas (2.6) and (2.7), in Figures 2, 3 and at Subsection **2.2**, we combine subcases **1.1** for $F = \begin{pmatrix} f_2^0 & f_2^1 \\ f_3^1 & f_3^1 \end{pmatrix}$ in (2.6) and **5.1** for $G = \begin{pmatrix} \sin\beta & 1-\cos\beta \\ 0 & 0 \end{pmatrix} = 2\sin\frac{\beta}{2}\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ 0 & 0 \end{pmatrix}$ in (2.7). The equation $\phi\psi\phi = \psi\phi\psi$ means, first in general form

$$\begin{pmatrix} \mathbf{1}_2 + GF & GM_\alpha \\ (\mathbf{1}_2 + FG + M_\alpha M_\beta)F & (FG + M_\alpha M_\beta)M_\alpha \end{pmatrix} = \\ = \begin{pmatrix} \mathbf{1}_2 + GF & G(\mathbf{1}_2 + FG + M_\alpha M_\beta) \\ M_\beta F & M_\beta(FG + M_\alpha M_\beta) \end{pmatrix}; \tag{5.9}$$

i. e. $FG + M_\alpha M_\beta = e\begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ \mp\sin\varepsilon & \pm\cos\varepsilon \end{pmatrix}$ follows first, then $FG = 0$ and $f_2^0 = 0 = f_3^0$. Furthermore, $\beta = \alpha \pmod{2\pi}$ holds from the equation $(5.9_{[22]})$. Moreover, we get $\mathbf{0}_2 = G(\mathbf{1}_2 - M_\alpha + M_\alpha M_\alpha)$ and $\mathbf{0}_2 = (\mathbf{1}_2 - M_\alpha + M_\alpha M_\alpha)F$ from equations $(5.9_{[02]})$and $(5.9_{[20]})$, respectively. From $(5.9_{[02]})$ follows $\alpha = \frac{\pi}{3} \pmod{2\pi}$, then $(5.9_{[20]})$ also holds.

The matrix presentation $\mathcal{T}$ of $\tau = (\phi\psi\phi)^2 = (\psi\phi\psi)^2 = (\phi\psi)^3 = (\psi\phi)^3$ can be computed step by step:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & f_2^1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & f_3^1 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & f_3^1 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Phi\Psi = \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & f_2^1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & f_3^1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \Psi\Phi = \begin{pmatrix} 1 & \frac{\sqrt{3}}{2}f_2^1 + \frac{1}{2}f_3^1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2}f_2^1 + \frac{\sqrt{3}}{2}f_3^1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2}f_2^1 + \frac{1}{2}f_3^1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

i.e. (see Fig. 3) $A_0(1,0,0,0)$, $A_{03}(1,0,0,1)$, $A_{03}^{\phi}(1,f_3^1,-\frac{\sqrt{3}}{2},\frac{1}{2})$, $A_0^{\psi}(1,0,\frac{\sqrt{3}}{2},\frac{1}{2})$, $A_{03}^{\phi\psi}(1,f_3^1,0,0)$, $A_0^{\psi\phi}(1,\frac{\sqrt{3}}{2}f_2^1+\frac{1}{2}f_3^1,0,1)$,

$$\Phi\Psi\Phi=\Psi\Phi\Psi=\begin{pmatrix}1 & \frac{\sqrt{3}}{2}f_2^1+\frac{1}{2}f_3^1 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & \frac{1}{2}f_2^1+\frac{\sqrt{3}}{2}f_3^1 & -1 & 0\\ 0 & -\frac{\sqrt{3}}{2}f_2^1+\frac{1}{2}f_3^1 & 0 & -1\end{pmatrix}, \tag{5.10}$$

$$\mathcal{T}=\begin{pmatrix}1 & \frac{\sqrt{3}}{2}f_2^1+\frac{3}{2}f_3^1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.11}$$

5.4.1. Discussion on the center. This center generator, the translation τ above, is trivial if and only if $f_2^1=-\sqrt{3}f_3^1$.

Then we have Euclidean plane triangle group $T(2,3,6)$ acting in plane $A_0A_2A_3$ by projecting the action of ϕ and ψ onto this plane.

If $(f_2^1,f_3^1)\neq(0,0)$ then $f_3^1=\frac{1}{2}$ can be chosen by our projective freedom. Then we get

$$\Phi=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}, \qquad \Psi=\begin{pmatrix}1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}, \tag{5.12}$$

$$\Phi\Psi=\begin{pmatrix}1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ 0 & 1 & 0 & 0\\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}, \qquad \Psi\Phi=\begin{pmatrix}1 & -\frac{1}{2} & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix},$$

$$\Phi\Psi\Phi=\Psi\Phi\Psi=\begin{pmatrix}1 & -\frac{1}{2} & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}.$$

Thus, by modification of Fig. 3 we can compute the possible vertices of polyhedron P.

$$A_0(1,0,0,0),\ A_{03}(1,0,0,1),\ A_{03}^{\phi}\left(1,\frac{1}{2},-\frac{\sqrt{3}}{2},\frac{1}{2}\right),$$
$$A_0^{\psi}\left(1,0,\frac{\sqrt{3}}{2},\frac{1}{2}\right),\ A_{03}^{\phi\psi}\left(1,\frac{1}{2},0,0\right),\ A_0^{\psi\phi}\left(1,-\frac{1}{2},0,1\right).$$

Namely, we see that $A_0^{\psi\phi}$ is "under" A_{03}, $A_{03}^{\phi\psi}$ lies "over" A_0, by the second (A_1) coordinate. We can check that $\phi\psi\phi=\psi\phi\psi$ is an additional halfturn with axis pair $(x^0,x^1,0,x^3=\frac{1}{2}x^0)$ and $(0,y^1,y^2,y^3=-2y^1)$. An extra 3-rotation $\phi\psi$ occurs with point axis $(x^0=2\sqrt{3}x^2,\ x^1,x^2,x^3=\sqrt{3}x^2)$, $\psi\phi$ is also a 3-rotation with point axis

$(y^0 = -2\sqrt{3}y^2, y^1, y^2, y^3 = -\sqrt{3}y^2)$. Now the horospheres of center $A_1(0,1,0,0)$ will be invariant, and the invariant polarity by (3.8) will be hyperbolic, as follows:

$$(b^{ij}) = \begin{pmatrix} 0 & -b & 0 & 0 \\ -b & b^{11} & 0 & -b \\ 0 & 0 & b & 0 \\ 0 & -b & 0 & b \end{pmatrix}, \quad \text{i.e.} \quad (b^{ij}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \bar{b}^{11} & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{5.13}$$

in case $b > 0$ by projective freedom. Introducing $b^{11}/b = \bar{b}^{11}$, the invariant quadratic form for planes $\boldsymbol{b}^i\xi_i$ will be

$$\xi_i b^{ij}\xi_j = (\xi_3 - \xi_1)^2 + \xi_2\xi_2 + (\bar{b}^{11} - 1)\left(\xi_1 - \frac{1}{\bar{b}^{11} - 1}\xi_0\right)^2 - \frac{1}{\bar{b}^{11} - 1}\xi_0\xi_0$$

if $\bar{b}^{11} - 1 \neq 0$, or else if $\bar{b}^{11} = 1$ then

$$\xi_i b^{ij}\xi_j = (\xi_3 - \xi_1)^2 + \xi_2\xi_2 + \frac{1}{2}(\xi_1 - \xi_0)^2 - \frac{1}{2}(\xi_1 + \xi_0)^2.$$

The signature is $(- + ++)$ in both cases, i.e. it is hyperbolic (Table 1). Its inverse $(b^{ij})^{-1} = a_{ij}$ describes the absolute point quadric:

$$(a_{ij}) = \begin{pmatrix} 1 - \bar{b}^{11} & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

$a_{11} = 0$ means that A_1 lies on the absolute point quadric of $\mathbb{H}^3$. Then either

$$\eta^i a_{ij}\eta^j = (\eta^3 - \eta^0)^2 + \eta^2\eta^2 - \bar{b}^{11}\left(\eta^0 + \frac{1}{\bar{b}^{11}}\eta^1\right)^2 + \frac{1}{\bar{b}^{11}}\eta^1\eta^1$$

if $\bar{b}^{11} \neq 0$, or

$$\eta^i a_{ij}\eta^j = (\eta^3 - \eta^0)^2 + \eta^2\eta^2 + \frac{1}{2}(\eta^0 - \eta^1)^2 - \frac{1}{2}(\eta^0 + \eta^1)^2$$

if $\bar{b}^{11} = 0$, will be an adequate absolute point quadric for $\mathbb{H}^3$, respectively. Our group generated by ϕ and ψ in (5.12) acts as a Euclidean plane triangle group $T(2,3,6)$ generated by rotations on π, $2\pi/3$, and $\pi/3$ at vertices of the triangle with angles $\pi/2$, $\pi/3$, and $\pi/6$, on the horospheres centred in A_1 at the absolute, We know that the intrinsic geometry on a horosphere is Euclidean.

We do not get adequate compact space, however we see interesting phenomena.

If $b = 0$ in (5.13) then $b^{11} > 0$ is still possible for Nil-realization. We get again the Euclidean plane triangle group $T(2,3,6)$ with non-compact fundamental domain in Nil, non-adequate for us now.

If $(f_2^1, f_3^1) = (0,0)$ in (5.11), i.e.

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \Psi = \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Phi\Psi = \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \Psi\Phi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\Phi\Psi\Phi = \Psi\Phi\Psi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

then we get

$$A_0(1,0,0,0),\ A_{03}(1,0,0,1),\ A_{03}^{\phi}\left(1,0,-\frac{\sqrt{3}}{2},\frac{1}{2}\right),$$
$$A_0^{\psi}\left(1,0,\frac{\sqrt{3}}{2},\frac{1}{2}\right),\ A_{03}^{\phi\psi}(1,0,0,0) = A_0,\ A_0^{\psi\phi}(1,0,0,1) = A_{03}.$$

The skech in Fig. 3 is degenerate, non-adequate for us, although (3.3) and (3.8) with

$$(b^{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b^{11} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

provides invariant (under ϕ, ψ) polarities e.g. Euclidean geometry as well.

5.4.2. The proper Nil-geometric realization. This occurs in formula (5.11) if the centre generated by $\tau = (\phi\psi)^3 = (\psi\phi)^3$ is not trivial, i.e. $f_2^1 \neq -\sqrt{3} f_3^1$.

As we have already seen, the projection of group action by $\langle \phi, \psi \rangle$ on the plane $A_0 A_2 A_3 = b^1$ is just the Euclidean plane triangle group $T(2,3,6)$. Now ϕ, ψ are 6-rotations (Figures 2, 3). By (5.10) and (5.11) we get

$$\begin{aligned} \left(1,0,0,\frac{1}{2}\right) \phi\psi\phi &= \left(1, \frac{\sqrt{3}}{4} f_2^1 + \frac{3}{4} f_3^1, 0, \frac{1}{2}\right), \\ \left(1,0,0,\frac{1}{2}\right) \tau &= \left(1, \frac{\sqrt{3}}{2} f_2^1 + \frac{3}{2} f_3^1, 0, \frac{1}{2}\right). \end{aligned}$$

Therefore $\phi\psi\phi = \psi\phi\psi$ is a 2_1 screw motion, i.e. 2-rotation about axis through A_1 and $(1,0,0,\frac{1}{2})$, composed by a half translational part of τ, according to Figures 2–3 indeed. Similarly,

$$\begin{aligned} \left(1,0,\frac{\sqrt{3}}{6},\frac{1}{2}\right) \phi\psi &= \left(1, \frac{\sqrt{3}}{6} f_2^1 + \frac{1}{2} f_3^1, \frac{\sqrt{3}}{6}, \frac{1}{2}\right), \\ \left(1,0,\frac{\sqrt{3}}{6},\frac{1}{2}\right) \tau &= \left(1, \frac{\sqrt{3}}{2} f_2^1 + \frac{3}{2} f_3^1, \frac{\sqrt{3}}{6}, \frac{1}{2}\right) \end{aligned}$$

shows $\phi\psi$ as a 3_1 screw motion, i.e. a 3-rotation about axis through A_1 and $(1,0,\frac{\sqrt{3}}{6},\frac{1}{2})$, composed by the third part of τ. The transform $\psi\phi$ is also a 3_1 srew motion with screw axis through A_1 and $(1,0,-\frac{\sqrt{3}}{6},\frac{1}{2})$, see Figures 2, 3 again. For the fundamental domain in Fig. 3 we choose e.g. $(f_2^1, f_3^1) = (0, 2)$. Then we have the vertices

$$A_0(1,0,0,0),\ A_{03}(1,0,0,1),\ A_{03}^{\phi}\left(1,2,-\frac{\sqrt{3}}{2},\frac{1}{2}\right),$$
$$A_0^{\psi}\left(1,0,\frac{\sqrt{3}}{2},\frac{1}{2}\right),\ A_{03}^{\phi\psi}(1,2,0,0),\ A_0^{\psi\phi}(1,1,0,1).$$

Let $U(1,1,\frac{\sqrt{3}}{6},\frac{1}{2})$ be chosen, then $U^{\phi}(1,2,-\frac{\sqrt{3}}{6},\frac{1}{2})$; let $V(1,\frac{3}{2},-\frac{\sqrt{3}}{6},\frac{1}{2})$ be placed under U^{ϕ}, then $V^{\psi}(1,\frac{3}{2},\frac{\sqrt{3}}{6},\frac{1}{2})$ will be over U. The screw component of $\phi\psi$ is $\frac{1}{2}+\frac{1}{2}=1$, indeed, as desired to an adequate fundamental domain P, bounded by triangulated star-shape faces. This construction was indicated to Fig. 3.

The invariant polarity or quadratic form by $B = (b^{ij})$ leads to $b^{11} \neq 0$ but zeros at other entries by (3.8), according to our *Nil* interpretation (see [12] and Table 1).

The case $\alpha = \frac{5\pi}{3}$ can be treated formally in "analogous way" (see [13] and [20] for a non-trivial model of Nil-geometry, not detailed here). Formally, the only change is that $\cos\frac{5\pi}{3} = \frac{1}{2},\ \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

Thus we have completely discussed $\mathcal{C}_{3/1}(\alpha)$.

Theorem 5.9. *There are geometric structures on $\mathcal{C}_{3/1}(\alpha)$ with a cone angle $0 < \alpha < 2\pi$. Namely, for $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$ there exists spherical geometry. For $\alpha = \frac{\pi}{3}$ we get Nil orbifold with a free distance parameter. For $0 < \alpha < \frac{\pi}{3}$ there exists $\widetilde{SL_2(\mathbb{R})}$-geometry. For $\alpha = \frac{5\pi}{3}$ we can introduce Nil-geometry (not detailed here) and for $\frac{5\pi}{3} < \alpha < 2\pi$ we get $\widetilde{SL_2(\mathbb{R})}$ geometry in a formal way by our projective-spherical model.*

This result implies the following geometric structures on orbifolds.

Corollary 5.10. *Orbifold $\mathcal{C}_{3/1}(\alpha)$ with $\alpha = \frac{2\pi}{n}$ is spherical for $n = 2,3,4,5$; Nil-orbifold for $n = 6$, and $\widetilde{SL_2(\mathbb{R})}$-orbifold for $n \geqslant 7$.*

By Lemma 5.6 in cases of spherical orbifolds we have the following angle-lengths pairs: $(\alpha,\ell) = (\pi, 2\pi)$; $(\alpha,\ell) = (\frac{2\pi}{3},\pi)$; $(\alpha,\ell) = (\frac{2\pi}{4},\frac{\pi}{2})$; $(\alpha,\ell) = (\frac{2\pi}{5},\frac{\pi}{5})$ for essential angle α and length ℓ of the singular set.

6. The double link $K(4,1)$

The double link $K(4,1)$ is presented in Fig. 9. By Minkus construction polyhedron $P_{4/1}$ looks at Fig. 10.

The double link $K(4,1)$ is presented in Fig. 9 and the corresponding Minkus polyhedron is described in Fig. 10. Applying Wirtinger algorithm to the link diagram in Fig. 9, it easy to see that $\pi_1(S^3 \setminus K(4,1))$ can be presented with four generators x, y, z, w, and four relations as follows:

$$y^{-1}w^{-1}yx = 1, \quad x^{-1}y^{-1}xz = 1, \quad z^{-1}x^{-1}zw = 1, \quad w^{-1}z^{-1}wy = 1$$

from which $w = yxy^{-1}$, $z = x^{-1}yx$ can be expressed. Thus, relations

$$(x^{-1}y^{-1}x)\,x^{-1}\,(x^{-1}yx)\,(yxy^{-1}) = 1 = (yx^{-1}y^{-1})\,(x^{-1}y^{-1}x)\,(yxy^{-1})\,y$$

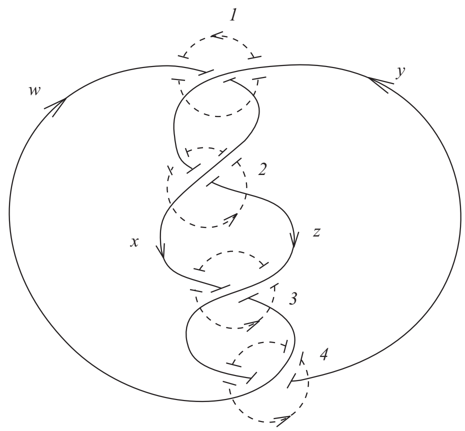

FIGURE 9. The double link $K(4,1)$

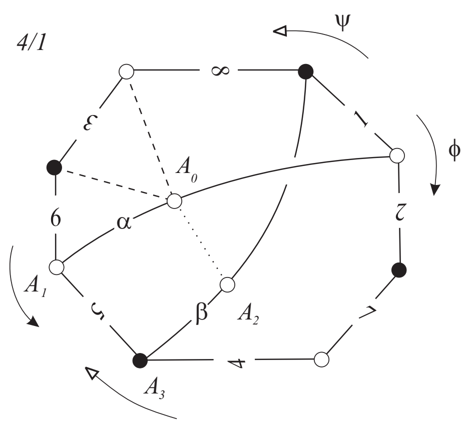

FIGURE 10. The polyhedron $P_{4/1}$

hold. Finally, we get one relation for the generators x and y: $xyxy = yxyx$, expressing the center generator in two forms.

This presentation for $\pi_1(S^3 \setminus K(4,1))$ is equivalent to that of Minkus polyhedron $P_{4/1}$ by identifying generators ϕ and ψ and defining relation

$$\phi\psi\phi\psi = \psi\phi\psi\phi. \tag{6.1}$$

Many steps of studying geometric structures on $\mathcal{C}_{4/1}(\alpha,\beta)$ are analogous to the above considered case $\mathcal{C}_{3/1}(\alpha)$. So, we will sketch details, emphasizing the differences only.

6.1. The case of skew axes

The projective-spherical interpretation of ϕ and ψ in the case of skew axes has been introduced by (2.1) and (2.2). The relation (6.1) provides the block equations:

$$M_\beta G + GFG + GM_\alpha = 0, \quad FM_\beta + FGF + M_\alpha F = 0 \tag{6.2}$$

and the consequences

$$FGM_\alpha - M_\alpha FG = 0, \quad M_\beta GF - GFM_\beta = 0. \tag{6.3}$$

6.1.1. Cases $\alpha \neq \pi \neq \beta\,(\mathbf{mod}\,2\pi)$. Then say, FG shall have the form

$$FG = c \cdot \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} = c \cdot M_\gamma \tag{6.4}$$

for some γ and $0 \leqslant c \in \mathbb{R}$. Hence the second equation of (6.2) for components of F by (2.1) has a matrix form

$$(f_2^0, f_2^1; f_3^0, f_3^1)\cdot$$
$$\cdot \begin{pmatrix} D & \sin\beta & -c\cdot\sin\gamma - \sin\alpha & 0 \\ -\sin\beta & D & 0 & -c\cdot\sin\gamma-\sin\alpha \\ c\cdot\sin\gamma+\sin\alpha & 0 & D & \sin\beta \\ 0 & c\cdot\sin\gamma+\sin\alpha & -\sin\beta & D \end{pmatrix} = \tag{6.5}$$
$$= (0,0;0,0).$$

with $D = \cos\alpha + \cos\beta + c\cdot\cos\gamma$.

For non-trivial solution of (6.5) we have either

$$\cos\alpha + \cos\beta + c\cdot\cos\gamma = 0, \quad \sin\alpha + \sin\beta + c\cdot\sin\gamma = 0, \tag{6.6}$$

or (just to opposite orientation $\beta \mapsto -\beta$)

$$\cos\alpha + \cos\beta + c\cdot\cos\gamma = 0, \quad \sin\alpha - \sin\beta + c\cdot\sin\gamma = 0, \tag{6.7}$$

to be fulfilled by the argument of zero determinant in (6.5). Thus, step-by-step follow from (6.6) and (6.5)

$$F = f\cdot M_\mu, \qquad G = g\cdot M_\nu, \tag{6.8}$$

$f, g > 0,\ f\cdot g = c = \pm 2\cos\frac{\alpha-\beta}{2},\ \gamma = \mu+\nu\,(\mathrm{mod}\,2\pi)$. In the case of (+) we have $\frac{\alpha+\beta}{2} = \gamma + \pi\,(\mathrm{mod}\,2\pi)$, and in the case of (-) we have $\frac{\alpha+\beta}{2} = \gamma\,(\mathrm{mod}\,2\pi)$.

Equations (6.7) and (6.5) lead analogously to

$$F = f\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot M_\mu, \qquad G = g\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot M_\mu, \tag{6.9}$$

$f, g > 0,\ f\cdot g = c = \pm 2\cos\frac{\alpha+\beta}{2},\ \gamma = \nu-\mu\,(\mathrm{mod}\,2\pi)$. In the case of (+) we have $\frac{\alpha-\beta}{2} = \gamma+\pi\,(\mathrm{mod}\,2\pi)$, and in the case of (-) we have $\frac{\alpha-\beta}{2} = \gamma\,(\mathrm{mod}\,2\pi)$, where only the second case $\frac{\alpha-\beta}{2} = \gamma = \nu-\mu\,(\mathrm{mod}\,2\pi)$ leads to solution of the opposite orientation $\beta \mapsto -\beta$.

6.1.2. Case of formulas (6.5) and (6.6). In (6.8) we consider the (+) case, first $c = 2\cos\frac{\alpha-\beta}{2} > 0$, and compute the matrix, presenting the generator of the center:

$$\Phi\Psi\Phi\Psi = \Psi\Phi\Psi\Phi = \begin{pmatrix} M_{\alpha+\beta-\pi} & \mathbf{0}_2 \\ \mathbf{0}_2 & M_{\alpha+\beta-\pi} \end{pmatrix}. \tag{6.10}$$

This is independent from F and G, moreover, the center is trivial if $\alpha+\beta = \pi \,(\mathrm{mod}\, 2\pi)$. We shall have a freedom in (6.13), later, to have economical generators ϕ and ψ in (2.1) and (2.2), and a nice fundamental polyhedron $P_{4/1}$ in Fig. 10. We may choose

$$F = \sqrt{2\cos\frac{\alpha-\beta}{2}} \cdot M_{(\frac{\alpha}{2}-\frac{\pi}{2})}, \qquad G = \sqrt{2\cos\frac{\alpha-\beta}{2}} \cdot M_{(\frac{\beta}{2}-\frac{\pi}{2})}. \tag{6.11}$$

Thus we concretize polyhedron $P_{4/1}$ as well, with its north pole N on axis of ϕ and south pole S on axis of ψ, as follows in (6.12).

Lemma 6.1 **($c = 2\cos\frac{\alpha-\beta}{2} > 0$).** *The vertices and poles N and S of fundamental polyhedron $P_{4/1}$ can be chosen as follows:*

$$A_1(0,1,0,0),\ A_1^{\psi^{-1}}\left(\sin\beta, \cos\beta, -\sqrt{2\cos\frac{\alpha-\beta}{2}}\cos\frac{\beta}{2}, \sqrt{2\cos\frac{\alpha-\beta}{2}}\sin\frac{\beta}{2}\right),$$
$$A_1^{\psi^{-1}\phi^{-1}}\left(-\sin\alpha, -\cos\alpha, -\sqrt{2\cos\frac{\alpha-\beta}{2}}\cos(\alpha+\frac{\beta}{2}), \sqrt{2\cos\frac{\alpha-\beta}{2}}\sin(\alpha+\frac{\beta}{2})\right),$$
$$A_1^{\psi^{-1}\phi^{-1}\psi^{-1}}(-\sin(\alpha+\beta), -\cos(\alpha+\beta), 0, 0),\ A_3(0,0,0,1),$$
$$A_3^{\phi^{-1}}\left(-\sqrt{2\cos\frac{\alpha-\beta}{2}}\cos\frac{\alpha}{2}, \sqrt{2\cos\frac{\alpha-\beta}{2}}\sin\frac{\alpha}{2}, \sin\alpha, \cos\alpha\right),$$
$$A_3^{\phi^{-1}\psi^{-1}}\left(-\sqrt{2\cos\frac{\alpha-\beta}{2}}\cos(\frac{\alpha}{2}+\beta), \sqrt{2\cos\frac{\alpha-\beta}{2}}\sin(\frac{\alpha}{2}+\beta), -\sin\beta, -\cos\beta\right),$$
$$A_3^{\phi^{-1}\psi^{-1}\phi^{-1}}(0,0,-\sin(\alpha+\beta), -\cos(\alpha+\beta)),$$
$$N\left(\sin\frac{\alpha+\beta-\pi}{2}, \cos\frac{\alpha+\beta-\pi}{2}, 0, 0\right), \quad S\left(0,0,\sin\frac{\alpha+\beta-\pi}{2}, \cos\frac{\alpha+\beta-\pi}{2}\right). \tag{6.12}$$

Now comes the invariant polarity under ϕ and ψ on the base of Section 3. Formulas (3.5) and (3.6) provide

$$B^{02} = \frac{\bar{b}}{2\sin\frac{\beta}{2}} \cdot M_{(-\frac{\pi}{2}-\frac{\beta}{2})} \cdot G = \frac{b}{2\sin\frac{\alpha}{2}} \cdot F^T \cdot M_{(\frac{\alpha}{2}+\frac{\pi}{2})}, \tag{6.13}$$

and (6.11) just implies easily

$$B = \begin{pmatrix} \sin\frac{\beta}{2} & 0 & -\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 \\ 0 & \sin\frac{\beta}{2} & 0 & -\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} \\ -\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 & \sin\frac{\alpha}{2} & 0 \\ 0 & -\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 & \sin\frac{\alpha}{2} \end{pmatrix}. \tag{6.14}$$

The invariant quadratic form for a "variable plane" $\boldsymbol{b}^i\xi_i$ is

$$\xi_i b^{ij}\xi_j = \sin\frac{\alpha}{2}\left(\xi_2 - \frac{1}{\sin\frac{\alpha}{2}}\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}}\,\xi_0\right)^2 + \sin\frac{\beta}{2}\left(\xi_1 - \frac{1}{\sin\frac{\beta}{2}}\sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}}\,\xi_3\right)^2$$
$$-\frac{1}{2\sin\frac{\alpha}{2}}\cos\frac{\alpha+\beta}{2}\xi_0\xi_0 - \frac{1}{2\sin\frac{\beta}{2}}\cos\frac{\alpha+\beta}{2}\xi_3\xi^3, \tag{6.15}$$

for $-\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{\pi}{2} \,(\mathrm{mod}\, 2\pi)$, i.e. (+) case in (6.8).

Hence we look for the signature of b^{ij} at (6.15):

(i) $\mathbb{S}^3(++++)$, $\frac{\pi}{2} < \frac{\alpha+\beta}{2} < \frac{3\pi}{2} \,(\mathrm{mod}\, 2\pi)$, $-\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{\pi}{2}$,

(ii) $\widetilde{SL_2(\mathbb{R})}(--++)$, $-\frac{\pi}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2} \,(\mathrm{mod}\, 2\pi)$, $-\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{\pi}{2}$,

(iii) $(0\,0\,++)$, $\frac{\alpha+\beta}{2} = \pm\frac{\pi}{2}$, i.e. with trivial center, needs extra discussions with intersecting axes of ϕ and ψ, later on.

Thus we can overview the spherical $\mathbb{S}^3$-realizations and $\widetilde{SL_2(\mathbb{R})}$-realizations for any $0 < \alpha, \beta < 2\pi$, $\frac{\alpha+\beta}{2} \neq \pm\frac{\pi}{2} \,(\mathrm{mod}\, 2\pi)$ above in the projective-spherical space $\mathcal{PS}^3(\mathbb{R})$. The vertices and poles of $P_{4/1}$ in Lemma 6.1, and the other data of $P_{4/1}$ need some discussions, of course. We have also freedom in choosing $P_{4/1}$.

*Now consider the (-) case in (*6.8*)*

$$c = -\cos\frac{\alpha-\beta}{2} > 0, \quad \gamma = \mu + \nu = \frac{\alpha+\beta}{2} \,(\mathrm{mod}\, 2\pi) \tag{6.16}$$

The generator of the center will be again as that in (6.10). Now we have to choose for Φ and Ψ, by (2.1) and (2.2), respectively: $F = f \cdot M_\mu$, and $G = g \cdot M_\nu$, with $f, g > 0$, $fg = c$, and angles μ, ν above.

But our requirement for invariant polarity (b^{ij}) in (6.13)

$$B^{02} = \frac{\bar{b}\cdot g}{2\sin\frac{\beta}{2}} M_{(-\frac{\pi}{2}-\frac{\beta}{2}+\nu)} = \frac{b\cdot f}{2\sin\frac{\alpha}{2}} M_{(\frac{\alpha}{2}+\frac{\pi}{2}-\mu)}$$

would imply $-\frac{\pi}{2} - \frac{\beta}{2} + \nu = \frac{\alpha}{2} + \frac{\pi}{2} - \mu \,(\mathrm{mod}\, 2\pi)$, i.e. $0 = \frac{\alpha}{2} + \frac{\beta}{2} + \pi - \mu - \nu = \pi \,(\mathrm{mod}\, 2\pi)$ – a *contradiction* to (6.16).

6.1.3. Case of formulas (6.5**) and (**6.7**).** . In (6.9): $c = -2\cos\frac{\alpha+\beta}{2} > 0$, $\gamma = \nu - \mu = \frac{\alpha-\beta}{2} \,(\mathrm{mod}\, 2\pi)$ leads to solution for opposite orientation $\beta \mapsto -\beta$.

Formally, we get the same solution (i)–(iii) as in case 6.1.1 above. For (iii) we need extra discussion as indicated.

6.1.4. Cases $\alpha = \pi$ or $\beta = \pi$. These cases can be considered as particular ones of (i) and (ii).

6.2. Spherical geometry

For the spherical realization we can start with our fundamental polyhedron $P_{4/1}$ given by Lemma 6.1 with $\alpha = \beta = \pi$ analogously to Fig. 7a. Then we have (see also Fig. 10):

$$N = A_0(1,0,0,0),\ S = A_2(0,0,1,0),\ A_1(0,1,0,0),\ A_1^{\psi^{-1}}(0,-1,0,\sqrt{2}),$$
$$A_1^{\psi^{-1}\phi^{-1}}(0,1,0,-\sqrt{2}),\ A_1^{\psi^{-1}\phi^{-1}\psi^{-1}}(0,-1,0,0),\ A_3(0,0,0,1),$$
$$A_3^{\phi^{-1}}(0,\sqrt{2},0,-1),\ A_3^{\phi^{-1}\psi^{-1}}(0,-\sqrt{2},0,1),\ A_3^{\phi^{-1}\psi^{-1}\phi^{-1}}(0,0,0,-1).$$

The last eight points lie on the line (main circle) A_1A_3 in the intersection of planes ($\boldsymbol{b}^0$), ($\boldsymbol{b}^2$). For their angle β^{02} we have by (6.14) as formerly

$$< \boldsymbol{b}^0, \boldsymbol{b}^2 >= \cos(\pi - \beta^{02}) = -\cos\beta^{02} = -\frac{1}{\sqrt{2}}, \quad \text{i.e. } \beta^{02} = \frac{\pi}{4}.$$

That means, the points in equivalent open segments (Fig. 10) have ball-like neighborhood by our "lens" construction.

Of cause, other α, β in (i) deform our $P_{4/1}$ above (see Fig. 7b) with star-like upper and lower faces formed out as triangulations from the corresponding poles N and S in axes of ϕ and ψ, respectively.

The inverse matrix $(a_{ij}) = (b^{ij^{-1}})$ from (6.14) is

$$(a_{ij}) = \frac{-2}{\cos\frac{\alpha+\beta}{2}} \cdot \begin{pmatrix} \sin\frac{\alpha}{2} & 0 & \sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 \\ 0 & \sin\frac{\alpha}{2} & 0 & \sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} \\ \sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 & \sin\frac{\beta}{2} & 0 \\ 0 & \sqrt{\frac{1}{2}\cos\frac{\alpha-\beta}{2}} & 0 & \sin\frac{\beta}{2} \end{pmatrix}. \quad (6.17)$$

The length of axis of ϕ with angle α is $\ell_\alpha = \widehat{A_1 N A_1^{\psi^{-1}\phi^{-1}\psi^{-1}}} = \alpha + \beta - \pi$ and $\ell_\beta = \widehat{A_3 S A_3^{\phi^{-1}\psi^{-1}\phi^{-1}}}$ is the same for axis length of ψ, both are between 0 and 3π.

Corollary 6.2. *The following orbifolds $\mathcal{C}_{4/1}(\alpha,\beta)$ admit spherical geometry:*
1_2) *$\alpha = \beta = \pi$; $\ell_\alpha = \ell_\beta = \pi$; the group is of order* 8.
1_3) *$\alpha = \pi$, $\beta = \frac{2\pi}{3}$; $\ell_\alpha = \ell_\beta = \frac{2\pi}{3}$; the group is of order* 24.
1_n) *$\alpha = \pi$, $\beta = \frac{2\pi}{n}$; $\ell_\alpha = \ell_\beta = \frac{2\pi}{n}$; the group is of order $8n$, $n \in \mathbb{N}$, $n \geqslant 4$.*
2) $\alpha = \frac{2\pi}{3} = \beta$; $\ell_\alpha = \ell_\beta = \frac{\pi}{3}$; the group is of order 24.
3) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{\pi}{2}$; $\ell_\alpha = \ell_\beta = \frac{\pi}{6}$; the group is of order 48.
4) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{2\pi}{5}$; $\ell_\alpha = \ell_\beta = \frac{\pi}{15}$, the group is of order 120.

6.3. $\widetilde{SL_2(\mathbb{R})}$-geometry

For $\widetilde{SL_2(\mathbb{R})}$ realizations, the signature of the quadratic form in (6.15) is $(--++)$, i.e.

$$-\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2} \pmod{2\pi}. \quad (6.18)$$

We can use the hyperboloid model as in Section 5.2, Fig. 8 with the machinery elaborated in [12]. This is on the base of projective metric induced by bilinear forms (scalar products) of (b^{ij}) in (6.14) and its inverse (a_{ij}) in (6.17).

Formally, we have the fundamental polyhedron $P_{4/1}$ in (6.12) with its triangulation by N and S there in the sense of universal covering space. That means a fiber line (over $\mathbb{H}^2$) through point $X(x^0, x^1, x^2, x^3)$ is defined by

$$(x^0 x^1, x^2, x^3) \cdot \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}$$

where $\phi \in \mathbb{R}$ and

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0,$$

in an appropriate orthogonal basis pairs (b^{ij}) and (a_{ij}) by formulas (6.14) and (6.17), respectively (Fig. 8).

By (6.18) we get the following.

Corollary 6.3. *There is an infinite series of orbifolds $\mathcal{C}_{4/1}(\alpha, \beta)$ admitting $\widetilde{SL_2(\mathbb{R})}$-geometry. The first examples are:*
1) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{2\pi}{7}$; $\ell_\alpha = \ell_\beta = |\frac{2\pi}{3} + \frac{2\pi}{7} - \pi| = \frac{\pi}{21}$;
2) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{2\pi}{8}$; $\ell_\alpha = \ell_\beta = \frac{\pi}{12}$;
3) $\alpha = \frac{2\pi}{4}$, $\beta = \frac{2\pi}{5}$; $\ell_\alpha = \ell_\beta = \frac{\pi}{10}$;
4) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{2\pi}{9}$; $\ell_\alpha = \ell_\beta = \frac{\pi}{9}$.

6.4. Intersecting axes, Nil-geometry

Now we choose ϕ and ψ by formulas (2.6) and (2.7), respectively (see Figures 2 and 3). We write $(\phi\psi)^2 = (\psi\phi)^2$ in block matrix form, first in general. Then we shall follow the arguments shortly as in case $\mathcal{C}_{3/1}$ of Section 5, but for $\mathcal{C}_{4/1}(\alpha, \beta)$ now.

Let us start with

$$(\Phi\Psi)^2 = \begin{pmatrix} \mathbf{1}_2 + GF & G(\mathbf{1}_2 + FG + M_\alpha M_\beta) \\ (\mathbf{1}_2 + FG + M_\alpha + M_\beta)F & FG(\mathbf{1}_2 + FG + 2\cdot M_\alpha M_\beta) + (M_\alpha + M_\beta)^2 \end{pmatrix}$$
$$(\Psi\Phi)^2 = \begin{pmatrix} (\mathbf{1}_2 + GF)^2 + GM_\alpha M_\beta F & (\mathbf{1}_2 + GF)GM_\alpha + GM_\alpha M_\beta M_\alpha \\ M_\beta F(\mathbf{1}_2 + GF) + M_\beta M_\alpha M_\beta F & M_\beta FGM_\alpha + (M_\beta M_\alpha)^2 \end{pmatrix}. \tag{6.19}$$

Step-by-step we get in $(6.19_{[02]})$

$$G(\mathbf{1}_2 + FG + M_\alpha M_\alpha)(\mathbf{1}_2 - M_\alpha) = 0, \quad \text{i.e.,} \quad G(\mathbf{1}_2 + FG + M_\alpha M_\alpha) = 0. \qquad (6.19_{[02]})$$

Then $(6.19_{[22]})$ yields

$$M_\beta(FGM_\alpha) = (FGM_\alpha)M_\beta.$$

(i). Either $\beta = \pi$, then FGM_α is arbitrary yet;

(ii). or $\beta \neq \pi$ and $FGM_\alpha = e \cdot \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ \mp\sin\varepsilon & \pm\cos\varepsilon \end{pmatrix}$, i.e. $FG = \mathbf{0}_2$ and so $f_2^0 = f_3^0 = 0$.

In the **(ii)** case $\alpha+\beta=\pi\,(\mathrm{mod}\,2\pi)$ follows, and we get equality in (6.19). Moreover, the central element in

$$(\Phi\Psi)^2=\begin{pmatrix}1 & \sin\beta\cdot f_2^1+(1-\cos\beta)\cdot f_3^1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\tau. \tag{6.20}$$

6.4.1. Trivial center. The center of $\pi_1(S^3\setminus K(4,1))$ is trivial if

$$2\sin\frac{\beta}{2}\left(\cos\frac{\beta}{2}\cdot f_2^1+\sin\frac{\beta}{2}\cdot f_3^1\right)=0,$$

i.e., $f_2^1/f_3^1=-\tan\frac{\beta}{2}$, or $f_2^1=f_3^1=0$.

In the same way as in Section 5, this trivial center does not lead to compact cone-manifold on $\mathcal{C}_{4/1}$, although phenomena with arising hyperbolic, *Nil* and Euclidean structures may occur.

6.4.2. Non-trivial center for proper Nil geometric realizations. Let us consider the non-trivial central element τ in (6.20) iff

$$f_2^1/f_3^1\neq\tan\frac{\beta}{2}=-\cot\frac{\alpha}{2}, \tag{6.21}$$

since $\alpha+\beta=\pi$ (mod2π). The projection of group action, generated by ϕ and ψ, is an Euclidean plane action on $A_0A_1A_3$ in Fig. 2. A picture, analogous to Fig. 3 would be difficult to draw. We have two orbifold candidates:

Corollary 6.4. *The following orbifolds $\mathcal{C}_{4/1}(\alpha,\beta)$ admit* Nil *geometry:*

1) $\alpha=\frac{\pi}{2}$, $\beta=\frac{\pi}{2}$, corresponding to Euclidean triangle group $T(2,4,4)$; a denotation for the orbifold group can be 2_144.

2) $\alpha=\frac{2\pi}{3}$, $\beta=\frac{\pi}{3}$, corresponding to Euclidean triangle group $T(2,3,6)$; denotation of the orbifold group can be 2_136.

(See also [20] for computations in *Nil* geometry, e.g. for geometric distances, etc.)

For a more detailed description of $P_{4/1}$ now, we have to follow strategy of Fig. 10 and of Lemma 6.1. But now, with Fig. 2 by intersecting axes, we choose $f_2^1=0$, $f_3^1=1$, with a freedom, and compute:

$$\begin{gathered}A_0(1,0,0,0),\ A_0^{\psi}(1,0,\sin\beta,1-\cos\beta),\ A_0^{\psi\phi\psi}(1,1-\cos\beta,0,0),\\ A_0^{\psi\phi}(1,1-\cos\beta,-\sin\alpha,1-\cos\alpha),\ A_{03}^{\phi}(1,1,-\sin\alpha,\cos\alpha),\\ A_{03}(1,0,0,1),\ A_{03}^{\phi\psi}(1,1,\sin\beta,-\cos\beta),\ A_{03}^{\phi\psi\phi}(1,1-\cos\beta,0,1).\end{gathered} \tag{6.22}$$

As we see at (6.20) for $f_2^1=0$, $f_3^1=1$, the translation τ has $(1-\cos\beta)$–component in "direction" A_1 (Fig. 2). The "midpoint" of axis $A_0A_0^{\psi\phi\psi}$ is $N(1,\frac{1}{2}(1-\cos\beta),0,0)$ as a former north pole, the analogous south pole is $S(1,\frac{1}{2}(1-\cos\beta),0,1)$ on $A_{03}A_{03}^{\phi\psi\phi}$. Then come adequate star-like triangulation from N and S, analogous to that of Fig. 3.

6.4.3. The case $\beta = \pi$ (in above (i)). In this case with arbitrary FGM_α cannot satisfy $G(\mathbf{1}_2 + FG + M_\alpha M_\beta) = 0$ in $(6.19_{[02]})$ with $\alpha \neq 0 \,(\mathrm{mod}\, 2\pi)$, as one can check. Thus, the equality in (6.17) would not hold.

Summarizing discussions for $P_{4/1}(\alpha, \beta)$ and $\mathcal{C}_{4/1}(\alpha, \beta)$ we conclude:

Theorem 6.5. *For $\alpha, \beta \in (0, 2\pi)$ there are the following geometric structures on $\mathcal{C}_{4/1}(\alpha, \beta)$:*
-spherical geometry if $-\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{\pi}{2}$ and $\frac{\pi}{2} < \frac{\alpha+\beta}{2} < \frac{3\pi}{2} \,(mod\, 2\pi)$;
-Nil *geometry if $\frac{\alpha+\beta}{2} = \pm\frac{\pi}{2} \,(mod\, 2\pi)$ (with a free distance parameter of both axis lengths);*
-$\widetilde{SL_2(\mathbb{R})}$ *geometry if $-\frac{\pi}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2}$.*

This implies formerly mentioned geometric structures on orbifolds, described above in Corollaries 6.2, 6.3, 6.4.

References

[1] Boileau, M., Leeb, B., Porti, J.: Geomtrization of 3-dimentional orbifolds. Ann. of Math. **162**, 195-290 (2005)

[2] Burde, G., Zieschang, H.: Knots. de Gruyter Studies in Mathematics, vol. 5, Berlin–New York (1985)

[3] Derevnin, D., Mednykh, A., Mulazzani, M.: Geometry of trefoil cone-manifolds. Preprint (2007)

[4] Dunbar, W. D.: Geometric orbifolds. Rev. Mat. Univ. Complut. Madrid **1**, 67-99 (1988)

[5] Hilden, H. M., Montesinos, J. M., Tejada, D. M., Toro, M. M.: Knots, butterflies and 3-manifolds. Disertaciones Matematicas del Seminario de Matematicas Fundamentales vol. 33, 1-22 (2004)

[6] Kerckhoff, S. P.: Deformations of hyperbolic cone manifolds. In: S. Kojima et al, (eds). Proc. 37th Taniguichi Sympos. on Topology and Teichmüller spaces, Finland, July 1995, 101-114. World Sci. Publ. Co., Singapore (1996)

[7] Kolpakov, A., Mednykh, A.: Spherical structures on torus knots and links. Siberian Math. J. **50**, 856-866 (2009)

[8] Mednykh, A., Rasskazov. A.: On the structure of the canonical fundamental set for the 2-bridge link orbifolds. Universitaet Bielefeld, SFB 343 "Discreete Structuren in der Matematik", preprint, 98-062 (1998)

[9] Mednykh, A., Rasskazov. A.: Volumes and degeneration of cone-structures on the figure-eight knot. Tokyo J. Math. **29**, 445-464 (2006)

[10] Minkus, J.: The branched cyclic coverings of 2-bridge knots and links. Mem. Amer. Math. Soc., vol. 35, 255 pp. (1982)

[11] Molnár, E.: Projective metrics and hyperbolic volume. Annales Univ. Sci. Budapest. Sect. Math. **32**, 127-157 (1990)

[12] Molnár, E.: The projective interpretation of the eight 3-dimensional homogeneous geometries. Beiträge zur Algebra und Geometrie **38**, 261-288 (1997)

[13] Molnár, E.: Combinatorial construction of tilings by barycentric simplex orbits (D-symbols) and their realizations in Euclidean and other homogeneous spaces. Acta Crystallographica **A61**, 542-552 (2005)

[14] Molnár, E., Prok, I., Szirmai, J.: Classification of tile-transitive 3-simplex tilings and their realizations in homogeneous spaces. In: A. Prékopa and E. Molnár, (eds.). Non-Euclidean Geometries, János Bolyai Memorial Volume, Mathematics and Its Applications vol. 581, 321-363. Springer (2006)

[15] Mulazzani, M., Vesnin, A.: The many faces of cyclic branched coverings of 2-bridge knots and links. Atti Sem. Mat. Fis. Univ. Modena, Supplemento al Vol. **IL**, 177-215 (2001).

[16] Petronio, C., Vesnin, A.: Two-sided asymptotic bounds for the complexity of cyclic branched coverings of two-bridge links. Osaka J. Math. **46**, 19 pp. (2009)

[17] Porti, J.: Spherical cone structures on 2-bridge knots and links. Kobe J. Math. **21**, 61-70 (2004)

[18] Riley, R.: Seven Excellent Knots. Proceedings of the Conference on Topology in Low Dimension, Cambridge University Press, Cambridge (1982)

[19] Shmatkov R.: Properties of Euclidean Whitehead link cone-manifolds. Siberian Adv. Math. **13**, 55-86 (2003)

[20] Szirmai, J.: The densest geodesic ball packing by a type of Nil lattices. Beiträge Alg. Geom. **48**, 383-397 (2007)

[21] Vesnin, A., Rasskazov, A.: Isometries of hyperbolic Fibonacci manifolds. Siberian Math. J. **40**, 9-22 (1999)

[22] Zhuk, I. K.: Fundamental tetrahedra in Euclidean and Lobachevsky spaces. Soviet Math. Dokl. **27**, 540-543 (1983)

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