Tohoku Math. J.
64 (2012), 125–140

PROJECTIVE NORMALITY OF TORIC 3-FOLDS WITH NON-BIG ADJOINT HYPERPLANE SECTIONS

SHOETSU OGATA

(Received February 18, 2009, revised August 23, 2011)

Abstract. Let L be an ample line bundle on a nonsingular toric 3-fold. We show that if the adjoint bundle of L has no global sections, then L is normally generated. Even if the adjoint bundle is effective, it is shown that L is normally generated if it is not big.

Introduction. It is known that any ample line bundle on a projective nonsingular toric variety is very ample (cf. [11, Corollary 2.15]). A line bundle L on a projective variety is called *normally generated* if the multiplication map $\Gamma(L)^{\otimes l} \to \Gamma(L^{\otimes l})$ is surjective for all $l \geq 1$. If an ample line bundle L is normally generated, then L is very ample. Furthermore, if the variety X is normal, then a normally generated ample line bundle L defines the embedding $\Phi_L : X \to \boldsymbol{P}(\Gamma(L))$ of X as a *projectively normal* variety, i.e., the homogeneous coordinate ring is a normal ring.

When we would ask questions about defining ideals of projective varieties, we often assume that the varieties are projectively normal. For example, Sturmfels [13] asked whether any projective nonsingular toric varieties embedded by normally generated ample line bundles are defined by only quadrics (see also Cox [2]). In practice, it is difficult to check the condition that the variety is projectively normal, or equivalently the very ample line bundle is normally generated.

Only few criteria of normal generation are known even for toric varieties. Koelman [7] showed that any ample line bundle on a toric surface is normally generated. Ewald and Wessels [3] showed that, for an ample line bundle L on a projective toric variety of dimension n, the twisted bundle $L^{\otimes l}$ is very ample for $l \geq n-1$, and Nakagawa [9] proved that $L^{\otimes l}$ is normally generated for these l (see also [10, Theorem 1]). More precisely, he proved that the multiplication map

$$\Gamma(L) \otimes \Gamma(L^{\otimes l}) \to \Gamma(L^{\otimes(l+1)})$$

is surjective for $l \geq n-1$. Ogata [12] showed that, if a toric 3-fold is the quotient of the projective 3-space $\boldsymbol{P}^3$ by an action of a finite abelian group, then a very ample line bundle on it is normally generated. Note that weighted projective 3-spaces are such toric varieties.

2000 *Mathematics Subject Classification.* Primary 14M25; Secondary 52B20.
Key words and phrases. Toric varieties, projectively normal.
Partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

A polarized toric variety (X, L) of dimension n corresponds to an integral convex polytope P of dimension n in $\boldsymbol{R}^n$. Then L is normally generated if and only if the equalities

$$(lP) \cap \mathbf{Z}^n + P \cap \mathbf{Z}^n = ((l+1)P) \cap \mathbf{Z}^n$$

hold for all positive integers l. If the condition holds, then P is called *normally generated* (cf. Definitions 1.2 and 1.3).

In this paper we shall prove the following theorems.

THEOREM 0.1. *Let X be a nonsingular projective toric variety of dimension three. Then any ample line bundle L on X satisfying $H^0(X, L \otimes \mathcal{O}_X(K_X)) = 0$ is normally generated.*

The theorem is proved by showing that a nonsingular integral convex polytope of dimension three without interior lattice points is normally generated, which is given as Proposition 2.7 in Section 2.

THEOREM 0.2. *Let L be an ample line bundle on a nonsingular projective toric variety X of dimension three. If $H^0(X, L \otimes \mathcal{O}_X(K_X)) \neq 0$ and $L \otimes \mathcal{O}_X(K_X)$ is not big, then L is normally generated.*

Theorem 0.2 is also interpreted as follow. A nonsingular integral convex polytope of dimension three with non-empty internal polytope of dimension less than three is normally generated. See Corollary 3.2 in Section 3.

For a proof of Theorems 0.1 and 0.2 we use the following result.

THEOREM 0.3 (Fakhruddin [4]). *Let X be a nonsingular projective toric surface. Then, for an ample line bundle A and a nef line bundle B on X, the multiplication map*

$$\Gamma(A) \otimes \Gamma(B) \to \Gamma(A \otimes B)$$

is surjective.

This theorem means that, for a nonsingular integral convex polygon P and an integral convex polygon Q whose inner fan is a subfan of that of P, the equality

$$P \cap \mathbf{Z}^2 + Q \cap \mathbf{Z}^2 = (P + Q) \cap \mathbf{Z}^2$$

holds, where it includes the case when Q is a line segment. Kondo and Ogata [8], and Haase, Nill, Paffenholz and Santos [6] generalized this to the case of singular toric surfaces.

Since the statements in the theorems can be interpreted into equalities on lattice points of integral convex polytopes, we need to investigate properties of convex polytopes of dimension three for the proof of Theorems 0.1 and 0.2.

The structure of this paper is as follows: In Section 1, we recall basic results about toric varieties and line bundles on them. In Section 2, we give a classification of nonsingular integral convex polytopes of dimension three without interior lattice points (Proposition 2.3). By using the classification, we prove Theorem 0.1. In Section 3, we treat the case that the adjoint bundle of L has a non-trivial global section and give a characterization of the internal polytope (Proposition 3.1). From this, we prove that P is normally generated if the internal polytope has dimension less than three (Corollary 3.2).

The author would like to thank Professor Masanori Ishida for his comments for improving the presentation and for his help in the last part of the proof of Proposition 3.1.

1. Projective toric varieties. In this section, we recall the facts on toric varieties which we need in this paper. For the proofs, see Oda's book [11] or Fulton's book [5]. For simplicity, we assume that toric varities are defined over the complex number field.

Let N be a free $\boldsymbol{Z}$-module of rank n, M its dual and $\langle , \rangle : M \times N \to \boldsymbol{Z}$ the canonical pairing. By the scalar extension to the field $\boldsymbol{R}$ of real numbers, we have real vector spaces $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $M_{\boldsymbol{R}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. We denote also by $\langle , \rangle$ the pairing of $M_{\boldsymbol{R}}$ and $N_{\boldsymbol{R}}$ defined by the scalar extension. Let $T_N := N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^* \cong (\boldsymbol{C}^*)^n$ be the algebraic torus over the field $\boldsymbol{C}$ of complex numbers, where $\boldsymbol{C}^*$ is the multiplicative group of $\boldsymbol{C}$. Then the character group $\mathrm{Hom}_{\mathrm{gr}}(T_N, \boldsymbol{C}^*)$ of T_N is identified with M and $T_N = \mathrm{Spec}\, \boldsymbol{C}[M]$. For $m \in M$ we denote $\mathbf{e}(m)$ as the character of T_N. Let Δ be a finite complete fan in N and $X = T_N\mathrm{emb}(\Delta)$ a complete toric variety of dimension n (see [11, Section 1.2] or [5, Section 1.4]). We note that a toric variety defined by a fan is always normal.

Let L be an ample line bundle on X. Then we have an integral convex polytope P in $M_{\boldsymbol{R}}$ with

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \boldsymbol{C}\mathbf{e}(m)\,, \tag{1}$$

where $\mathbf{e}(m)$ are considered as rational functions on X because they are functions on an open dense subset T_N of X (see [11, Section 2.2] or [5, Section 3.5]). Here an integral convex polytope P in $M_{\boldsymbol{R}}$ is the convex hull $\mathrm{Conv}\{u_1, u_2, \dots, u_s\}$ in $M_{\boldsymbol{R}}$ of a finite subset $\{u_1, u_2, \dots, u_s\} \subset M$. We note that $\dim_{\boldsymbol{R}} P = \dim X$. The l-th power $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\boldsymbol{R}}; x \in P\}$.

DEFINITION 1.1. An integral convex polytope P in $M_{\boldsymbol{R}}$ of dimension n is called *nonsingular* if, for each vertex u of P, the cone $\boldsymbol{R}_{\geq 0}(P-u) := \{\lambda(x-u) \in \boldsymbol{R}^n; x \in P \text{ and } \lambda \geq 0\}$ is nonsingular, that is, there exists a $\boldsymbol{Z}$-basis $\{m_1, \dots, m_n\}$ of M such that

$$\boldsymbol{R}_{\geq 0}(P-u) = \boldsymbol{R}_{\geq 0}m_1 + \cdots + \boldsymbol{R}_{\geq 0}m_n\,.$$

A face $F \subset P$ is said to be *nonsingular* if it is nonsingular with respect to the sublattice $\boldsymbol{R}(F) \cap M$, where $\boldsymbol{R}(F)$ is the smallest affine subspace of $M_{\boldsymbol{R}}$ containing F.

We note that a nonsingular polytope P is *simple*, that is, each vertex of P is contained in just n faces of dimension $n-1$, or equivalently contained in just n faces of dimension one.

The ample line bundle L is very ample if and only if, for each vertex u of P, the semigroup $\boldsymbol{R}_{\geq 0}(P-u) \cap M$ is generated by $(P-u) \cap M$, i.e., all lattice points x in the semigroup are represented as a finite sum of elements $y_1, \dots, y_s$ in $(P-u) \cap M$. An ample line bundle on a nonsingular complete toric variety is very ample (see [11, Corollary 2.15]).

DEFINITION 1.2. An ample line bundle L on a projective variety X is called *normally generated* if the multiplication map $\mathrm{Sym}^l H^0(X, L) \to H^0(X, L^{\otimes l})$ is surjective for all $l \geq 1$.

DEFINITION 1.3. An integral convex polytope in $M_{\boldsymbol{R}}$ is called *normally generated* if for the corresponding polarized toric variety (X, L) the ample line bundle L is normally generated.

REMARK 1.4. If X is toric and if (X, L) corresponds to an integral convex polytope P in $M_{\boldsymbol{R}}$ satisfying (1), then the normal generation of L is equivalent to the condition that for all $l \geq 1$ every element $v \in lP \cap M$ is written as a sum $v = u_1 + \cdots + u_l$ of l lattice points $u_i \in P \cap M$. This is equivalent to the condition

$$(lP) \cap M + P \cap M = ((l+1)P) \cap M \quad \text{for all } l \geq 1 . \tag{2}$$

2. Convex polytopes without interior lattice points. In this section we prove Theorem 0.1. In the theorem we assume that $\Gamma(L \otimes \mathcal{O}_X(K_X)) = 0$.

Let X be a nonsingular projective toric 3-fold and L an ample line bundle on X. Let P be the integral convex polytope of dimension three corresponding to the polarized toric variety (X, L). From [11, Theorem 3.6] we have

$$\Gamma(X, L \otimes \mathcal{O}_X(K_X)) \cong \bigoplus_{m \in \mathrm{Int}(P) \cap M} \boldsymbol{C}\mathbf{e}(m) . \tag{3}$$

Hence we see that $\Gamma(L \otimes \mathcal{O}_X(K_X)) = 0$ is equivalent to $\mathrm{Int}(P) \cap M = \emptyset$. In this section, we consider an integral convex polytope P of dimension three satisfying the condition $\mathrm{Int}(P) \cap M = \emptyset$.

Before investigating convex polytopes of dimension three, we have to classify nonsingular integral convex polytopes of dimension two without lattice points in the interior. Set $G_0 := \mathrm{Conv}\{(0,0), (1,0), (0,1)\}$ and $G_{a,b} := \mathrm{Conv}\{(0,0), (0,1), (a,1), (b,0)\}$ for $a \geq b \geq 1$.

The following lemma is checked easily (cf. [1]).

LEMMA 2.1. *A nonsingular integral convex polytope of dimension two without lattice points in its interior coincides with one of G_0, $2G_0$ and $G_{a,b}$ up to affine transformations of $\boldsymbol{Z}^2$.*

See Figure 1. The convex polygons G_0 and $2G_0$ correspond to the projective plane $\boldsymbol{P}^2$ with $\mathcal{O}(1)$ and $\mathcal{O}(2)$, respectively. $G_{a,b}$ corresponds to the Hirzebruch surface $\boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$ of degree $a - b$ with a suitable ample line bundle.

First we introduce typical examples of nonsingular integral convex polytope P of dimension three with $\mathrm{Int}(P) \cap M = \emptyset$. Set $P_1 := \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ the basic 3-simplex. Then P_1 defines the polarized toric variety $(\boldsymbol{P}^3, \mathcal{O}(1))$. Since the canonical line bundle of $\boldsymbol{P}^3$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^3}(-4)$, we see that lP_1 does not contain lattice points in its interior for $l = 1, 2, 3$. Set $P_2 := 2P_1$ and $P_3 := 3P_1$. We note that lP_1 is normally generated for all $l \geq 1$.

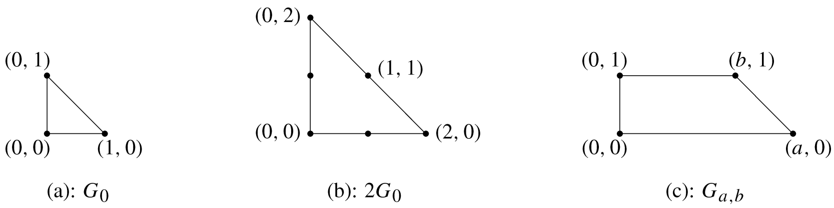


FIGURE 1. Nonsingular integral polygons without lattice points in their interior.

Set $P_2^{(1)} := \mathrm{Conv}\{(0,0,0), (2,0,0), (0,2,0), (1,0,1), (0,1,1), (0,0,1)\}$ and $P_3^{(1)} :=$ Conv $\{(0,0,0), (3,0,0), (0,3,0), (1,0,2), (0,1,2), (0,0,2)\}$. The polytope $P_2^{(1)}$ is obtained by cutting off the top of P_2 at $z = 1$. $P_3^{(1)}$ is obtained by cutting P_3 at $z = 2$. In particular, we have that $\mathrm{Int}(P_2^{(1)}) \cap M = \mathrm{Int}(P_3^{(1)}) \cap M = \emptyset$. The convex polytopes $P_2^{(1)}$ and $P_3^{(1)}$ define the blowup of $\boldsymbol{P}^3$ at a T_N-invariant point. This is also a toric $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$, that is, $X \cong \boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(1))$. We also have convex polytopes defining the blowup of $\boldsymbol{P}^3$ at several points. We write as $P_3^{(1)} = (P_3) \cap (0 \le z \le 2)$. Then we set $P_3^{(2)} := P_3^{(1)} \cap (0 \le x \le 2)$, $P_3^{(3)} := P_3^{(2)} \cap (0 \le y \le 2)$ and $P_3^{(4)} := P_3^{(3)} \cap (1 \le x + y + z \le 3)$. See Figure 2 (a).

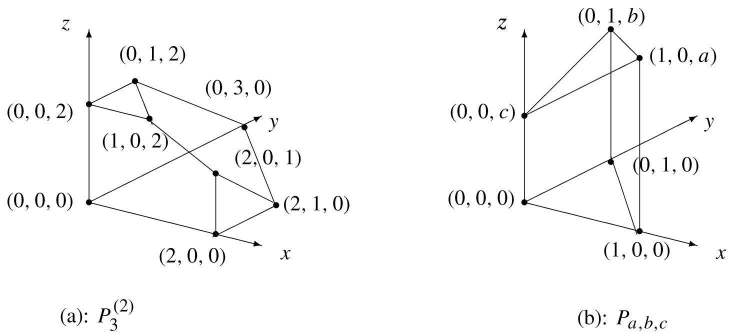


FIGURE 2. Typical P with $(\mathrm{Int}P) \cap M = \emptyset$.

For integers $a, b, c \ge 1$, set

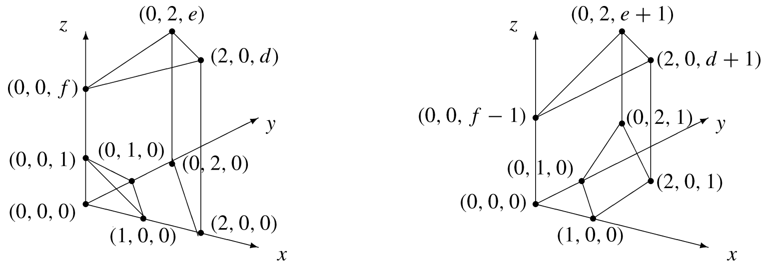
$$P_{a,b,c} := \mathrm{Conv}\{(0,0,0), (1,0,0), (0,1,0), (1,0,a), (0,1,b), (0,0,c)\}.$$

This is a bounded triangular prism over the basic triangle with the three edges of lengths a, b and c. See Figure 2 (b). The convex polytope $P_{a,b,c}$ defines a toric $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$, that is, $X \cong \boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$. For integers $d, e, f \ge 1$, set

$$Q_{d,e,f} := \mathrm{Conv}\{(0,0,0), (2,0,0), (0,2,0), (2,0,d), (0,2,e), (0,0,f)\}.$$

Then we see that $Q_{2a,2b,2c} = 2P_{a,b,c}$ and that $Q_{2a-1,2b-1,2c-1}$ also corresponds to an ample line bundle on $\boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$. We note that $d + f$ and $e + f$ are even integers if

$Q_{d,e,f}$ is nonsingular, for the points $(1, 0, (d + f)/2)$ and $(0, 1, (e + f)/2)$ must be lattice points. See Figure 3 (a).



(a): $Q^{(1)}_{d,e,f}$ cut off at $(0, 0, 0)$ (b): $Q^{(1)}_{d,e,f}$ with a triangular bottom

FIGURE 3. The shape of $Q^{(1)}_{d,e,f}$.

We obtain nonsingular polytopes from $Q_{d,e,f}$ by cutting off one or two basic 3-simplices. For $f \geq 2$, set $Q^{(1)}_{d,e,f} := Q_{d,e,f} \cap (x + y + z \geq 1)$. See Figure 3 (a). Since $Q^{(1)}_{d,e,f}$ is nonsingular, we see that $d + f$ and $e + f$ are also even integers. If f is even, then set $d = 2a$, $e = 2b$ and $f = 2c$. If f is odd, then set $d + 1 = 2a$, $e + 1 = 2b$ and $f + 1 = 2c$. Then the polytope $Q^{(1)}_{d,e,f}$ defines the blowup of $\boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ at a point on a T_N-invariant fiber. We say that $Q^{(1)}_{d,e,f}$ is obtained from $Q_{d,e,f}$ by cutting off one basic 3-simplex at the vertex $(0, 0, 0)$. Note that $Q^{(1)}_{d,e,f}$ coincides with $Q_{d+1,e+1,f-1} \cap (x + y - z \leq 1)$ by a suitable affine transform of M. See Figure 3 (b).

If one cuts off one more basic 3-simplex from $Q^{(1)}_{d,e,f}$ at the vertex $(2, 0, 0)$ or $(0, 2, 0)$ in Figure 3 (a), then the resulting polytope is singular. Set $Q^{(2)}_{d,e,f}$ the polytope obtained from $Q^{(1)}_{d,e,f}$ by cutting off one basic 3-simplex at one of the vertices $(2, 0, d)$, $(0, 2, e)$ (if $d, e \geq 2$) and $(0, 0, f)$ (if $f \geq 3$) of the top triangle. Then the polytope $Q^{(2)}_{d,e,f}$ defines the blowup of $\boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ at two points contained in the distinct two T_N-invariant fibers.

We call a face of dimension two a *facet* and a face of dimension one an *edge*.

LEMMA 2.2. *Let P is a nonsingular polytope of dimension three contained in the prism $\{x \geq 0, y \geq 0, z \geq 0, x + y \leq 2\}$. Assume that P has three facets F_1, F_2 and F_3 contained in the planes $(x = 0)$, $(y = 0)$ and $(x + y = 2)$, respectively. Furthermore, we assume that P has three edges E_1, E_2 and E_3 contained in the lines $(x = 2, y = 0)$, $(x = 0, y = 2)$ and $(x = y = 0)$, respectively. Then, by a suitable affine transformation of M, P coincides with one of $Q_{d,e,f}$, $Q^{(1)}_{d,e,f}$ and $Q^{(2)}_{d,e,f}$ such that $d - f$ and $e - f$ are even.*

PROOF. We may assume that the origin $(0, 0, 0)$ is a vertex of P and the two edges at the origin, besides E_3, contain $(1, 0, 0)$ and $(0, 1, 0)$, respectively, since P is nonsingular.

Since F_2 (resp. F_1) is nonsingular in the plane $(y = 0)$ (resp. $(x = 0)$), the shape of F_2 (resp. F_1) near the line $(z = 0)$ is one of the Figure 4.

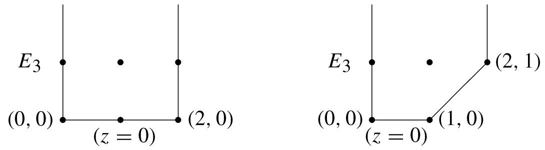


FIGURE 4. The shape of F_2 or F_1 near the line $(z = 0)$.

If both of the points $(2, 0, 0)$ and $(0, 2, 0)$ are vertices of P, then $P \cap (z = 0)$ is $2G_0$ (see Figure 1).

If both of the points $(2, 0, 0)$ and $(0, 2, 0)$ are not vertices of P, then both of the points $(1, 0, 0)$ and $(0, 1, 0)$ are vertices and the bottom facet $F_0 = P \cap (z = 0)$ is the basic triangle. In this case, $P \cap (z = 1)$ is a triangle isomorphic to $2F_0$ since $P \cap (x + y = 2) \neq \emptyset$ and since the points $(1, 0, 0)$ and $(0, 1, 0)$ are nonsingular vertices of P. The shape of P near the bottom is as Figure 3 (b).

If the point $(2, 0, 0)$ is a vertex and the point $(0, 2, 0)$ is not a vertex of P, then the points $(0, 1, 0)$ and $(1, 1, 0)$ are vertices of P and the bottom facet F is a tetragon. Since $(0, 1, 0)$ is a nonsingular vertex, the edge from $(0, 1, 0)$ on the facet F_1 has the direction $(0, b, 1)$. Since the facet $F_3 = P \cap (x + y = 2)$ is not empty, we have $0 \leq b \leq 1$. If $b = 0$, then P is contained in $\{0 \leq y \leq 1\}$. This contradicts to the assumption that P contains the edge E_2. If $b = 1$, then the point $(0, 2, 1)$ is a vertex of P. The shape of P near the bottom is as Figure 3 (a) by a suitable affine transform of M.

Since the condition of P near the top is the same, if P satisfies our assumption, then it is one of $Q_{d,e,f}$, $Q^{(1)}_{d,e,f}$ and $Q^{(2)}_{d,e,f}$ with even $d - f$ and $e - f$. □

Let F_0 be a facet of P. We take a coordinate of M such that $P \subset (z \geq 0)$ and $F_0 \subset (z = 0)$. Since P is nonsingular, F_0 is nonsingular in the plane $(z = 0)$. We fix a notation of lattice points in P near F_0. Denote $\{u_0, u_1, \dots, u_r\}$ the set of vertices of F_0. Assume that u_i is adjacent to u_{i+1} for $i = 0, 1, \dots, r$ (set $u_{r+1} = u_0$). Take $m_1 \in M$ on the edge $\overline{u_0u_1}$ of F_0 and $m_2 \in M$ on $\overline{u_0u_r}$ so that $\{m_1 - u_0, m_2 - u_0\}$ be a $\mathbf{Z}$-basis of $M \cap (z = 0)$. Since P is nonsingular, we can take the lattice point $m_3 \in M \cap P$ on the third edge at u_0 so that $\{m_1 - u_0, m_2 - u_0, m_3 - u_0\}$ is a $\mathbf{Z}$-basis of M. Let (x, y, z) be the coordinates of $M_{\boldsymbol{R}} \cong \boldsymbol{R}^3$ with respect to this basis. For each u_i we take the point $w_i \in P \cap M$ with the coordinate $z = 1$ on the third edge at u_i. See Figure 5.

Now set $P(F_0) := (0 \leq z \leq 1) \cap P$ and $G := (z = 1) \cap P \subset P(F_0)$. Then $P(F_0)$ is an integral convex polytope with the parallel faces F_0 and G. If $\dim G \leq 1$, then $P(F_0) = P$ and G is a face of P. When $\dim G = 0$, that is, when $w_0 = w_1 = \cdots = w_r$, we see that $r = 2$ and $P = P_1$ since P is nonsingular. When $\dim G = 1$, we see $r = 3$ since $\{w_0, \dots, w_r\}$ are vertices of G and P is simple. In this case, we may assume $w_0 = w_1$, then we see that

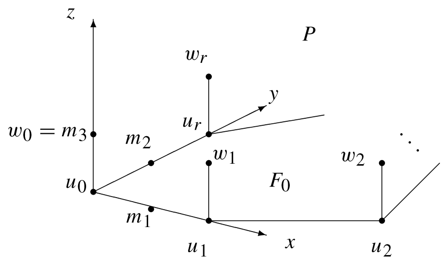


FIGURE 5. P and F_0 centered at u_0.

$u_1 = m_1$ since P is nonsingular. The point $w_0 = w_1$ is one end of G. Since the other end of G is $w_2 = w_3$, the vertex u_2 has the coordinate of the form $(1, a, 0)$. If we write as $u_3 = (0, b, 0)$, $w_2 = w_3 = (0, c, 1)$, then we see that $P \cong P_{a,b,c}$ by a change of coordinates.

We assume that $\dim G = 2$. If G is a facet of P, then all w_i's are distinct since P is simple. On the other hand, we note that if all w_i's are distinct, then G has the same number of vertices as that of F_0 and G is nonsingular. Furthermore, $P(F_0)$ defines a toric 3-fold which is a toric $\boldsymbol{P}^1$-bundle over a toric surface Y defined by F_0.

When $G = (z = 1) \cap P$ is not a face of P, it may happen that $w_0 = w_1$. In this case, we see that $u_1 = m_1$ because the facet $\mathrm{Conv}\{u_0, u_1, w_0\}$ of P is nonsingular. If $w_0 = w_1 = w_2$, then $r = 2$ and $u_2 = m_2$, that is, $P = P_1$. Thus we see that if $\dim G = 2$ and if $w_0 = w_1$, then $w_2 \neq w_1$. See Figure 6.

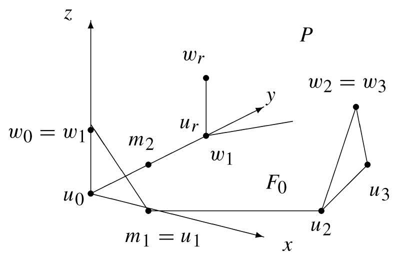


FIGURE 6. P and F_0 centered at u_0.

In general, if $w_i \neq w_{i+1}$, then the edge $\overline{w_i w_{i+1}}$ of G is parallel to the edge $\overline{u_i u_{i+1}}$ of F_0.

PROPOSITION 2.3. *Let P be a nonsingular integral convex polytope in $M_{\boldsymbol{R}}$ of dimension three. We assume that P has no lattice points in its interior. Then, by a suitable affine transformation of M, P coincides with one of the following.*

(1) *The convex hull P of parallel two nonsingular facets F_0 and F_1 of distance one such that they define a same 2-dimensional nonsingular fan. This P defines a toric $\boldsymbol{P}^1$-bundle over the nonsingular toric surface defined by this fan.*

(2) *P_1, P_2 or P_3. The convex polytope $P_l = lP_1$ corresponds to $(\boldsymbol{P}^3, \mathcal{O}(l))$.*

(3) *$P_{a,b,c}$ or $Q_{d,e,f}$ with even $d-f$ and $e-f$. The convex polytopes $P_{a,b,c}$ and $Q_{d,e,f}$ define the same toric $\boldsymbol{P}^2$-bundle $\boldsymbol{P}(\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c))$ over $\boldsymbol{P}^1$ if $d-f=2(a-c)$, $e-f=2(b-c)$.*

(4) *$P_3^{(i)}$ for $i=1,\dots,4$. The convex polytope $P_3^{(i)}$ defines the blowup of $\boldsymbol{P}^3$ at T_N-invariant i points. In this case, we have $P=P(F_0)\cup P(F_1)$ by taking the parallel two facets F_0 and F_1 of distance two.*

(5) *$Q_{d,e,f}^{(i)}$ with $d-f=2(a-c)$, $e-f=2(b-c)$ for $i=1,2$. The convex polytope $Q_{d,e,f}^{(1)}$ defines the blowup of $\boldsymbol{P}(\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c))$ at a T_N-invariant point on a T_N-invariant fiber. $Q_{d,e,f}^{(2)}$ defines the blowup of $Q_{d,e,f}^{(1)}$ at a T_N-invariant point on the other T_N-invariant fiber. $Q_{d,e,f}^{(i)}$ has three facets F_1, F_2 and F_3 contained in the planes $(x=0)$, $(y=0)$ and $(x+y=2)$, respectively. We have $P=P(F_1)\cup P(F_2)$.*

PROOF. First we assume that P is contained in the prism $R:=\{x\geq 0, y\geq 0, z\geq 0, x+y\leq 2\}$ and P has three facets F_1, F_2 and F_3 contained in the planes $(x=0)$, $(y=0)$ and $(x+y=2)$, respectively. We may assume that the origin $(0,0,0)$ is a vertex of P and the points $(1,0,0)$ and $(0,1,0)$ are contained in the boundary of P since P is nonsingular.

If P has three edges E_1, E_2 and E_3 in the sense of Lemma 2.2, then it is one of $Q_{d,e,f}$, $Q_{d,e,f}^{(1)}$ and $Q_{d,e,f}^{(2)}$ with even $d-f$ and $e-f$ from Lemma 2.2.

If P does not have the edge E_1, then it is contained in $\{0\leq x\leq 1\}$. In this case, the facet contained in the plane $(x=1)$ is a tetragon isomorphic to $G_{a,b}$ for some $a,b\geq 1$. Since P is nonsingular, the facet F_1 is a tetragon whose edges are parallel to those of $G_{a,b}$.

If P does not have the edges E_1 and E_2, then it does not have the facet F_3.

Next we use the notation described just before Proposition 2.3. The vertex u_0 of P is the origin, P is contained in the region $\{x\geq 0, y\geq 0, z\geq 0\}$, the facet $F_0=P\cap(z=0)$ has the vertices $\{u_0,u_1,\dots,u_r\}$ and P has three edges from u_0 with the directions $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Set $G=P\cap(z=1)$. If $\dim G\leq 1$, then we see that $P=P_1$, or $P\cong P_{a,b,c}$ as discussed before this proposition. In the following we assume $\dim G=2$.

(I) We treat the case that F_0 and G have the same number of edges. Then G is nonsingular as we noted before. If G is a facet of P, then it is in the case (1).

Assume that G is not a facet of P. Then the interior lattice points $\mathrm{Int}(G)\cap M$ are contained in the interior of P. Thus by our assumption G does not contain lattice points in its interior. From Lemma 2.1, G coincides with one of G_0, $2G_0$ and $G_{a,b}$.

(a) The case that $G\cong G_0$: We claim that $P=P_2$, or $P_{a,b,c}$.
To see this, note that $F_0\cong kG_0$ for a positive integer k since F_0 and G have parallel edges. In this case, $P(F_0)$ is given as $0\leq z\leq 1$, $0\leq x$, $0\leq y$ and $x+y+(k-1)z\leq k$. We divide it into the cases $k=1$ and $k\geq 2$.

If $k=1$, then $P=P_{a,b,c}$.

If $k\geq 2$, then the three affine hyperplanes defined by the last three inequalities intersect in the point $(0,0,k/(k-1))$, whose z-coordinate is less than or equal to 2, with equality only for $k=2$. Since G is not a facet of P, there has to exist a vertex of P whose z-coordinate is greater than or equal to 2. Hence this implies $k=2$ and $P=2P_1=P_2$.

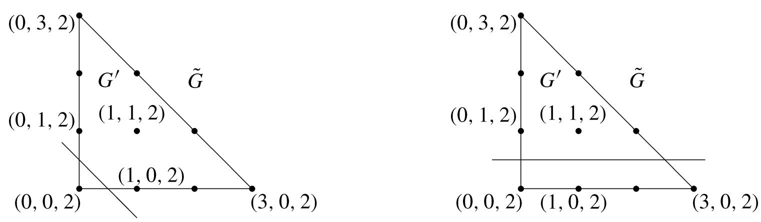
(b) The case that $G \cong 2G_0$: In this case, we note that $F_0 \cong kG_0$ for a positive integer k. By the same reason above, we have $k \leq 4$. If $k = 4$, then the point $(0, 0, 2)$ is a singular vertex of the cone over F_0. Hence, $1 \leq k \leq 3$. We consider each of the cases.

If $k = 3$, then P is contained in $3P_1 = P_3$. Set $F_1 := P \cap (x = 0)$. Then F_1 is contained in the triangle $\mathrm{Conv}\{(0, 0, 0), (0, 3, 0), (0, 0, 3)\}$. If the point $(0, 0, 1)$ is a vertex of F_1, then there has to exist an edge connecting $(0, 0, 1)$ with $(0, 1, 2)$ or $(0, 2, 1)$. If the edge connects with $(0, 1, 2)$, then the point $(0, 1, 2)$ is a singular vertex of F_1. The situation is the same in the facet $P \cap (y = 0)$. Thus, if the point $(0, 0, 1)$ is a vertex of P, then there have to exist two edges connecting $(0, 0, 1)$ with $(0, 2, 1)$ and $(2, 0, 1)$, hence, G is a facet of P. This contradicts to the assumption. None of points $(0, 0, 1)$, $(2, 0, 1)$, $(0, 2, 1)$ is a vertex of P. If the point $(0, 0, 2)$ is a vertex of P, then it has to be connected with $(1, 0, 2)$ by an edge, hence, we have $P = P_3^{(1)}$, otherwise $P = P_3$.

If $k = 2$, then P is contained in the prism $R = \{x \geq 0, y \geq 0, z \geq 0, x + y \leq 2\}$, which is contained in the case considered first.

If $k = 1$, then we claim that P is of the form (5), or $P_3^{(i)}$ for $i = 1, \dots, 4$ in (4). We assume that P is not of the form (5), that is, P is not contained in the prism $R = \{x \geq 0, y \geq 0, z \geq 0, x + y \leq 2\}$. See Figure 3 (b). Set $G' := P \cap (z = 2)$. Then G' is a rational polygon. We will prove that G' contains the point $(1, 1, 2)$ in its interior. We note that G' is contained in the triangle $\tilde{G} := \{0 \leq x, 0 \leq y, x + y \leq 3, z = 2\} \cong 3G_0$. The point $(1, 1, 2)$ is the center of $\tilde{G}$ and G' is obtained by cutting $\tilde{G}$ several times.

If the point $(0, 0, 1)$ is a vertex of P, then there have to exist two edges connecting $(0, 0, 1)$ with $(1, 0, a)$ and $(0, 1, b)$. Since G is not a facet of P, we see $a, b \geq 1$ and one of them is greater than 1. If both a and b are greater than 1, then G' is obtained by cutting off a triangle with the vertex $(0, 0, 2)$ and with two edges of lengths $1/(a-1)$ (≤ 1) and $1/(b-1)$ (≤ 1). See Figure 7 (a). In this case, G' contains $(1, 0, 2)$ and $(0, 1, 2)$, and the point $(1, 1, 2)$ is in the interior of G'. If $a = 1$ and $b = 2$, then G' is obtained by cutting $\tilde{G}$ at the line $\{y = 1\}$ and $G' \cong 2G_0$, and this implies that P has the form (5) after an affine transform of $\boldsymbol{Z}^3$, since P is contained in the prism $\{x \geq 0, z \geq 0, y \geq z - 1, x + y - z \leq 1\}$ with the cross-section isomorphic to $2G_0$. Hence, if $a = 1$, then $b \geq 3$. In this case G' is obtained by cutting $\tilde{G}$ at the line $\{y = 1/(b-1)\}$. See Figure 7 (b). The point $(1, 1, 2)$ remains in the interior of G'



(a): cut at $(a-1)x + (b-1)y = 1$ (b): cut at $(b-1)y = 1$

FIGURE 7. $\tilde{G}$ containing G'.

since $b-1 \geq 2$. Even if all three points $(0,0,1)$, $(2,0,1)$ and $(0,2,1)$ are vertices of P, then $(1,1,2)$ is the interior point of G' unless P is of the form (5).

Since P has no lattice points in its interior, G' is a facet of P. This implies that $a = b = 2$. This corresponds to $P_3^{(2)}$. If $(2,0,1)$ or $(0,2,1)$ is a vertex of P, then we have $P \cong P_3^{(3)}$ or $P \cong P_3^{(4)}$.

(c) The case that $G \cong G_{a,b}$: In this case, F_0 is a tetragon with two parallel edges. If P is contained in the region $\{0 \leq y \leq 1\}$, then it is in the case (1). If F_0 is a tetragon of the form $\mathrm{Conv}\{(0,0),(0,k),(a',k),(b',0)\}$ with $k \geq 2$, then $k = 2$ since G is not a facet of P. In this case, P is contained in the prism $\{x \geq 0, y \geq 0, z \geq 0, x+z \leq 2\}$, hence, we see that P is $Q_{d,e,f}$ or of the form (5) by exchanging the role of F_0 with the facet of P contained in the plane $(x = 0)$.

(II) Next we consider the case that $w_i = w_j$ for some $i \neq j$. By our numbering we may set $j = i+1$. By an affine transform of M, we can set $u_i = u_0$ and $u_{i+1} = u_1$. Then we have $w_0 = w_1$ as in Figure 6. Since P is nonsingular, we see that $m_1 = u_1 = (1,0,0)$ and that w_0 is a vertex of P. By exchanging the role of F_0 with the facet $\mathrm{Conv}\{u_0, u_1, w_0\}$, we can reduce to the case that $F_0 \cong G_0$ in the cases (a) and (b) of (I) treated above. □

We will use the following Lemmas for the proof of the normal generation of polytopes.

LEMMA 2.4. *Let P be an integral convex polytope in $M_{\boldsymbol{R}}$. If P is a union of normally generated integral convex polytopes, then P is also normally generated.*

PROOF. Let $P = \bigcup_{i=1}^r Q_i$ be a decomposition into a union of integral convex polytopes such that each Q_i is normally generated. For an positive integer l, take a lattice point in lP, i.e., $m \in (lP) \cap M$. Then we can choose i so that $m \in lQ_i$ because $lP = \bigcup_{i=1}^r lQ_i$. Since Q_i is normally generated, there exist $m_1, \ldots, m_l \in Q_i \cap M \subset P \cap M$ such that $m = m_1 + \cdots + m_l$ from Remark 1.4. □

LEMMA 2.5. *The integral convex polytope $P(F_0)$ is normally generated.*

PROOF. We show that $P(F_0) \cap M + P(F_0) \cap M = (2P(F_0)) \cap M$. We note that F_0 and G are normally generated because they are of dimension two. From the result of Fakhruddin (Theorem 0.3), we see that

$$F_0 \cap M + G \cap M = (F_0 + G) \cap M \tag{4}$$

because F_0 and G define an ample and a nef line bundles on the nonsingular toric surface Y, respectively.

Take $m \in (2P(F_0)) \cap M$. If the z-coordinate of m is 0, 1 or 2, then m is in $2F_0$, $F_0 + G$ and $2G$, respectively. Thus we can find $m_1, m_2 \in P(F_0) \cap M$ with $m = m_1 + m_2$. □

REMARK 2.6. In the proof of lemma 2.5 the equality (4) is essential. The result of Fakhruddin [4] says that if each edge of G has the same inner normal direction as that of some edge of F_0, then the equality (4) holds. The condition contains the case when G is a line segment E and F_0 is a tetragon with two edgs parallel to E.

From Proposition 2.3 and Lemmas 2.4 and 2.5 we prove Theorem 0.1.

PROPOSITION 2.7. *Let X be a projective nonsingular toric variety of dimension three and let L be an ample line bundle on X. If $\Gamma(X, L \otimes \mathcal{O}_X(K_X)) = 0$, then L is normally generated.*

PROOF. Let P be the integral convex polytope corresponding to the polarized toric variety (X, L). By the assumption $\Gamma(X, L \otimes \mathcal{O}_X(K_X)) = 0$, the polytope P does not contain lattice points in its interior. We have a classification of such polytopes in Proposition 2.3.

We can apply Lemmas 2.4 and 2.5 to the polytopes in (1), (4) and (5) of Proposition 2.3 for the normal generation of P and also to $P = Q_{d,e,f}$ in (3) in the same way of (5).

The basic 3-simplex P_1 is trivially normally generated. If $P = kQ$ for some integral convex polytope Q and $k \geq 2$, then P is normally generated from [9]. (2) is in this case. When $P = P_{a,b,c}$, we may set $F_1 := P_{a,b,c} \cap (x = 0)$ and $E := P_{a,b,c} \cap (x = 1)$. Then E is a line segment and F_1 is a tetragon with two edges parallel to E. From Remark 2.6 we see that $P_{a,b,c}$ is normally generated. This completes the proof. □

3. Adjoint bundles. In this section we investigate properties of the adjoint bundle $L \otimes \mathcal{O}_X(K_X)$ of an ample line bundle L on X.

Let L be an ample line bundle on a nonsingular projective toric variety $X = T_N\mathrm{emb}(\Delta)$ of dimension n. Let $\{\rho_1, \dots, \rho_r\} = \Delta(1)$, i.e., the set of all cones of dimension one in Δ, and D_i the T_N-invariant prime divisor on X corresponding to ρ_i for each i. Then there exists a divisor $D = \sum_i a_i D_i$ with $L \cong \mathcal{O}_X(D)$. We may assume $a_i \geq 0$.

In this case,

$$P := \{u \in M_{\boldsymbol{R}};\ \langle u, n(\rho_i)\rangle \geq -a_i \text{ for all } i\}$$

is the corresponding polytope, where $n(\rho_i) \in \rho_i \cap M$ is the primitive element for each i. Recalling that the canonical divisor K_X is $-\sum_i D_i$, we set

$$P_K := \{u \in M_{\boldsymbol{R}};\ \langle u, n(\rho_i)\rangle \geq -a_i + 1 \text{ for all } i\}\,. \tag{5}$$

Assume that $\Gamma(L \otimes \mathcal{O}_X(K_X)) \neq 0$, equivalently that $\mathrm{Int}(P) \cap M \neq \emptyset$. We know $P_K \cap M = \mathrm{Int}(P) \cap M$. Set $Q := \mathrm{Conv}(\mathrm{Int}(P) \cap M)$. We call Q the *internal integral polytope* of P. We see that $Q \subset P_K$ because P_K is convex and $\mathrm{Int}(P) \cap M \subset P_K$.

Let $u_0 \in P$ be a vertex of P. Then there is the n-dimensional cone $\sigma \in \Delta(n)$ such that $\sigma^\vee \cong \boldsymbol{R}_{\geq 0}(P - u_0)$. Since $\sigma = \rho_{i_1} + \cdots + \rho_{i_n}$ is nonsingular, $\{n(\rho_{i_1}), \dots, n(\rho_{i_n})\}$ is a $\boldsymbol{Z}$-basis of N and there are $m_1, \dots, m_n \in P \cap M$ such that $\{m_1 - u_0, \dots, m_n - u_0\}$ is the dual basis of M and that $\boldsymbol{R}_{\geq 0}(P - u_0) = \sum_{i=1}^n \boldsymbol{R}_{\geq 0}(m_i - u_0)$. From this notation, we see

$$\begin{aligned} &\langle u, n(\rho_{i_k})\rangle = \langle u - u_0, n(\rho_{i_k})\rangle + \langle u_0, n(\rho_{i_k})\rangle \geq \langle u_0, n(\rho_{i_k})\rangle \\ &\qquad \text{for } u \in P \text{ and } k = 1, \dots, n\,. \end{aligned} \tag{6}$$

By definition we see $\langle u_0, n(\rho_{i_k})\rangle = -a_{i_k}$. Set $\bar{l}_\sigma := u_0 + \sum_{i=1}^n (m_i - u_0)$. Then the lattice point $\bar{l}_\sigma - u_0 = \sum_{i=1}^n (m_i - u_0)$ is in the interior of $\sigma^\vee = \boldsymbol{R}_{\geq 0}(P - u_0)$ and $(\mathrm{Int}\,\sigma^\vee) \cap M = (\bar{l}_\sigma - u_0) + \sigma^\vee \cap M$.

Since the set of all vertices of P bijectively corresponds to $\Delta(n)$, we can define $\bar{l}_\sigma \in M$ for all $\sigma \in \Delta(n)$. We note that it may happen $\bar{l}_\sigma = \bar{l}_\tau$ for $\sigma, \tau \in \Delta(n)$ with $\sigma \neq \tau$. If $\sigma = \rho_{i_1} + \cdots + \rho_{i_n}$, then from (6) we see that

$$\langle \bar{l}_\sigma, n(\rho_{i_k}) \rangle = -a_{i_k} + 1 \quad \text{for } k = 1, \ldots, n .$$

If there is a $\rho_i \in \Delta(1)$ with $\langle \bar{l}_\sigma, n(\rho_i) \rangle \leq -a_i$, then $\bar{l}_\sigma$ is not contained in P_K. If all $\bar{l}_\sigma$ are contained in P_K, then the line bundle $\mathcal{O}_X(D + K_X)$ is generated by global sections from [5, Section 3.4] and hence P_K is the convex hull of $\{\bar{l}_\sigma; \sigma \in \Delta(n)\}$ (see [11, Theorem 2.7]). In this case, we have $P_K = Q$ because $P_K \cap M = Q \cap M$.

Even if not the case, we will see $P_K = Q$ when $\dim X = 3$ in the following Proposition.

PROPOSITION 3.1. *Let P be a nonsingular integral convex polytope of dimension three in $M_{\boldsymbol{R}}$ corresponding to a polarized toric variety $(X, \mathcal{O}_X(D))$. We assume that $\Gamma(X, \mathcal{O}_X(D + K_X)) \neq 0$, that is,* $\mathrm{Int}(P) \cap M \neq \emptyset$. *Set P_K the rational convex polytope of the adjoint divisor $D + K_X$ defined by* (5), *and set $Q =$* $\mathrm{Conv}(\mathrm{Int}(P) \cap M)$ *the internal polytope of P. Then we have $P_K = Q$.*

PROOF. Let $u_0 \in P$ be a vertex and F_0 a facet containing u_0. The two edges of F_0 meeting at u_0 have the lattice points m_1 and m_2 respectively so that $\{m_1 - u_0, m_2 - u_0\}$ is a $\boldsymbol{Z}$-basis of $(\boldsymbol{R}F_0) \cap M \cong \boldsymbol{Z}^2$. Then we have the same figure as Figure 5 and the coordinate system (x, y, z) of $M \cong \boldsymbol{Z}^3$.

Consider the point $(1, 1, 1)$, which is $\bar{l}_\sigma - u_0$ of $\sigma^\vee = \boldsymbol{R}_{\geq 0}(P - u_0)$ as described above. If $(1, 1, 1)$ is contained in $P_K - u_0$, then it is a vertex because $P_K - u_0$ is contained in $\bar{l}_\sigma + \sigma^\vee$, hence, $(1, 1, 1)$ is also a vertex of $Q - u_0$.

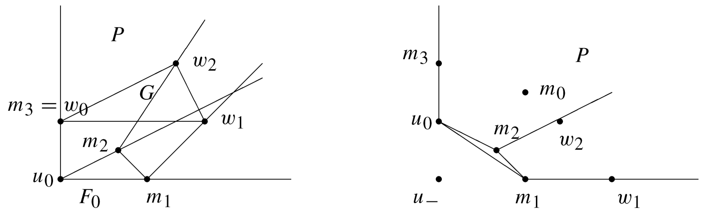
We assume that the point $(1, 1, 1)$ is not contained in $P_K - u_0$. As in the proof of Proposition 2.3, we set $G := (P - u_0) \cap (z = 1)$. We note that G is not a facet of P since P contains interior lattice points. Then the assumption implies that $(1, 1, 1)$ is not contained in the interior of G. We may assume that G contains the points $(1, 0, 1)$ and $(0, 1, 1)$. If $(1, 0, 1)$ is not contained in G, then $(0, 0, 1)$ is a vertex of P and the facet $P \cap (y = 0)$ is the basic triangle $\mathrm{Conv}\{(0, 0, 0), (1, 0, 0), (0, 0, 1)\}$. In this case, if we exchange the role of F_0 with the facet $P \cap (y = 0)$, then new G satisfies the assumption.

If the lattice polygon G containing $(0, 0, 1)$, $(1, 0, 1)$ and $(0, 1, 1)$ does not contain $(1, 1, 1)$, then we may assume that it is a triangle with vertices $(1, 0, 1)$ and $(0, a, 1)$ $(a \geq 1)$. Let (p, q, r) be a lattice point in the interior of P. Then p, q and r are positive integers. The line segment connecting (p, q, r) and $(1, 0, 0)$ crosses the plane $(z = 1)$ at $(1 + (p - 1)/r, q/r, 1)$, which is not contained in G. This contradicts that P is convex. Thus the point $(1, 1, 1)$ is contained in the boundary of G.

If G is a tetragon, then it has two parallel edges with the distance one, hence, F_0 also has two parallel edges. In this case, P belongs to (1) or (5) in Proposition 2.3. Then P cannot contain lattice points in its interior.

From this argument, we see that if $\mathrm{Int}(P) \cap M \neq \emptyset$, then G is the triangle $\mathrm{Conv}\{(0, 0, 1), (2, 0, 1), (0, 2, 1)\} \cong 2G_0$ containing the point $(1, 1, 1)$ in its boundary.

Since $\bar{l}_\sigma \in P$, we have $\langle \bar{l}_\sigma, n(\rho_j)\rangle \geq -a_j$ for all $\rho_j \in \Delta(1)$. If $(1,1,1) = \bar{l}_\sigma - u_0$ is not contained in $P_K - u_0$, then there exists a $\rho_i \in \Delta(1)$ with $\langle \bar{l}_\sigma, n(\rho_i)\rangle = -a_i$, that is, the point $(1,1,1)$ is contained in the plane $H = \{u \in M_{\boldsymbol{R}};\ \langle u + u_0, n(\rho_i)\rangle = -a_i\}$ bounding $P - u_0$. If H is defined by the equation $x + y = 2$, then P is one of (5) in Proposition 2.3 and P does not contain lattice points in its interior. Thus in the defining equation $x + y - az = b$ with $a + b = 2$ of the plane H, the assumption $\mathrm{Int}(P) \cap M \neq \emptyset$ implies that a is positive. Since $F_0 \subset \{0 \leq x, 0 \leq y, x + y \leq b\}$ and since $b < 2$ we see that $F_0 \cong G_0$. See Figure 8 (a).



(a): P around F_0 (b): P in a new coordinates

FIGURE 8. Local shapes of P.

We claim that $(1,1,2)$ is an interior lattice point of $P - u_0$. We denote $(1,1,2) = m_0 - u_0$ in $P - u_0$. Take the lattice point $u_- = (0,0,-1)$ outside $P - u_0$. By taking an affine transformation of $M \cong \boldsymbol{Z}^3$, we may set $u_0 = (0,0,1), m_1 = (1,0,0), m_2 = (0,1,0), u_- = (0,0,0)$. Then the point $(1,1,2)$ in $P - u_0$ is transformed to the point $(1,1,1)$. See Figure 8 (b). Let $\tilde{P}$ be the convex hull of u_- and $P - u_0$. Then $\tilde{P}$ is nonsingular and $\mathrm{Int}(\tilde{P}) \cap M = \mathrm{Int}(P - u_0) \cap M$. We note that three facets of $\tilde{P}$ with the vertex u_- are not isomorphic to the basic triangle G_0. The lattice point $(m_0 - u_0) - u_-$ is $\bar{l}_{\tilde{\sigma}}$ of the cone $\tilde{\sigma}^\vee := \boldsymbol{R}_{\geq 0}(\tilde{P} - u_-)$. From above if $m_0 - u_0$ is not contained in $\mathrm{Int}(\tilde{P}) \cap M = \mathrm{Int}(P - u_0) \cap M = (P_K - u_0) \cap M$, then one of three facets with vertex u_- is isomorphic to G_0. This is a contradiction.

In the new coordinates, since $P_K - u_0$ is contained in the region $\{1 \leq x, 1 \leq y, 1 \leq z\}$ and since the point $m_0 - u_0 = (1,1,1)$ is contained in $(Q - u_0) \cap M = (P_K - u_0) \cap M$, the point m_0 is a vertex of P_K. We also know that the cone $\boldsymbol{R}_{\geq 0}(P_K - m_0)$ is contained not only in the cone $\boldsymbol{R}_{\geq 0}(P - u_0)$ but also in the cones $\boldsymbol{R}_{\geq 0}(P - m_1)$ and $\boldsymbol{R}_{\geq 0}(P - m_2)$.

We claim that a vertex of P_K is an $\bar{l}_\sigma \in P_K$ or these m_0. If $\bar{l}_\sigma \in P_K$, then the cone $\sigma = \boldsymbol{R}_{\geq 0}(P - u_0)^\vee$ is contained in the cone $\boldsymbol{R}_{\geq 0}(P_K - \bar{l}_\sigma)^\vee$. On the other hand, if $\bar{l}_\sigma$ is not contained in P_K, then the cone $\sigma = \boldsymbol{R}_{\geq 0}(P - u_0)^\vee$ is contained in $\boldsymbol{R}_{\geq 0}(P_K - m_0)^\vee$ in the above manner. Since $\cup_{\sigma \in \Delta(3)} \sigma = N_{\boldsymbol{R}}$, the union of the dual cones defined by these vertices of P_K covers whole $N_{\boldsymbol{R}}$. This implies that all vertices of P_K are these kinds. $\square$

Set $\{F_i \subset P; i \in I\}$ the set of all facets of P. We define the integral polytope $P(F_i)$ similarly as we defined $P(F_0)$ for F_0. Then Proposition 3.1 shows that we have a decomposition

$$P = Q \cup \bigcup_{i \in I} P(F_i). \tag{7}$$

If $\dim Q \leq 2$, then we can delete Q in the decomposition (7), since $\bigcup_{i \in I} P(F_i)$ is closed and $P = \overline{P \setminus Q} \subset \bigcup_{i \in I} P(F_i)$. Thus we have the following corollary.

COROLLARY 3.2. *Let P be a nonsingular integral convex polytope of dimension three in $M_{\boldsymbol{R}}$. We assume that P has lattice points in its interior and that the internal polytope $Q = \mathrm{Conv}(\mathrm{Int}(P) \cap M)$ has dimension less than three. Then P is normally generated.*

Let $(X, \mathcal{O}_X(D))$ be the nonsingular polarized toric 3-fold corresponding to a nonsingular integral convex polytope P. If $H^0(X, \mathcal{O}_X(D + K_X)) \neq 0$ and if the internal polytope Q of P has dimension three, then the dimension of $\Gamma(X, \mathcal{O}_X(l(D + K_X)))$ is equal to the cardinality $\sharp\{(lP_K) \cap M\} = \sharp\{(lQ) \cap M\}$ for each positive integer l, which is asymptotically proportional to l^3 times the volume of Q for large l. This implies that $D + K_X$ is big. Thus Theorem 0.2 follows from Corollary 3.2.

REFERENCES

[1] J. R. ARKINSTALL, Minimal requirements for Minkowski's Theorem in the plane I, Bull. Austral. Math. Soc. 22 (1980), 259–274.

[2] D. COX, Recent developments in toric geometry, In: Algebraic Geometry–Santa Cruz 1995, Proc. Sympos. Pure Math. 62 (1997), 389–436.

[3] G. EWALD AND U. WESSELS, On the ampleness of invertible sheaves in complete projective toric varieties, Results Math. 19 (1991), 275–278.

[4] N. FAKHRUDDIN, Multiplication maps of linear systems on smooth toric surfaces, preprint, arXiv:math. AG/0208178.

[5] W. FULTON, Introduction to toric varieties, Ann. of Math. Stud. 131, Princeton Univ. Press, Princeton, NJ, 1993.

[6] C. HAASE, B. NILL, A. PAFFENHOLZ AND F. SANTOS, Lattice points in Minkowski sums, Electron. J. Combin. 15 (2008), 5 pp.

[7] B. J. KOELMAN, Generators for the ideal of a projectively embedded toric surface, Tohoku Math. J. 45 (1993), 385–392.

[8] D. KONDO AND S. OGATA, On multiplication maps of ample bundles with nef bundles on toric surfaces, Interdiscip. Inform. Sci. (2008), 183–190.

[9] K. NAKAGAWA, Generators for the ideal of a projectively embedded toric varieties, master's thesis at Tohoku University, 1994.

[10] K. NAKAGAWA AND S. OGATA, On generators of ideals defining projective toric varieties, Manuscripta Math. 108 (2002), 33–42.

[11] T. ODA, Convex bodies and algebraic geometry, Ergebnisse der Math. 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[12] S. OGATA, k-normality of weighted projective spaces, Kodai Math. J. 28 (2005), 519–524.

[13] B. STURMFELS, Equations defining toric varieties, In: Algebraic Geometry–Santa Cruz 1995, Proc. Sympos. Pure Math. 62 (1997), 437–449.

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980–8578
JAPAN

E-mail address: ogata@math.tohoku.ac.jp