Open access • Journal Article • DOI:10.1017/S1446788700005085

Projective-symmetric spaces — Source link

R. F. Reynolds, A. H. Thompson

Institutions: University of Pittsburgh

Published on: 01 Feb 1967 - Journal of The Australian Mathematical Society (Cambridge University Press)

Topics: Riemann curvature tensor, Space (mathematics), Scalar curvature, Ricci curvature and Weyl tensor

SPACE RESEARCH COORDINATION CENTER

UNIVERSITY OF PITTSBURGH
1787
VERITAS ET VIRTUS

PROJECTIVE-SYMMETRIC SPACES

BY

R. F. REYNOLDS AND A. H. THOMPSON

DEPARTMENT OF MATHEMATICS

SRCC REPORT NO. 16

UNIVERSITY OF PITTSBURGH

PITTSBURGH, PENNSYLVANIA

22 OCTOBER 1965

The Space Research Coordination Center, established in May, 1963, coordinates space-oriented research in the various schools, divisions and centers of the University of Pittsburgh. Members of the various faculties of the University may affiliate with the Center by accepting appointments as Staff Members. Some, but by no means all, Staff Members carry out their researches in the Space Research Coordination Center building. The Center's policies are determined by an SRCC Faculty Council.

The Center provides partial support for space-oriented research, particularly for new faculty members; it awards annually a number of postdoctoral fellowships and NASA predoctoral traineeships; it issues periodic reports of space-oriented research and a comprehensive annual report. In concert with the University's Knowledge Availability Systems Center it seeks to assist in the orderly transfer of new space-generated knowledge into industrial application.

The Center is supported by a Research Grant (NsG-416) from the National Aeronautics and Space Administration, strongly supplemented by grants from The A. W. Mellon Educational and Charitable Trust, the Maurice Falk Medical Fund, the Richard King Mellon Foundation and the Sarah Mellon Scaife Foundation. Much of the work described in SRCC reports is financed by other grants, made to individual faculty members.

PROJECTIVE-SYMMETRIC SPACES*

R. F. Reynolds and A. H. Thompson

Introduction.

Gy. Soos [1] and B. Gupta [2] have discussed the properties of Riemannian spaces V_n $(n>2)$ in which the first covariant derivative of Weyl's projective curvature tensor is everywhere zero; such spaces they call Projective-Symmetric spaces. In this paper we wish to point out that all Riemannian spaces with this property are symmetric in the sense of Cartan [3]; that is the first covariant derivative of the Riemann curvature tensor of the space vanishes. Further sections are devoted to a discussion of projective-symmetric affine spaces A_n with symmetric affine connexion. Throughout, the geometrical quantities discussed will be as defined by Eisenhart [4] and [5].

1. Projective-Symmetric Riemannian Spaces.

For a V_n, Weyl's projective curvature tensor $W^a{}_{bcd}$ is

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{2}{n-1} \{\delta^a{}_{[d} R_{c]b}\} ,$$

where $R^a{}_{bcd}$ is the curvature tensor, and $R_{bc} = R^a{}_{bca}$ the Ricci

---

*Supported in part by the National Aeronautics and Space Administration under grant Ns G-416; University of Pittsburgh. Accepted for publication in Australian Journal of Mathematics.

tensor, of the space. The V_n is a projective-symmetric space if and only if

$$W^a{}_{bcd;e} = 0. \tag{1.1}$$

We define the tensor $U^a{}_d$ by

$$U^a{}_d = g^{bc} W^a{}_{bcd} = \frac{n}{n-1} \{R^a{}_d - \frac{1}{n} R\, \delta^a{}_d\} ,$$

where $R = R^a{}_a$, and from (1.1) it follows that if the space is projective-symmetric, then

$$U^a{}_{d;e} = 0 . \tag{1.2}$$

For n>2, equation (1.2) and the twice-contracted Bianchi Identity $R^a{}_{b;a} = \frac{1}{2} R_{,b}$ imply

$$R = \text{constant} ,$$

and thus we have $R^a{}_{b;e} = 0$. With (1.1) this gives the result

$$0 = W^a{}_{bcd;e} \Longleftrightarrow R^a{}_{bcd;e} = 0 ,$$

from which follows:-

<u>Theorem</u> 1.

<u>A Riemannian space</u> V_n (n>2) <u>is a projective-symmetric space if and only if it is symmetric in the sense of Cartan</u> [3].

For n = 2, $W^a{}_{bcd}$ is identically zero and (1.1) is a degenerate condition in a V_2. We remark however that a V_2 is a symmetric space if and only if it has constant scalar curvature R.

The results of Gupta [2] follow immediately since they are trivially true for symmetric spaces. The paper of Soos [2] contains theorems for projective-symmetric spaces which are generalisations of results found by Sinjukow [6] for symmetric spaces.

2. Affine Spaces With Symmetric Connexion.

For the remainder of this paper we consider the application of the preceding theorem in an Affine space with symmetric connexion. Such a space we will denote by A_n, its connexion by $\Gamma^a{}_{bc}$, and covariant differentiation with respect to this connexion by ";".

The curvature tensor of A_n is defined

$$B^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} + 2\Gamma^h{}_{b[d}\Gamma^a{}_{c]h}, \tag{2.1}$$

for which the identities

$$B^a{}_{b(cd)} = B^a{}_{[bcd]} = 0, \tag{2.2}$$

and Bianchi's identity

$$B^a{}_{b[cd;e]} = 0, \tag{2.3}$$

hold. The analogue of the Ricci tensor for an A_n is $B_{bc} = B^a{}_{bca}$, but in this case it is not necessarily symmetric; it follows from (2.2) that

$$S_{cd} = -2B_{[cd]}, \tag{2.4}$$

where $S_{cd} = B^a{}_{acd}$. From (2.3) we have also

$$S_{[cd;e]} = 0 ,$$

(2.5)

$$B^{a}{}_{bcd;a} = 2B_{b[c;d]} .$$

Weyl's projective curvature tensor for an A_n is

(2.6) $$W^{a}{}_{bcd} = B^{a}{}_{bcd} - \frac{1}{n+1} \delta^{a}{}_{b} S_{cd} - \frac{2}{n-1} B_{b[c}\delta^{a}{}_{d]} - \frac{2}{n^2-1} S_{b[c}\delta^{a}{}_{d]} .$$

This tensor is invariant for projective transformations of the space and its vanishing implies that the A_n has the same paths as flat space [5]. By a projective-symmetric affine space we will mean an A_n $(n>2)$ such that

(2.7) $$W^{a}{}_{bcd;e} = 0 ,$$

throughout; an A_n is symmetric [3] if and only if

(2.8) $$B^{a}{}_{bcd;e} = 0 ,$$

at all points.

Equation (2.8) implies that every symmetric A_n is a projective-symmetric A_n. Such projective-symmetric spaces we will call degenerate, and from Theorem 1 we see that all projective-symmetric Riemannian spaces V_n $(n>2)$ are degenerate in this sense. We will show that this is not true for a general A_n and will consider its validity in relation to certain sub-classes of Affine spaces.

3. A Non-Degenerate Projective-Symmetric A_n.

Consider the A_n with connexion coefficients

(3.1) $$\Gamma^a_{bc} = 2\,\delta^a_{(b}\psi_{c)}\ ,$$

in a coordinate system $\{x^a\}$ such that

$$\frac{\partial}{\partial x^a}\psi_c = 0\ .$$

The latter condition is expressed covariantly as

(3.2) $$\psi_{c;d} + 2\psi_c\psi_d = 0\ .$$

The A_n is projectively related to flat space; its projective curvature tensor vanishes and therefore it is a projective-symmetric space. From (2.1) we have for this A_n

$$B^a_{\ bcd} = 2\psi_b\,\delta^a_{[c}\psi_{d]}\ ,$$

and using (3.2)

$$B^a_{\ bcd;e} = -4\psi_e\,B^a_{\ bcd}\ .$$

For $\psi_e \neq 0$, the curvature tensor of the space is non-zero and we have the result:-

<u>Theorem</u> 2.

<u>There exist projective-symmetric</u> A_n's <u>which are non-degenerate</u>.

3. <u>The Decomposable</u> A_n.

If two spaces A_m and A_{n-m} are given with coordinates x^α: $(\alpha,\beta,\gamma = 1,2,\ldots,m)$ and x^A: $(A,B,C = m+1,\ldots,n)$ and the connexions $\Gamma^\alpha_{\beta\gamma}$ and Γ^A_{BC}, then the A_n with coordinates x^a: $(a,b,c = 1,2,\ldots,n)$

and connexion $\Gamma^a_{bc} \equiv \{\Gamma^\alpha_{\beta\gamma}, \Gamma^A_{BC}\}$, is called the product of A_m and A_{n-m}. An A_n that is a product space is said to be decomposable. A geometric object in a decomposable A_n is decomposable if and only if its components with respect to the special coordinates are always zero when they have indices from both ranges, and the components belonging to the subspace A_m (A_{n-m}) are functions of x^α (x^A) only. In a decomposable A_n, B^a_{bcd}, B_{bc} and their covariant derivatives are decomposable; W^a_{bcd} and $W^a_{bcd;e}$ are not in general decomposable.

Theorem 3.

A projective-symmetric A_n which is decomposable is necessarily degenerate.

We assume that $A_n \equiv \{A_m X A_{n-m}\}$ where indices $\alpha,\beta,\gamma = 1....m$ relate to A_m, and $A,B,C = m+1,....n$ relate to A_{n-m}. From the definition of the projective-curvature tensor we have for the decomposable A_n.

$$W^\alpha_{\beta CD} = -\frac{1}{n+1}\delta^\alpha_\beta S_{CD}, \tag{3.1}$$

and

$$W^\alpha_{B\gamma D} = \frac{1}{n-1}\delta^\alpha_\gamma \{B_{BD} + \frac{1}{n+1} S_{BD}\}. \tag{3.2}$$

The assumption that A_n is a projective-symmetric space gives with (3.1)

$$S_{CD;E} = 0,$$

and therefore in (3.2)

$$B_{BD;E} = 0 .$$

Similarly we have

$$B_{\beta\delta;\varepsilon} = 0 ,$$

and since $B_{bd;e}$ is a decomposable tensor of the A_n it follows that

$$B_{bd;e} = 0 .$$

With the above, the differentiation of (2.5) gives

$$0 = W^a{}_{bcd;e} = B^a{}_{bcd;e} ,$$

and the decomposable A_n is a symmetric space. Q.E.D.

4. The projective-Symmetric W_n.

An A_n in which there exists a symmetric two index tensor g_{ab} of rank n such that

(4.1) $$g_{ab;d} = -2\phi_c \, g_{ab} ,$$

for some covariant vector ϕ_c is called a W_n and was first discussed by Weyl [7]. Define the contravariant tensor g^{ab} by $g^{ab} g_{bc} = \delta^a{}_c$, then from (4.1)

(4.1) $$g^{ab}{}_{;c} = 2\phi_c \, g^{ab} .$$

We can use g_{ab} (g^{ab}) to define a correspondence between covariant and contravariant quantities in A_n; in fact if ϕ_c is a gradient vector $\phi_{,c}$ the W_n is a Riemannian space V_n with metric tensor $\bar{g}_{ab} = e^{2\phi} g_{ab}$.

With $W_{abcd} = g_{ae} W^e{}_{bcd}$ and $B_{abcd} = g_{ae} B^e{}_{bcd}$, we define

(4.2) $$T_{ad} = g^{bc} W_{abcd} ,$$

and

(4.3) $$Q_{ad} = g^{bc} B_{abcd} .$$

From the Ricci Identity applied to g_{ab}, and the use of (4.1) and (4.1a) we have

$$B_{(ab)cd} = -2\, g_{ab}\, \phi_{[c;d]} ,$$

which yields after contraction

$$Q_{ad} = B_{ad} - 4\, \phi_{[a;d]} ,$$

and

$$S_{cd} = -2n\phi_{[c;d]} .$$

We extract the symmetric and anti-symmetric parts of these equations to obtain

$$Q_{(ad)} = B_{(ad)} ,$$

(4.4) $$B_{[ad]} = n\phi_{[a;d]} ,$$

$$Q_{[ad]} = (n-4)\phi_{[a;d]} .$$

With equation (4.2), the definition of the projective curvature tensor gives

$$T_{ab} = Q_{ab} - \frac{n-2}{n^2-1} S_{ab} + \frac{1}{n-1}\{B_{ab} - Bg_{ab}\} ,$$

where $B = g^{bc} B_{bc} = g^{bc} Q_{bc}$. Frequent use of the relations (2.4) and (4.4) give the decomposition;

$$T_{(ab)} = \frac{n}{n-1}\{B_{(ab)} - \frac{1}{n} g_{ab} B\} ,$$

(4.5)

$$T_{[ab]} = \frac{n^2-4}{n(n-1)} B_{[ab]} .$$

Lemma 1.

In a projective-symmetric W_n $(n>2)$ $T_{ab;c} = 0$.

Proof.

$$T_{ab;c} = g_{ad}\, g^{ef}\, W^{d}{}_{efb;c} + g_{ad;c}\, T^{d}{}_{b} + g^{ef}{}_{;c}\, W_{aefb} .$$

From (4.1) and (4.1a) the sum of the second and third terms on the right hand side of the above equation is zero. Hence

$$T_{ab;c} = g_{ad}\, g^{ef}\, W^{d}{}_{efb;c} ,$$

which vanishes if W_n is projective-symmetric. Q.E.D.

Lemma 1 applied to the second equation of (4.5) gives

(4.6) $$B_{[ab];c} = 0 ,$$

and with (4.6) in the first equation of (4.5)

(4.7) $$B_{ab;c} = \frac{1}{n} g_{ab}\{B_{,c} - 2B\phi_c\} .$$

Lemma 2.

In a projective-symmetric W_n $(n>2)$ $B_{a[b;c]} = 0$.

Proof.

We have

$$0 = W^a{}_{bcd;a} = B^a{}_{bcd;a} - \frac{n-2}{n^2-1} S_{cd;b} - \frac{2}{n-1} B_{b[c;d]} .$$

From (2.3) and (4.6) $S_{cd;e} = 0$, and we see that

$$B^a{}_{bcd;a} = \frac{2}{n-1} B_{b[c;d]} .$$

However from the contracted Bianchi Identity (2.5) we have

$$B^a{}_{bcd;a} = 2B_{b[c;d]} ,$$

and the result of the lemma follows.

From lemma 2 and (4.7) we have

$$g_{a[b} B_{,c]} - 2Bg_{a[b}\phi_{c]} = 0 ,$$

and after contraction

$$(4.8) \qquad (n-1)\{B_{,c} - 2B\phi_c\} = 0 .$$

Referring to equation (4.7) we deduce that $B_{ab;c} = 0$, and therefore for a W_n

$$0 = W^a{}_{bcd;e} \Longleftrightarrow B^a{}_{bcd;e} = 0 .$$

<u>Theorem</u> 4.

<u>Every projective-symmetric</u> W_n <u>is degenerate</u>.

We also remark that if $B \neq 0$ in (4.3) then ϕ_c is necessarily a gradient:-

<u>Theorem</u> 5.

<u>The "scalar curvature"</u> B <u>of a projective-symmetric</u> W_n <u>which is not a Riemannian space is necessarily zero</u>.

REFERENCES

[1] Soos, Gy., Ueber die Geodaetischen Abbildung von Riemannsche Raeumen auf Projectiv-Symmetrische Riemannsche Raeume. Acta. Math. Acad. Sci. Hung., 9, (1958) 349-361.

[2] Gupta, B., On Projective-Symmetric Spaces. J. Austr. Math. Soc. IV, (1964), 113-121.

[3] Schouten, J. A., Ricci Calculus, 2nd Ed., p. 163. Springer-Verlag, Berlin 1954.

[4] Eisenhart, L. P., Riemannian Geometry, Princeton U. Press, Princeton. (1926).

[5] Eisenhart, L. P., Non-Riemannian Geometry, Amer. Math. Soc. New York (1927).

[6] Sinjukow, N. S., On the Geodesic Correspondence of a Riemannian Space with a Symmetric Space (Russian) D. A. N. (U.S.S.R.). 98 (1954) 21-23.

[7] Weyl, H., Reine Infinitesimal Geometrie. Math. Zeitschrift, 2, (1918) 384-411.

Department of Mathematics
University of Pittsburgh
Pittsburgh, Pennsylvania
U.S.A.