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Abstract

We consider the q-Painlevé III equation arising from the birational representation of the affine Weyl group of type $(A_2 + A_1)^{(1)}$. We study the reduction of the q-Painlevé III equation to the q-Painlevé II equation from the viewpoint of affine Weyl group symmetry. In particular, the mechanism of apparent inconsistency between the hypergeometric solutions to both equations is clarified by using factorization of difference operators and the τ functions.

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1 Introduction

The discrete Painlevé equations have been studied actively from various points of view. Together with the Painlevé equations, they are now regarded as one of the most important classes of equations in the theory of integrable systems (see, for example, [6]). Originally, the discrete Painlevé equations had been identified as single second-order equations [1–3, 33, 37] and then were generalized to simultaneous first-order equations. A typical example is the following equation known as a discrete Painlevé II equation [33, 37]:

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c}{1 - {x_n}^2}, \tag{1.1}$$

where x_n is the dependent variable, n is the independent variable, and $a, b, c \in \mathbb{C}$ are parameters. By applying the singularity confinement criterion [7], (1.1) is generalized to

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c + (-1)^n d}{1 - {x_n}^2}, \tag{1.2}$$

where d is a parameter, with its integrability preserved. Introducing the dependent variables X_n and Y_n by

$$X_n = x_{2n}, \quad Y_n = x_{2n-1}, \tag{1.3}$$

then (1.2) can be rewritten as

$$Y_{n+1} + Y_n = \frac{(2an + b)X_n + c + d}{1 - {X_n}^2}, \quad X_{n+1} + X_n = \frac{(a(2n + 1) + b)Y_{n+1} + c - d}{1 - {Y_{n+1}}^2}. \tag{1.4}$$

Equation (1.4) is known as a discrete Painlevé III equation since it admits a continuous limit to the Painlevé III equation [5]. Conversely, (1.1) can be recovered from (1.4) by putting $d = 0$ and (1.3). We call this procedure "*symmetrization*" of (1.4), which comes from the terminology of the Quispel–Roberts–Thompson (QRT) mapping [34, 35]. After this terminology, (1.4) is sometimes called the "*asymmetric*" discrete Painlevé II equation, and (1.1) is called the "*symmetric*" discrete Painlevé III equation [21].

It looks that the symmetrization is a simple specialization of parameters at the level of the equation, but some strange phenomena have been reported as to their particular solutions expressed in terms of hypergeometric functions (*hypergeometric solutions*). The hypergeometric solutions to (1.1) have been constructed as follows [9, 19]:

Proposition 1.1 *For each $N \in \mathbb{N}$, let τ_N^n be an $N \times N$ determinant defined by*

$$\tau_N^n = \begin{vmatrix} H_n & H_{n+1} & \cdots & H_{n+N-1} \\ H_{n+2} & H_{n+3} & \cdots & H_{n+N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+2N-2} & H_{n+2N-1} & \cdots & H_{n+3N-3} \end{vmatrix}, \tag{1.5}$$

where H_n is a function satisfying the three-term relation:

$$H_{n+1} - zH_n + nH_{n-1} = 0. \tag{1.6}$$

Then,

$$x_n = \frac{2}{z}\,\frac{\tau_{N+1}^{n+1}\tau_N^n}{\tau_{N+1}^n\tau_N^{n+1}} - 1, \tag{1.7}$$

satisfies (1.1) *with the parameters*

$$a = \frac{8}{z^2}, \quad b = \frac{4(1+2N)}{z^2}, \quad c = -\frac{4(1+2N)}{z^2}. \tag{1.8}$$

On the other hand, since (1.4) appears as the Bäcklund transformation of the Painlevé V equation [28, 38], its hypergeometric solutions are essentially the same as those to the Painlevé V equation [22, 31]. The explicit form of the hypergeometric solutions to (1.4) are given as follows:

Proposition 1.2 *For each $N \in \mathbb{N}$, let $\tau_N^{n,m}$ be an $N \times N$ determinant defined by*

$$\tau_N^{n,m} = \begin{vmatrix} K_n^m & K_{n+1}^m & \cdots & K_{n+N-1}^m \\ K_{n+1}^m & K_{n+2}^m & \cdots & K_{n+N}^m \\ \vdots & \vdots & \ddots & \vdots \\ K_{n+N-1}^m & K_{n+N}^m & \cdots & K_{n+2N-2}^m \end{vmatrix}, \tag{1.9}$$

where K_n^m is a function satisfying

$$K_{n+1}^m - K_n^m - tK_{n+1}^{m+1} = 0, \quad nK_{n+1}^m - (n+t)K_n^m - (n-m)tK_n^{m+1} = 0. \tag{1.10}$$

Then,

$$X_n = 2(n+2N-1)\frac{\tau_{N+1}^{n,m}\tau_N^{n,m}}{\tau_{N+1}^{n-1,m-1}\tau_N^{n+1,m+1}} - 1, \quad Y_n = \frac{2}{t}\,\frac{\tau_{N+1}^{n-1,m-1}\tau_N^{n,m+1}}{\tau_{N+1}^{n-1,m}\tau_N^{n,m}} - 1, \tag{1.11}$$

satisfy (1.4) *with the parameters*

$$a = -\frac{4}{t}, \quad b = \frac{-4(-m + 2N - 1)}{t}, \quad c = \frac{2(1 + 2N)}{t}, \quad d = \frac{2(2m + 2N - 3)}{t}. \tag{1.12}$$

It is obvious that substituting $d = 0$ into the hypergeometric solutions to (1.4) in Proposition 1.2 do not yield those to (1.1) in Proposition 1.1. In particular, we remark the following differences between the two solutions:

(1) The hypergeometric functions are different. Equation (1.6) can be solved by considering the parabolic cylinder function (Weber function), while (1.10) can be solved by considering the confluent hypergeometric function. In fact, the former function is expressed as a specialization of the latter, but this specialization is not consistent with the symmetrization.

(2) Structures of the determinant are different. The determinant (1.5) has asymmetry in the shift of index: the shift in the vertical direction is two while that in the horizontal direction is one. On the other hand, the determinant (1.9) is an ordinary Hankel determinant.

We note that similar phenomena have been reported also for some other discrete Painlevé equations [8, 18, 25]. Many integrable systems admit particular solutions expressed in terms of determinants, but such an asymmetric structure of the determinant solutions has been seen only in the hypergeometric solutions to the discrete Painlevé equations. Note here that these phenomena cannot be seen for the algebraic (or rational) solutions. For example, it is known that substituting $d = 0$ into the determinant expression of the rational solutions to (1.4) yields those to (1.1); see [20, 23, 24].

The τ function is one of the most important objects in the theory of integrable systems and is regarded as carrying the underlying fundamental mathematical structures. Concerning the discrete Painlevé equations, investigation of the τ functions started [18, 19] through the search for the explicit formulae of the hypergeometric and algebraic solutions. In fact, the above mysterious asymmetric structure has been one motivation of further study.

It is now known that theory of birational representations of affine Weyl groups provides us with an algebraic tool to study the Painlevé systems [27, 29–32]. Moreover, a geometric framework of the two-dimensional Painlevé systems has been presented based on certain rational surfaces [15,39]. Combining these results enables us to study the Painlevé systems effectively. For instance, it played a crucial role in the identification of hypergeometric functions that appear as the particular solutions to the Painlevé systems in Sakai's classification [12–14].

The purpose of this paper is to clarify the mechanism of the phenomena of hypergeometric solutions from the viewpoint of the affine Weyl group symmetry. We shall take the q-Painlevé equation of type $(A_2 + A_1)^{(1)}$ as an example, which is the simplest non-trivial discrete Painlevé system [39]. The key is to formulate the symmetrization in terms of the birational representation of the affine Weyl group, where the discrete Painlevé equation arises from the action of the translational subgroup. In fact, the discrete time evolution of the symmetric case can be regarded as a "half-step" of a translation of the affine Weyl group with restricted to a certain line in the parameter space. Conversely, we can derive various discrete Painlevé equations from elements of infinite order that are not only translations by taking a projection on a certain subspace of the parameters. We call such a procedure to obtain a "smaller" discrete time evolution of Painlevé type a *projective reduction*.

This paper is organized as follows: in Section 2, we introduce a q-Painlevé III equation and derive a q-Painlevé II equation by applying the symmetrization. Then we give a brief review on their hypergeometric solutions. In Section 3, we first introduce the family of Bäcklund transformations of the q-Painlevé III equation, which is a birational representation of the affine Weyl group of type $(A_2+A_1)^{(1)}$. We next lift the representation on the level of τ functions and derive various bilinear equations. We then clarify the mechanism of the inconsistency among the hypergeometric solutions by using this framework. Some concluding remarks are given in Section 4.

2 q-P$_{\text{III}}$ and q-P$_{\text{II}}$

We consider the following system of q-difference equations [11,17,39]:

$$g_{n+1}=\frac{q^{2N+1}c^2}{f_ng_n}\,\frac{1+a_0q^nf_n}{a_0q^n+f_n},\quad f_{n+1}=\frac{q^{2N+1}c^2}{f_ng_{n+1}}\,\frac{1+a_2a_0q^{n-m}g_{n+1}}{a_2a_0q^{n-m}+g_{n+1}}, \tag{2.1}$$

for the unknown functions $f_n=f_n(m,N)$ and $g_n=g_n(m,N)$ and the independent variable $n\in\mathbb{Z}$. Here $m,N\in\mathbb{Z}$ and $a_0,a_2,c,q\in\mathbb{C}^\times$ are parameters. Equation (2.1) has the (extended) affine Weyl group symmetry of type $(A_2+A_1)^{(1)}$ and is known as a q-Painlevé III equation (q-P$_{\text{III}}$) since the continuous limit yields the Painlevé III equation. We also consider the following q-difference equation [25,36]:

$$X_{k+1}=\frac{q^{2N+1}c^2}{X_kX_{k-1}}\,\frac{1+a_0q^{\frac{k}{2}}X_k}{a_0q^{\frac{k}{2}}+X_k}, \tag{2.2}$$

for the unknown function $X_k=X_k(N)$ and the independent variable $k\in\mathbb{Z}$. Equation (2.2) is a q-Painlevé II equation (q-P$_{\text{II}}$) and actually it admits a continuous limit to the Painlevé II equation.

Note that substituting

$$m=0,\quad a_2=q^{\frac{1}{2}}, \tag{2.3}$$

and putting

$$f_k(0,N)=X_{2k}(N),\quad g_k(0,N)=X_{2k-1}(N), \tag{2.4}$$

in (2.1) yield (2.2).

We shall briefly review the hypergeometric solutions to q-P$_{\text{III}}$ and q-P$_{\text{II}}$ following [11,25].

2.1 Hypergeometric solutions to q-P$_{\text{III}}$

First, we review the hypergeometric solutions to q-P$_{\text{III}}$. For each $N\in\mathbb{Z}_{\geq 0}$, let $\psi_N^{n,m}$ be an $N\times N$ determinant defined by

$$\psi_N^{n,m}=\begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m}\\ F_{n-1,m} & F_{n,m} & \cdots & F_{n+N-2,m}\\ \vdots & \vdots & \ddots & \vdots\\ F_{n-N+1,m} & F_{n-N+2,m} & \cdots & F_{n,m}\end{vmatrix},\quad \psi_0^{n,m}=1, \tag{2.5}$$

where $F_{n,m}$ satisfies

$$\begin{aligned} F_{n+1,m}-F_{n,m}&=-{a_0}^2q^{2n}F_{n,m-1},\\ F_{n,m+1}-F_{n,m}&=-{a_2}^{-2}q^{2m+2}F_{n-1,m}. \end{aligned} \tag{2.6}$$

Lemma 2.1 ([11]) $\psi_N^{n,m}$ *satisfies the following bilinear difference equations*:

$$a_0{}^2 q^{2n-2}\psi_{N+1}^{n-1,m-1}\psi_N^{n,m} - q^{2N}\psi_N^{n,m-1}\psi_{N+1}^{n-1,m} + \psi_N^{n-1,m-1}\psi_{N+1}^{n,m} = 0, \tag{2.7}$$
$$\psi_{N+1}^{n,m}\psi_N^{n,m-1} - q^{-2N}\psi_N^{n-1,m}\psi_{N+1}^{n+1,m+1} - a_0{}^2 q^{2n}\psi_N^{n,m}\psi_{N+1}^{n,m-1} = 0, \tag{2.8}$$
$$\psi_{N+1}^{n,m}\psi_N^{n-1,m-1} - \psi_{N+1}^{n,m-1}\psi_N^{n-1,m} + a_2{}^{-2}q^{2m}\psi_N^{n,m}\psi_{N+1}^{n-1,m-1} = 0, \tag{2.9}$$
$$\psi_{N+1}^{n,m-1}\psi_N^{n,m} - a_2{}^{-2}q^{2m}\psi_{N+1}^{n-1,m-1}\psi_N^{n+1,m} - \psi_N^{n,m-1}\psi_{N+1}^{n,m} = 0. \tag{2.10}$$

Proposition 2.2 ([11]) *The hypergeometric solutions to q-P_{III}*, (2.1), *with $c = 1$ are given by*

$$f_n = -a_0 q^n \frac{\psi_{N+1}^{n,m-1}\psi_N^{n,m}}{\psi_{N+1}^{n,m}\psi_N^{n,m-1}}, \quad g_n = \frac{a_2}{a_0}q^{-n-m+1}\frac{\psi_{N+1}^{n,m}\psi_N^{n-1,m-1}}{\psi_{N+1}^{n-1,m-1}\psi_N^{n,m}}. \tag{2.11}$$

Proposition 2.2 follows from Lemma 2.1.

Remark 2.3 (1) $F_{n,m}$ satisfies the three-term relation:

$$F_{n+1,m} + \left(a_0{}^2 q^{2n} - a_2{}^{-2}q^{2m+2} - 1\right)F_{n,m} + a_2{}^{-2}q^{2m+2}F_{n-1,m} = 0, \tag{2.12}$$

whose general solution is given by

$$\begin{aligned} F_{n,m} =& \frac{A}{(a_2{}^{-2}q^{2m+2};q^2)_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ a_2{}^2q^{-2m}\end{matrix};q^2, a_2{}^2a_0{}^2q^{2n-2m}\right)\\ &+ B\,(a_2{}^{-2}q^{2m+4};q^2)_\infty\left(a_2{}^{-1}q^{m+1}\right)^{2n}\,{}_1\varphi_1\left(\begin{matrix}0\\ a_2{}^{-2}q^{2m+4}\end{matrix};q^2, a_0{}^2q^{2n+2}\right). \end{aligned} \tag{2.13}$$

Here, A and B are arbitrary constants, and ${}_1\varphi_1$ is the basic hypergeometric function defined by [4]

$${}_1\varphi_1\left(\begin{matrix}a\\ b\end{matrix};q,z\right) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(b;q)_k(q;q)_k}(-1)^k q^{\frac{k(k-1)}{2}}z^k, \quad (a;q)_k = \prod_{i=1}^{k}(1-aq^{i-1}). \tag{2.14}$$

(2) $\psi_N^{n,m}$ satisfies the discrete Toda equation:

$$\psi_{N+1}^{n,m}\psi_{N-1}^{n,m} - \left(\psi_N^{n,m}\right)^2 + \psi_N^{n+1,m}\psi_N^{n-1,m} = 0. \tag{2.15}$$

In general, (2.15) admits a solution expressed in terms of the Toeplitz type determinant

$$\psi_N^{n,m} = \det\left(c_{n-i+j,m}\right)_{i,j=1,\dots,N} \quad (N>0), \tag{2.16}$$

for an arbitrary function $c_{n,m}$ under the boundary conditions

$$\psi_0^{n,m} = 1, \quad \psi_N^{n,m} = 0 \quad (N<0). \tag{2.17}$$

Since the hypergeometric solutions to q-P$_{\mathrm{III}}$ satisfy the conditions (2.17), the bilinear equation (2.15) is regarded as to fix the determinant structure of the solutions.

2.2 Hypergeometric solutions to q-P$_{\rm II}$

Next, we review the hypergeometric solutions to q-P$_{\rm II}$. For each $N \in \mathbb{Z}_{\geq 0}$, let ϕ_N^k be an $N \times N$ determinant defined by

$$\phi_N^k = \begin{vmatrix} G_k & G_{k-1} & \cdots & G_{k-N+1} \\ G_{k+2} & G_{k+1} & \cdots & G_{k-N+3} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k+2N-2} & G_{k+2N-3} & \cdots & G_{k+N-1} \end{vmatrix}, \quad \phi_0^k = 1, \tag{2.18}$$

where G_k satisfies

$$G_{k+1} - G_k + \frac{1}{{a_0}^2 q^k} G_{k-1} = 0. \tag{2.19}$$

Lemma 2.4 ([25]) ϕ_N^k *satisfies the following bilinear difference equations*:

$$a_0{}^{-2} q^{-k+1} \phi_{N+1}^{k-2} \phi_N^{k+1} + \phi_{N+1}^k \phi_N^{k-1} - q^{-N} \phi_{N+1}^{k-1} \phi_N^k = 0, \tag{2.20}$$

$$q^N \phi_{N+1}^{k+1} \phi_N^{k-2} + a_0{}^{-2} q^{-k-N} \phi_{N+1}^{k-1} \phi_N^k - \phi_{N+1}^k \phi_N^{k-1} = 0. \tag{2.21}$$

Proposition 2.5 ([25]) *The hypergeometric solutions to q-P$_{\rm II}$,* (2.2)*, with $c = 1$ are given by*

$$X_k = -a_0 q^{\frac{k}{2}+N} \frac{\phi_{N+1}^k \phi_N^{k-1}}{\phi_{N+1}^{k-1} \phi_N^k}. \tag{2.22}$$

Proposition 2.5 follows from Lemma 2.4.

Remark 2.6 (1) The general solution to (2.19) is given by

$$\begin{aligned} G_k =& \Theta(-a_0 q^{\frac{2k+1}{4}}; q^{\frac{1}{2}}) \\ &\times \left\{ A e^{\frac{k\pi i}{2}} {}_1\varphi_1 \left(\begin{matrix} 0 \\ -q^{\frac{1}{2}} \end{matrix}; q^{\frac{1}{2}}, -i a_0 q^{\frac{3+2k}{4}} \right) + B e^{-\frac{k\pi i}{2}} {}_1\varphi_1 \left(\begin{matrix} 0 \\ -q^{\frac{1}{2}} \end{matrix}; q^{\frac{1}{2}}, i a_0 q^{\frac{3+2k}{4}} \right) \right\}. \end{aligned} \tag{2.23}$$

Here, $\Theta(a;q)$ denotes the Jacobi theta function, which is defined by

$$\Theta(a;q) = (a;q)_\infty (qa^{-1};q)_\infty, \tag{2.24}$$

and satisfies

$$\Theta(qa;q) = -a^{-1} \Theta(a;q). \tag{2.25}$$

(2) ϕ_N^k also satisfies the bilinear equation

$$\phi_{N+1}^k \phi_{N-1}^{k+1} - \phi_N^k \phi_N^{k+1} + \phi_N^{k+2} \phi_N^{k-1} = 0, \tag{2.26}$$

which is a variant of the discrete Toda equation. Under the conditions

$$\phi_0^k = 1, \quad \phi_N^k = 0 \quad (N < 0), \tag{2.27}$$

(2.26) admits a solution expressed by

$$\phi_N^k = \det \left(c_{k+2i-j-1} \right)_{i,j=1,\ldots,N} \quad (N > 0), \tag{2.28}$$

for an arbitrary function c_k. Hence, (2.26) can be regarded as the bilinear equation that fixes the determinant structure of the hypergeometric solutions to q-P$_{\rm II}$.

2.3 Comparing the hypergeometric solutions

By comparing the hypergeometric solutions to q-$\mathrm{P}_{\mathrm{III}}$ and q-P_{II} (see Propositions 2.2 and 2.5, respectively) one may immediately notice that a naïve application of the specialization, (2.3), to the former does not yield the latter. As analogous to the phenomena seen in Section 1, we find the following differences between the two solutions:

(i) the hypergeometric functions are different. In fact, substituting $a_2 = q^{\frac{1}{2}}$ into (2.12) and (2.13) do not yield (2.19) and (2.23), respectively;

(ii) the determinant structures are different.

Remark 2.7 The correspondence between the rational solutions to q-$\mathrm{P}_{\mathrm{III}}$ (see [10]) and that to q-P_{II} (see [25]) are straightforward. It is easily verified that substituting $a_2 = q^{\frac{1}{2}}$ into the former yields the latter.

3 Projective reduction from q-$\mathrm{P}_{\mathrm{III}}$ to q-P_{II}

3.1 Birational representation of $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$

We formulate the family of Bäcklund transformations of q-$\mathrm{P}_{\mathrm{III}}$ as a birational representation of the extended affine Weyl group of type $(A_2 + A_1)^{(1)}$ [11, 17]. We refer to [27] for basic ideas of this formulation.

We define the transformations s_i $(i = 0, 1, 2)$ and π on the variables f_j $(j = 0, 1, 2)$ and parameters a_k $(k = 0, 1, 2)$ by

$$s_i(a_j) = a_j a_i^{-a_{ij}}, \qquad s_i(f_j) = f_j \left(\frac{a_i + f_i}{1 + a_i f_i}\right)^{u_{ij}}, \tag{3.1}$$

$$\pi(a_i) = a_{i+1}, \qquad \pi(f_i) = f_{i+1}, \tag{3.2}$$

for $i, j \in \mathbb{Z}/3\mathbb{Z}$. Here the symmetric 3×3 matrix

$$A = (a_{ij})_{i,j=0}^{2} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{3.3}$$

is the Cartan matrix of type $A_2^{(1)}$, and the skew-symmetric one

$$U = (u_{ij})_{i,j=0}^{2} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{3.4}$$

represents an orientation of the corresponding Dynkin diagram. We also define the transformations

w_j ($j = 0, 1$) and r by

$$w_0(f_i) = \frac{a_i a_{i+1}(a_{i-1}a_i + a_{i-1}f_i + f_{i-1}f_i)}{f_{i-1}(a_i a_{i+1} + a_i f_{i+1} + f_i f_{i+1})}, \qquad w_0(a_i) = a_i, \tag{3.5}$$

$$w_1(f_i) = \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1}}{a_i a_{i+1} f_{i+1}(1 + a_{i-1} f_{i-1} + a_{i-1} a_i f_{i-1} f_i)}, \qquad w_1(a_i) = a_i, \tag{3.6}$$

$$r(f_i) = \frac{1}{f_i}, \qquad r(a_i) = a_i, \tag{3.7}$$

for $i \in \mathbb{Z}/3\mathbb{Z}$.

Proposition 3.1 ([17]) *$\langle s_0, s_1, s_2, \pi, w_0, w_1, r\rangle$ forms the extended affine Weyl group of type $(A_2 + A_1)^{(1)}$. Namely, the transformations satisfy the fundamental relations*

$$s_i{}^2 = (s_i s_{i+1})^3 = \pi^3 = 1,\ \pi s_i = s_{i+1}\pi\ (i \in \mathbb{Z}/3\mathbb{Z}), \quad w_0{}^2 = w_1{}^2 = r^2 = 1,\ r w_0 = w_1 r, \tag{3.8}$$

and the actions of $\widetilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi\rangle$ and $\widetilde{W}(A_1^{(1)}) = \langle w_0, w_1, r\rangle$ commute with each other.

In general, for a function $F = F(a_i, f_j)$ we let an element $w \in \widetilde{W}\left((A_2 + A_1)^{(1)}\right)$ act as $w.F(a_i, f_j) = F(a_i.w, f_j.w)$, that is, w acts on the arguments from the right. Note that $a_0 a_1 a_2 = q$ and $f_0 f_1 f_2 = qc^2$ are invariant under the actions of $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$ and $\widetilde{W}(A_2^{(1)})$, respectively. We define the translations T_i ($i = 1, 2, 3, 4$) by

$$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2, \quad T_3 = s_2 s_1 \pi, \quad T_4 = r w_0, \tag{3.9}$$

whose actions on parameters a_i ($i = 0, 1, 2$) and c are given by

$$\begin{aligned} T_1 &: (a_0, a_1, a_2, c) \mapsto (q a_0, q^{-1} a_1, a_2, c),\\ T_2 &: (a_0, a_1, a_2, c) \mapsto (a_0, q a_1, q^{-1} a_2, c),\\ T_3 &: (a_0, a_1, a_2, c) \mapsto (q^{-1} a_0, a_1, q a_2, c),\\ T_4 &: (a_0, a_1, a_2, c) \mapsto (a_0, a_1, a_2, qc). \end{aligned} \tag{3.10}$$

Note that T_i ($i = 1, 2, 3, 4$) commute with each other and $T_1 T_2 T_3 = 1$. The action of T_1 on f-variables can be expressed as

$$T_1(f_1) = \frac{qc^2}{f_1 f_0}\,\frac{1 + a_0 f_0}{a_0 + f_0}, \quad T_1(f_0) = \frac{qc^2}{f_0 T_1(f_1)}\,\frac{1 + a_2 a_0 T_1(f_1)}{a_2 a_0 + T_1(f_1)}. \tag{3.11}$$

Or, applying $T_1{}^n T_2{}^m T_4{}^N$ ($n, m, N \in \mathbb{Z}$) on (3.11) and putting

$$f_{i,N}^{n,m} = T_1{}^n T_2{}^m T_4{}^N (f_i) \quad (i = 0, 1, 2), \tag{3.12}$$

we obtain

$$f_{1,N}^{n+1,m} = \frac{q^{2N+1}c^2}{f_{1,N}^{n,m} f_{0,N}^{n,m}}\,\frac{1 + a_0 q^n f_{0,N}^{n,m}}{a_0 q^n + f_{0,N}^{n,m}}, \quad f_{0,N}^{n+1,m} = \frac{q^{2N+1}c^2}{f_{0,N}^{n,m} f_{1,N}^{n+1}}\,\frac{1 + a_2 a_0 q^{n-m} f_{1,N}^{n+1,m}}{a_2 a_0 q^{n-m} + f_{1,N}^{n+1,m}}, \tag{3.13}$$

which is equivalent to q-$\mathrm{P_{III}}$, (2.1). Then, T_i ($i = 1, 2, 3, 4$) are regarded as Bäcklund transformations of q-$\mathrm{P_{III}}$.

In order to formulate the symmetrization to q-P$_{\rm II}$, it is crucial to introduce the transformation R_1 defined by

$$R_1 = \pi^2 s_1, \tag{3.14}$$

which satisfies

$$R_1{}^2 = T_1. \tag{3.15}$$

The actions of R_1 are given by

$$R_1 : \ (a_0, a_1, a_2, c) \mapsto (a_2 a_0, a_0{}^{-1}, a_1 a_0, c), \tag{3.16}$$

$$R_1(f_0) = \frac{qc^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0}, \quad R_1(f_1) = f_0, \tag{3.17}$$

which describe the zig-zag motion around the line $a_2 = q^{\frac{1}{2}}$ on the parameter space. However, if we put $a_2 = q^{\frac{1}{2}}$, then R_1 becomes the translation on the line $a_2 = q^{\frac{1}{2}}$ with the step $q^{\frac{1}{2}}$ (see Figure 1). In fact, the actions of R_1 are now given by

$$R_1 : \ (a_0, a_1, c) \mapsto (q^{\frac{1}{2}} a_0, q^{-\frac{1}{2}} a_1, c), \tag{3.18}$$

$$R_1(f_0) = \frac{qc^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0}, \quad R_1(f_1) = f_0. \tag{3.19}$$

Applying $R_1{}^k T_4{}^N$ on (3.19) and putting

$$f^k_{i,N} = R_1{}^k T_4{}^N (f_i) \quad (i = 0, 1, 2), \tag{3.20}$$

we have

$$f^{k+1}_{0,N} = \frac{q^{2N+1} c^2}{f^k_{0,N} f^{k-1}_{0,N}} \frac{1 + a_0 q^{\frac{k}{2}} f^k_{0,N}}{a_0 q^{\frac{k}{2}} + f^k_{0,N}}, \tag{3.21}$$

which is equivalent to q-P$_{\rm II}$, (2.2). Then, R_1 and T_4 are regarded as Bäcklund transformations of q-P$_{\rm II}$.

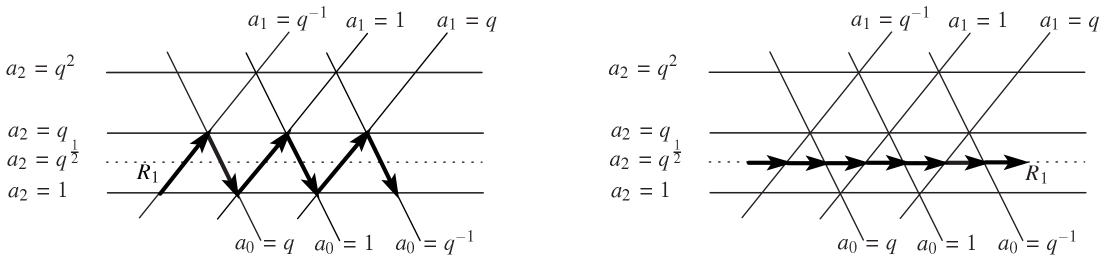

Figure 1. Action of R_1 on the parameter space $\boldsymbol{a} = (a_0, a_1, a_2) \in (\mathbb{C}^\times)^3$ with $a_0 a_1 a_2 = q$. Left: generic case. Right: $a_2 = q^{\frac{1}{2}}$.

In general, it is possible to obtain various discrete dynamical systems of Painlevé type from elements of infinite order that are not necessarily translations in the affine Weyl group by taking a projection on an appropriate sublattice of corresponding root lattice. We call such a procedure a *projective reduction*.

By using the above formulation, we can now explain why the difference of hypergeometric solutions to q-P$_{\rm III}$ and that to q-P$_{\rm II}$ occurs.

3.2 Hypergeometric functions

First, we explain about the difference of hypergeometric functions. For convenience, we define the function $H_{n,m}$ by

$$H_{n,m} = \Theta({a_0}^2 q^{2n+1}; q^2) F_{n+\frac{1}{2},m}, \tag{3.22}$$

where $F_{n,m}$ is given in Remark 2.3. Then, we obtain from (2.12) with $a_2 = q^{\frac{1}{2}}$ the three-term relation for $H_{n,0}$:

$$\left[{T_1}^2 + \left(q^{-3-2n} {a_0}^{-2} + q^{-2-2n} {a_0}^{-2} - 1 \right) T_1 + q^{-3-4n} {a_0}^{-4} \right] H_{n,0} = 0. \tag{3.23}$$

Let $n = 0$. Since ${R_1}^2 = T_1$, the linear difference operator in (3.23) is fourth order with respect to R_1. Moreover, it admits the following factorization into the second order linear difference operators:

$${T_1}^2 + \left({a_0}^{-2} q^{-3} + {a_0}^{-2} q^{-2} - 1 \right) T_1 + {a_0}^{-4} q^{-3} = \left({R_1}^2 + R_1 + {a_0}^{-2} q^{-2} \right) \left({R_1}^2 - R_1 + {a_0}^{-2} q^{-1} \right). \tag{3.24}$$

On the other hand, the three-term relation for G_0 (see (2.19)) can be expressed as

$$\left({R_1}^2 - R_1 + {a_0}^{-2} q^{-1} \right) G_0 = 0. \tag{3.25}$$

Note that the second factor in the right-hand side of (3.24) is exactly the operator in (3.25), thus, G_0 also satisfies (3.23) with $n = 0$. This factorization (3.24) implies that G_k can not be obtained simply from $F_{n,m}$ by a specialization of parameters (2.3).

3.3 Determinant structure

Next, in order to discuss the difference of determinant structures, we need to introduce the τ functions and lift the representation to the Weyl group on the level of τ functions [17,40]. We introduce τ_i and $\overline{\tau}_i$ ($i \in \mathbb{Z}/3\mathbb{Z}$) with

$$f_i = q^{\frac{1}{3}} c^{\frac{2}{3}} \frac{\overline{\tau}_{i+1} \tau_{i-1}}{\tau_{i+1} \overline{\tau}_{i-1}}. \tag{3.26}$$

Proposition 3.2 ([40]) *We define the action of* s_i $(i = 0, 1, 2)$, π, w_j $(j = 0, 1)$, *and* r *on* τ_k *and* $\overline{\tau}_k$ $(k = 0, 1, 2)$ *by the following formulae*:

$$\begin{cases} s_i(\tau_i) = \dfrac{u_i \tau_{i+1} \overline{\tau}_{i-1} + \overline{\tau}_{i+1} \tau_{i-1}}{u_i^{\frac{1}{2}} \overline{\tau}_i}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j), \\ s_i(\overline{\tau}_i) = \dfrac{v_i \overline{\tau}_{i+1} \tau_{i-1} + \tau_{i+1} \overline{\tau}_{i-1}}{v_i^{\frac{1}{2}} \tau_i}, \quad s_i(\overline{\tau}_j) = \overline{\tau}_j \quad (i \neq j), \end{cases} \tag{3.27}$$

$$\pi(\tau_i) = \tau_{i+1}, \quad \pi(\overline{\tau}_i) = \overline{\tau}_{i+1}, \tag{3.28}$$

$$\begin{cases} w_0(\overline{\tau}_i) = \dfrac{{a_{i+1}}^{\frac{1}{3}} (\overline{\tau}_i \tau_{i+1} \tau_{i+2} + u_{i-1} \tau_i \overline{\tau}_{i+1} \tau_{i+2} + u_{i+1}^{-1} \tau_i \tau_{i+1} \overline{\tau}_{i+2})}{{a_{i+2}}^{\frac{1}{3}} \overline{\tau}_{i+1} \overline{\tau}_{i+2}}, \\ w_0(\tau_i) = \tau_i, \end{cases} \tag{3.29}$$

$$\begin{cases} w_1(\tau_i) = \dfrac{a_{i+1}{}^{\frac{1}{3}}(\tau_i\overline{\tau}_{i+1}\overline{\tau}_{i+2} + v_{i-1}\overline{\tau}_i\tau_{i+1}\overline{\tau}_{i+2} + v_{i+1}^{-1}\overline{\tau}_i\overline{\tau}_{i+1}\tau_{i+2})}{a_{i+2}{}^{\frac{1}{3}}\tau_{i+1}\tau_{i+2}}, \\ w_1(\overline{\tau}_i) = \overline{\tau}_i, \end{cases} \tag{3.30}$$

$$r(\tau_i) = \overline{\tau}_i, \quad r(\overline{\tau}_i) = \tau_i, \tag{3.31}$$

with

$$u_i = q^{-\frac{1}{3}}c^{-\frac{2}{3}}a_i, \quad v_i = q^{\frac{1}{3}}c^{\frac{2}{3}}a_i, \tag{3.32}$$

where $i, j \in \mathbb{Z}/3\mathbb{Z}$*. Then,* $\langle s_0, s_1, s_2, \pi, w_0, w_1, r\rangle$ *realizes the affine Weyl group* $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$.

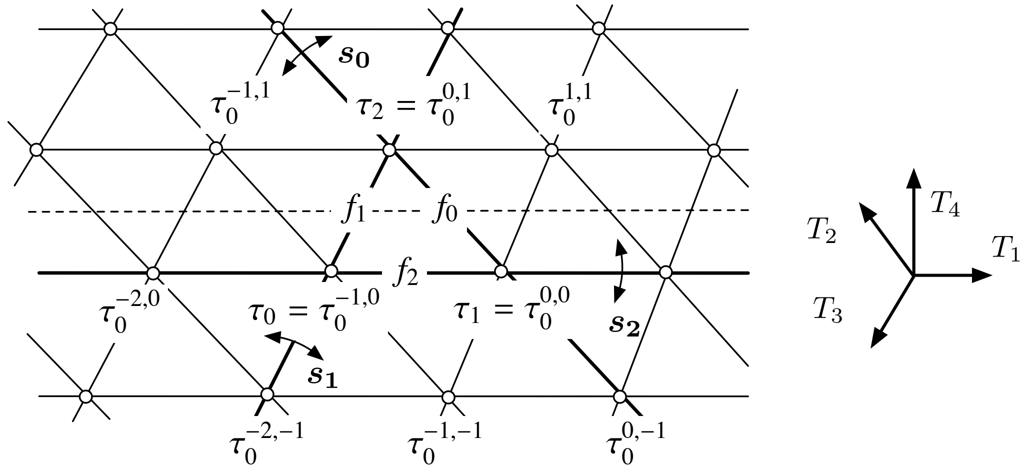

Figure 2. Configuration of the τ functions on the lattice with $N = 0$.

Then, we define the τ functions $\tau_N^{n,m}$ ($n, m, N \in \boldsymbol{Z}$) by

$$\tau_N^{n,m} = T_1{}^nT_2{}^mT_4{}^N(\tau_1). \tag{3.33}$$

We note that $\tau_0 = \tau_0^{-1,0}$, $\tau_1 = \tau_0^{0,0}$, $\tau_2 = \tau_0^{0,1}$, $\overline{\tau}_0 = \tau_1^{-1,0}$, $\overline{\tau}_1 = \tau_1^{0,0}$, and $\overline{\tau}_2 = \tau_1^{0,1}$.

Proposition 3.3 *The action of* $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$ *on* $\tau_N^{n,m}$ *is*

$$s_0(\tau_N^{n,m}) = \tau_N^{-n,m-n}, \quad s_1(\tau_N^{n,m}) = \tau_N^{m-1,n+1}, \quad s_2(\tau_N^{n,m}) = \tau_N^{n-m,-m}, \quad \pi(\tau_N^{n,m}) = \tau_N^{-m,n-m+1}, \tag{3.34}$$

$$w_0(\tau_N^{n,m}) = \tau_{-N}^{n,m}, \quad w_1(\tau_N^{n,m}) = \tau_{2-N}^{n,m}, \quad r(\tau_N^{n,m}) = \tau_{1-N}^{n,m}. \tag{3.35}$$

For convenience, we put

$$\alpha_i = a_i{}^{\frac{1}{6}}, \quad \gamma = c^{\frac{1}{6}}, \quad Q = q^{\frac{1}{6}}. \tag{3.36}$$

Though it is possible to derive more various bilinear difference equations from Proposition 3.2, we present here only the equations that are directly relevant to q-P$_{\mathrm{III}}$, (3.11).

Proposition 3.4 *The following bilinear equations hold*:

$$\tau_{N+1}^{n,m}\tau_N^{n+1,m+1} - Q^{-3n+3m+2N-2}\gamma^2{\alpha_1}^3\tau_N^{n+1,m}\tau_{N+1}^{n,m+1} + Q^{-6n+6m+4N-4}\gamma^4{\alpha_1}^6\tau_N^{n,m}\tau_{N+1}^{n+1,m+1} = 0, \tag{3.37}$$

$$\tau_{N+1}^{n+1,m+1}\tau_N^{n+1,m} - Q^{3n+2N+4}\gamma^2{\alpha_0}^3\tau_N^{n,m}\tau_{N+1}^{n+2,m+1} + Q^{6n+4N+8}\gamma^4{\alpha_0}^6\tau_N^{n+1,m+1}\tau_{N+1}^{n+1,m} = 0, \tag{3.38}$$

$$\tau_{N+1}^{n+1,m+1}\tau_N^{n,m} - Q^{-3n+3m-2N-4}\gamma^{-2}{\alpha_1}^3\tau_{N+1}^{n+1,m}\tau_N^{n,m+1} + Q^{-6n+6m-4N-8}\gamma^{-4}{\alpha_1}^6\tau_N^{n+1,m+1}\tau_{N+1}^{n,m} = 0, \tag{3.39}$$

$$\tau_{N+1}^{n+1,m}\tau_N^{n+1,m+1} - Q^{3n-2N+2}\gamma^{-2}{\alpha_0}^3\tau_{N+1}^{n,m}\tau_N^{n+2,m+1} + Q^{6n-4N+4}\gamma^{-4}{\alpha_0}^6\tau_N^{n+1,m}\tau_{N+1}^{n+1,m+1} = 0, \tag{3.40}$$

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n,m} + Q^{-8n+4m-4}{\alpha_0}^{-4}{\alpha_1}^4\left(\tau_N^{n,m}\right)^2 - Q^{-2n+m-1}{\alpha_0}^{-1}\alpha_1\tau_N^{n+1,m}\tau_N^{n-1,m} = 0. \tag{3.41}$$

The proof of Proposition 3.4 will be given in the appendix A.1.

As seen below q-$\mathrm{P_{III}}$, (3.11) or (3.13), can be obtained from the bilinear equations. Noticing that

$$f_{0,N}^{n,m} = Q^{4N+2}\gamma^4\frac{\tau_{N+1}^{n,m}\tau_N^{n,m+1}}{\tau_N^{n,m}\tau_{N+1}^{n,m+1}}, \quad f_{1,N}^{n,m} = Q^{4N+2}\gamma^4\frac{\tau_{N+1}^{n,m+1}\tau_N^{n-1,m}}{\tau_N^{n,m+1}\tau_{N+1}^{n-1,m}}, \quad f_{2,N}^{n,m} = Q^{4N+2}\gamma^4\frac{\tau_{N+1}^{n-1,m}\tau_N^{n,m}}{\tau_N^{n-1,m}\tau_{N+1}^{n,m}}, \tag{3.42}$$

we can rewrite (3.37) and (3.39) as

$$1 + Q^{-6n+6m-6}{\alpha_1}^6 f_{1,N}^{n+1,m} = Q^{-3n+3m+2N-2}\gamma^2{\alpha_1}^3\frac{\tau_N^{n+1,m}\tau_{N+1}^{n,m+1}}{\tau_{N+1}^{n,m}\tau_N^{n+1,m+1}}, \tag{3.43}$$

$$1 + Q^{6n-6m+6}{\alpha_1}^{-6} f_{1,N}^{n+1,m} = Q^{3n-3m+2N+4}\gamma^2{\alpha_1}^{-3}\frac{\tau_{N+1}^{n+1,m}\tau_N^{n,m+1}}{\tau_{N+1}^{n,m}\tau_N^{n+1,m+1}}, \tag{3.44}$$

respectively. Dividing (3.44) by (3.43), we have

$$\frac{1 + Q^{6n-6m+6}{\alpha_1}^{-6} f_{1,N}^{n+1,m}}{1 + Q^{-6n+6m-6}{\alpha_1}^6 f_{1,N}^{n+1,m}} = Q^{6n-6m+6}{\alpha_1}^{-6}\frac{\tau_{N+1}^{n+1,m}\tau_N^{n,m+1}}{\tau_N^{n+1,m}\tau_{N+1}^{n,m+1}} = Q^{6n-6m+6}{\alpha_1}^{-6}\frac{f_{0,N}^{n,m}}{f_{2,N}^{n+1,m}}, \tag{3.45}$$

which is equivalent to the second equation of (3.13). Similarly, (3.38) and (3.40) yield the first equation of (3.13).

For the hypergeometric solutions, we relate the τ functions to the determinants $\psi_N^{n,m}$, (2.5), by multiplication of appropriate "gauge" factor. Set

$$\begin{aligned}\tau_N^{n,m} =& (-1)^{\frac{N(N+1)}{2}} Q^{-2(2n-m)N^2+6Nn}{\alpha_0}^{-4N^2+6N}{\alpha_2}^{-2N^2}\left(\frac{\Theta(-Q^{-6n}{\alpha_0}^{-6}, Q^6)\Theta(-Q^{6m}{\alpha_2}^{-6}, Q^6)}{\Theta(Q^{-6(n-m)}{\alpha_0}^{-6}{\alpha_2}^{-6}, Q^6)}\right)^N \\ &\times \Gamma(Q^{2n-m+1}{\alpha_0}^2\alpha_2; Q, Q)\Gamma(Q^{-n+2m-1}{\alpha_1}^2\alpha_0; Q, Q)\Gamma(Q^{-n-m}{\alpha_2}^2\alpha_1; Q, Q)\,\psi_N^{n,m-1},\end{aligned} \tag{3.46}$$

where $\Gamma(a; p, q)$ denotes the Elliptic gamma function, which is defined by

$$\Gamma(a; p, q) = \frac{(q^2a^{-1}; p, q)_\infty}{(a; p, q)_\infty}, \tag{3.47}$$

and satisfies

$$\Gamma(qa; q, q) = \Theta(a, q)\Gamma(a; q, q). \tag{3.48}$$

Let $\gamma = 1$. Then the bilinear equations (3.37)–(3.41) can be rewritten as

$$\psi^{n,m}_{N+1}\psi^{n+1,m+1}_{N} - Q^{-12n+12N}{\alpha_0}^{-12}\psi^{n+1,m}_{N}\psi^{n,m+1}_{N+1} + Q^{-12n}{\alpha_0}^{-12}\psi^{n,m}_{N}\psi^{n+1,m+1}_{N+1} = 0, \tag{3.49}$$
$$\psi^{n+1,m+1}_{N+1}\psi^{n+1,m}_{N} - Q^{-12N}\psi^{n,m}_{N}\psi^{n+2,m+1}_{N+1} - Q^{12n+12}{\alpha_0}^{12}\psi^{n+1,m+1}_{N}\psi^{n+1,m}_{N+1} = 0, \tag{3.50}$$
$$\psi^{n+1,m+1}_{N+1}\psi^{n,m}_{N} - \psi^{n+1,m}_{N+1}\psi^{n,m+1}_{N} + Q^{12m+12}{\alpha_2}^{-12}\psi^{n+1,m+1}_{N}\psi^{n,m}_{N+1} = 0, \tag{3.51}$$
$$\psi^{n+1,m}_{N+1}\psi^{n+1,m+1}_{N} - Q^{12m+12}{\alpha_2}^{-12}\psi^{n,m}_{N+1}\psi^{n+2,m+1}_{N} - \psi^{n+1,m}_{N}\psi^{n+1,m+1}_{N+1} = 0, \tag{3.52}$$
$$\psi^{n,m}_{N+1}\psi^{n,m}_{N-1} - \left(\psi^{n,m}_{N}\right)^2 + \psi^{n+1,m}_{N}\psi^{n-1,m}_{N} = 0, \tag{3.53}$$

respectively. Equations (3.49)–(3.52) are equivalent to (2.7)–(2.10). Note that (3.53) is exactly the discrete Toda equation, (2.15), which fixes the determinant structure of the hypergeometric solutions as mentioned in Remark 2.3.

Remark 3.5 The gauge factor $\tau^{n,m}_N/\psi^{n,m-1}_N$ in (3.46) is obtained by solving the overdetermined system of the bilinear difference equations with $\gamma = 1$ under the boundary conditions $\tau^{n,m}_N = 0$ ($N \in \mathbb{Z}_{<0}$) [26].

Let us consider the bilinear equations for q-P$_{\rm II}$. Since we need R_1, τ_i, and $\overline{\tau}_i$ ($i \in \mathbb{Z}/3\mathbb{Z}$), the lattice is restricted to the "unit-strip" (see Figure 3). Therefore, we have only to consider $\tau^{n,0}_N$ and $\tau^{n,1}_N$ ($n, N \in \mathbb{Z}$). We set

$$\tau^k_N = {R_1}^k {T_4}^N(\tau_1). \tag{3.54}$$

Note that

$$\tau_0 = \tau^{-2}_0, \quad \tau_1 = \tau^0_0, \quad \tau_2 = \tau^{-1}_0, \quad \overline{\tau}_0 = \tau^{-2}_1, \quad \overline{\tau}_1 = \tau^0_1, \quad \overline{\tau}_2 = \tau^{-1}_1. \tag{3.55}$$

In general, it follows that

$$\tau^{n,0}_N = \tau^{2n}_N, \quad \tau^{n,1}_N = \tau^{2n-1}_N, \tag{3.56}$$

and

$$f^k_{0,N} = Q^{4N+2}\gamma^4 \frac{\tau^k_{N+1}\tau^{k-1}_N}{\tau^k_N\tau^{k-1}_{N+1}}. \tag{3.57}$$

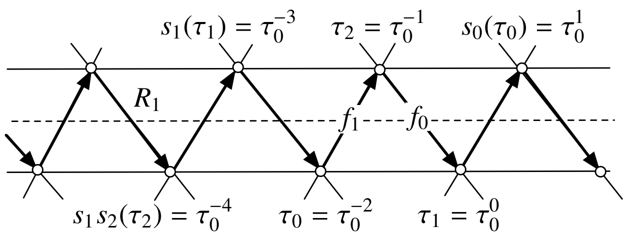

Figure 3. The actions of R_1 on τ_i ($i = 0, 1, 2$).

Proposition 3.6 *The following bilinear equations hold*:

$$Q^{\frac{-3k+4N+2}{2}}\gamma^2{\alpha_0}^{-3}\tau^{k+1}_N\tau^{k-2}_{N+1} - Q^{-3k+4N+2}\gamma^4{\alpha_0}^{-6}\tau^{k-1}_N\tau^k_{N+1} - \tau^{k-1}_{N+1}\tau^k_N = 0, \tag{3.58}$$
$$Q^{\frac{-3k-4N-2}{2}}\gamma^{-2}{\alpha_0}^{-3}\tau^{k+1}_{N+1}\tau^{k-2}_N - Q^{-3k-4N-2}\gamma^{-4}{\alpha_0}^{-6}\tau^k_N\tau^{k-1}_{N+1} - \tau^k_{N+1}\tau^{k-1}_N = 0, \tag{3.59}$$
$$\tau^k_{N+1}\tau^{k+1}_{N-1} - Q^{\frac{k-4N+1}{2}}\gamma^{-2}\alpha_0\tau^{k+2}_N\tau^{k-1}_N - Q^{-k+4N-1}\gamma^4{\alpha_0}^{-2}\tau^k_N\tau^{k+1}_N = 0. \tag{3.60}$$

The proof of Proposition 3.6 will be given in the appendix A.2.

One can obtain q-$\mathrm{P_{II}}$, (3.19), from Proposition 3.6 as follows. Equations (3.58) and (3.59) can be rewritten as

$$1 + Q^{-3k}{\alpha_0}^{-6} f_{0,N}^k = Q^{\frac{-3k+4N+2}{2}} \gamma^2 {\alpha_0}^{-3} \frac{\tau_N^{k+1}\tau_{N+1}^{k-2}}{\tau_{N+1}^{k-1}\tau_N^k}, \tag{3.61}$$

$$1 + Q^{3k}{\alpha_0}^{6} f_{0,N}^k = Q^{\frac{3k+4N+2}{2}} \gamma^2 {\alpha_0}^{3} \frac{\tau_{N+1}^{k+1}\tau_{N}^{k-2}}{\tau_{N+1}^{k-1}\tau_N^k}. \tag{3.62}$$

Dividing (3.62) by (3.61), we have

$$\frac{1 + Q^{3k}{\alpha_0}^{6} f_{0,N}^k}{1 + Q^{-3k}{\alpha_0}^{-6} f_{0,N}^k} = Q^{3k}{\alpha_0}^{6} \frac{\tau_{N+1}^{k+1}\tau_{N}^{k-2}}{\tau_{N}^{k+1}\tau_{N+1}^{k-2}} = Q^{3k-12N-6}\gamma^{12}{\alpha_0}^{6} f_{0,N}^{k+1} f_{0,N}^{k} f_{0,N}^{k-1}, \tag{3.63}$$

which is equivalent to (3.21).

For hypergeometric solutions, by putting $\gamma = 1$ and

$$\tau_N^k = (-1)^{\frac{N(N-1)}{2}} Q^{N(N-1)(k+n)} {\alpha_0}^{2N(N-1)} \frac{\Gamma(Q^{\frac{2k+3}{2}}{\alpha_0}^2; Q, Q)\Gamma(Q^{-\frac{k}{2}}{\alpha_0}^{-1}; Q, Q)\Gamma(Q^{\frac{-k+3}{2}}{\alpha_0}^{-1}; Q, Q)}{\Theta(Q^{3k+1}{\alpha_0}^6, Q^3)^N} \phi_n^k, \tag{3.64}$$

we can rewrite the bilinear equations (3.58), (3.59), and (3.60) as

$$Q^{6N-6k+6}{\alpha_0}^{-12}\phi_N^{k+1}\phi_{N+1}^{k-2} + Q^{6N}\phi_N^{k-1}\phi_{N+1}^{k} - \phi_{N+1}^{k-1}\phi_N^k = 0, \tag{3.65}$$

$$Q^{6N}\phi_{N+1}^{k+1}\phi_N^{k-2} + Q^{-6N-6k}{\alpha_0}^{-12}\phi_N^{k}\phi_{N+1}^{k-1} - \phi_{N+1}^{k}\phi_N^{k-1} = 0, \tag{3.66}$$

$$\phi_{N+1}^{k}\phi_{N-1}^{k+1} - \phi_N^{k}\phi_N^{k+1} + \phi_N^{k+2}\phi_N^{k-1} = 0, \tag{3.67}$$

which are equivalent to (2.20), (2.21), and (2.26), respectively. The determinant structure of the hypergeometric solutions is fixed by (3.67) as was explained in Remark 2.6.

Therefore, the difference of the determinant structures of the hypergeometric solutions to q-$\mathrm{P_{III}}$ and that to q-$\mathrm{P_{II}}$ originates from the following procedures:

(i) the specialization $a_2 = q^{\frac{1}{2}}$ and the restriction of τ functions on the "unit-strip";

(ii) taking the half-step translation R_1 instead of T_1 as a time evolution.

These result in the difference of the bilinear equations (3.41) (or (3.53)) and (3.60) (or (3.67)), which fix the determinant structure of the hypergeometric solutions.

4 Concluding remarks

In this paper, we have clarified the mechanism that gives rise to the apparent "inconsistency" in the hypergeometric solutions to q-$\mathrm{P_{III}}$ and that to q-$\mathrm{P_{II}}$ by using their underlying affine Weyl group symmetry. In general, it is also possible to explain the inconsistency among the hypergeometric solutions to other symmetric and asymmetric discrete Painlevé equations (see, for example, Propositions 1.1 and 1.2).

We shall make a slightly technical remark on the solutions to q-$\mathrm{P_{II}}$. Besides the hypergeometric solution to q-$\mathrm{P_{II}}$ in Proposition 2.5, one can also obtain another solution to q-$\mathrm{P_{II}}$ from that to q-$\mathrm{P_{III}}$ in Proposition 2.2 through a naïve specialization (2.3). This solution, however, takes different expressions according to the parity of the time variable k of q-$\mathrm{P_{II}}$, (2.2). On the other hand, the solution in Proposition 2.5 forms a smooth function in k. In this sense it is more natural as a solution to q-$\mathrm{P_{II}}$.

Before closing, we demonstrate another example of the projective reductions. Let us consider the following system of difference equations [28]:

$$Z_n + X_n = \frac{3na + b_1}{Y_n} + t, \quad X_{n+1} + Y_n = \frac{(3n+1)a + b_2}{Z_n} + t, \quad Y_{n+1} + Z_n = \frac{(3n+2)a + b_3}{Z_n} + t, \tag{4.1}$$

where X_n, Y_n, and Z_n are the dependent variables, $n \in \mathbb{Z}$ is the independent variable, and $a, b_1, b_2, b_3, t \in \mathbb{C}$ are parameters. Equation (4.1) is one of the discrete Painlevé systems of type $A_3^{(1)}$. Namely, it arises from a Bäcklund transformation of the Painlevé V equation, which describes a translation in a different direction from (1.4). Putting $b_1 = b_2 = b_3 = b$, $X_n = x_{3n-1}$, $Y_n = x_{3n}$, and $Z_n = x_{3n+1}$, we can reduce (4.1) to

$$x_{n+1} + x_{n-1} = \frac{an + b}{x_n} + t, \tag{4.2}$$

which is known as a discrete Painlevé I equation [36]. This reduction from (4.1) to (4.2) is a typical example of the projective reductions other than a symmetrization.

It seems that various projective reductions of the discrete Painlevé systems change the underlying symmetry and yield a number of intriguing problems. One interesting project is to make a list of the hypergeometric functions that appear as the solutions to all the symmetric discrete Painlevé equations in Sakai's classification [13,14,39]. These will be discussed in forthcoming papers [16].

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A Derivation of bilinear equations

In this appendix, we derive various bilinear equations for τ functions from the birational representations of $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$ given in Proposition 3.2.

A.1 Bilinear equations for q-$\mathrm{P_{III}}$

We use the notations introduced in (3.33) and (3.36). For convenience, we classify the bilinear equations into six types so that any equations which belong to the same type can be transformed into each other by the action of $\widetilde{W}\left((A_2 + A_1)^{(1)}\right)$.

Proposition A.1 (Type I: Discrete Toda type) *The following bilinear equations hold*:

$$\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{4n-8m+4}\alpha_1{}^{-4}\alpha_2{}^{4}\left(\tau^{n,m}_{N}\right)^2 - Q^{n-2m+1}\alpha_1{}^{-1}\alpha_2\tau^{n,m+1}_{N}\tau^{n,m-1}_{N} = 0, \tag{A.1}$$

$$\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{4n+4m}\alpha_0{}^{4}\alpha_2{}^{-4}\left(\tau^{n,m}_{N}\right)^2 - Q^{n+m}\alpha_0\alpha_2{}^{-1}\tau^{n+1,m+1}_{N}\tau^{n-1,m-1}_{N} = 0, \tag{A.2}$$

$$\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{-8n+4m-4}\alpha_0{}^{-4}\alpha_1{}^{4}\left(\tau^{n,m}_{N}\right)^2 - Q^{-2n+m-1}\alpha_0{}^{-1}\alpha_1\tau^{n+1,m}_{N}\tau^{n-1,m}_{N} = 0. \tag{A.3}$$

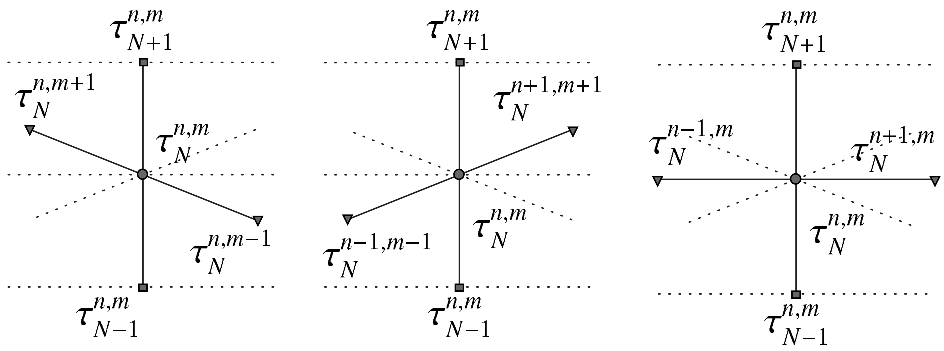

Figure 4. Configuration of τ functions for the bilinear equations of type I. Left: (A.1), center: (A.2), right: (A.3).

Proof. Application of $T_4 = rw_0$ on $\overline{\tau}_0$ yields

$$T_4(\overline{\tau}_0) = c^{-\frac{2}{3}}a_0{}^{-\frac{1}{3}}a_1{}^{-1}a_2{}^{-\frac{2}{3}}\frac{\overline{\tau}_0\overline{\tau}_1}{\tau_1} + c^{\frac{2}{3}}a_0{}^{\frac{1}{3}}a_1{}^{\frac{2}{3}}a_2\frac{\overline{\tau}_0\overline{\tau}_2}{\tau_2} + a_1{}^{\frac{1}{3}}a_2{}^{-\frac{1}{3}}\frac{\tau_0\overline{\tau}_1\overline{\tau}_2}{\tau_1\tau_2}, \tag{A.4}$$

which is rearranged as

$$T_4(\overline{\tau}_0) - c^{-\frac{2}{3}}a_0{}^{-\frac{1}{3}}a_1{}^{-1}a_2{}^{-\frac{2}{3}}\frac{\overline{\tau}_0\overline{\tau}_1}{\tau_1}\left(\frac{q^{\frac{1}{3}}c^{\frac{2}{3}}a_1\tau_0\overline{\tau}_2 + \overline{\tau}_0\tau_2}{\overline{\tau}_0\tau_2}\right)\left(\frac{q^{\frac{1}{3}}c^{\frac{2}{3}}a_2\tau_1\overline{\tau}_0 + \overline{\tau}_1\tau_0}{\overline{\tau}_1\tau_0}\right) + a_1{}^{-\frac{2}{3}}a_2{}^{\frac{2}{3}}\frac{\overline{\tau}_0{}^2}{\tau_0} = 0. \tag{A.5}$$

Applying $T_2 = s_2\pi s_1$ and $T_3 = s_2s_1\pi$ on $\overline{\tau}_0$ and $\overline{\tau}_1$, respectively, we obtain

$$q^{\frac{1}{6}}c^{\frac{1}{3}}a_1{}^{\frac{1}{2}}\tau_1T_2(\overline{\tau}_0) = q^{\frac{1}{3}}c^{\frac{2}{3}}a_1\tau_0\overline{\tau}_2 + \overline{\tau}_0\tau_2, \tag{A.6}$$

$$q^{\frac{1}{6}}c^{\frac{1}{3}}a_2{}^{\frac{1}{2}}\tau_2T_3(\overline{\tau}_1) = q^{\frac{1}{3}}c^{\frac{2}{3}}a_2\tau_1\overline{\tau}_0 + \overline{\tau}_1\tau_0. \tag{A.7}$$

Using (A.6) and (A.7), we can rewrite (A.5) as

$$T_4{}^2(\tau_0)\tau_0 + a_1{}^{-\frac{2}{3}}a_2{}^{\frac{2}{3}}T_4(\tau_0)^2 - a_1{}^{-\frac{1}{6}}a_2{}^{\frac{1}{6}}T_2T_4(\tau_0)T_3T_4(\tau_1) = 0. \tag{A.8}$$

Then by applying $T_1{}^{l+1}T_2{}^{m}T_4{}^{n-1}$, $T_1{}^{l}T_2{}^{m}T_4{}^{n-1}\pi$, and $T_1{}^{l}T_2{}^{m-1}T_4{}^{n-1}\pi^2$ on (A.8), we obtain (A.1), (A.2), and (A.3), respectively. □

Figure 4 shows the configuration of τ functions in the bilinear equations. Each bilinear equation takes the form of a linear combination of the three quadratic terms in τ functions. In the left figure, we mark the first, the second, and the third multiplication of τ functions of (A.1) with the square, the circle, and the triangle, respectively. In the rest of this paper, we use similar representations as above.

Proposition A.2 (Type II: Discrete 2d-Toda type) *The following bilinear difference equations hold*:

$$(1 - Q^{-12m}\alpha_2{}^{12})\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{n-11m}\alpha_0\alpha_2{}^{11}\tau^{n+1,m+1}_N\tau^{n-1,m-1}_N - Q^{n-2m}\alpha_0\alpha_2{}^2\tau^{n,m+1}_N\tau^{n,m-1}_N = 0, \quad (A.9)$$

$$(1 - Q^{12n}\alpha_0{}^{12})\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{10n+m}\alpha_0{}^{10}\alpha_2{}^{-1}\tau^{n+1,m}_N\tau^{n-1,m}_N - Q^{n+m}\alpha_0\alpha_2{}^{-1}\tau^{n+1,m+1}_N\tau^{n-1,m-1}_N = 0, \quad (A.10)$$

$$(1 - Q^{12n-12m}\alpha_0{}^{12}\alpha_2{}^{12})\tau^{n,m}_{N+1}\tau^{n,m}_{N-1} + Q^{10n-11m}\alpha_0{}^{10}\alpha_2{}^{11}\tau^{n+1,m}_N\tau^{n-1,m}_N - Q^{n-2m}\alpha_0\alpha_2{}^2\tau^{n,m+1}_N\tau^{n,m-1}_N = 0. \quad (A.11)$$

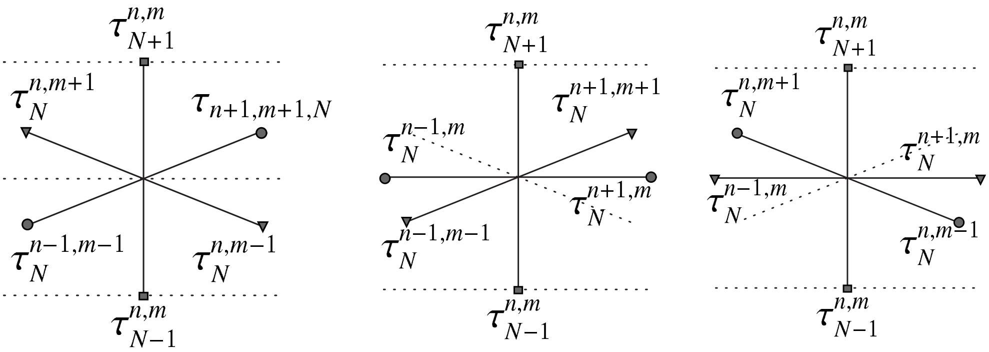

Figure 5. Configuration of τ functions for the bilinear equations of type II. Left: (A.9), center: (A.10), right: (A.11).

Proof. Equation (A.9) is derived by eliminating $\tau_{l,m,n}$ from (A.1) and (A.2). We obtain (A.10) and (A.11) in a similar manner. □

Proposition A.3 (Type III) *The following bilinear equations hold*:

$$(Q^{4n-8m+4}\alpha_1{}^{-4}\alpha_2{}^4 - Q^{4l+4m}\alpha_0{}^4\alpha_2{}^{-4})\left(\tau^{n,m}_N\right)^2 + Q^{n+m}\alpha_0\alpha_2{}^{-1}\tau^{n+1,m+1}_N\tau^{n-1,m-1}_N - Q^{n-2m+1}\alpha_1{}^{-1}\alpha_2\tau^{n,m+1}_N\tau^{n,m-1}_N = 0, \quad (A.12)$$

$$(Q^{4n+4m}\alpha_0{}^4\alpha_2{}^{-4} - Q^{-8n+4m-4}\alpha_0{}^{-4}\alpha_1{}^4)\left(\tau^{n,m}_N\right)^2 + Q^{-2n+m-1}\alpha_0{}^{-1}\alpha_1\tau^{n+1,m}_N\tau^{n-1,m}_N - Q^{n+m}\alpha_0\alpha_2{}^{-1}\tau^{n+1,m+1}_N\tau^{n-1,m-1}_N = 0, \quad (A.13)$$

$$(Q^{-8n+4m-4}\alpha_0{}^{-4}\alpha_1{}^4 - Q^{4n-8m+4}\alpha_1{}^{-4}\alpha_2{}^4)\left(\tau^{n,m}_N\right)^2 - Q^{-2n+m-1}\alpha_0{}^{-1}\alpha_1\tau^{n+1,m}_N\tau^{n-1,m}_N + Q^{n-2m+1}\alpha_1{}^{-1}\alpha_2\tau^{n,m+1}_N\tau^{n,m-1}_N = 0. \quad (A.14)$$

Proof. We obtain (A.12) by eliminating $\tau_{l,m,n+1}\tau_{l,m,n-1}$ from (A.1) and (A.2). Other equations can be derived in a similar manner. □

Proposition A.4 (Type IV) *The following bilinear equation holds*:

$$Q^{-3n}\alpha_0{}^{-3}(1 - Q^{-12m}\alpha_2{}^{-12})\tau^{n+1,m}_N\tau^{n-1,m}_N - Q^{-3m}\alpha_2{}^3(1 - Q^{-12l}\alpha_0{}^{-12})\tau^{n,m+1}_N\tau^{n,m-1}_N + (Q^{-12m}\alpha_2{}^{-12} - Q^{-12l}\alpha_0{}^{-12})\tau^{n+1,m+1}_N\tau^{n-1,m-1}_N = 0. \quad (A.15)$$

Proof. Equation (A.15) can be derived by eliminating $\tau^{n,m}_N$ from (A.12) and (A.13). □

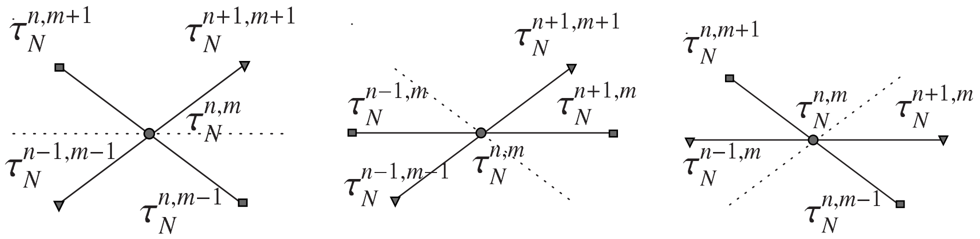

Figure 6. Configuration of τ functions for the bilinear equations of type III. Left: (A.12), center: (A.13), right: (A.14).

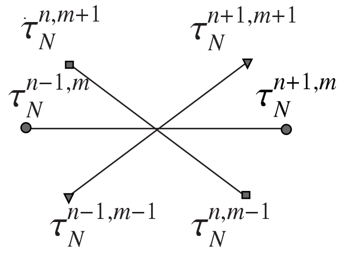

Figure 7. Configuration of τ functions for the bilinear equations of type IV.

Proposition A.5 (Type V) *The following bilinear equations hold*:

$$\tau^{n,m}_{N+1}\tau^{n+1,m+1}_{N-1} - Q^{n+m-2N}\gamma^{-2}\alpha_0{}^2\alpha_1\tau^{n+1,m}_N\tau^{n,m+1}_N - Q^{-2n+2m-4N}\gamma^4\alpha_0{}^{-4}\alpha_1{}^{-2}\tau^{n,m}_N\tau^{n+1,m+1}_N = 0, \tag{A.16}$$

$$\tau^{n+1,m}_{N+1}\tau^{n,m}_{N-1} - Q^{-2n+m-2N}\gamma^{-2}\alpha_0{}^{-3}\alpha_1{}^{-1}\alpha_2{}^{-2}\tau^{n+1,m+1}_N\tau^{n,m-1}_N - Q^{4n-2m+4N}\gamma^4\alpha_0{}^6\alpha_1{}^2\alpha_2{}^4\tau^{n+1,m}_N\tau^{n,m}_N = 0, \tag{A.17}$$

$$\tau^{n+1,m+1}_{N+1}\tau^{n+1,m}_{N-1} - Q^{n-2m-2N+1}\gamma^{-2}\alpha_1{}^{-1}\alpha_2\tau^{n,m}_N\tau^{n+2,m+1}_N - Q^{-2n+4m+4N-2}\gamma^4\alpha_1{}^2\alpha_2{}^{-2}\tau^{n+1,m+1}_N\tau^{n+1,m}_N = 0, \tag{A.18}$$

$$\tau^{n+1,m+1}_{N+1}\tau^{n,m}_{N-1} - Q^{n+m+2N}\gamma^2\alpha_0{}^2\alpha_1\tau^{n,m+1}_N\tau^{n+1,m}_N - Q^{-2n-2m-4N}\gamma^{-4}\alpha_0{}^{-4}\alpha_1{}^{-2}\tau^{n+1,m+1}_N\tau^{n,m}_N = 0, \tag{A.19}$$

$$\tau^{n,m}_{N+1}\tau^{n+1,m}_{N-1} - Q^{-2n+m+2N}\gamma^2\alpha_0{}^{-3}\alpha_1{}^{-1}\alpha_2{}^{-2}\tau^{n,m-1}_N\tau^{n+1,m+1}_N - Q^{4n-2m-4N}\gamma^{-4}\alpha_0{}^6\alpha_1{}^2\alpha_2{}^4\tau^{n,m}_N\tau^{n+1,m}_N = 0, \tag{A.20}$$

$$\tau^{n+1,m}_{N+1}\tau^{n+1,m+1}_{N-1} - Q^{n-2m+2N}\gamma^2\alpha_0\alpha_2{}^2\tau^{n+2,m+1}_N\tau^{n,m}_N - Q^{-2n+4m-4N}\gamma^{-4}\alpha_0{}^{-2}\alpha_2{}^{-4}\tau^{n+1,m}_N\tau^{n+1,m+1}_N = 0. \tag{A.21}$$

Proof. First, we prove (A.16)–(A.18). We rewrite (A.4) as

$$T_4(\overline{\tau}_0) - c^{-\frac{2}{3}}a_0{}^{-\frac{1}{3}}a_1{}^{-1}a_2{}^{-\frac{2}{3}}\frac{\overline{\tau}_1}{\tau_1\tau_2}\left(q^{\frac{1}{3}}c^{\frac{2}{3}}a_1\tau_0\overline{\tau}_2 + \overline{\tau}_0\tau_2\right) - c^{\frac{2}{3}}a_0{}^{\frac{1}{3}}a_1{}^{\frac{2}{3}}a_2\frac{\overline{\tau}_0\overline{\tau}_2}{\tau_2} = 0. \tag{A.22}$$

By using (A.6), we have from (A.22) that

$$T_4(\overline{\tau}_0)\tau_2 - c^{-\frac{1}{3}}a_0{}^{-\frac{1}{6}}a_1{}^{-\frac{1}{3}}a_2{}^{-\frac{1}{2}}\overline{\tau}_1T_2(\overline{\tau}_0) - c^{\frac{2}{3}}a_0{}^{\frac{1}{3}}a_1{}^{\frac{2}{3}}a_2\overline{\tau}_0\overline{\tau}_2 = 0, \tag{A.23}$$

which is equivalent to

$$T_1{}^{-1}T_4{}^2(\tau_1)T_2(\tau_1) - c^{-\frac{1}{3}}a_0{}^{-\frac{1}{6}}a_1{}^{-\frac{1}{3}}a_2{}^{-\frac{1}{2}}T_4(\tau_1)T_1{}^{-1}T_2T_4(\tau_1) - c^{\frac{2}{3}}a_0{}^{\frac{1}{3}}a_1{}^{\frac{2}{3}}a_2T_1{}^{-1}T_4(\tau_1)T_2T_4(\tau_1) = 0. \tag{A.24}$$

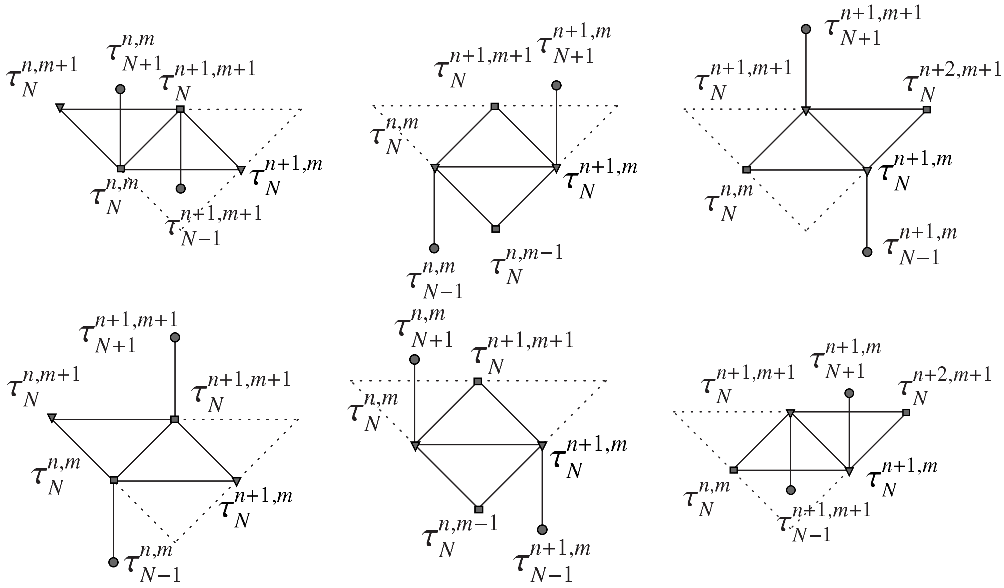

Figure 8. Configuration of τ functions for the bilinear equations of type V. Upper left: (A.16), upper center: (A.17), upper right: (A.18), lower left: (A.19), lower center: (A.20), lower right: (A.21).

We obtain (A.16), (A.17), and (A.18) by applying ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}$, ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}\pi$, and ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}\pi^2$ on (A.24), respectively.

Next, we prove (A.19)–(A.21). We rewrite (A.4) as

$$T_4(\overline{\tau}_0) - {a_1}^{\frac{1}{3}}{a_2}^{-\frac{1}{3}}\frac{\overline{\tau}_2}{\tau_1\tau_2}\left(q^{\frac{1}{3}}c^{\frac{2}{3}}a_2\tau_1\overline{\tau}_0 + \overline{\tau}_1\tau_0\right) - c^{-\frac{2}{3}}{a_0}^{-\frac{1}{3}}{a_1}^{-1}{a_2}^{-\frac{2}{3}}\frac{\overline{\tau}_0\overline{\tau}_1}{\tau_1} = 0. \tag{A.25}$$

By using (A.7), we have from (A.25) that

$$ {T_4}^2(\tau_0)\tau_1 - c^{\frac{1}{3}}{a_0}^{\frac{1}{6}}{a_1}^{\frac{1}{2}}{a_2}^{\frac{1}{3}}T_3T_4(\tau_1)T_4(\tau_2) - c^{-\frac{2}{3}}{a_0}^{-\frac{1}{3}}{a_1}^{-1}{a_2}^{-\frac{2}{3}}T_4(\tau_0)T_4(\tau_1) = 0. \tag{A.26}$$

We obtain (A.19), (A.20), and (A.21) by applying ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}\pi^2$, ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}$, and ${T_1}^{l+1}{T_2}^m{T_4}^{n-1}\pi$ on (A.26), respectively. □

Proposition A.6 (Type VI) *The following bilinear equations hold*:

$$\tau_{N+1}^{n,m}\tau_N^{n+1,m+1} - Q^{-3n+3m+2N-2}\gamma^2{\alpha_1}^3\tau_N^{n+1,m}\tau_{N+1}^{n,m+1} + Q^{-6n+6m+4N-4}\gamma^4{\alpha_1}^6\tau_N^{n,m}\tau_{N+1}^{n+1,m+1} = 0, \tag{A.27}$$
$$\tau_{N+1}^{n+1,m}\tau_N^{n,m} - Q^{-3m+2N+1}\gamma^2{\alpha_2}^3\tau_N^{n+1,m+1}\tau_{N+1}^{n,m-1} + Q^{-6m+4N+2}\gamma^4{\alpha_2}^6\tau_N^{n+1,m}\tau_{N+1}^{n,m} = 0, \tag{A.28}$$
$$\tau_{N+1}^{n+1,m+1}\tau_N^{n+1,m} - Q^{3n+2N+4}\gamma^2{\alpha_0}^3\tau_N^{n,m}\tau_{N+1}^{n+2,m+1} + Q^{6n+4N+8}\gamma^4{\alpha_0}^6\tau_N^{n+1,m+1}\tau_{N+1}^{n+1,m} = 0, \tag{A.29}$$
$$\tau_{N+1}^{n+1,m+1}\tau_N^{n,m} - Q^{-3n+3m-2N-4}\gamma^{-2}{\alpha_1}^3\tau_{N+1}^{n+1,m}\tau_N^{n,m+1} + Q^{-6n+6m-4N-8}\gamma^{-4}{\alpha_1}^6\tau_N^{n+1,m+1}\tau_{N+1}^{n,m} = 0, \tag{A.30}$$
$$\tau_{N+1}^{n,m}\tau_N^{n+1,m} - Q^{-3m-2N-1}\gamma^{-2}{\alpha_2}^3\tau_{N+1}^{n+1,m+1}\tau_N^{n,m-1} + Q^{-6m-4N-2}\gamma^{-4}{\alpha_2}^6\tau_N^{n,m}\tau_{N+1}^{n+1,m} = 0, \tag{A.31}$$
$$\tau_{N+1}^{n+1,m}\tau_N^{n+1,m+1} - Q^{3n-2N+2}\gamma^{-2}{\alpha_0}^3\tau_{N+1}^{n,m}\tau_N^{n+2,m+1} + Q^{6n-4N+4}\gamma^{-4}{\alpha_0}^6\tau_N^{n+1,m}\tau_{N+1}^{n+1,m+1} = 0. \tag{A.32}$$

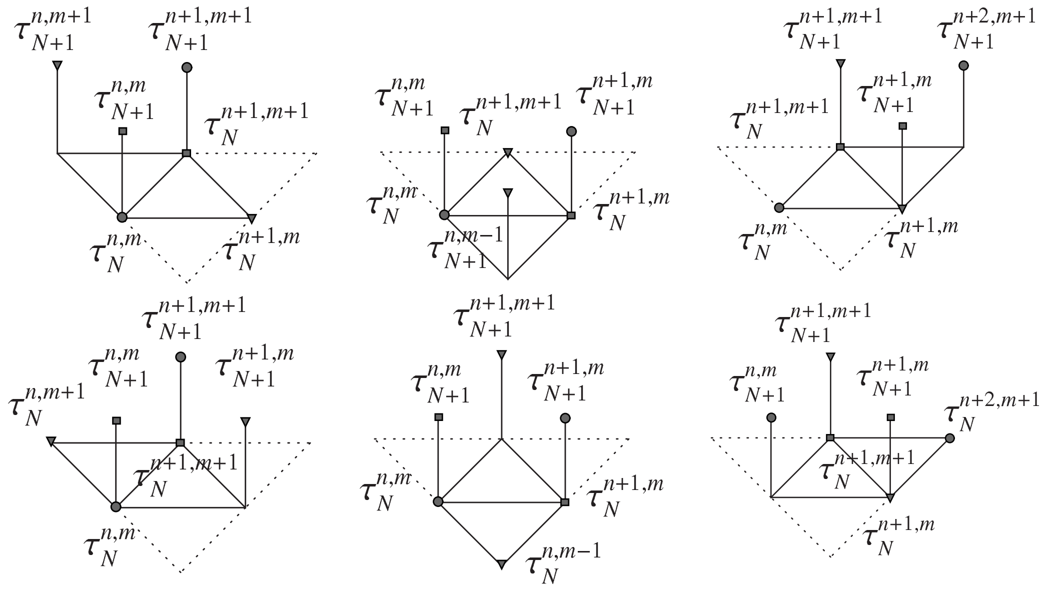

Figure 9. Configuration of τ functions for the bilinear equations of type VI. Upper left: (A.27), upper center: (A.28), upper right: (A.29) lower left: (A.30), lower center: (A.31), lower right: (A.32).

Proof. First, we prove (A.27)–(A.29). Equations (A.27), (A.28), and (A.29) can be derived by applying ${T_1}^{l+1}{T_2}^m{T_4}^n$, ${T_1}^{l+1}{T_2}^m{T_4}^n\pi$, and ${T_1}^{l+1}{T_2}^m{T_4}^n\pi^2$ on (A.7), respectively.

Next, we prove (A.30)–(A.32). By applying T_2 on τ_0, we obtain

$$q^{-\frac{1}{6}}c^{-\frac{1}{3}}{a_1}^{\frac{1}{2}}\overline{\tau}_1 T_2(\tau_0) - q^{-\frac{1}{3}}c^{-\frac{2}{3}}a_1\tau_2\overline{\tau}_0 - \overline{\tau}_2\tau_0 = 0. \tag{A.33}$$

Equations (A.30), (A.31), and (A.32) can be derived by applying ${T_1}^{l+1}{T_2}^m{T_4}^n$, ${T_1}^{l+1}{T_2}^m{T_4}^n\pi$, and ${T_1}^{l+1}{T_2}^m{T_4}^n\pi^2$ on (A.33), respectively. □

Remark A.7 The bilinear equations in Proposition 3.4 correspond to (A.27), (A.29), (A.30), (A.32), and (A.3).

A.2 Bilinear equations for q-$\mathrm{P_{II}}$

The bilinear equations for q-$\mathrm{P_{II}}$ are derived from the equations in Section A.1. Since the parameter space and τ functions are restricted, we only have to pick up the bilinear equations that consist of the τ functions on the "unit-strip," and to rewrite them in terms of R_1 instead of T_1 (see Figure 3). Therefore, only the bilinear equations of type V and VI are relevant. We use the notation in (3.54).

Proposition A.8 *The following bilinear equations hold*:

$$\tau^{k+1}_{N+1}\tau^{k+2}_{N-1} - Q^{\frac{k-4N+2}{2}}\gamma^{-2}\alpha_0\tau^{k+3}_N\tau^k_N - Q^{-k+4N-2}\gamma^4{\alpha_0}^{-2}\tau^{k+1}_N\tau^{k+2}_N = 0, \tag{A.34}$$

$$\tau^{k+2}_{N+1}\tau^{k+1}_{N-1} - Q^{\frac{k+4N+2}{2}}\gamma^{2}\alpha_0\tau^{k+3}_N\tau^k_N - Q^{-k-4N-2}\gamma^{-4}{\alpha_0}^{-2}\tau^{k+2}_N\tau^{k+1}_N = 0, \tag{A.35}$$

$$Q^{-\frac{3k-4N+4}{2}}\gamma^2{\alpha_0}^{-3}\tau^{k+3}_N\tau^k_{N+1} - Q^{-3k+4N-4}\gamma^4{\alpha_0}^{-6}\tau^{k+1}_N\tau^{k+2}_{N+1} - \tau^{k+1}_{N+1}\tau^{k+2}_N = 0, \tag{A.36}$$

$$Q^{-\frac{3k+4N+8}{2}}\gamma^{-2}{\alpha_0}^{-3}\tau^{k+3}_{N+1}\tau^k_N - Q^{-3k-4N-8}\gamma^{-4}{\alpha_0}^{-6}\tau^{k+2}_N\tau^{k+1}_{N+1} - \tau^{k+2}_{N+1}\tau^{k+1}_N = 0. \tag{A.37}$$

Proof. Noticing (3.55), we obtain from (A.23)

$$R_1{}^{-2}T_4{}^2(\tau_1)R_1{}^{-1}(\tau_1) - q^{-\frac{5}{12}}c^{-\frac{1}{3}}a_0{}^{\frac{1}{6}}T_4(\tau_1)R_1{}^{-3}T_4(\tau_1) - q^{\frac{5}{6}}c^{\frac{2}{3}}a_0{}^{-\frac{1}{3}}R_1{}^{-2}T_4(\tau_1)R_1{}^{-1}T_4(\tau_1) = 0, \quad (A.38)$$

from which (A.34) is derived by applying $R_1{}^{m+3}T_4{}^{n-1}$. Similarly, we have

$$T_4{}^2(\tau_1)R_1{}^{-1}(\tau_1) - q^{\frac{1}{3}}c^{\frac{1}{3}}a_0{}^{\frac{1}{6}}R_1T_4(\tau_1)R_1{}^{-2}T_4(\tau_1) - q^{-\frac{2}{3}}c^{-\frac{2}{3}}a_0{}^{-\frac{1}{3}}T_4(\tau_1)R_1{}^{-1}T_4(\tau_1) = 0. \quad (A.39)$$

by applying π on (A.26). Then we obtain (A.35) by applying $R_1{}^{m+2}T_4{}^{n-1}$ on (A.39). Equation (A.36) is derived by applying $R_1{}^{m+3}T_4{}^{n}$ on

$$q^{\frac{1}{6}}c^{\frac{1}{3}}a_0{}^{-\frac{1}{2}}\tau_1R_1{}^{-3}T_4(\tau_1) - q^{\frac{1}{3}}c^{\frac{2}{3}}a_0{}^{-1}R_1{}^{-2}(\tau_1)R_1{}^{-1}T_4(\tau_1) - R_1{}^{-2}T_4(\tau_1)R_1{}^{-1}(\tau_1) = 0, \quad (A.40)$$

which follows from (A.6). Finally, we obtain (A.37) by applying $R_1{}^{m+3}T_4{}^{n}$ on

$$q^{-\frac{1}{6}}c^{-\frac{1}{3}}a_0{}^{-\frac{1}{2}}T_4(\tau_1)R_1{}^{-3}(\tau_1) - q^{-\frac{1}{3}}c^{-\frac{2}{3}}a_0{}^{-1}R_1{}^{-1}(\tau_1)R_1{}^{-2}T_4(\tau_1) - R_1{}^{-1}T_4(\tau_1)R_1{}^{-2}(\tau_1) = 0, \quad (A.41)$$

which is follows from (A.33). □

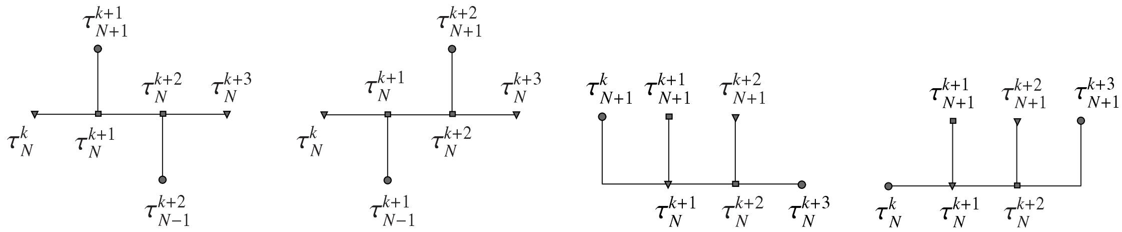

Figure 10. Configuration of τ functions for the bilinear equations in Proposition A.8. The figures correspond to (A.34), (A.35), (A.36), and (A.37), respectively, from the left to the right.

Remark A.9 The bilinear equations in Proposition 3.6 correspond to (A.36), (A.37), and (A.34).

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