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Projective normality of flag varieties and Schubert varieties

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1. Introduction

Let G be a reductive group over an algebraically closed field and B a Borel subgroup of G. The flag variety $X=G/B$ is a projective variety and any line bundle L on X is homogeneous and the space of sections $\Gamma(X,L)$ and more generally the cohomology groups $H^i(X,L)$ are G-modules and have been the subject of a lot of study. When L is effective, i.e. $\Gamma(X,L)\neq 0$, one knows that $H^i(X,L)=0$ for $i>0$ and that $\dim \Gamma(X,L)$ may be computed by the dimension formula of Weyl. This vanishing theorem is a simple consequence of the Kodaira vanishing theorem over $\mathbb{C}$ and was proved by Kempf in characteristic $p>0$. Simpler proofs were given by Haboush, Andersen and Mehta-Ramanathan.

A problem of interest in this connection is whether $\Gamma(X,L)\otimes\Gamma(X,L')\to\Gamma(X,L\otimes L')$ is surjective when L and L' are effective. In particular this would imply that the complete linear system of L imbeds G/B as a projectively normal variety whenever L is ample. This is a simple consequence of the irreducibility of G-modules of the type $\Gamma(X,L)$ in characteristic zero. In positive characteristics this was known in several cases as a consequence of the standard monomial theory of Seshadri and his school. We prove it here in full generality.

Let T be a maximal torus in B and $W=N(T)/T$ the Weyl group. Then the B-orbits of ωB, $\omega\in W$ give an affine cellular decomposition of G/B. The closures X_ω of $B\omega B$ in G/B are called Schubert varieties. The cohomology vanishing theorem for L over X_ω was proved recently by Mehta-Ramanathan [4], when L is ample. They also show that in this case $\Gamma(G/B,L)\to\Gamma(X_\omega,L)$ is surjective. Their principal contribution consists in the notion of a Frobenius split variety and compatible splitting and a criterion for it proved by exploiting the local nature of the duality theorem for the Frobenius morphism. This way they show that G/B is Frobenius split, compatibly with any Schubert variety.

In this note we improve these results to the case when L is effective and as a consequence prove that Schubert varieties are normal. In a recent manuscript [6] Seshadri has proved the normality of Schubert varieties.

Our observation is simply that in the case of the flag variety G/B there is a splitting $F_*\mathcal{O}\to\mathcal{O}$ which actually factors through $F_*(\mathcal{O}(D))\to\mathcal{O}$ where D is an effective divisor linearly equivalent to L_ρ^{p-1} where ρ is the product of all the fundamental weights, $p>0$ is the characteristic of the base field and F is the absolute Frobenius morphism. This extra information yields the improved result mentioned above.

2. Flag varieties

First we introduce some notation. The base field k will always be an algebraically closed field of characteristic $p>0$, unless otherwise mentioned. For any variety X over k, $F\colon X\to X$ is the absolute Frobenius morphism. For a smooth variety X we denote by K_X (or K) its canonical bundle. If V is an $\mathcal{O}_X$-Module we denote the pull back F^*V sometimes by $V^{(p)}$. The character on B given by the product of fundamental characters is denoted ρ. In additive terminology, we may also say ρ is the half sum of positive roots.

We start with a supplement to the result of Metha-Ramanathan ([4], Proposition 1) on the vanishing of cohomology of ample line bundles on Frobenius split varieties. Recall that a variety is Frobenius split ([4], Definition 2) if the exact sequence

$$0\to\mathcal{O}\to F_*\mathcal{O}\to C\to 0$$

splits, where $\mathcal{O}\to F_*\mathcal{O}$ is the pth power map. The refinement consists in assuming that the splitting, namely a section of $(F_*\mathcal{O})^*$ is actually a section of $F_*(\mathcal{O}(D))^*$ where D is an effective divisor. The result gives vanishing of cohomology for many line bundles which are not necessarily ample.

Lemma 1. *Let X be a variety which admits a splitting of*

$$0\to\mathcal{O}\to F_*\mathcal{O}\to C\to 0$$

given by a map $F_*\mathcal{O}\to F_*\mathcal{O}(D)\to\mathcal{O}$ where D is some effective divisor. Then for any vector bundle V on X, we have that $H^i(X,V^{(p)}\otimes\mathcal{O}(D))=0$ implies $H^i(X,V)=0$. If $V\to W$ is a homomorphism then $H^i(X,V)\to H^i(X,W)$ is injective (resp. surjective) if $H^i(X,V^{(p)}\otimes\mathcal{O}(D))\to H^i(X,W^{(p)}\otimes\mathcal{O}(D))$ is so.

Proof. Tensor the Frobenius sequence with V and note the isomorphisms $V\otimes F_*\mathcal{O}=F_*(V^{(p)})$, $V\otimes F_*(\mathcal{O}(D))=F_*(V^{(p)}\otimes\mathcal{O}(D))$. Thus we have a diagram

$$\begin{array}{ccc} H^i(X,V) & \longrightarrow & H^i(X,F_*V^{(p)}) \\ & \nwarrow & \downarrow \\ & & H^i(X,F_*(V^{(p)}\otimes\mathcal{O}(D))) \end{array}$$

where the composite on $H^i(X,V)$ is identity. But

$$H^i(X,F_*(V^{(p)}\otimes\mathcal{O}(D)))\approx H^i(X,V^{(p)}\otimes\mathcal{O}(D)),$$

since F is an affine morphism. The first assertion is then obvious. The second assertion is equally clear after observing that in the diagram

$$\begin{array}{ccc} H^i(X,V) & \longrightarrow & H^i(X,V^{(p)}\otimes\mathcal{O}(D)) \\ \downarrow & & \downarrow \\ H^i(X,W) & \longrightarrow & H^i(X,W^{(p)}\otimes\mathcal{O}(D)) \end{array}$$

the induced splittings of the horizontal rows are compatible.

We would like to use the above result in the following situation. Let L be a line bundle and $\Gamma(L)_X\to L$ be the natural evaluation homomorphism of the trivial bundle to L. We tensor this by another line bundle M and apply Γ to it. We get the natural map

$$\Gamma(L)\otimes\Gamma(M)\to\Gamma(L\otimes M).$$

By Lemma 1 this is surjective if

$$\Gamma((\Gamma(L)_X\otimes M)^{(p)}\otimes\mathcal{O}(D))\to\Gamma((L\otimes M)^{(p)}\otimes\mathcal{O}(D))$$

is surjective. Thus we get the following corollary.

Corollary. *Under the assumptions on X as in Lemma* 1 *we have: the natural map*

$$\Gamma(X,L)\otimes\Gamma(X,M)\to\Gamma(X,L\otimes M)$$

is surjective if

$$\Gamma(X,L)^{(p)}\otimes\Gamma(X,M^p\otimes\mathcal{O}(D))\to\Gamma(X,L^p\otimes M^p\otimes\mathcal{O}(D))$$

is surjective.

Remark. Similar result for G/B has been proved by Andersen [1].

We wish to apply this principle now to the flag variety G/B. This is possible due to the following proposition.

Proposition 1. *Let $X=G/B$. The exact sequence*

$$0\to\mathcal{O}_X\to F_*\mathcal{O}_X\to C\to 0$$

admits a splitting $F_*\mathcal{O}_X\to\mathcal{O}_X$ which factors through $F_*(\mathcal{O}_X(D))$ where D is an effective divisor with $\mathcal{O}_X(D)\approx L_\rho^{p-1}$, L_ρ being the line bundle associated to the character ρ (=the product of fundamental weights) of B to the B-bundle $G\to G/B$.

Proof. This is a simple consequence of Corollary to Proposition 10 in [4]. We will recall how this splitting is given, indicating how the improvement is actually implicit in the construction. In order to give a map $F_*\mathcal{O}_X\to\mathcal{O}_X$ of the desired type or what is the same a section of $F_*(\mathcal{O}_X(D))^*$ one uses the relative duality for the Frobenius morphism. This gives the isomorphism $F_*(\mathcal{O}_X(D))^*\approx F_*(K^{1-p}\otimes\mathcal{O}_X(-D))$. A section of the latter can be viewed as a section of $K^{1-p}\otimes\mathcal{O}_X(-D)$. For X we know that $K^{-1}\approx L_\rho^2$ and hence $K^{1-p}\otimes\mathcal{O}_X(-D)$

$\approx L_\rho^{p-1}$. The composite of a map $F_*(\mathcal{O}_X(D)) \to \mathcal{O}_X$ with the inclusion $\mathcal{O}_X \to F_* \mathcal{O}_X \to F_*(\mathcal{O}_X(D))$ is a constant function and so it is enough to check that it is nonzero at a point, say $(B) \in G/B$. In [4] (corollary to Proposition 10) however, this section is given by considering a well-known birational morphism ψ: $Z \to G/B$ of a nonsingular variety Z onto G/B. A section of K^{-1} is then constructed which vanishes on a divisor which we may assume to be of the form $\psi^{-1}(D)$ with D linearly equivalent to L_ρ^{p-1}. The differential of ψ gives a nonzero map $K^{-1} \to \psi^* K_X^{-1}$ and hence we get a section of $\psi^* K_X^{-1}$ or what is the same a section of $\psi_*(\psi^* K_X^{-1}) = K_X^{-1} \otimes \psi_* \mathcal{O} = K_X^{-1}$. This gives the desired splitting.

Now we will apply the corollary to the case of G/B to prove the following theorem.

Theorem 1. *Let k be an algebraically closed field of arbitrary characteristic, G a reductive group over k and Q a parabolic subgroup of G. Let L, L' be effective line bundles on G/Q,* i.e. *$H^0(G/Q, L) \neq 0$ and $H^0(G/Q, L') \neq 0$. Then we have*

i) *$H^i(G/Q, L) = 0$ for $i > 0$.*

ii) *The natural map*

$$\Gamma(G/Q, L) \times \Gamma(G/Q, L') \to \Gamma(G/Q, L \otimes L')$$

is surjective.

iii) *If moreover L is ample then the complete linear system of L imbeds G/Q as a projectively normal variety.*

Proof. The variety G/Q and the line bundles all can be constructed as schemes flat over $\mathbb{Z}$. Then a simple application of semicontinuity shows that if we prove the theorem for fields of positive characteristic then it also follows for fields of zero characteristic. (Of course one uses the Lefschetz principle to go from one field to another of the same characteristic.) Thus we can assume that the base field is of characteristic $p > 0$.

We can also reduce to the case $Q = B$, a Borel subgroup. For, assume by induction that we have proved i) for lower dimensional reductive groups. Then the fibration of π: $G/B \to G/Q$ has as fibre Q/B, which is the flag variety of a lower dimensional group viz. the reductive part of Q. Hence we have $H^i(Q/B, \mathcal{O}_{Q/B}) = 0$, so that $R^i \pi_* \mathcal{O} = 0$ for $i > 0$. Since $\pi_* \mathcal{O} = \mathcal{O}$, $\pi_* \pi^* L = L$. Thus by Leray's spectral sequence we have, for any line bundle L on G/Q, $H^i(G/Q, L) = H^i(G/B, \pi^* L)$. Thus we need deal with only G/B.

i) This of course is well known (cf. e.g. [1], [3], [4]). In the present set up we can argue as follows. By Lemma 1 we have only to show that $H^i(L^p \otimes L_\rho^{p-1}) = 0$ or by interation that $H^i((L \otimes L_\rho)^{p^r} \otimes L_\rho^{-1}) = 0$. This is true since on G/B, $L \otimes L_\rho$ is ample whenever L is effective.

ii) We will need the following property of the Steinberg module $\Gamma(L_\rho^{p-1})$.

Lemma 2. *The natural map*

$$\Gamma(L)^{(p)} \otimes \Gamma(L_\rho^{p-1}) \to \Gamma(L^p \otimes L_\rho^{p-1})$$

is an isomorphism, for all effective line bundles $L = L_\chi$.

Proof. From 1), we conclude that $\dim \Gamma(L_\lambda)$ is the same as that given in characteristic 0 and hence given by Weyl's dimension formula, namely

$$\frac{\prod_{\alpha>0} (\lambda+\rho, \alpha)}{\prod_{\alpha>0} \langle \rho, \alpha \rangle}.$$

Substituting in turn $\lambda=\chi$, $(p-1)\rho$ and $p\chi+(p-1)\rho$ in this formula, one sees that the dimensions on the left and the right sides are the same. We first note that when L is of the form $L_\rho^{p^r-1}$ for some $r \geqq 0$, the right side is $\Gamma(L_\rho^{p^{r+1}-1})$ which is an irreducible G-module. Hence the G-linear map in the lemma is surjective and consequently an isomorphism. In general, $\Gamma(L_\rho^{p^r-1} \otimes L^{-1}) \neq 0$ for large enough r and let s be a nonzero section of $L_\rho^{p^r-1} \otimes L^{-1}$. Tensorising with s^p gives an injection $\Gamma(L)^{(p)} \to \Gamma(L_\rho^{p^r-1})^{(p)}$ and hence an injection $\Gamma(L)^{(p)} \otimes \Gamma(L_\rho^{p-1}) \to \Gamma(L_\rho^{p^{r+1}-1})$. Since this factors through the map in the lemma, we see it is also an injection and hence an isomorphism.

To return to the proof of Theorem 1, it is enough to show by corollary to Lemma 1 that

$$\Gamma(L)^{(p)} \otimes (\Gamma((L')^p \otimes L_\rho^{p-1}) \to \Gamma(L^p \otimes (L')^p \otimes L_\rho^{p-1})$$

is onto. We will show that

$$\Gamma(L)^{(p)} \otimes \Gamma((L')^p) \otimes \Gamma(L_\rho^{p-1}) \to \Gamma((L \otimes L')^p \otimes L_\rho^{p-1})$$

is itself surjective. Now by Lemma 2, $\Gamma(L)^{(p)} \otimes \Gamma(L_\rho^{p-1}) \approx \Gamma(L^p \otimes L_\rho^{p-1})$ and hence it is enough to show that

$$\Gamma(L^p) \otimes \Gamma((L')^p) \otimes \Gamma(L_\rho^{p-1}) \to \Gamma((L \otimes L')^p \otimes L_\rho^{p-1})$$

is onto. Again by Lemma 2, it is enough to show that

$$\Gamma(L^p) \otimes \Gamma((L')^p \otimes L_\rho^{p-1}) \to \Gamma((L \otimes L')^p \otimes L_\rho^{p-1})$$

is onto. By iterating this result replacing L by L^p and L' by $L'^p \otimes L_\rho^{p-1}$ and so on, we see that it is enough to show that

$$\Gamma(L^{p^r}) \otimes \Gamma((L')^{p^r} \otimes L_\rho^{p^r-1}) \to \Gamma((L \otimes L')^{p^r} \otimes L_\rho^{p^r-1})$$

is onto for large r. If L is ample, this is obvious since on $X \times X$, the bundle $\tilde{L} = p_1^* L \otimes p_2^*(L' \otimes L_\rho)$ is ample and hence we have only to choose r large enough so that $H^1(\tilde{L}^{p^r} \otimes p_2^* L_\rho^{-1} \otimes I_\Delta) = 0$ where I_Δ is the ideal sheaf of the diagonal. Thus we have proved the assertion when one of L, L' is ample. In general, we have only to show that

$$\Gamma(L^p) \otimes \Gamma((L')^p \otimes L_\rho^{p-1}) \to \Gamma(L^p \otimes (L')^p \otimes L_\rho^{p-1})$$

is onto. Since $(L')^p \otimes L_\rho^{p-1}$ is ample, this follows from the earlier assertion.

iii) Since G/Q is nonsingular this follows at once from ii).

3. Schubert varieties

We now turn to similar questions on Schubert varieties. We will prove the following theorem.

Theorem 2. *Let k be a field of arbitrary characteristic, zero or positive. Let G be a reductive group over k and Q be a parabolic subgroup. Let X be a Schubert variety in G/Q,* i.e. *the closure of a B-orbit in G/Q. Let L, L' be effective line bundles on G/Q. Then we have*

i) $H^i(X, L)=0$ *for* $i>0$

ii) $\Gamma(G/Q, L)\to\Gamma(X, L)$ *is surjective*

iii) $\Gamma(X, L)\otimes\Gamma(X, L')\to\Gamma(X, L\otimes L')$ *is surjective.*

Proof. The Schubert variety X can be constructed as a scheme flat over (at least) a nonempty open subset of Spec $\mathbb{Z}$ (see Lemma 3, p. 22 in [4]). Thus as we have remarked in the proof of Theorem 1 we can assume the field k is of characteristic $p>0$.

Next we show how to reduce to the case $Q=B$. Consider the fibration π: $G/B\to G/Q$. Let $X'=\pi^{-1}(X)$. Then π: $X'\to X$ is a Q/B fibration and X' is irreducible, closed and B-invariant. Hence it is a Schubert variety in G/B. Moreover as in the proof of Theorem 1 Leray spectral sequence gives that $H^i(X, L)=H^i(X', \pi^* L)$. Then it follows easily that the theorem for G/B implies it for G/Q. So we only deal with the case G/B below.

We have given a splitting $F_*(\mathcal{O}(D))\to\mathcal{O}$ on G/B (see Proposition 1 above). The induced splitting $F_*\mathcal{O}\to\mathcal{O}$ on G/B has been shown in [4] to be compatible with the Schubert varieties $X_i(=\bar{\psi}(Z_{[1,\ldots,i]})$ in the notation of [4], [2]). Since by suitably choosing the reduced expression for the longest element of the Weyl group we can get our given X as one of the X_i (cf. [2], §3, Lemma 3, p. 71) we can assume that this splitting is compatible with X. (In fact one can show that this splitting is compatible with *any* Schubert variety. See [5]). Thus there is a splitting $F_*(\mathcal{O}(D))\to\mathcal{O}$ on G/B such that F_*I maps on I, where I is the ideal sheaf of X. Since this factors through $F_*(I(D))\to\mathcal{O}$, its image is an ideal containing I. If D is so chosen as not to contain X (which is possible since D is very ample) then the image of $F_*(I(D))$ equals I outside the support of D and hence everywhere (see proof of Lemma 1 in [4]). In other words, the splitting $F_*(\mathcal{O}(D))\to\mathcal{O}$ takes $F_*(I(D))$ into I giving a splitting of the inclusion $I\to F_*(I(D))$.

Now tensor the split sequence

$$0\to I\to F_*(I(D))\to C'\to 0$$

by L and take cohomology. Then we get that $H^i(G/B, L\otimes I)$ is a direct summand of $H^i(G/B, L\otimes F_*(I(D)))\simeq H^i(G/B, L^p\otimes L_\rho^{p-1}\otimes I)$. Replacing L by $L^{p^{r-1}}\otimes L_\rho^{p^{r-1}-1}$ we see that $H^i(G/B, L^{p^r}\otimes L_\rho^{p^r-1}\otimes I)=0$ implies that $H^i(G/B, L^{p^{r-1}}\otimes L_\rho^{p^{r-1}-1}\otimes I)=0$ and hence by induction that $H^i(G/B, L\otimes I)=0$. But $L^{p^r}\otimes L_\rho^{p^r-1}\otimes I=(L\otimes L_\rho)^{p^r}\otimes(L_\rho^{-1}\otimes I)$ so that, $L\otimes L_\rho$ being ample, we do have $H^i(G/B, L^{p^r}\otimes L_\rho^{p^r-1}\otimes I)=0$ for $i>0$ and large enough r. Hence we conclude that $H^i(G/B, L\otimes I)=0$ for $i>0$. Using this in the cohomology sequence of

$$0\to L\otimes I\to L\to L|X\to 0$$

Theorem 2 follows from Theorem 1.

Theorem 3. *Let k be a field of arbitrary characteristic. Let G be a reductive group over k and Q a parabolic subgroup. Then any Schubert variety X in G/Q is normal. Moreover the linear system on X given by an ample line bundle on G/Q imbeds X as a projectively normal variety.*

Proof. We only have to prove the normality of X. The projective normality then follows from Theorem 2, part iii).

As in the proof of Theorem 2 for any Schubert variety $X \subset G/Q$, $\pi^{-1}(X)$ is a Schubert variety in G/B, where $\pi\colon G/B \to G/Q$ is the canonical map. Since $\pi\colon \pi^{-1}(X) \to X$ is a fibration with the nonsingular Q/B as fibre, if we prove $\pi^{-1}(X)$ is normal it then follows that X is normal. Thus we are reduced to the case $Q = B$.

Let X be the Schubert variety in G/B corresponding to $\omega \in W$. We proceed by induction on $\dim X$ $(= l(\omega)$, the length of $\omega)$. If $l(\omega)=0$ there is nothing to prove. Otherwise we can find a simple root α such that $\omega = \omega' s_\alpha$, with $l(\omega') = l(\omega) - 1$. Let P_α be the parabolic subgroup $B \cup B s_\alpha B$. Let Y be the Schubert variety corresponding to ω'. Then under $\pi\colon G/B \to G/P_\alpha$ both X and Y have the same image, say Y', with $X = \pi^{-1}(Y')$ a $\mathbb{P}^1$-bundle over Y' and Y birational to Y' (see e.g. [3] §2, Lemma 1, p. 562). Thus to prove X is normal it is enough to prove Y' is normal.

Let L be an ample line bundle on G/P_α. Consider the diagram

$$\begin{array}{ccc} \Gamma(G/B, \pi^* L^n) & \twoheadrightarrow & \Gamma(Y, \pi^* L^n) \\ \wr\big\uparrow & & \big\uparrow \\ \Gamma(G/P_\alpha, L^n) & \twoheadrightarrow & \Gamma(Y', L^n) \end{array}$$

Since the top horizontal arrow is surjective by Theorem 2 and bottom arrow is surjective because of ampleness it follows that

$$\Gamma(Y', L^n) \to \Gamma(Y, \pi^* L^n) = \Gamma(Y', L^n \otimes \pi_* \mathcal{O}_Y)$$

is surjective (and hence bijective). Since this is true for all n, and L is ample on Y', this implies that $\pi_* \mathcal{O}_Y = \mathcal{O}_{Y'}$ (where $\pi\colon Y \to Y'$ is given by natural map $G/B \to G/P_\alpha$). Since $\dim Y = l(\omega') = \dim X - 1$, by the induction hypothesis Y is normal. Then $\pi_* \mathcal{O}_Y = \mathcal{O}_{Y'}$ implies Y' is normal, as was to be proved.

Remark. In [5] the following results will be proved.

i) The diagonal $G/B \to G/B \times G/B$ is compatibly split. This easily yields the projective normality. In fact for any Schubert variety X, the diagonal is compatibly split.

ii) It is easy to see that the splitting of G/B given in [4] compatibly splits all Schubert varieties. Then it would follow that the intersection of any set of Schubert varieties is reduced.

iii) Schubert varieties are Cohen-Macaulay.

References

1. Andersen, H.H.: The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B. Ann. Math. **112**, 113–121 (1980)
2. Demazure, M.: Désingularisations des variétés de Schubert généralisés, Ann. E.N.S. **7**, 53–88 (1974)
3. Kempf, G.: Linear systems on homogeneous spaces. Ann. Math. **103**, 557–591 (1976)
4. Mehta, V.B., Ramanathan, A.: Frobenius splitting and cohomology vanishing for Schubert varieties.
5. Ramanathan, A.: Schubert varieties are arithmetically Cohen-Macaulay. (To appear)
6. Seshadri, C.S.: Line bundles on Schubert varieties. (To appear)

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