Pacific Journal of Mathematics

PROJECTIVE MODULES OVER SUBRINGS OF $k[X, Y]$ GENERATED BY MONOMIALS

DAVID FENIMORE ANDERSON

Vol. 79, No. 1 May 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 1, 1978

PROJECTIVE MODULES OVER SUBRINGS OF $k[X, Y]$ GENERATED BY MONOMIALS

DAVID F. ANDERSON

In this paper we study finitely generated projective modules over affine subrings A of $k[X, Y]$ generated by monomials. If A is normal, then all finitely generated projective A-modules are free. If A is not normal, we show that finitely generated projective A-modules stably have the form free $\oplus$ rank one

1. Introduction. In this paper we study projective modules over subrings A of $k[X, Y]$ generated by monomials. We study conditions on A so that all finitely generated projective A-modules have the form free $\oplus$ rank one. In §IV we use Seshadri's localization technique to show that all finitely generated projective A-modules are free when A is an affine normal subring of $k[X, Y]$ generated by monomials. If we drop the assumption that A is normal it need not be true that all finitely generated projective A-modules are free. However, in §V we show that finitely generated projective A-modules stably have the form free $\oplus$ rank one. We also give sufficient conditions on k for finitely generated projective A-modules to have the form free $\oplus$ rank one. These results do not generalize to arbitrary subrings of $k[X, Y]$.

This paper constitutes part of the author's dissertation at the University of Chicago. The author would like to thank his advisor, Professor M. Pavaman Murthy, for his many helpful suggestions.

2. Preliminaries. All rings A will be commutative with 1. Spec (A) is the set of all prime ideals of A and max (A) is the subset of spec (A) consisting of maximal ideals. We give spec (A) the Zariski topology. If X is a topological space, the combinatorial dimension of X will be denoted by dim X. If A is a ring, the group of units of A is A^*. SL (n, A) is the group of $n \times n$ matrices over A with determinant 1, and $E(n, A)$ is the subgroup of SL (n, A) generated by elementary matrices. The Krull dimension of A will be denoted by dim A. k will always be a field. Let P be a finitely generated projective A-module and $Q \in$ spec (A). We define $\text{rank}_Q P$ to be $\dim_{A_Q/QA_Q} P_Q/QP_Q$. If $\text{rank}_Q P$ is constant, we will denote it by rank P. Our K-theory notation will follow Bass [4].

$\tilde{K}_0(A)$ is the subgroup of $K_0(A)$ generated by $[A^{\text{rank} P}] - [P]$ for finitely generated projective A-modules P, and Pic (A) is the group of isomorphism classes of finitely generated projective A-modules of rank one. There is a natural determinant epimorphism det: $\tilde{K}_0(A) \to \mathrm{Pic}(A)$ defined by $\det([P]) = \Lambda^n(P)$ where $n = \operatorname{rank} P$. We denote the kernel of this map by $SK_0(A)$. Clearly $SK_0(A) = 0$ iff every finitely generated projective A-module stably has the form free $\oplus$ rank one. In this case P is stably isomorphic to $A^{n-1} \oplus \Lambda^n(P)$.

A commutative square of rings

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ A_2 & \xrightarrow{g_2} & B \end{array}$$

is cartesian if $g_1(x) = g_2(y)$ implies there is a unique $z \in A$ with $f_1(z) = x$ and $f_2(z) = y$.

THEOREM 2.1 (*Milnor* [10]). *Given a cartesian square of rings with* g_1 *surjective, the following ("Mayer-Vietoris") sequences are exact*

$$(1)\quad \begin{array}{l} 0 \longrightarrow A^* \longrightarrow A_1^* \oplus A_2^* \longrightarrow B^* \xrightarrow{\partial} \mathrm{Pic}(A) \\ \longrightarrow \mathrm{Pic}(A_1) \oplus \mathrm{Pic}(A_2) \longrightarrow \mathrm{Pic}(B) \end{array}$$

$$(2)\quad \begin{array}{l} K_1(A) \longrightarrow K_1(A_1) \oplus K_1(A_2) \longrightarrow K_1(B) \xrightarrow{\partial} \tilde{K}_0(A) \\ \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow \tilde{K}_0(B)\,. \end{array}$$

Moreover, if $\mathrm{GL}(n, A_1) \to \mathrm{GL}(n, B)$ *is surjective for all* n *and all finitely generated projective* A_1 *and* A_2*-modules are free, then all finitely generated projective* A*-modules are free.*

Using the natural determinant maps, sequences (1) *and* (2) *may be connected to obtain the following commutative diagram with exact rows and columns.*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & SK_1(A) & \longrightarrow & SK_1(A_1) \oplus SK_1(A_2) & \longrightarrow & SK_1(B) & \xrightarrow{\partial} & SK_0(A) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(B) & \xrightarrow{\partial} & \tilde{K}_0(A) \\ & & \downarrow & & {\scriptstyle h}\downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^* & \longrightarrow & A_1^* \oplus A_2^* & \longrightarrow & B^* & \xrightarrow{\partial} & \mathrm{Pic}(A) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 & & 0 \end{array}$$

$$\begin{array}{ccc} & 0 & 0 \\ & \downarrow & \downarrow \\ \longrightarrow & SK_0(A_1) \oplus SK_0(A_2) \longrightarrow & SK_0(B) \\ & \downarrow & \downarrow \\ \longrightarrow & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow & \tilde{K}_0(B) \\ & \downarrow & \downarrow \\ \longrightarrow & \mathrm{Pic}\,(A_1) \oplus \mathrm{Pic}\,(A_2) \longrightarrow & \mathrm{Pic}\,(B) \\ & \downarrow & \downarrow \\ & 0 & 0 \end{array}$$

The following lemma is obvious.

LEMMA 2.2. *Suppose that* $SK_0(A_1) = SK_0(A_2) = 0$, *then*
(1) $SK_0(A) = \partial(SK_1(B))$.
(2) $SK_1(B) = 0$ *implies* $SK_0(A) = 0$.
(3) *If* h *is an isomorphism, then* $SK_1(B) \approx SK_0(A)$.

We review a localization technique due to Seshadri which will be used in §IV. For details one may consult [4]. A set S of ideals of A is multiplicative if $I, J \in S$ implies $IJ \in S$. A prime ideal P is special if P is invertible and A/P is a PID for which $E(n, A/P) = \mathrm{SL}\,(n, A/P)$ for all n. A multiplicative set of ideals is special if it is generated by special prime ideals. If S is any multiplicative set of invertible ideals, we define $S^{-1}A = \bigcup_{I \in S} I^{-1}$. For M an A-module, $S^{-1}M = S^{-1}A \otimes_A M$.

THEOREM 2.3 (*Seshadri*). *Let A be a commutative noetherian ring and S a special multiplicative set of invertible ideals. Let P be a finitely generated projective A-module and suppose that $S^{-1}P \approx L'_1 \oplus \cdots \oplus L'_n$ where each L'_i is a finitely generated projective $S^{-1}A$-module of rank one. Then*

(1) *There are finitely generated projective A-modules L_i of rank one with $L'_i \approx S^{-1}A \otimes_A L_i$ for $i = 1, \cdots, n$.*

(2) *For each choice of L_i in* (1) *there is an I inthe group of invertible ideals generated by S such that $P \approx IL_1 \oplus L_2 \oplus \cdots \oplus L_n$.*

COROLLARY 2.4. *Let A, S, and P be as above. If $S^{-1}P$ is the direct sum of a free $S^{-1}A$-module and a projective $S^{-1}A$-module of rank one, then P is also the direct sum of a free A-module and a projective A-module of rank one.*

The next two lemmas will also be used in §IV. I do not know a reference for Lemma 2.6, however compare [16, p. 7].

LEMMA 2.5 ([11]). *Let* $A = A_0 \oplus A_1 \oplus \cdots$ *be a graded affine normal domain with* A_0 *a field, then* $\operatorname{Pic}(A) = 0$.

LEMMA 2.6. *Let* A *be a commutative ring with* $\max(A)$ *noetherian and* $V(I) = F \subset \max(A)$ *closed with* $\dim(\max(A)\backslash F) \leqq 1$. *Let* P *be a finitely generated projective* A*-module with rank* $P = n \geqq 2$, *and assume that* P/IP *is a free* A/I*-module. Then* $P \approx A^{n-1} \oplus \Lambda^n(P)$.

Proof. It is sufficient to show that if P is a finitely generated projective A-module with rank $P \geqq 2$ and P/IP a free A/I-module, then $P \approx A \oplus P'$ with $P'/IP'A/I$-free.

Let $\max(A)\backslash F = U_1 \cup \cdots \cup U_t$ be a decomposition into closed irreducible components and pick $M_i \in U_i$. For $s \in P$ and $N \in \max(A)$, let $s(N)$ be the image of s in P_N/NP_N. P/IP is free, so by the Chinese Remainder Theorem there is a $s \in P$ with $s(M_i) \neq 0, 1 \leqq i \leqq t$, and $\bar{s}$ a basis element for P/IP. Clearly $s(M) \neq 0$ for $M \supset I$.

Let $Z(s) = \{J \in \max(A) \mid s(J) = 0\}$; then $Z(s) \subset \max(A)\backslash F$ and $Z(s)$ is closed [16, p. 6]. Each $M_i \notin Z(s)$, so $Z(s)$ is 0-dimensional and hence finite, say $Z(s) = \{I_1, \cdots, I_l\}$. Rank $P \geqq 2$, so as above we may choose $t \in P$ such that (1) $t(I_i) \neq 0$ for $1 \leqq i \leqq l$, (2) $\bar{t}$ and $\bar{s}$ form part of a basis for P/IP, and (3) $s(M_i)$ and $t(M_i)$ are linearly independent for $1 \leqq i \leqq t$.

As above $Z(s, t) = \{M \in \max(A) \mid s(M)$ and $t(M)$ are linearly dependent$\}$ is finite. Let $Y = Z(s, t)\backslash Z(s) = \{J_1, \cdots, J_m\}$, then pick $0 \neq a \in (J_1 \cap \cdots \cap J_m)\backslash(I_1 \cup \cdots \cup I_l)$, or let $a = 1$ if $Y = \phi$. Let $u = s + at$, then $u(M) \neq 0$ for all $M \in \max(A)$, so Au is a direct summand of P [16, p. 6]. Note that $\bar{u} = \overline{s + at}$ is part of a basis for P/IP.

$(a_0, \cdots, a_n) \in A^{n+1}$ is unimodular if $Aa_0 + \cdots + Aa_n = A$. $U_{n+1}(A)$ is the set of all unimodular elements in A^{n+1}. The stable range of A, denoted by $\operatorname{sr}(A)$, is $\leqq d$ if given any unimodular row $(a_0, \cdots, a_d)$, there exist $c_0, \cdots, c_{d-1} \in A$ so that $(a_0 + c_0a_d, \cdots, a_{d-1} + c_{d-1}a_d) \in A^d$ is unimodular. It is well-known that $\operatorname{sr}(A) \leqq 1 + \dim A$ and that $A^{n+1} \approx A \oplus P$ implies $A^n \approx P$ whenever $n \geqq \operatorname{sr}(A)$ ([4, p. 239]).

3. Subrings of $k[X, Y]$ generated by monomials.

Subrings A of $B = k[X, Y]$ generated by monomials arise naturally as either the ring of invariants of an automorphism of B of finite order or as the kernel of a k-derivation of B when char $k = p \neq 0$. Clearly if A is as above, then $A \subset B$ is integral, so not all affine normal subrings of B generated by monomials are one of these two types. However, over an algebraically closed field of char $k = 0$, any affine normal subring of B generated by monomials is isomorphic to B^G where G is the cyclic group generated by an automorphism of the form $\phi: X \mapsto aX,\ Y \mapsto bY,\ a, b \in k$. We state the following three propositions without proof; for details see [1] or [2].

PROPOSITION 3.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $A \approx A'$ where $A' = k[X, Y]$ or $A' = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where* $0 < j < n$, $\gcd(j, n) = 1$, *and "$-$" denotes* mod n.

PROPOSITION 3.2. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials. If* $\dim A = 1$, *then $A \approx A'$ where A' is an affine subring of $k[X]$ generated by monomials. If* $\dim A = 2$, *then $A \approx A''$ where A'' is an affine subring of B generated by monomials with $A'' \subset k[X, Y]$ integral.*

PROPOSITION 3.3. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $\bar{A}$, the integral closure of A, is also an affine subring of B generated by monomials. The conductor of $\bar{A}/A$ contains a nonzero monomial.*

We recall that $\operatorname{sing}(A) = \{P \in \operatorname{spec}(A) \mid A_p \text{ is not regular}\}$. If A is an affine normal domain of dim 2, then $\operatorname{sing}(A)$ is a closed subset of $\operatorname{spec}(A)$ of dim 0, and hence finite [9, p. 245]. If in addition $A \subset k[X, Y]$ is generated by monomials, we can explicitly describe $\operatorname{sing}(A)$.

PROPOSITION 3.4. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and A not regular. Then the origin is the only singularity of A, that is,* $\operatorname{sing}(A) = \{M = (X, Y)B \cap A\}$.

Proof. The proof of Proposition 3.1 [2, Thm. 2.5] shows that the isomorphism is just a change of variables which does not change the origin. Thus we may assume that $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. It is sufficient to show that for each of the generators $f_1 = X^n, f_2 = XY^j, \cdots, f_{n+1} = Y^n$ of M, $A[1/f_i]$ is regular. For if N is any other maximal ideal, then some $f_i \notin N$, and thus A_N is regular since it is a localization of the regular ring $A[1/f_i]$.

Clearly $A[1/Y^n] = k[XY^j, Y^n][1/Y^n]$ which is regular. Similarly $A[1/X^n] = k[X^n, YX^i][1/X^n]$ where $YX^i \in A$. If $a, b \neq 0$, then $A[1/X^aY^b]$

contains $1/X^n$ and $1/Y^n$ and thus is a localization of $A[1/Y^n]$. So $A[1/X^a Y^b]$ is also regular.

We note that a subring of $B = k[X, Y]$ generated by monomials is a graded ring with the natural grading it inherits from B.

4. Projective modules over affine normal subrings of $k[X, Y]$ generated by monomials.

THEOREM 4.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials, then all projective A-modules are free.*

Proof. Let P be a projective A-module. If P is not finitely generated, then P is free by a result of Bass [5] or Hinohara [8]. So we may assume that P is finitely generated.

If $\dim A = 1$, then by Serre's theorem [4, p. 173], P has the form free $\oplus$ rank one. But $\operatorname{Pic}(A) = 0$ by Lemma 2.5, so P is free. Thus, we may assume that $\dim A = 2$. By Propositions 3.1 and 3.2 we may assume that $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$.

Let $\bar{B} = \bar{k}[X, Y]$ where $\bar{k}$ is the algebraic closure of k. The maximal ideals of $\bar{B}$ are of the form $M_{a,b} = (X - a, Y - b)$, and thus the maximal ideals of A are of the form $A \cap M_{a,b}$ because $A \subset \bar{B}$ is integral.

For each $0 \neq b \in \bar{k}$ let $Q_b = (Y - b)\bar{B} \cap A$. Clearly

$$A/Q_b \approx k[T^n, b^jT, b^{\overline{2j}}T^2, \cdots, b^{\overline{(n-1)j}}T^{n-1}, b^n] .$$

$b \in \bar{k}$ is algebraic over k, so $k' = k[b^n]$ is a field. Each $b^{\overline{ij}}T^i = (b^n)^l(b^jT)^i$ for some integer l. Thus $A/Q_b \approx k'[b^jT] = k'[S]$ is a euclidean ring because $S = b^jT$ is transcendental over k'.

Next we show that $Q_b (b \neq 0)$ is invertible. It is sufficient to show that $(Q_b)_N$ is principal for each maximal ideal N of A. If $Q_b \not\subset N$, then $(Q_b)_N = A_N$. If $Q_b \subset N$, then clearly $N \neq (X, Y)\bar{B} \cap A$. So by Proposition 3.4, A_N is a regular local ring and hence factorial. Thus $(Q_b)_N$ is a *ht* one prime ideal in a factorial ring and hence principal. So Q_b is locally principal and thus invertible. Q_b is actually principal because $\operatorname{Pic}(A) = 0$ by Lemma 2.5. In fact $Q_b = (Y - b)\bar{B} \cap A = fA$ where $f \in k[Y^n]$ is the polynomial of least degree satisfying $f(b) = 0$.

For each $0 \neq b \in \bar{k}$ Q_b is an invertible ideal such that A/Q_b is a euclidean ring, and thus $E(n, A/Q_b) = \mathrm{SL}(n, A/Q_b)$ for all n. Thus each Q_b is a special prime ideal. Let S be the multiplicative set generated by the Q_b for $0 \neq b \in \bar{k}$. We show that all finitely generated projective $S^{-1}A$-modules have the form free $\oplus$ rank one.

Let $I = (Y\bar{B} \cap A)S^{-1}A$ and $Z = \max(S^{-1}A \backslash V(I))$. S kills all the maximal ideals $(X - a, Y - b) \cap A$ when $b \neq 0$; so $\dim Z \leqq 1$.

$$A/(Y\bar{B} \cap A) \approx k[T] \quad \text{and} \quad S^{-1}A/I \approx S^{-1}(A/Y\bar{B} \cap A)\,,$$

so $\operatorname{Pic}(S^{-1}A/I) = 0$ also. Thus all finitely generated projective $S^{-1}A/I$-modules are free. By Lemma 2.6 all finitely generated projective $S^{-1}A$-modules have the form free $\oplus$ rank one. Thus all finitely generated projective A-modules have the form free $\oplus$ rank one by Corollary 2.4. But $\operatorname{Pic}(A) = 0$, so all finitely generated projective A-modules are free.

Seshadri [17] first showed that all finitely generated projective $k[X, Y]$-modules are free. Murthy and Pedrini [12] showed that all finitely generated projective A-modules are free if $A = k[X^n, XY, Y^n]$ or $A = k[X^n, XY^{n-1}, \cdots, X^{n-1}Y, Y^n]$. Our result generalizes these. Quillen [15] and Suslin have recently, and independently, proved Serre's problem. That is, all finitely generated projective $k[X_1, \cdots, X_n]$-modules are free. The following conjecture thus seems reasonable.

Conjecture. Let A be an affine normal subring of $k[X_1, \cdots, X_n]$ generated by monomials, then all finitely generated projective A-modules are free.

We can however prove a weaker version of this conjecture. First a result which follows from Quillen's work [15]. The author learned of this result in a course given by R. G. Swan.

PROPOSITION 4.2. *Let A be a commutative ring and $f \in A[X]$ a monic polynomial. Let P and Q be finitely generated projective $A[X]$-modules with*

(1) *Q is extended from A.*

(2) *$fQ \subset P \subset Q$.*

Then P and Q are isomorphic.

Proof (sketch). The proof is similar to that of [15, Thm. 3]. Let $A(X)$ denote the localization of $A[X]$ with respect to the multiplicative system of monic polynomials. Let $Q \approx Q_0 \otimes_A A[X]$. Since $f \in A[X]$ is monic, by (2), $P \otimes_{A[X]} A(X) \approx Q \otimes_{A[X]} A(X) \approx Q_0 \otimes_A A(X)$. Then as in [15, Thm. 3], P is extended from A, say $P \approx P_0 \otimes_A A[X]$. Thus $P_0 \approx Q_0$, and so $P \approx Q$.

THEOREM 4.3. *Let A be an affine normal subring of $k\ [X, Y]$ generated by monomials, then all finitely generated projective $A[X_1, \cdots, X_n]$-modules are free.*

Proof. By induction on n, the case $n = 0$ is just Theorem 4.1. Suppose $A = k[f_1, \cdots, f_r]$ with $f_i \in k[X, Y]$ and let $B = A[X_1, \cdots, X_n]$. Let S be the set of monic polynomials in $k[X_{n+1}]$. Then $B[X_{n+1}]_S = k(X_{n+1})[f_1, \cdots, f_r][X_1, \cdots, X_n]$. Clearly $k(X_{n+1})[f_1, \cdots, f_r]$ is still an affine normal subring of $k(X_{n+1})[X, Y]$ generated by monomials. So by induction all finitely generated projective $B[X_{n+1}]_S$-modules are free. Let P be a finitely generated projective $B[X_{n+1}]$-module; then P_S is free, say $P_S \approx F_S$ where F is a finitely generated free $B[X_{n+1}]$-module. Thus there exists a $g \in S$ so that $P_g \approx F_g$ and hence $g^m F \subset P \subset F$ for some m. By Proposition 4.2 $P \approx F$, and hence is free.

Thus affine normal subrings A of $k[X, Y]$ generated by monomials are nontrivial examples of nonregular rings for which $NK_0(A) = 0$, where $NK_0(A) = \ker(K_0(A[T]) \to K_0(A))$, induced by $T \mapsto 0$. In fact, it is an open question if $NK_0(A) = 0$ for all normal domains.

5. **Projective modules over subrings of $k[X, Y]$ generated by monomials.** If we drop the assumption that A is normal, all finitely generated projective A-modules need not be free.

EXAMPLE 5.1 ([7]). Let $A = k[X^2, X^3, Y]$, then not all finitely generated projective A-modules are free. $P = (1 + XY, 1 + XY + X^2Y^2)$ is a rank one projective A-module (invertible ideal in $k(X, Y)$) which is not free. In fact, $K_0(A) \approx \boldsymbol{Z} \oplus \operatorname{Pic}(A) \approx \boldsymbol{Z} \oplus k[Y]$. That $K_0(A) \approx \boldsymbol{Z} \oplus \operatorname{Pic}(A)$ is just Theorem 5.5. We show that $\operatorname{Pic}(A) \approx k[Y]$. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, X^3, Y] & \hookrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[Y] & \hookrightarrow & B/I = k[\varepsilon][Y] \end{array}$$

Here $I = (X^2, X^3)B$ is contained in the conductor ideal and $\varepsilon^2 = 0$. By (1) of Theorem 2.1, $\operatorname{Pic}(A) \approx k[\varepsilon][Y]^*/k^*$. But, as abelian groups, $k[\varepsilon][Y]^*/k^* \approx k[Y]$.

Of course all finitely generated projective A-modules may be free even though A is not normal.

EXAMPLE 5.2. Let $A = k[X^2, XY, Y]$, then all finitely generated projective A-modules are free. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, XY, Y] & \longrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[X^2] & \longrightarrow & B/I = k[X]\,. \end{array}$$

Here $I = YB$ is the conductor ideal. All finitely generated projective B and A/I-modules are free and all $\mathrm{GL}(n, B) \to \mathrm{GL}(n, B/I)$ are surjective. Thus by Theorem 2.1, all finitely generated projective A-modules are free.

We show that if A is an affine subring of $B = k[X, Y]$ generated by monomials, then $SK_0(A) = 0$, that is, the natural map det: $\tilde{K}_0(A) \to \mathrm{Pic}(A)$ is an isomorphism. This just means that stably any finitely generated projective A-module has the form free $\oplus$ rank one.

Let A be an affine subring of $B = k[X, Y]$ generated by monomials, $\bar{A}$ the integral closure of A, and I the conductor ideal. $I \neq 0$, so let J be any nonzero ideal contained in I. Thus $\dim A/J$, $\dim \bar{A}/J \leq 1$, so $SK_0(A/J) = SK_0(\bar{A}/J) = 0$ by Serre's theorem [4, p. 173]. By Theorems 3.3 and 4.1 $SK_0(\bar{A}) = 0$, so by Lemma 2.2 it is sufficient to show that $SK_1(\bar{A}/J) = 0$. Again we may assume that $\dim A = 2$, and thus by Propositions 3.1, 3.2, and 3.3 we may assume that $\bar{A} = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. Also I contains a nonzero monomial $f = X^aY^b$ with $a, b \neq 0$; let $J = f\bar{A}$.

LEMMA 5.3. *Let A and B be commutative rings with $A \subset B$ integral. If $I \subset A$ is an ideal of A, then* $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.

Proof. Clearly $\sqrt[A]{I} \subset \sqrt[B]{IB} \cap A$. Let $P \in \mathrm{spec}(A)$ with $I \subset P$. $A \subset B$ is integral, so there is a $\bar{P} \in \mathrm{spec}(B)$ with $P = \bar{P} \cap A$. $I \subset P \subset \bar{P}$, so $IB \subset \bar{P}$. Thus $\sqrt[B]{IB} \subset \bar{P}$ and $\sqrt[B]{IB} \cap A \subset \bar{P} \cap A = P$. So $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.

LEMMA 5.4. $SK_1(\bar{A}/f\bar{A}) = 0$.

Proof. By above $\bar{A}/f\bar{A} = k[X^n, XY^j, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]/X^aY^b\bar{A}$. Let $B = k[X, Y]$, then $\sqrt[B]{fB} = XYB$. By Lemma 5.3 $\sqrt[\bar{A}]{f\bar{A}} = XYB \cap \bar{A}$. By [4, p. 469], $SK_1(\bar{A}/f\bar{A}) \approx SK_1((\bar{A}/f\bar{A})/(\sqrt[\bar{A}]{f\bar{A}}/f\bar{A})) \approx SK_1(\bar{A}/\sqrt[\bar{A}]{f\bar{A}})$. Clearly $\bar{A}/\sqrt[\bar{A}]{f\bar{A}} \approx k[X, Y]/(XY)$. But $SK_1(k[X, Y]/(XY)) = 0$, so $SK_1(\bar{A}/f\bar{A}) = 0$.

THEOREM 5.5. *Let A be an affine subring of $k[X, Y]$ generated by monomials, then $SK_0(A) = 0$. Thus all finitely generated projective A-modules stably have the form free $\oplus$ rank one.*

COROLLARY 5.6. *Let A be a subring of $k[X, Y]$ generated by monomials, then all finitely generated projective A-modules stably have the form free $\oplus$ rank one.*

Proof. This follows from the following well-known result. Let M be a finitely presented A-module, then there is a noetherian subring R of A and a finitely presented R-module M' with $M \approx M' \otimes_R A$. If M is projective, M' may also be chosen to be projective.

Theorem 5.5 is rather unsatisfying because it does not say that any finitely generated projective A-module has the form free $\oplus$ rank one, but only that this is stably true. Since $\dim A \leqq 2$, by Bass' Cancellation Theorem [4, p. 184], if $\operatorname{rank} P = n \geqq 3$, then actually $P \approx A^{n-1} \oplus \Lambda^n(P)$. If rank $P = 2$, we only have $P \oplus A \approx A^2 \oplus \Lambda^2(P)$. If k is algebraically closed, by a cancellation theorem of Murthy and Swan [13], $P \approx A^{n-1} \oplus \Lambda^n(P)$. I know of no examples where $P \not\approx A^{n-1} \oplus \Lambda^n(P)$.

If $\operatorname{Pic}(A) = 0$, then all finitely generated projective A-modules are stably free. If $\operatorname{sr}(A) \leqq 2$, then $E(3, A)$ acts transitively on $U_3(A) = \{(a_0, a_1, a_2) \in A^3 \mid (a_0, a_1, a_2) \text{ unimodular}\}$, so all finitely generated projective A-modules are free. This happens when k is algebraic over a finite field [18, p. 45]. We next show that this also happens whenever $1/2 \in k$.

We recall a few definitions. $KSp_0(A)$ is the Grothendieck group with generators $[P]$ for each symplectic A-module P and relations $[P] = [Q]$ if $P \approx Q$ and $[P \perp Q] = [P] + [Q]$. $W(A)$ is the kernel of the natural map $KSp_0(A) \to K_0(A)$ given by $[P] \mapsto [P]$ which forgets the symplectic structure. W is a functor from rings to abelian groups. For more details one is referred to [6] or [18].

LEMMA 5.7 (*C. Weibel and R. G. Swan*). *Let* $A = A_0 \oplus A_1 \oplus \cdots$ *be a graded ring and* F *a functor on rings. If the natural map induces an isomorphism* $F(A) \to F(A[T])$, *then the natural map* $A_0 \to A$ *also induces an isomorphism* $F(A_0) \to F(A)$.

Proof. Define $f: A \to A[T]$ by $f: \Sigma a_i \to \Sigma a_i T^i$. By hypothesis the two maps $F(A[T]) \to F(A)$ induced by $T \mapsto 0$ and $T \mapsto 1$ are both isomorphisms. Consider the composition $A \xrightarrow{f} A[T] \to A$. If $T \mapsto 1$ we obtain the identity, while $T \mapsto 0$ gives the natural augmentation $A \to A_0$. Thus the natural map $A \to A[T] \to A_0$ induces a monomorphism $F(A) \to F(A_0)$. But this map is always surjective, so $F(A_0) \approx F(A)$.

PROPOSITION 5.8. *Let* A *be an affine subring of* $k[X, Y]$ *generated by monomials. If* $\operatorname{Pic}(A) = 0$ *and* $1/2 \in k$, *then all finitely generated projective* A*-modules are free.*

Proof. By a theorem of Karoubi [6, p. 8], when R is a commutative ring with $1/2 \in R$, $R \to R[T]$ induces an isomorphism $W(R) \to$

$W(R[T])$. A is a graded ring with $A_0 = k$, so $W(A) \approx W(k)$ by Lemma 5.7. It is well-known than $W(k) = 0$ [6, p. 8], so also $W(A) = 0$.

By a result of Vaserstein [6, p. 7], there is a natural map

$$\phi\colon \mathrm{SL}\,(3, R)\backslash U_3(R) \longrightarrow W(R)$$

which is bijective if $E(r, R)$ acts transitively on $U_r(R)$ for all $r \geqq 4$.

In our case $W(A) \approx W(k) = 0$ and $E(r, A)$ acts transitively on $U_r(A)$ for all $r \geqq 4$ since $\mathrm{sr}\,(A) \leqq 3$. Thus $\mathrm{SL}\,(3, A)\backslash U_3(A) = 1$; that is, $\mathrm{SL}\,(3, A)$ acts transitively on $U_3(A)$. So all finitely generated projective A-modules which are stably free are actually free. But $\mathrm{Pic}\,(A) = 0$, so all finitely generated projective A-modules are free by Theorem 5.5.

6. **Subrings A of $k[X, Y]$ with $\mathrm{Pic}\,(A) = 0$.** It is not hard to determine precisely which subrings A of $k[X, Y]$ generated by monomials have $\mathrm{Pic}\,(A) = 0$. If $\dim A = 1$, clearly $A \approx k[X]$ iff $\mathrm{Pic}\,(A) = 0$. If $\dim A = 2$, by Proposition 3.2 we may assume that $A \subset k[X, Y]$ is integral.

PROPOSITION 6.1. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and let $\bar{A}$ be the integral closure of A. Then* $\mathrm{Pic}\,(A) = 0$ *iff*

(1) *Let X^m and Y^n be the lowest powers of X and Y in A, then X^i, $Y^j \in A$ imply $m \mid i$ and $n \mid j$.*

(2) *$XYB \cap \bar{A}$ is contained in the conductor of $\bar{A}/A$.*

Proof. We prove the notationally easier case with $\bar{A} = B = k[X, Y]$. Otherwise we may assume $\bar{A} = k[X^n, XY^j, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ and the proof is similar.

($\Leftarrow$) Let $I = XYB$, then $A/I \approx k[X, Y]/(XY)$. Clearly $A^* = B^* = (A/I)^* = (B/I)^* = k^*$. Also $\mathrm{Pic}\,(A/I) = 0$. This follows from Theorem 2.1 applied to following cartesian square.

$$\begin{array}{ccc} k[X, Y]/(XY) & \longrightarrow & k[X, Y]/(X) \\ \downarrow & & \downarrow \\ k[X, Y]/(Y) & \longrightarrow & k[X, Y]/(X, Y) \end{array}$$

Thus also $\mathrm{Pic}\,(A) = 0$ by Theorem 2.1.

($\Rightarrow$) Conversely assume that $\mathrm{Pic}\,(A) = 0$. Suppose that (1) fails. Say that not all powers of X are multiples of m. There is a retract of rings $R \to A \overset{\theta}{\to} R$ where R is the image in $k[X]$ of the map

$\theta: X \mapsto X$, $Y \mapsto 0$. Thus $\mathrm{Pic}(R) \subset \mathrm{Pic}(A)$. By Theorem 2.1 it is easy to see that $\mathrm{Pic}(R) \neq 0$, and hence also $\mathrm{Pic}(A) \neq 0$. So we may assume that (1) holds.

Pick $f = X^i Y^i$ in the conductor of B/A with $i > m, n$; this is possible by Proposition 3.3. Since $\mathrm{Pic}(A) = 0$, also $(A/fB)^* = (B/fB)^*$ by Theorem 2.1. For each $g = X^a Y^b$ with $1 \leqq a \leqq m$ and $1 \leqq b \leqq n$, $1 + g + fB$ is a unit in B/fB, and hence also in A/fB. But thus $X^a Y^b \in A$, so XYB is contained in the conductor, and the proposition is proved.

For example, the affine subrings A of $B = k[X, Y]$ generated by monomials with integral closure B for which $\mathrm{Pic}(A) = 0$ are precisely those of the form

$$A = k[X^m, \{X^i Y^j \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n\}, Y^n] .$$

For these rings $K_0(A) \approx \boldsymbol{Z}$. Since this does not depend on the field k, by an argument similar to that of Theorem 4.3 we see that all finitely generated projective $A[X_1, \cdots, X_n]$-modules are stably free. So these rings provide many examples of nonnormal rings for which $NK_0(A) = 0$.

We note that even though these rings are not normal, they are "power closed" in the sense that if f is in the quotient field of A and $f^n \in A$ for all large n, then actually $f \in A$. This condition is in fact necessary, for if A is not "power closed", then $\mathrm{Pic}(A) \to \mathrm{Pic}(A[T])$ is not an isomorphism (see Example 5.1).

One can also see that $NK_0(A) = 0$ for the rings of Proposition 6.1 by using the Mayer-Vietoris K-theory sequence for NK_1 and NK_0 ([14]). So the rings of Proposition 6.1 are precisely the nonnormal affine subrings of $k[X, Y]$ generated by monomials for which $NK_0(A) = 0$. Thus if A is an affine subring of $k[X, Y]$ generated by monomials, $\mathrm{Pic}(A) = 0$ iff $NK_0(A) = 0$.

7. **The general case.** One can ask if these results generalize to more general subrings of $k[X, Y]$. This is studied in more detail *in [1] or [3]. The analogue of Theorem 5.5 fails in general because* there exist $f \in k[X, Y]$ with $SK_1(k[X, Y]/(f)) \neq 0$. This also depends on the field k. We close with one example.

EXAMPLE 7.1. Let $A = k[X, Y(X^2 - Y), Y^2(X^2 - Y)]$, then

(1) If k is algebraic over a finite field all finitely generated projective A-modules are free.

(2) If k is an algebraically closed field of char 0, then $SK_0(A) \approx \Omega^1_{k/Z} \neq 0$ and $\mathrm{Pic}(A) = 0$. Thus there exist indecomposable finitely generated projective A-modules of rank 2.

REFERENCES

1. D. F. Anderson, *Projective modules over subrings of* $k[X, Y]$, dissertation, University of Chicago, 1976.

2. ———, *Subrings of* $k[X, Y]$ *generated by monomials*, Canad. J. Math., **30** (1978), 215-224.

3. ———, *Projective modules over subrings of* $k[X, Y]$, Trans. Amer. Math. Soc., **240** (1978), 317-328.

4. H. Bass, *Algebraic K-theory*, W. A. Benjamin, New York, 1968.

5. ———, *Big projective modules are free*, Illinois J. Math., **7** (1963), 24-31.

6. ———, *Libération des modules projectifs sur certains anneaux de polynômes*, Sem. Bourbaki, **448** (1973-74).

7. ———, *Torsion free and projective modules*, Trans. Amer. Math. Soc., **102** (1962), 319-327.

8. Y. Hinohara, *Projective modules over weakly noetherian rings*, J. Math. Soc. Japan, **15** (1963), 75-88.

9. H. Matsumura, *Commutative Algebra*, W. A. Benjamin, New York, 1970.

10. J. Milnor, *Introduction to Algebraic K-theory*, Princeton University Press, Princeton, 1971.

11. M. P. Murthy, *Vector bundles over affine surfaces birationally equivalent to a ruled surface*, Ann. of Math., **89** (1969), 242-253.

12. M. P. Murthy and C. Pedrini, K_0 *and* K_1 *of polynomial rings*, in *Algebraic K-theory II*, Springer Lecture Notes in Mathematics, No. 342, Berlin, 1973, 109-121.

13. M. P. Murthy and R. G. Swan, *Vector bundles over affine surfaces*, Inventiones Math., **36** (1976), 125-165.

14. C. Pedrini, *On the* K_0 *of certain polynomial extensions*, in *Algebraic K-theory II*, Springer Lecture Notes in Mathematics, No. 342, Berlin, 1973, 92-108.

15. D. Quillen, *Projective modules over polynomial rings*, Inventiones Math., **36** (1976), 167-171.

16. J. P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, Sem. Dubriel-Pisot, **11** (1957-1958), No. 2.

17. C. S. Seshadri, *Triviality of vector bundles over the affine space* K^2, Proc. Natl. Acad. Sci. U. S. A., **44** (1958), 456-458.

18. R. G. Swan, *Serre's problem*, in Conference on commutative algebra, 1975, ed. A. Geramita, Queen's Papers in Pure and Applied Math., 42.

Received January 21, 1977 and in revised form May 3, 1978.

UNIVERSITY OF TENNESSEE
KNOXVILLE, TN 37916

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 79, No. 1 May, 1978