J. DIFFERENTIAL GEOMETRY
32 (1990) 235-268

PROJECTIVITY OF COMPLETE MODULI

JÁNOS KOLLÁR

1. Introduction

The aim of this paper is to continue developing a method to prove the projectivity of certain moduli spaces. Classically there have been two approaches to this problem. Mumford's geometric invariant theory was specifically designed with this aim in mind, and Griffiths' period maps can be used in a similar manner in certain cases.

The approach of [20] and [22] starts with the observation that it is fairly easy to construct moduli spaces which are a priori only algebraic spaces. Frequently these spaces come endowed with a natural line bundle, and one might hope to check that this line bundle is ample. If the moduli space is proper, then ampleness can be checked using the Nakai-Moishezon criterion.

Assume for simplicity that we have a smooth family of algebraic varieties $f\colon X \to Y$, and that for every fiber X_y the canonical line bundle ω_{X_y} is very ample. The natural line bundles on Y are the bundles $\lambda_k = \det f_*(\omega^k_{X/Y})$. From Hodge theory one has a natural metric with positive semidefinite curvature on $f_*(\omega_{X/Y})$; hence λ_1 is at least nonnegative. To get something strictly positive one can consider the multiplication map

$$S^k(f_*(\omega_{X/Y})) \to f_*(\omega^k_{X/Y}),$$

where S^k denotes the kth symmetric power. At each $y \in Y$ the kernel of this map is exactly the space of degree k equations satisfied by the canonical image of the fiber X_y. Thus one can recover the fibers from these maps. Using the curvature form on $f_*(\omega_{X/Y})$, this leads to the ampleness of λ_k for k large [22, 6.14]. Moreover, the method works even if f is only generically smooth, but breaks down when applied to the boundary of the moduli space where $f_*(\omega_{X/Y})$ is only an *extension* of bundles with natural metrics. For curves these difficulties can be circumvented,

Received August 5, 1988 and, in revised form, May 15, 1989. Partial financial support was provided by the National Science Foundation under grant number DMS-8707320.

though the arguments are very artificial. This leads to a new proof of the projectivity of the moduli space of stable curves (unpublished).

Recently Viehweg [34] realized that the method of the hard covering trick (cf. [33], [22]) can be used more effectively to exploit the multiplication map. This eliminates the necessity of curvature considerations, making the proof more general and considerably simpler. He applied this in the context of geometric invariant theory and proved that the smooth locus of the moduli space of surfaces of general type is quasi-projective [34].

Here a different version of the same idea is used. For the applications it has the advantage that it relies less on resolution of singularities, and therefore it is applicable in positive characteristic or even over $\mathbb{Z}$. The precise formulation of the projectivity criterion is given in Lemma 3.9.

Among the assumptions necessary for this result, the condition on "semi-positivity" is the most restrictive. This is checked in some cases in §4—first for curves in any characteristic (Theorem 4.3) and then for stable surfaces in characteristic zero (Theorem 4.12). These results are combined in §5 to give a new proof of the projectivity of the moduli of stable curves over $\mathbb{Z}$. The moduli problem of stable surfaces is separated and satisfies the valuative criterion of properness [23, Chapter 5]. However, it is not known to be bounded, and thus we prove projectivity only for the proper subvarieties (conjecturally everything).

In §6 the compactified Picard schemes defined by Altman and Kleiman [1] are considered. This is the moduli space of torsion free rank one sheaves. We prove that they are projective under some mild conditions (Theorem 6.4).

For noncomplete moduli spaces the method of Viehweg [34] using "weak positivity" is much better. Even in that case however it is simpler to avoid using geometric invariant theory and applying his techniques directly to the moduli space instead of the Hilbert scheme. The reader familiar with his article should have no problem combining the two methods.

I would like to thank E. Viehweg for pointing out several inaccuracies.

Notation and conventions. The notation and terminology follow Hartshorne's book. There are three notions that are standard in higher dimensional geometry which may not be generally known.

1. A line bundle on a scheme (or algebraic space) is called *nef* if it has nonnegative degree on any complete curve. This is usually used only if the underlying space is proper.

2. A line bundle L on an irreducible scheme X is called *big* if the following equivalent conditions are satisfied:

(i) $h^0(X, L^m) > \text{constant} \cdot m^{\dim X}$ for $m \gg 0$;

(ii) for any divisor H there is an $m > 0$ such that mL is linearly equivalent to $H + E$, where E is effective.

3. A vector bundle E on a scheme (or algebraic space) X is called *semipositive* if for every smooth complete curve C and every map $f: C \to X$, any quotient bundle of f^*E has nonnegative degree (see also 3.3).

2. Projectivity criterion

First we set up axiomatically the type of moduli problems that we will be able to handle. The guiding principle is that this class should contain the (compactified) moduli problem for surfaces of general type.

We will work over a fixed algebraically closed field K.

2.1. Moduli problem for $\mathbb{Q}$-polarized varieties.

2.1.1. First we specify the objects which we want to parametrize. This will be a class $\mathscr{C}$ consisting of pairs (X, L) satisfying the following assumptions:

X is projective and satisfies Serre's condition S_2.

L is a reflexive sheaf on X which is locally free in codimension one.

L is ample; i.e., for some $r > 0$ the sheaf $L^{[r]}$ is locally free and ample, where as usual $[r]$ denotes the double dual of the rth tensor power.)

2.1.2. The class $\mathscr{C}$ is said to be *open* if the following condition is satisfied:

For any pointed scheme $s \in S$, any scheme X/S flat over S, and any sheaf L/X that satisfy the following three conditions:

(i) $(X_s, L|X_s) \in \mathscr{C}$,

(ii) $L^{[k]}$ is flat over S for any $k > 0$,

(iii) $(L|X_s)^{[k]} = L^{[k]}|X_s$ for every k,

there is an open set $s \in U \subset S$ such that if $u \in U$ then $(X_u, L|X_u) \in \mathscr{C}$. (Condition (iii) is specifically put here because it is the right condition for surfaces.)

2.1.3. The *moduli functor* $\mathscr{M}\mathscr{C}$ associated to $\mathscr{C}$ is the following functor:

Given a scheme S, $\mathscr{M}\mathscr{C}(S)$ is the set of all pairs (X, L), where X is a flat and proper scheme over S, and L is a reflexive sheaf on X

such that for every closed point $s \in S$ the conditions of 2.1.2 are satisfied. Two such pairs (X, L) and (Y, M) are said to be equivalent if there is an isomorphism $f: X/S \to Y/S$ which induces an *isomorphism* $(X_s, f^*M|X_s) \cong (X_s, L|X_s)$ for every $s \in S$. (Note that usually one requires only that $f^*M|X_s$ and $L|X_s$ be numerically equivalent. Requiring isomorphism makes things technically easier though it gives a slightly too large moduli functor.)

If all the X_s are reduced and connected, then the equivalence condition can be reformulated as follows: Two such pairs (X, L) and (Y, M) are said to be isomorphic if $X/S \cong Y/S$ and there is a line bundle B on S such that $L \cong \pi^*B \otimes M$, where $\pi: X \to S$ is the structure map.

2.1.4. The function $H_{(X,L)}(t) = \chi(X, L^{[t]})$ is called the *Hilbert function* of (X, L). Given a family $(X, L) \in \mathcal{M}\mathcal{C}(S)$ the Hilbert function of the fibers is locally constant. Thus if we decompose the class $\mathcal{C}$ into subclasses of pairs having the same Hilbert function, then the moduli functor of $\mathcal{C}$ becomes the direct sum of the moduli functors of the subclasses. Hence for most purposes it is sufficient to study classes with constant Hilbert function. Then we call $H(t) = H_{(X,L)}(t)$ the Hilbert function of $\mathcal{C}$.

2.1.5. The moduli functor $\mathcal{M}\mathcal{C}$ associated to $\mathcal{C}$ is said to be *separated* if the following condition is satisfied:

Given the spectrum T of a DVR and two pairs $(X_i, L_i) \in \mathcal{M}\mathcal{C}(T)$ $(i = 1, 2)$, every isomorphism of (X_1, L_1) and (X_2, L_2) over the generic point of T extends to an isomorphism of the pairs over T.

(This condition also might be called properness of the relation "isomorphism".)

2.1.6. The moduli functor $\mathcal{M}\mathcal{C}$ associated to $\mathcal{C}$ is said to be *bounded* if there is a scheme of finite type S and a pair $(X, L) \in \mathcal{M}\mathcal{C}(S)$ such that for every $(X, L) \in \mathcal{M}\mathcal{C}(\operatorname{Spec} K)$ there is a point $s \in S$ such that (X, L) and $(X_s, L|X_s)$ are isomorphic.

2.1.7. If a moduli functor is bounded, then there is a (nonunique) particularly nice choice of the above scheme S. Let $H(t)$ be the Hilbert function of $\mathcal{C}$. By boundedness one can choose a fixed N such that for every $(X, L) \in \mathcal{M}\mathcal{C}(\operatorname{Spec} K)$ the sheaf $L^{[N]}$ is locally free, very ample, and without higher cohomologies. Let $\mathbb{P}$ be the projective space of dimension $H(N) - 1$ with fixed coordinate system.

Let $\mathbf{H}$ be the Grothendieck-Hilbert scheme of $\mathbb{P}$ parametrizing closed subvarieties of $\mathbb{P}$ and a sequence of N coherent sheaves on them. If we embed (X, L) via the global sections of $L^{[N]}$, then the images are subvarieties with Hilbert polynomials $H(Nt)$, and the sheaves $L^{[i]}$, $i = 1, \cdots, N$, have Hilbert polynomials $H(i+Nt)$. Thus there is a quasiprojective open subvariety **HC** of the Grothendieck-Hilbert scheme of $\mathbb{P}$ which parametrizes exactly the images of the pairs $(X, L) \in \mathcal{M}\mathcal{C}(\operatorname{Spec} K)$ under the imbedding given by a choice of basis in $L^{[N]}$. There is a so-called universal family $X_{\mathbf{HC}}$ over **HC**. A priori our conditions guarantee only that the sheaves $L^{[i]}$ are flat for $i = 1, \cdots, N$. However since $L^{[N]}$ is locally free, we have $L^{[i+N]} \cong L^{[i]} \otimes L^{[N]}$, and therefore all the sheaves $L^{[t]}$ are flat over **HC**.

Furthermore, if $(X, L) \in \mathcal{M}\mathcal{C}(Z)$ and $f\colon X \to Z$ is the structure map, then $f_*L^{[N]}$ is locally free and we get an embedding of (X, L) into $\operatorname{Proj}_Z f_*L^{[N]}$. Replacing Z with an open cover $\bigcup U_i$, we may assume that $f_*L^{[N]}$ is in fact free. Fixing a basis in the space of sections then gives a map $U_i \to \mathbf{HC}$ for every i such that the original family over U_i is just the pull-back of the universal family over **HC**.

2.1.8. The class $\mathcal{C}$ is said to be *complete* if given the spectrum T of a DVR with general point T_{gen} and $(X_{\text{gen}}, L_{\text{gen}}) \in \mathcal{M}\mathcal{C}(T_{\text{gen}})$, there is a finite cover $g\colon T' \to T$ and a pair $(X', L') \in \mathcal{M}\mathcal{C}(T')$ such that $(X', L')|T'_{\text{gen}}$ and $g^*(X_{\text{gen}}, L_{\text{gen}})$ are isomorphic.

2.1.9. If an algebraic space **MC** coarsely represents the functor $\mathcal{M}\mathcal{C}$ (see [27, p. 99]), then **MC** is called the *coarse moduli space* of the class $\mathcal{C}$. In terms of the space **HC** the moduli space can be obtained as the quotient by the equivalence relation "isomorphism" (if it exists).

2.1.10. We say that $\mathcal{C}$ has *tame automorphisms* if every pair (X, L) in $\mathcal{C}$ has a compact and reduced automorphism group. In fact, since L is ample this implies that the automorphism groups are finite and reduced.

A fairly general existence theorem essentially due to Artin is the following [3, 6.3] (cf. also [27, p. 172] or [20, 4.1.1]).

2.2. Theorem. *Let $\mathcal{C}$ be an open class with Hilbert polynomial $H(t)$. Assume that the corresponding moduli problem is separated* (*resp. separated and bounded, resp. separated, bounded, and complete*). *Assume furthermore that $\mathcal{C}$ has tame automorphisms. Then there is a separated algebraic space* **MC** *locally of finite type* (*resp. of finite type, resp. proper*) *which coarsely represents the functor $\mathcal{M}\mathcal{C}$.*

Before formulating the main result we need some more definitions.

2.3. Definition. The moduli functor $\mathcal{M}\mathcal{C}$ associated to $\mathcal{C}$ is said to be *functorially polarizable* if the following condition is satisfied:

For any S and $(X, L) \in \mathcal{M}\mathcal{C}(S)$ there is a pair $(X, L^c) \in \mathcal{M}\mathcal{C}(S)$ (note that X is unchanged!) such that

(i) if (X, L) and (Y, M) are equivalent, then (X, L^c) and (Y, M^c) are isomorphic;

(ii) for any base change $g: S' \to S$, we have an isomorphism $g^*(X, L^c) \cong (g^*X, (g^*L)^c)$.

If the condition is satisfied, then L^c is called a functorial (or canonical) polarization of X/S. If the choice of L^c is specified, then we say that $\mathcal{M}\mathcal{C}$ is functorially (or canonically) polarized.

There are several examples of functorially polarizable moduli functors.

2.3.1. If $L|X_s = \omega_{X_s}$ for every $s \in S$, then $L^c = \omega_{X/S}$ is a canonical polarization.

2.3.2. If the moduli functor under consideration is such that every family $(X, L) \in \mathcal{M}\mathcal{C}(S)$ has a natural section $\sigma: S \to X$, then we can take $L^c = L \otimes (\sigma^* L)^{-1}$. This applies for instance for polarized abelian varieties or the moduli of polarized pointed varieties.

2.3.3. Essentially every bounded moduli functor parametrizing reduced, projective, and connected schemes admits a functorial polarization. To see this choose an $m > 0$ such that for every (X, L) we have $H^i(X, L^{\otimes m}) = 0$ for $i > 0$. Thus $k = h^0(X, L^{\otimes m})$ is independent of the choice of (X, L). Now given any $(X, L) \in \mathcal{M}\mathcal{C}(S)$ with structure map $f: X \to S$ consider the line bundle $L^c = L^k \otimes (\det f_*(L^{\otimes m}))^{-1}$. If L is changed to $L \otimes f^*B$, where B is a line bundle on S, then $(L \otimes f^*B)^c = L^c$. Thus L^c is a functorial polarization of the moduli functor

$$\mathcal{M}'\mathcal{C}(S) = \{(X, L^k) | (X, L) \in \mathcal{M}\mathcal{C}(S)\}.$$

This is essentially the same functor as the one we had before.

2.4. Definition. The functorial polarization of the moduli functor $\mathcal{M}\mathcal{C}$ associated to $\mathcal{C}$ is said to be *semipositive* if the following condition holds:

There is a fixed k_0 such that whenever C is a complete smooth curve and $(X, L) \in \mathcal{M}\mathcal{C}(C)$ with structure map $f: X \to C$, then for all $k \geq k_0$ the vector bundle $f_*((L^c)^{[k]})$ is semipositive on C.

2.5. Line bundles on moduli spaces. Given a functional polarization of the moduli functor $\mathcal{M}\mathcal{C}$, and $(X, L) \in \mathcal{M}\mathcal{C}(S)$ defines a series of line bundles on S for $k \gg 0$ via $\lambda_k(X) = \det(f_*(L^{[k]}))$. These line bundles are clearly functorial. Because of the presence of automorphisms, they do not descend to the moduli space **MC**, but if the functor is bounded and

$\mathrm{Aut}(X, L)$ operates on $\det(H^0(X, L^{[k]}))$ through a finite quotient, then λ_k^N descends to a line bundle on **MC** for sufficiently divisible N. Thus λ_k exists as a $\mathbb{Q}$-linebundle on **MC**.

The following is the main result of this section.

2.6. Theorem. *Let $\mathscr{C}$ be an open class of $\mathbb{Q}$-polarized varieties with Hilbert function $H(t)$. Assume that the corresponding moduli functor is separated, functorially polarized, semipositive, and has tame automorphisms. Then λ_k is ample on every complete subspace of* **MC** *for k sufficiently large. In particular, if $\mathscr{C}$ is also bounded and complete then* **MC** *is projective.*

Proof. It is somewhat inconvenient that the line bundle λ_k is not defined directly on **MC**. To get around this problem, for any given proper subspace $Z \subset \mathbf{MC}$ we will find a scheme Y, a family $(X, L) \in \mathscr{M}\mathscr{C}(Y)$, and a finite and surjective map $p: Y \to Z$ such that for any $y \in Y$ the moduli point of the pair $(X_y, L|X_y)$ is exactly $p(y)$. Thus $\lambda_k(X)^N \cong p^*(\lambda_k^N)$. Since Z and Y are proper, the ampleness of λ_k^N is equivalent to the ampleness of $\lambda_k(X)$ (see 3.11). Thus it is sufficient to prove that $\lambda_k(X)$ is ample on Y.

2.7. Proposition. *Let $\mathscr{C}$ be an open class of $\mathbb{Q}$-polarized varieties with Hilbert function $H(t)$. Assume that every pair (X, L) has a finite automorphism group. Let Z be a subspace of finite type of* **MC**. *Then there is a scheme of finite type Y, a family $(X, L) \in \mathscr{M}\mathscr{C}(Y)$, and a finite and surjective map $p: Y \to Z$ such that for any $y \in Y$ the moduli point of the pair $(X_y, L|X_y)$ is exactly $p(y)$.*

Proof. It is sufficient to prove this for irreducible subspaces. **MC** has an open subspace of finite type that contains Z. Replacing $\mathscr{C}$ by the subfamily given by this subspace we get a new functor which is bounded. Thus we can assume that we start with a bounded moduli functor.

As a first step we will find an algebraic space Y' with the required properties.

Let **HC** be the corresponding Hilbert scheme parametrizing the embeddings of elements of $\mathscr{C}$ into $\mathbb{P}$ with the universal family X_{HC}. We have a natural map $m: \mathbf{HC} \to \mathbf{MC}$. Let H_0 be the preimage of Z under m. Let H_1 be the normalization of H_0, and let X/H_1 be the universal family over H_1.

Given any $z \in Z$ one can take a subvariety $U_z \subset H_1$ of dimension $\dim H_1 - \dim m^{-1}(z)$ which intersects $m^{-1}(z)$ in at least one isolated point. One can also assume, by shrinking U_z, that U_z is irreducible and that $m: U_z \to Z$ is quasifinite. By construction we also have a specified family over U_z. Finitely many of the U_z cover Z; let them be U_i. The function fields $\mathbb{C}(U_i)$ are finite extensions of $\mathbb{C}(Z)$ and we can consider their composite F. If Y_1 denotes the normalization of Z in F, then there is an open cover V_i of Y_1 and finite and surjective maps $p_i: V_i \to U_i$. Via these maps we can pull back the families to get $(X_i', L_i')/V_i$. By this construction the V_i are quasiprojective, thus Y_1 is a scheme.

Over $V_i \cap V_j$ we have two families: $(X_i', L_i')|V_i \cap V_j$ and $(X_j', L_j')|V_i \cap V_j$. We define a space $\mathrm{Iso}(Y_1)_{ij}$ representing the isomorphism functor as follows. This is the subset of $V_i \cap V_j \times \mathrm{Aut}\,\mathbb{P}$ consisting of pairs (y, g) such that $g(m_i(y)) = m_j(y)$. The fiber of the natural map $\mathrm{Iso}(Y_1)_{ij} \to V_i \cap V_j$ over y is a principal homogeneous space over the automorphisms of (X_y, L_y). It is also a closed subvariety of the linear group $\mathrm{Aut}\,\mathbb{P}$, thus it is finite. The separatedness of the moduli problem implies that $\mathrm{Iso}(Y_1)_{ij} \to V_i \cap V_j$ is proper, and thus finite. Let $\mathrm{Iso}(Y_1)^{\circ}_{ij}$ be the union of the components that dominate $V_i \cap V_j$ and let $\mathrm{Iso}(Y_1)^{+}_{ij}$ be a compactification of $\mathrm{Iso}(Y_1)^{\circ}_{ij}$ which is finite over Y_1.

Let now Y' be a component of the normalization of the fiber products of all the spaces $\mathrm{Iso}(Y_1)^{+}_{ij}$ that dominate Y_1. It has a natural map onto Y_1 and let W_i be the preimage of V_i. We also have the pull-back families $(X_i, L_i)/W_i$. Let Y'_{gen} be the generic point of Y', it is also a generic point of $W_i \cap W_j$. By construction any isomorphism of

$$(X_i, L_i)|X_{\mathrm{gen}} \quad \text{and} \quad (X_j, L_j)|X_{\mathrm{gen}}$$

extends to an isomorphism of

$$(X_i, L_i)|W_i \cap W_j \quad \text{and} \quad (X_j, L_j)|W_i \cap W_j.$$

Now fix an i and for every $j \neq i$ pick an isomorphism of

$$(X_i, L_i)|W_i \cap W_j \quad \text{and} \quad (X_j, L_j)|W_i \cap W_j.$$

For every j and j' this determines an isomorphism of

$$(X_{j'}, L_{j'})|W_{j'} \cap W_j \quad \text{and} \quad (X_j, L_j)|W_{j'} \cap W_j.$$

Using this set of isomorphisms we can patch together the families (X_j, L_j) to a single family over Y'. It clearly has the required properties.

The following lemma completes the proof of 2.7. The author heard it first in a lecture of M. Artin.

2.8. Lemma. *Let Z' be an algebraic space of finite type. Then there is a scheme Z and a finite and surjective map $p: Z \to Z'$. If Z is normal and irreducible, then one can choose Z and p such that p is the quotient map by a finite group action.*

Proof. We can normalize Z' and therefore it is sufficient to prove the second part.

Let $p_i\colon U_i \to Z'$ be an affine étale covering of Z', and Z be the normalization of Z' in the Galois-closure of $\langle k(U_i)\colon i = 1, \dots \rangle$. Then we have $p\colon Z \to Z'$ and the required group action. Given any $z' \in Z'$ there is at least one point $z \in Z$ such that $z \to z' \in Z'$ factors through $z \to U_i \to Z$. Thus z has a neighborhood which is a scheme. By the group action any point in $p^{-1}(z')$ is such. This shows that Z is a scheme.

2.9. *Proof of* 2.6. Consider the family $f\colon X \to Y$ with a relatively ample line bundle L constructed in 2.7. Now choose integers k and j satisfying the following properties:

(i) L^k is f-very ample,

(ii) $R^i f_*(L^{k\cdot m}) = 0$ for $i, j > 0$,

(iii) every fiber X_y embedded via $L^k|X_y$ is defined (set theoretically) by degree $\leq j$ equations,

(iv) the multiplication map $S^j(f_*(L^k)) \to f_*(L^{jk})$ is surjective.

We claim that under these assumptions $\det f_*(L^{k\cdot j})$ is ample on Y.

We will check the conditions of the Ampleness Lemma 3.9.

To set notation let $V = f_*(L^k)$. Let $\rho = S^j$; then $W = S^j(f_*(L^k))$. For the quotient bundle we take $f_*(L^{jk})$ via the multiplication map given in (iv).

V is semipositive by assumption and ρ is also semipositive. The classifying map (3.8) is finite by the following reason.

For every $y \in Y$, $H^0(X_y, L^k|X_y)$ embeds X_y to a projective space; call it $\mathbb{P}_y$. Then $H^0(X_y, L^k|X_y) \cong H^0(\mathbb{P}_y(\mathscr{O}(1))$ in a natural way and $S^j(H^0(X_y, L^k|X_y)) \cong H^0(\mathbb{P}_y, \mathscr{O}(j))$. Thus the multiplication map at the point y is the same as the restriction map

$$m\colon S^j H^0(\mathbb{P}_y \mathscr{O}(1)) \cong H^0(\mathbb{P}_y, \mathscr{O}(j)) \to H^0(X_y, \mathscr{O}(j)|X_y).$$

The kernel of this map is exactly the set of degree j equations of X_y. By assumption (iii), X_y is defined by degree j equations, and therefore X_y can be recovered from the kernel of the multiplication map m. Thus the classifying map has finite fibers since these fibers are exactly the same as the fibers of the map $p\colon Y \to \mathbf{MC}$ which is finite by construction. Hence $\det f_*(L^{jk})$ is ample, and the proof of 2.6 is complete.

2.10. Remark. One can easily see that condition (iv) is in fact not necessary, but it makes the proof simpler.

3. The ampleness lemma

In this section we prove the ampleness lemma that was used in the proof of 2.6. We try to formulate and prove a fairly general version which will hopefully be useful in similar situations. Therefore we start with a general discussion about semipositive vector bundles and semipositive representations of linear groups. Everything at the beginning is known to experts but the author does not know of any convenient reference.

3.1. Definition. (i) Let $G = \times_i GL(E_i)$ be a product of general linear groups. A representation $\rho\colon G \to GL(F)$ is called *semipositive* (or polynomial) if it extends to a morphism $\overline{\rho}\colon \times_i \operatorname{End}(E_i) \to \operatorname{End}(F)$.

3.2. Examples. (i) Tensor products, symmetric products, exterior products, and direct sums are semipositive. Quotients and subrepresentations of semipositive representations are semipositive.

(ii) If $*$ denotes the dual representation, and ρ is semipositive, then $*\rho*$ is also semipositive. This is interesting mainly in characteristic p.

(iii) If ρ_i is a representation of G_i, then $\rho_1 \otimes \rho_2$, as a representation of $G_1 \times G_2$, is semipositive iff the ρ_i are semipositive. Thus in characteristic zero one has to know only the irreducible semipositive representations of $GL(E)$, which are exactly the subrepresentations of tensor products and the trivial representation.

(iv) Let $V_1, \cdots, V_n$ be vector bundles with typical fibers E_i and let ρ be any representation of G. One can define a vector bundle $\rho(V_1, \cdots, V_n)$ as follows. If the V_i are defined by transition functions g^i_{jk} (in the same covering), then we construct a new bundle whose typical fiber is F and has transition functions $\rho(g^1_{jk}, \cdots, g^n_{jk})$. Clearly, $\rho(V_1, \cdots, V_n)$ has structure group G.

(v) Assume that in (iv) every V_i is a direct sum of line bundles $\sum_j L_{ij}$. Then the structure group of V_i can be reduced to a torus. Thus the structure group of $\rho(V_1, \cdots, V_n)$ is also a torus and is a direct sum of line bundles of the form $\bigotimes L_{ij}^{a_{ij}}$. Since ρ is semipositive, each of the a_{ij} are nonnegative.

(vi) Let $V_1, \cdots, V_n$ and $W_1, \cdots, W_n$ be vector bundles such that $\operatorname{rk} V_i = \operatorname{rk} W_i$, and let $f_i\colon V_i \to W_i$ be sheaf homomorphisms. If ρ is semipositive, there exists a natural map

$$\rho(f_1, \cdots, f_n)\colon \rho(V_1, \cdots, V_n) \to \rho(W_1, \cdots, W_n).$$

In fact it is clear that the existence of the above map characterizes semipositive representations.

3.3. Definition-Proposition. *A locally free sheaf V on a scheme X is semipositive if the following equivalent conditions are satisfied:*

(i) $\mathscr{O}_{\mathrm{Proj}\, V}(1)$ *is nef on* $\mathrm{Proj}_X V$.

(ii) *For every map from a proper curve $f: C \to X$ every quotient bundle of $f^* V$ has nonnegative degree.*

(iii) *For every map from a proper curve $f: C \to X$ every quotient line bundle of $f^* V$ has nonnegative degree.*

(iv) *For every map from a proper curve $f: C \to X$ and for every ample line bundle H on C the bundle $H \otimes f^* V$ is ample* [11].

Proof. Clearly (i) $\Rightarrow$ (iii). If $C \subset \mathbb{P}_X V$ is a curve, then $\mathscr{O}_{\mathbb{P}V}(1)|C$ is a quotient of $\pi^* V|C$, where π is the projection map. Thus (iii) $\Rightarrow$ (i). If W is a locally free quotient sheaf of $f^* V$ over a curve C of rank k, then the k-fold selfintersection of $\mathscr{O}_{\mathbb{P}W}(1)$ is $\deg W$. $\mathbb{P}_C W$ is a closed subvariety of $\mathbb{P}_C V$ and we have $\deg W = [\mathbb{P}_C W] \cap [\mathscr{P}_{\mathbb{P}V}(1)]^k$. Thus if $\deg W < 0$, then $\mathscr{O}_{\mathbb{P}V}(1)$ is not nef, proving (i) $\Rightarrow$ (ii).

If V satisfies (i), then it is easy to check that $\mathscr{O}_{\mathbb{P}V}(1) \otimes \pi^* H$ satisfies Seshadri's ampleness criterion. Thus (i) $\Rightarrow$ (iv).

A quotient of an ample vector bundle is ample and has positive degree [11, 2.2 and 2.6]. Now choose H of degree one and conclude that any line bundle quotient F of $f^* V$ has degree at least $1 - \deg H = 0$. This shows that (iv) $\Rightarrow$ (iii).

3.4. Corollary. (i) *Quotients and extensions of semipositive vector bundles are semipositive.*

(ii) *Semipositivity is an open condition in flat families.*

Proof. (i) follows from 3.3(iii), and (ii) from 3.3(i).

3.5. Proposition. *Assume that the V_i are semipositive vector bundles over X, and let ρ be a semipositive representation as above. Then $\rho(V_1, \cdots, V_n)$ is again semipositive.*

Proof. We will use this only for direct sums, tensor products, and symmetric powers in which cases this is a special case of [11, 5.2 and 6, 3.3]. The proof of the general case can be done along the lines of [11, 6.6] as follows. By the definition of semipositivity it is sufficient to prove this when the base is a curve C.

Let E be a semipositive vector bundle over C. If L is a line bundle and $\deg L > 2g(C) - 2$, then ω_C cannot be a quotient of $E \otimes L$. Hence $H^1(C, E \otimes L) = 0$ which implies that if $\deg L > 2g(C)$, then $E \otimes L$ is generated by global sections. Thus we have a generically surjective map

$$f_E \colon \overbrace{L^{-1} + \cdots + L^{-1}}^{\mathrm{rk}\, E} \to E.$$

If we apply this to the vector bundles V_i, then using 3.2(v) and (vi) we get a generically surjective map

$$\rho(f_1, \cdots, f_n): \sum_i L^{-b_i} \to \rho(V_1, \cdots, V_n).$$

Let $\min \deg L^{-b_i} = -N(g(C), \rho)$, and note that N depends only on the genus of C and on ρ but not on the V_i. The above map shows that any quotient of $\rho(V_1, \cdots, V_n)$ has degree at least $-N$.

Assume now that $\rho(V_1, \cdots, V_n)$ has a quotient line bundle of degree e. If $p: C' \to C$ is a map of degree d, then pulling everything back by p we obtain that $\rho(p^* V_1, \cdots, p^* V_n)$ has a quotient of degree de. In particular $de \geq -N(g(C'), \rho)$. If we are in positive characteristic, then we can use a power of the Frobenius map to get a large degree self-map of C. Hence $de \geq -N(g(C), \rho)$ for d large, and $e \geq 0$ as desired.

The characteristic zero case can be reduced to the positive characteristic case by using 3.4(ii).

3.6. Proposition. *Let X be a scheme, and let W be a vector bundle with typical fiber W_x and structure group $G \to GL(W_x)$. If V_x is a G-invariant subspace of W_x, then it defines a subbundle $V \subset W$. Assume that W is semipositive. Then V is also semipositive if one of the following conditions is satisfied:*

(Δ)
- (i) *G is reductive and the characteristic is zero, or*
- (ii) *W is of the form $\rho(V_1, \cdots, V_n)$, where ρ and the V_i are semipositive and $G = \times GL_{r_i}$.*

Proof. In characteristic zero V_x has a G-invariant complement; thus V is a direct summand of W.

If W is of the form $\rho(V_1, \cdots, V_n)$ then V is also of the form $\sigma(V_1, \cdots, V_n)$, where σ is a subrepresentation of ρ. Since ρ is semipositive, the same holds for σ by 3.2(i). Thus V is semipositive by 3.5.

3.7. Remarks. (i) It is quite possible that G reductive is sufficient in any characteristic. Geometric reductivity of G implies that $\deg V \geq 0$, but I do not know how to get more.

(ii) One can weaken condition 3.6(ii) slightly to the following:

(Δ') For every proper curve $C \subset X$ there is a line bundle L of degree zero on C such that $W \otimes L$ is of the form $\rho(V_1, \cdots, V_n)$, where ρ and the V_i are semipositive and $G = \times GL_{r_i}$.

This is the form that will be used in §6.

3.8. Definition. Let X be a scheme and let W be a vector bundle of rank w with structure group $\rho: G \to GL_w$. Let $q: W \to Q$ be a quotient vector bundle of rank k. Let $Gr(w, k)/G$ denote the *set* of G-orbits on the k-dimensional quotients of a w-dimensional vector space. We call the natural map

$$u_{Gr}: \{\text{closed points of } X\} \to Gr(w, k)/G$$

the *classifying map.* Note that this is a map of sets and usually it does not correspond to any algebraic morphism.

We say that the classifying map is *finite* if

(i) every fiber of u_{Gr} is finite, and

(ii) for every $x \in X$ only finitely many elements of G leave $\ker q_x$ invariant.

3.9. Ampleness Lemma. *Let X be a proper algebraic space and let W be a semipositive vector bundle with structure group G. Let Q be a quotient vector bundle of W. Assume that*

(i) *W satisfies one of conditions (Δ) in 3.6 or (Δ') in 3.7, and*

(ii) *the classifying map is finite.*

Then $\det Q$ *is ample. In particular X is projective.*

3.10. Remarks. (i) This is the statement where we escape geometric invariant theory. We do not have to assume that the map goes to the stable points of $Gr(n, k)/G$. Note however that geometric invariant theory proves more if it applies: the ampleness of $(\det Q)^{\operatorname{rk} W} \otimes (\det W)^{-\operatorname{rk} Q}$. This is obviously harder to get than the ampleness of $\det Q$.

(ii) In characteristic zero Viehweg [34] proved a version of this statement that works for nonproper X too. He considered the case where W is a symmetric power of a semipositive vector bundle. He has to impose a different positivity condition on the V_i, called "weak positivity". The ample line bundle which he obtains is of the form $(\det Q)^q \otimes (\det W)^b$, where $a, b > 0$. This is somewhat weaker than the ampleness of $\det Q$. The reason for the difference is that there is no good ampleness criterion for nonproper varieties, therefore he has to construct sections with bare hands.

(iii) The idea of the proof in [22, §6] is the following. Assume that W is a trivial vector bundle. The structure group reduces to the trivial group and we get a finite morphism from X to a Grassmanian; hence X is projective.

Here we assume that W is semipositive, so we have more chance to get ampleness. Technically however this makes things more complicated.

Thus in [22] the analogous result is proved assuming that W admits a metric with semipositive curvature form.

Viehweg's idea [34] is to force triviality by considering the space of universal bases over X and pulling back everything. Descent is then achieved by a beautifully simple argument.

In this last step resolution of singularities of maps is heavily used. This causes some problems in finite characteristics; therefore we have to proceed in a different way (see [33], [22]).

The proof will rely on the Nakai-Moishezon criterion which we will need in a form that is usually not stated.

3.11. Theorem (*Nakai-Moishezon criterion* [28], [24], *and* [16]). *Let Z be a proper algebraic space and let H be a line bundle on Z. Then H is ample on Z iff for every irreducible closed subspace $X \subset Z$ the* $\dim X$*-fold selfintersection of $H|X$ is positive.*

Proof. The proof given in [16, III.1] works in this context as well. Alternatively, one can characterize ampleness via Serre vanishing as usual. Using [19, III,1.4] the same proof as for algebraic varieties shows that if $p: \overline{Z} \to Z$ is finite and surjective, then H is ample on Z iff p^*H is ample on $\overline{Z}$. By 2.8 we can assume that $\overline{Z}$ is a scheme. Now one can use [16, III.1] to get ampleness on $\overline{Z}$.

3.1.2. *Proof of* 3.9. By 3.11 we can harmlessly normalize X and so we can assume that X is irreducible. Next we must prove that $\det Q$ has positive selfintersection on any irreducible subvariety. If Z is a subvariety of X, then all the conditions of 3.9 hold if we restrict everything to Z. Thus it is sufficient to prove that $\det Q$ has positive selfintersection on X. Let Y be a projective scheme and $r: Y \to X$ be a morphism which is birational. We can pull-back everything to Y. This will not change the selfintersection of $\det Q$, but the classifying map will no longer be finite. Therefore we need to prove the following:

3.13. Lemma. *Let Y be a normal projective variety and let W be a semipositive vector bundle with structure group G. Let Q be a quotient vector bundle of W. Assume that*

(i) *W satisfies one of conditions* (Δ) *in* 3.6 *or* (Δ') *in* 3.7, *and*

(ii) *there is a nonempty open subset of Y such that the classifying map restricted to that subset is finite.*

Then the self intersection number of $\det Q$ *is positive.*

Proof. Let $n = \operatorname{rk} W$. Define

$$\mathbb{P} = \mathbb{P}_Y \overbrace{(W^* + \cdots + W^*)}^{n \text{ times}} .$$

Thus $\mathbb{P}$ is the projectivized space of matrices whose columns belong to W. Let $\pi: \mathbb{P} \to Y$ be the projection. We have the universal basis map:

$$\overbrace{\mathscr{O}_{\mathbb{P}}(-1) + \cdots + \mathscr{O}_{\mathbb{P}}(-1)}^{n \text{ times}} \to \pi^* W,$$

or equivalently

$$\overbrace{\mathscr{O}_{\mathbb{P}} + \cdots + \mathscr{O}_{\mathbb{P}}}^{n \text{ times}} \to \pi^* W \otimes \mathscr{O}_{\mathbb{P}}(1),$$

which sends a matrix to its columns. This map can also be described via the following isomorphisms:

$$\begin{aligned} H^0(\mathbb{P}, \pi^* W \otimes \mathscr{O}_{\mathbb{P}}(1)) &\cong H^0(Y, \pi_*(\pi^* W \otimes \mathscr{O}_{\mathbb{P}}(1))) \\ &\cong H^0(Y, W \otimes \overbrace{(W^* + \cdots + W^*)}^{n \text{ times}}). \end{aligned}$$

Let $D \subset \mathbb{P}$ be the divisor of matrices of determinant zero. The universal basis map is surjective outside D and gives a global trivialization of $\pi^* W|\mathbb{P} - D$.

Let W_x be a typical fiber of W. The (noncanonical) G-action $\rho: G \to GL(W_x)$ induces a G-action

$$\overline{\rho}: G \to \operatorname{Aut} \mathbb{P}\, \overbrace{(W_x + \cdots + W_x)}^{n \text{ times}}.$$

Let $P_x \subset \mathbb{P}(W_x + \cdots + W_x)$ be a general orbit. P_x is isomorphic to $G/\ker \overline{\rho}$. The G-structure transports the closure of P_x around; this way we obtain $\mathbf{P} \subset \mathbb{P}$. Thus $\mathbf{P}$ is a locally trivial fiber bundle over Y whose fiber is the closure of a projective G-orbit. Let $p: \mathbf{P} \to Y$ be the restriction of π. We denote $D \cap \mathbf{P}$ again by D.

We can compose the restriction of the universal basis map with $p^* q$ to obtain:

$$U_{Gr}: \sum_1^n \mathscr{O}_{\mathbf{P}} \to p^* W \otimes \mathscr{O}_{\mathbf{P}}(1) \to p^* Q \otimes \mathscr{O}_{\mathbf{P}}(1).$$

This map is surjective over $\mathbf{P} - D$.

For $k = \operatorname{rk} Q$ we take the kth exterior power of U_{Gr}. This gives

$$U: \sum \mathscr{O}_{\mathbf{P}} \to p^* \left(\bigwedge^k W \right) \otimes \mathscr{O}_{\mathbf{P}}(k) \to p^*(\det Q) \otimes \mathscr{O}_{\mathbf{P}}(k),$$

which is again surjective outside D. We expressed $p^*(\det Q) \otimes \mathscr{O}_{\mathbf{P}}(k)$ as the quotient of a trivial bundle of rank $\binom{n}{k}$. This gives a rational map

$$u: \mathbf{P} \dashrightarrow Gr(n, k) \subset \mathbb{P}\left(\bigwedge^k W \right),$$

which is a morphism on $\mathbf{P} - D$.

By construction this map is obtained by lifting the classifying map $u_{Gr}\colon X \to Gr(n, k)/G$ to $u_{Gr}^{\text{lift}}\colon \mathbf{P} - D \to Gr(n, k)$ and composing u_{Gr}^{lift} with the Plücker embedding.

Given any G-orbit T in $Gr(n, k)$ there are only finitely many $x \in X$ such that P_x is mapped to T by 3.8(i). On any given P_x the map u has finite fibers by 3.8(ii). In particular u is quasifinite on a suitable open subset of $\mathbf{P}$.

The map u need not be a morphism since the image of U need not be locally free. To remedy this we blow up the ideal sheaf of the image of U to obtain $g\colon \mathbf{P}' \to \mathbf{P}$. Now u extends to a morphism $u'\colon \mathbf{P}' \to \mathbb{P}(\bigwedge^k W)$. Thus there is an effective divisor $E \subset \mathbf{P}'$ such that

$$g^*(p^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(k)) \cong u'^* \mathscr{O}_{Gr}(1) \otimes \mathscr{O}_{\mathbf{P}'}(E).$$

Now let H be an ample divisor on Y. Since u' is generically finite, the line bundle $u'^*\mathscr{O}_{Gr}(1)$ is big on $\mathbf{P}'$. Thus there is a positive integer m such that we have a nontrivial section of $u'^*\mathscr{O}_{Gr}(m) \otimes g^*p^*H^{-1}$ and hence a nontrivial section

$$\sigma\colon \mathscr{O}_{\mathbf{P}'} \to g^*((p^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(k))^{\otimes m} \otimes p^* H^{-1}).$$

Pushing this down to Y we get a nonzero map

$$(g \circ p)_* \sigma\colon \mathscr{O}_Y \to (\det Q)^m \otimes H^{-1} \otimes p_* \mathscr{O}_{\mathbf{P}}(mk).$$

We can rearrange this to give a map

$$\tau\colon (p_* \mathscr{O}_{\mathbf{P}}(mk))^* \otimes H \to (\det Q)^m.$$

By construction,

$$\pi_* \mathscr{O}_{\mathbb{P}}(mk) \cong S^{mk} \left(\sum_1^n W^* \right).$$

If m is sufficiently large, then the natural map

$$\pi_* \mathscr{O}_{\mathbb{P}}(mk) \to p_* \mathscr{O}_{\mathbb{P}}(mk)$$

is surjective. Thus we get a subbundle

$$(p_* \mathscr{O}_{\mathbf{P}}(mk))^* \hookrightarrow (\pi_* \mathscr{O}_{\mathbb{P}}(mk))^*.$$

By construction it is also a G-subbundle. $(\pi_* \mathscr{O}_{\mathbb{P}}(mk))^*$ is semipositive by 3.5 and 3.2(ii), therefore $(p_* \mathscr{O}_{\mathbf{P}}(mk))^*$ is also semipositive by 3.6.

If we blow up the image sheaf of τ, then we get a birational map $s\colon Y' \to Y$ such that

$$s^*(\det Q)^m \cong s^* H \otimes \mathscr{O}_{Y'}(N) \otimes \mathscr{O}_{Y'}(F),$$

where F is effective, and $\mathcal{O}_{Y'}(N)$ is a quotient of $s^*(p_*\mathcal{O}_{\mathbf{P}}(mk))^*$; in particular it is semipositive by 3.4(i).

Now we can compute the selfintersection using the following formula:

$$((\det Q)^{\otimes m})^{(t)} = (H+N+F)^{(t)}$$
$$= H^{(t)} + \sum_{i=1}^{t} H^{(t-i)} \cdot (H+N+F)^{(i-1)} \cdot (N+F),$$

where $t = \dim Y$, and $^{(d)}$ denotes d-fold selfintersection. Note that $H+N+F$ is nef since it is numerically equivalent to $m \cdot c_1(Q)$, the first term is positive since H is the pull-back of an ample divisor by a birational morphism, and the sum is nonnegative since $H+N+F$ and N are nef, and F is effective.

This proves 3.13 and also 3.9.

4. Semipositivity results

In this section we check various semipositivity results that are needed in order to apply 2.6. Before we do this we need a technical result about partial resolutions that do not affect the normal crossing locus of a variety.

4.1. Definition. (i) An algebraic variety (or an algebraic space) X is called *semismooth* if all of its closed points are analytically isomorphic to one of the following:

a smooth point;
a double normal crossing point: $x_1 x_2 = 0 \in \mathbb{A}^n$; or
a pinch point: $x_1^2 - x_2^2 x_3 = 0 \in \mathbb{A}^n$.

In this case the singular locus is smooth, and we will call it the double divisor of X.

(ii) Let Y be a pure dimensional algebraic variety (or an algebraic space) such that outside a codimension-two set Z it has only smooth points or double normal crossing points (we will say that X is semismooth in codimension one). A proper map $f: X \to Y$ is called a semiresolution if X is semismooth, f is an isomorphism in codimension one on Y, and every component of the double divisor of X maps birationally onto the closure of a component of the double divisor of $X - Z$.

4.2 Proposition [31, 1.4.3]. (i) *Let Y be a two-dimensional algebraic space over any field which is semismooth in codimension one. Then Y has a semiresolution.*

(ii) *Let Y be a pure dimensional algebraic variety over a field of characteristic zero which is semismooth in codimension one. Then Y has a semiresolution $f: X \to Y$. (In our construction X will be an algebraic space even though Y is a variety.)*

Proof. The two-dimensional case is treated in [31].

(ii) can be done as follows. By Hironaka [12] we can perform a series of blow-ups centered in the locus where the multiplicity is at least three to obtain $g: Y_1 \to Y$ such that Y_1 has double points only as singularities and that D_1, the double point locus of Y_1, is smooth. Let $Y_2 \to Y_1$ be the normalization map and let D_2 be the preimage of D_1. This gives a natural involution τ on D_2. Easy local computation shows that D_2 is a Cartier divisor. Now perform a series of permissible blow-ups to resolve the singularities of D_2 in a τ-equivariant way. This gives us Y_3 and D_3, and D_3 is a smooth Cartier divisor. Thus Y_3 is smooth along D_3. Now we can resolve the singularities of Y_3 to get Y_4 which still contains D_3. The fixed-point set of τ is smooth on D_3. If we blow it up, then we get a Y_4 and D_4, and the fixed-point set of τ on D_4 has codimension one. Now we can pinch together D_4 in Y_4 via τ [4, 6.1] to get an algebraic space X. It is smooth outside the image of D_4, has normal crossing points at the image of the nonfixed points of τ, and has pinch points at the images of τ-fixed points. Clearly X is a semiresolution of Y.

The next statement is the first semipositivity result.

4.3. Theorem. *Let S be a complete Cohen-Macaulay surface over a field. Assume that S is Gorenstein outside finitely many points. Let $f: S \to C$ be a surjective map onto a smooth curve C. If the general fiber of f has only nodes as singularities, then $f_*(\omega_{S/C}^{[k]})$ is semipositive for $k \geq 2$.*

4.4. Remarks. (i) In characteristic zero this holds even for $k = 1$. The proof of 4.10 works with minor simplifications.

(ii) Ekedahl pointed out to me that for $k = 1$ the above result is false in positive characteristic. An example was given by Moret-Bailly [25, 3.2] (with S smooth and f generically smooth). This is interesting since in characteristic zero the $k = 1$ case is the usual starting point.

Proof. If $g: S' \to S$ is a semiresolution, then there is a natural injection $g_*(\omega_{S'/C}^{[k]}) \to \omega_{S/C}^{[k]}$ which is an isomorphism along the general fiber. Thus it is sufficient to prove the result for semismooth surfaces. If a double curve of S maps to a point in C, then we can blow it up. Since the relative dualizing sheaf commutes with base change, we can prove the result after some base change. Thus we can also assume that the genus of C is at least two, that the double curves of S are sections, and that they split into two components under the normalization of S.

The following is the most important part of the proof. It can be viewed as a generalization of some results of Arakelov [2] and Szpiro [32]. In characteristic zero it is a special case of a result of Kawamata [14].

4.5. Proposition. *Let S be a smooth surface and let $f\colon S \to C$ be a surjective map onto a smooth curve C. If the general fiber of f is smooth of genus at least two, then $f_*(\omega_{S/C}^{[k]})$ is semipositive for $k \geq 2$.*

Proof. As before we can assume that the genus of C is at least two. Thus S is of general type and we can assume that it is minimal. Using standard reduction mod p techniques (see e.g. [32]), it is sufficient to prove the claim for positive characteristic. Assuming that the statement is false, there is a negative quotient $f_*(\omega_{S/C}^{[k]}) \to M^{-1}$. Using base change by the Frobenius map $F\colon C \to C$ we can assume that in fact $f_*(\omega_{S/C}^{[k]})$ has a quotient of very negative degree. In particular, we may assume that $M \cong \omega_C^{k-1} \otimes L$, where L is very ample. Therefore we have a map

$$\omega_C^k \otimes M \otimes f_*(\omega_{S/C}^{[k]}) \to \omega_C,$$

which in turn implies that

$$H^1(C, \omega_C^k \otimes M \otimes f_*(\omega_{S/C}^{[k]})) \neq 0.$$

By looking at the Leray spectral sequence for f, this implies that

$$H^1(S, \omega_S^k \otimes f^*L) \neq 0.$$

By the main result of Ekedahl [9] this is impossible if $k \geq 2$ except in characteristic two, where this space is at most one-dimensional. The same technique can be used to make H^1 as large as we want. Thus we are done even in characteristic 2.

4.6. Corollary (*Arakelov, Szpiro, Raynaud* [32, p. 56]). *Assume the notation is as in 4.5 and that there are no -1 curves in the fibers of f. Then $\omega_{S/C}$ is nef.*

Proof. Assume that $D \subset S$ is an irreducible curve such that $\omega_{S/C}$ has negative degree on D. D cannot be contained in a fiber. If we have -2 curves in the fibers, then they can be contracted, $c\colon S \to S'$, and $\omega_{S/C}$ is nef iff $\omega_{S'/C}$ is nef. Thus we may assume that $\omega_{S/C}$ is f-ample. Let $h\colon D' \to D$ be the normalization of D and let $g\colon D' \to C$ be the natural map. For large k, $R^1 f_*(\omega_{S/C}^k(-D)) = 0$, and therefore we have a surjective map

$$g^*(f_*(\omega_{S/C}^k)) \to h^*\omega_{S/C}^k,$$

which contradicts semipositivity.

The next result is a kind of "log-generalisation" of 4.5.

4.7. Proposition. *Let $S \to C$ be a map from a smooth complete surface to a smooth curve. Assume that the general fiber of f is smooth. Let C_i be a set of distinct sections of f. Then $f_*(\omega^k_{S/C}(\sum a_i C_i))$ is semipositive provided that $k \geq 2$ and $a_i \leq k$ for every i.*

Proof. Let $g: S' \to S$ be a blow-up and let C'_i be the proper transform of C_i. Then $f_*(\omega^k_{S'/C}(\sum a_i C'_i))$ is a subsheaf of $f_*(\omega^k_{S/C}(\sum a_i C_i))$, and the two agree generically. Therefore it is sufficient to prove the claim after some blow-ups, and we may assume that the C_i are disjoint.

We prove the claim by induction on $\sum a_i$. If the general fiber of f has genus at least two, then 4.5 settles the case when $a_i = 0$. Assume that the general fiber is smooth elliptic. In this case $\omega_{S/C}$ is nef by using [7, p. 27] and Igusa's inequality $\chi(\mathscr{O}_S) \geq 0$ [13]. If the general fiber is smooth rational and $\sum a_i \leq 2k - 1$, then $f_*(\omega^k_{S/C}(\sum a_i C_i))$ is zero and we are done. There can be only one section with positive selfintersection by the Hodge index theorem; let this be C_0. Let $D_0 = a_0 C_0 + \sum_{i>0} b_i C_i$, where $b_i \leq a_i$ and $a_0 + \sum_{i>0} b_i = 2k - 1$. Then $f_*(\omega^k_{S/C}(D_0))$ is zero and hence semipositive. Starting with these cases we will add one section at a time and prove semipositivity step by step.

If the result is already proved for a sum of sections D_{j-1}, then pick another section C_t and let $D_j = D_{j-1} + C_t$. Assume first that the general fiber of f has genus at least one. Even though S need not be minimal, 4.6 implies that $\omega_{S/C} \cdot C_t \geq 0$. We claim that the same holds for rational general fiber. By adjunction, $\omega_{S/C}(C_t)|C_t \cong \mathscr{O}_{C_t}$. Therefore $\omega_{S/C} \cdot C_t = -C_t \cdot C_t \geq 0$ since $C_t \neq C_0$.

Now consider the exact sequence

$$0 \to \omega^k_{S/C}(D_{j-1}) \to \omega^k_{S/C}(D_j) \to \omega^k_{S/C}(D_j)|C_t \to 0.$$

If p is the coefficient of C_t in D_j, then $\omega^k_{S/C}(D_j)|C_t \cong \omega^{k-p}_{S/C}|C_t$, and so it has positive degree. Since $R^1 f_* \omega^k_{S/C}(D_{j-1}) = 0$ for $k \geq 2$, we get that $f_*(\omega^k_{S/C}(\sum a_i C_i))$ is successive extension of $f_*(\omega^k_{S/C})$ and the bundles $f_*(\omega^{k-p}_{S/C}|C_t)$. All these are semipositive and therefore we are done.

The result remains true if instead of assuming that the C_i are sections we require that the natural maps $C_i \to C$ be separable. It is probably false in the inseparable case.

4.8. *Proof of* 4.3. As we noted before, we are reduced to the case where S is semismooth and D, the double curve of S, consists of sections of $f: S \to C$. If $g: S' \to S$ is the normalization and D' is the preimage of D, then we can also assume that each component of D' is again a section.

Assume first that the general fiber has a component which is a smooth rational curve that intersects the rest in one point only. Let S_1 denote the surface which we obtain by throwing away the component containing that rational curve. Let G be the section corresponding to the removed component on S_1. Then $f_*(\omega^k_{S/C}) \cong f_*(\omega^k_{S_1/C}((k-1)G))$. Thus it is sufficient to prove that the latter is semipositive.

Next assume that the general fiber has a component which is a smooth rational curve that intersects the rest in two points only. Let S_1 denote the surface which we obtain by throwing away the component containing that rational curve and glueing together the remaining surface along the two sections corresponding to the removed component. Then $f_*(\omega^k_{S/C}) \cong f_*(\omega^k_{S_1/C})$.

If S has a component T such that the general fiber of T/C is a nodal rational curve, then let T' be the normalization of T and let D be the preimage of the double curve. Then $f_*(\omega^k_{T/C}) \cong f_*(\omega^k_{T'/C}(kD))$; thus $f_*(\omega^k_{T/C})$ is semipositive by 4.7.

Therefore 4.3 follows if we prove the following:

4.9. Proposition. *Let S be a semismooth surface and let $f\colon S \to C$ be a surjective map. Assume that the general fiber of f is a stable curve (hence with at most nodes). If D is the double curve of S, assume that it consists of sections of $f\colon S \to C$. If $g\colon S' \to S$ is the normalization and D' is the preimage of D, then assume further that each component of D' is again a section. Let G be any union of some disjoint sections that do not meet D. Then $f_*(\omega^k_{S/C}((k-1)G))$ is semipositive.*

Proof. By 4.7 we know that $(f \circ g)_*(\omega^k_{S'/C}((k-1)(G+D')))$ is semipositive.

Now observe that $g^*\omega_{S/C} \cong \omega_{S'/C}(D')$, and that we have an exact sequence

$$0 \to g_*(\omega_{S'/C}((k-1)G)) \to \omega_{S/C}((k-1)G) \to \omega_{S/C}|D \cong \mathcal{O}_D \to 0.$$

Tensoring the latter with $\omega^{k-1}_{S/C}$ and using the projection formula, we get the following sequence:

$$0 \to g_*\omega^k_{S'/C}((k-1)(D'+G)) \to \omega^k_{S/C}((k-1)G) \to \mathcal{O}_D \to 0.$$

Taking into account that $R^1(f \circ g)_*\omega^k_{S'/C}((k-1)(D'+G)) = 0$ for $k \geq 2$, this gives the sequence

$$0 \to (f \circ g)_*\omega^k_{S'/C}((k-1)(D'+G)) \to f_*\omega^k_{S/C}((k-1)G) \to f_*\mathcal{O}_D \to 0.$$

$f_*\mathcal{O}_D$ is isomorphic to the sum of several copies of $\mathcal{O}_C$, therefore $f_*(\omega^k_{S/C}((k-1)G))$ is semipositive for $k \geq 2$. This completes the proof of 4.9 and therefore also the proof of 4.3.

The proof of the next result goes pretty much along the lines developed by Fujita [10] and Viehweg [33]. They always treated smooth varieties whereas we need to consider nonnormal varieties with semi-log-canonical singularities. The basic philosophy states that this should not make too much difference; in fact one could expect all the relevant proofs to go through without change. In our case this indeed happens. However, this is not always true. The very important result of [33, I,5.4] fails to hold even for families of curves with nodes.

4.10. Definition [23, 4.17]. A singularity (S, S) is called *semi-log-canonical* if the following conditions are satisfied:

(i) (s, S) is Cohen-Macaulay;
(ii) (s, S) is semismooth in codimension one;
(iii) $\omega_S^{[k]}$ is locally free for some $k > 0$;
(iv) if $f\colon X \to S$ is a semiresolution with exceptional divisors C_i and $\omega_X^k \cong f^*(\omega_S^{[k]} \otimes \mathcal{O}_X(\sum a_i C_i))$, then $a_i \geq -k$ for every i.

We will be mainly interested in the surface case where a complete classification is known:

4.11. Proposition [15], [23, 4.24]. *The semi-log-canonical surface singularities are the following:*

semismooth points,
DuVal singularities,
simple elliptic singularities,
cusps and degenerate cusps,
quotients of the above by certain cyclic group actions.

4.12. Theorem. *Let Z be a complete variety over a field of characteristic zero. Assume that Z satisfies Serre's condition S_2 and that it is Gorenstein in codimension one. Let $f\colon Z \to C$ be a map onto a smooth curve. Assume that the general fiber of f has only semi-log-canonical singularities, and further that ω of the general fiber is ample. Then $f_*(\omega^k_{Z/C})$ is semipositive for every $k \geq 1$.*

The proof will be done in three steps.

4.13. Lemma. *Let Z be a complete semismooth variety over a field of characteristic zero, and let $f\colon Z \to C$ be a surjective map. Assume that the double locus is flat over C. Then $f_*(\omega_{Z/C})$ is semipositive.*

Proof. Let D be the double locus of Z. Let $g: Z' \to Z$ be the normalization and let D' be the preimage of D. We have an exact sequence

$$0 \to g_*\omega_{Z'/C} \to \omega_{Z/C} \to \omega_{Z/C}|D \to 0.$$

This gives a long exact sequence

$$0 \to (f \circ g)_*\omega_{Z'/C} \to f_*(\omega_{Z/C}) \to f_*(\omega_{Z/C}|D) \xrightarrow{\delta} R^1(f \circ g)_*\omega_{Z'/C}.$$

By Fujita [10] $(f \circ g)_*\omega_{Z'/C}$ is semipositive. Therefore we are done if we can prove that the kernel of the map

$$f_*(\omega_{Z/C}|D) \xrightarrow{\delta} R^1(f \circ g)_*\omega_{Z'/C}$$

is semipositive. To show this note that

$$g^*(\omega_{Z/C}|D) = (g^*\omega_{Z/C})|D' = \omega_{Z'/C}(D')|D' = \omega_{D'/C}.$$

Therefore $f_*(\omega_{Z/C}|D)$ is a direct summand of $f_*\omega_{D'/C}$ and δ factors through the natural map

$$\delta': (f \circ g)_*\omega_{D'/C} \to R^1(f \circ g)_*\omega_{Z'/C}.$$

By [21, 2.6] the sheaves $(f \circ g)_*\omega_{D'/C}$ and $R^1(f \circ g)_*\omega_{Z'/C}$ are determined by some variations of Hodge structures, and the connecting map δ is induced by a map between these variations of Hodge structures. Therefore the kernel of δ is a direct summand of $f_*(\omega_{Z/C}|D)$. The latter is semipositive as before; hence any of its direct summands are semipositive. This proves 4.13.

4.14. *Proof of* 4.12 *for* $k = 1$. Here we do not need the ampleness assumption. We take a minimal semiresolution (see [23, 4.9]) Z'_{gen} of the general fiber Z_{gen} of f. This extends to a semiresolution over the preimage of an open set in C. We complete this somehow and then take a semiresolution of this space to get $g: X \to Z$. As earlier we may assume that the double locus of X is flat over C.

If z is a point of the general fiber of f which is not semirational, then it is a Gorenstein point (cf. e.g.[23, 4.24]). If E_{gen} denotes the reduced preimage of z in Z'_{gen}, then E_{gen} is a divisor with normal crossings only, and in a neighborhood of E_{gen} the pull-back of $\omega_{Z_{\text{gen}}}$ equals $\omega_{Z'_{\text{gen}}}(E_{\text{gen}})$. We can take the closure E of E_{gen} in X. By further blowing up in some fibers we may assume that E is a Cartier divisor. By our construction the natural map $g_*(\omega_{X/C}(E)) \to \omega_{Z/C}$ is an isomorphism over the general fiber. Furthermore, the short exact sequence

$$0 \to \omega_{X/C} \to \omega_{X/C}(E) \to \omega_{E/C} \to 0$$

gives rise to a long exact sequence

$$0 \to (f\circ g)_*\omega_{X/C} \to (f\circ g)_*\omega_{X/C}(E) \to (f\circ g)_*\omega_{E/C} \xrightarrow{\delta} R^1(f\circ g)_*\omega_{X/C}.$$

Again we know that $(f\circ g)_*(\omega_{X/C})$ and $(f\circ g)_*(\omega_{E/C})$ are semipositive. As in the proof of 4.13 the kernel of δ is a direct summand and thus semipositive. This completes the proof of the special case $k = 1$.

Before completing the proof we need the following:

4.15. Lemma. *Let Y be a variety with semi-log-canonical singularities. Assume that $\omega_Y^{[k]}$ is very ample. For a general section σ of $\omega_Y^{[k]}$ take a k th root of σ. Then the resulting variety Y' again has semi-log-canonical singularities.*

Proof. This can be proved in the same way as [29, 1.7 and 1.13].

4.16. *Proof of* 4.12. This will be done very much along the lines of [33]. As before we can assume that the nonnormal locus of Z is flat over C.

Let us fix k such that on the general fiber $\omega^{[k]}$ is Cartier and very ample. Fix an ample line bundle H on C. Then $f_*(\omega_{Z/C}^{[k]})\otimes H^{ks-1}$ is semipositive if s is sufficiently large. We fix one such s. This implies that

$$S^m(f_*(\omega_{Z/C}^{[k]})\otimes H^{ks}) = S^m(f_*(\omega_{Z/C}^{[k]})\otimes H^{ks-1})\otimes H^m$$

is generated by global sections if $m \geq 2\cdot(\text{genus of } C)$. The natural map

$$f^*(f_*\omega_{Z/C}^{[k]}\otimes H^{ks}) \to (\omega_{Z/C}\otimes f^*H^s)^{[k]}$$

is surjective over the general fiber. By blowing up away from the general fiber we may assume that its image M is locally free and then we have

$$(\omega_{Z/C}\otimes f^*H^s)^{[k]} = M\otimes\mathcal{O}(E),$$

where E is a effective Cartier divisor supported in some fibers of f.

By construction M is generated by global sections. We pick one sufficiently general; its zero locus is denoted by T. If we take the k th roots of $T+E$, then we get a variety Z' and a cyclic covering map $p\colon Z'\to Z$. (We emphasize that Z' is the nonnormalized cover.) Next we claim that

$$p_*\omega_{Z'/C} \cong \sum_{i=0}^{k-1}(\omega_{Z/C}\otimes(\omega_{Z/C}\otimes f^*H^s)^{[i]})^{[1]}.$$

(The [1] of course just denotes double dual.)

Indeed, by easy general results this holds over the locus where everything is Cartier, i.e., in codimension one in our case. Since both sides are reflexive, the isomorphism is automatic from this.

By 4.14 and 4.15 we know that $(f \circ p)_* \omega_{Z'/C}$ is semipositive, hence so are its direct summands. Therefore we know that

$$f_*((\omega_{Z/C} \otimes (\omega_{Z/C} \otimes f^* H^s)^{[k-1]})^{[1]}) = f_*(\omega_{Z/C}^{[k]}) \otimes H^{ks-s}$$

is semipositive.

If we pick the smallest possible value for s such that $f_*(\omega_{Z/C}^{[k]}) \otimes H^{ks-1}$ is semipositive, then the above yields a contradiction unless $s \leq k$. Thus we obtain that $f_*(\omega_{Z/C}^{[k]}) \otimes H^{k^2}$ and also $f_*(\omega_{Z/C}^{[j]}) \otimes H^{k^2}$ for $j \leq k$ are always semipositive. We can choose H to have degree one, and therefore we see that any quotient line bundle of $f_*(\omega_{Z/C}^{[j]})$ for $j \leq k$ has degree at least $-k^2$. This implies that there cannot be any negative quotients at all since a suitable base change $t: C' \to C$ produces a more negative quotient of $f_*(\omega_{Z \times_C C'/C}^{[j]})$. This completes the proof of 4.12.

5. Projectivity of certain moduli spaces

In this section we put together the previous results to give a new proof of the projectivity of the moduli space of stable curves over $\mathbb{Z}$ and to investigate the moduli space of surfaces of general type.

5.1. Theorem [17], [18], [26]. *Let $\overline{\mathscr{M}}_g$ be the moduli space of stable curves of genus $g \geq 2$ over $\mathbb{Z}$. Then $\overline{\mathscr{M}}_g$ is projective over $\mathbb{Z}$.*

Proof. First we have to check the existence as an algebraic space. Being a stable curve is an open condition. Boundedness holds since ω_C^3 is very ample. Completeness and separatedness follow from the stable reduction theorem [8], [5]. The automorphism group is finite and reduced since $H^0(T_C) = 0$. The relative dualizing sheaf provides a functorial polarization which is semipositive by 4.3. Thus $\overline{\mathscr{M}}_g$ is projective over any field.

The image of the three-canonical map is a curve of degree $6g-6$, thus it is defined by equations of degree at most $6g-6$. Hence λ_{18g-18} is ample. Using a little more about curves we know that the three-canonical images are defined by quadratic equations, thus in fact already λ_6 is ample.

To show projectivity over $\mathbb{Z}$ we need to remark that the $\mathbb{Q}$-line bundles λ_m exist over $\overline{\mathscr{M}}_g$. Thus λ_6 is relatively ample over $\mathbb{Z}$. This completes the proof.

For surfaces, first we have to make precise the moduli problem which we are considering.

5.2. Definition. (i) A *stable surface* is a proper two-dimensional reduced scheme S over a field of characteristic zero such that S has only semi-log-canonical singularities (4.10–4.11), and $\omega_S^{[k]}$ is locally free and ample for some $k > 0$.

(ii) A *family* of stable surfaces over a scheme Y is a proper flat scheme X/Y such that:

(a) every fiber is a stable surface,
(b) for every closed point $y \in Y$ and every $k > 0$ we have a natural isomorphism $\omega_{X/Y}^{[k]}|X_y \cong (\omega_{X/Y}|X_y)^{[k]}$.

5.3. Remarks. (i) This notion of stability has nothing to do with geometric invariant theory. In fact, stable surfaces are frequently asymptotically unstable in the sense of [26] with respect to the canonical line bundle (see e.g. [30, p. 37]).

(ii) The second condition on the families can be explained as follows. First, since ω_S is usually too small we also need its powers to be flat over any base. This is exactly the condition. Also, without it separatedness of the moduli space would fail.

5.4. Definition. (i) Given a rational number K^2 and an integer χ, let $\overline{\mathscr{M}}_{K^2,\chi}$ be the functor that associates to a scheme Y the families of stable surfaces where every closed fiber S_y satisfies $\omega_{S_y} \cdot \omega_{S_y} = K^2$ and $\chi(\mathscr{O}_{S_y}) = \chi$. Here of course we define

$$\omega_{S_y} \cdot \omega_{S_y} = \frac{1}{k^2}\omega_{S_y}^{[k]} \cdot \omega_{S_y}^{[k]},$$

where $\omega_{S_y}^{[k]}$ is locally free, $k > 0$, and "$\cdot$" denotes the intersection product.

This makes good sense since the numbers $\omega_{S_y} \cdot \omega_{S_y}$ and $\chi(\mathscr{O}_{S_y})$ are deformation invariant.

(ii) A stable surface S is called *smoothable* if there is a one-parameter family of stable surfaces such that the central fiber is S and the general fiber has only DuVal singularities. (This is clearly a closed condition.)

(iii) Let $\overline{\mathscr{M}}^{\mathrm{sm}}_{K^2,\chi}$ be the subfunctor of $\overline{\mathscr{M}}_{K^2,\chi}$ where all the fibers are smoothable stable surfaces.

5.5. Proposition [23, Chapter 5]. *The functor $\overline{\mathscr{M}}^{\mathrm{sm}}_{K^2,\chi}$ is coarsely represented by a separated algebraic space locally of finite type. It also satisfies the valuative criterion of properness.*

5.6. Corollary. *If for some K^2 and some χ the functor $\overline{\mathscr{M}}^{\mathrm{sm}}_{K^2,\chi}$ is bounded, then it is coarsely represented by a projective algebraic scheme.*

Proof. The canonical polarization is semipositive by 4.12.

6. Compactified Picard schemes

If X is a singular projective variety then its Picard scheme is in general not compact. Considerable attention was given to its compactifications, especially in the case of curves. In the general case a geometrically meaningful compactification was described by Altman and Kleiman [1]. They proved that this compactification is a proper scheme; however the question of projectivity remains open. Here we will study this question.

6.1. Definition. (i) Let X be a proper scheme and let L be an ample line bundle. The *Hilbert polynomial* of a sheaf F is the polynomial

$$H(t) = \chi(X, F \otimes L^t).$$

(ii) Let X/S be a flat relatively projective scheme with a relatively ample line bundle L. For a polynomial $H(t)$ we define the relative compactified Picard functor $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$ as follows:

$$\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}(Z/S) = \left\{ \begin{array}{l} F \text{ is a sheaf over } X \times_X Z \text{ flat over } Z \text{ such that for} \\ \text{every } z \in Z\ F|X \times_S \{z\} \text{ is torsion free or rank one} \\ \text{with Hilbert polynomial } H\text{; modulo the equivalence} \\ \text{relation } F \sim F \otimes \pi^* M \text{ where } M \text{ is a line bundle on} \\ Z \text{ and } \pi\colon X \times_S Z \to Z \text{ is the natural projection.} \end{array} \right.$$

6.2. Theorem (*Altman-Kleiman* [1]). *Assume that X/S is projective, flat, and has geometrically integral fibers. Let L be a relatively ample line bundle and let H be a polynomial. Then $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$ is coarsely represented by a proper and separated scheme $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$.*

6.3. Remark. In their first paper [1, I,7.9], they prove that $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$ exists as an algebraic space. We will not need that it is a scheme.

6.4. Theorem. *Let X/S, H, and L be as in 6.2. Assume furthermore that S is of finite type. Then $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$ is projective over S.*

Proof. We can obviously assume that S is connected, in particular various numerical invariants will be constant over S. We will use this without mentioning it again.

Again we try to apply the Ampleness Lemma. To do this we have to come up with a vector bundle on $\overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$. Assume for a moment that there exists a universal sheaf F over $f\colon X \times_S \overline{\overline{\mathcal{P}ic}}{}^H_{X/S} \to \overline{\overline{\mathcal{P}ic}}{}^H_{X/S}$. Then we can try to use f_*F. This might not be a vector bundle but the main problem is that the universal sheaf is not unique. If M is any line bundle on $\overline{\overline{\mathcal{P}ic}}^H_{X/S}$, then $F \otimes f^*M$ is also a universal sheaf. Thus we have to find a way to rigidify F. The approach of [1, II,1.2] is the following. Assume that f has a section $s\colon \overline{\overline{\mathcal{P}ic}}^H_{X/S} \to X \times_S \overline{\overline{\mathcal{P}ic}}^H_{X/S}$ such that F is locally free along the image of s. Then we can replace F by $F \otimes (s^*F)^{-1}$, i.e., we take the unique universal sheaf which is trivial along s. Unfortunately such a section almost never exists globally, and different sections give different rigidification, thus patching is a problem. We will go around this obstacle using the following.

6.5. Proposition. *Assume the notation used above. Let* $n = \dim X/S$. *Let* $g\colon U \to \overline{\overline{\mathcal{P}ic}}^H_{X/S}$ *be a morphism. Assume that a universal sheaf* F *exists over* $U \times_S X$ *and that* L *is relatively very ample over* $U \times_S X$. *Let* $L_1, \cdots, L_n \subset U \times_S X$ *be the zero sets of* n *sections* $t_i\colon \mathcal{O}_{U\times_S X} \to L$. *Assume the following:*

(i) $f\colon L_1 \cap \cdots \cap L_n \to U$ *is finite and étale,*

(ii) F *is locally free along* $L_1 \cap \cdots \cap L_n$,

(iii) $R^i f_*(F \otimes L^{-j}) = 0$ *for* $i > 0$ *and* $0 \le j \le n$.

Then $\det f_*(F|L_1 \cap \cdots \cap L_n)$ *is independent of the choice of the* L_i, *and will be denoted by* $N(F, L)$. *If* M *is a line bundle on* U, *then*

$$N(F, L \otimes f^*M) \cong N(F, L) \otimes M^d,$$

where d *is the selfintersection number of* L *on the fibers.*

Proof. Using the sections $t_i^{-1}\colon L^{-1} \to \mathcal{O}$ we can build the Koszul complex:

$$0 \to \binom{n}{n} F\otimes L^{-n} \to \cdots \to \binom{n}{1} F\otimes L^{-1} \to \binom{n}{0} F\otimes L^{0} \to F|L_1 \cap \cdots \cap L_n \to 0.$$

This complex is exact since F is locally free along $L_1 \cap \cdots \cap L_n$. By assumption (iii) it stays exact if we apply f_*. Thus

$$\det f_*(F|L_1 \cap \cdots \cap L_n) \cong \prod_{i=0}^{n} [\det f_*(F \otimes L^{-i})]^{(-1)^i \binom{n}{i}},$$

which clearly shows independence of the choice of the L_i. The last claim follows from the observation that $f_*(F|L_1 \cap \cdots \cap L_n)$ is a vector bundle of rank d.

6.6. Notation. To formulate our next result we need some notation. Assume that we are given $g\colon Y \to \overline{\overline{\mathcal{P}ic}}^H_{X/S}$ and a section $s\colon Y/S \to Y \times_S X$ such that for every $y \in Y$ the sheaf $F_{g(y)}$ is locally free at $s(y) \in X_{g(y)}$.

Then there is a universal sheaf F_Y on $Y \times_S X$ by [1, II, 1.6]. Let $f_Y: Y \times_S X \to Y$ be the projection. For large k the sheaf $F'_Y = F_Y \otimes L^k$ satisfies the conditions of 6.5 locally on Y. Thus locally on Y we can construct the vector bundles $f_{Y*}(F_Y \otimes L^k)$ and $N(F_Y \otimes L^k, L)$. By 6.5 these glue together to global vector bundles.

6.7. Proposition. *We use the notation and assumptions as in* 6.4, 6.5, *and* 6.6. *Then for every* $k \gg 0$ *and* $m \geq 0$ *there is a vector bundle* $W(k, m)$ *on* $\overline{\overline{\mathcal{P}ic}}^H_{X/S}$ *with the following property:*

For every Y *as above there is a canonical isomorphism*

$$f_{Y^*}(F_Y \otimes L^{k+m}) \otimes (f_{Y^*}(F_Y \otimes L^k))^{\otimes d-1} \otimes N(F_Y \otimes L^k, L)^{-1} \cong g^* W(k, m).$$

Proof. First we construct $W(k, m)$ locally. Any point of $\overline{\overline{\mathcal{P}ic}}^H_{X/S}$ has an open neighborhood U such that over U there is an étale section $g: Y/U \to U \times_S X$ such that the conditions of 6.7 are satisfied. Thus there is a universal sheaf over $Y \times_S X$, and so we have the vector bundle

$$f_{Y^*}(F_Y \otimes L^{k+m}) \otimes ((f_{Y^*}F_Y \otimes L^k))^{\otimes d-1} \otimes N(F_Y \otimes L^k, L)^{-1},$$

which is independent of the choice of F. Indeed, if F is replaced by $F \otimes f^* M$ where M is a line bundle on Y, then the first factor changes by M, the second by M^{d-1}, and the third by M^{-d}. Thus the product is unchanged and can be descended to a bundle on U. Because of the unicity we can then patch them together to a single $W(k, m)$.

6.8. Proposition. *The above* $W(k, m)$ *is seminegative, i.e.,* $W(k, m)^*$ *is semipositive on every fiber of* $\overline{\overline{\mathcal{P}ic}}^H_{X/S}/S$.

Proof. Pick any $s \in S$ and let X_s be the corresponding variety. Pick a complete smooth curve C mapping to the fiber of $\overline{\overline{\mathcal{P}ic}}^H_{X/S}$ over s. For every $c \in C$ there is the corresponding rank one sheaf F_c on $X_s \times \{c\}$. Let $V \subset X_s$ be the smooth locus. Then F_c is locally free on V in codimension one, and there is a point $x \in X_s$ such that F_c is locally free at x for every c. This point gives a section $s: C \to C \times X_s$ which satisfies the conditions of 6.6. Thus there is a universal sheaf F on $C \times X_s$.

Now let us look at the definition of $N(F, L)$. The subscheme $L_1 \cap \cdots \cap L_n$ consists of finitely many disjoint trivial sections of $C \times X_s$. We can rigidify F such that $s^* F \cong \mathcal{O}_C$. If t is any other section such that F is locally free along the image of t, then $t^* F$ is algebraically equivalent to $s^* F$ (X_s is integral, hence irreducible). Thus $N(F, L)$ is a degree zero line bundle on C.

Therefore it is sufficient to prove that $f_*(F \otimes L^j)$ is seminegative. For some large r consider the rth order infinitesimal neighborhood $s(C)_r$

of $s(C)$. Since $F \otimes L^j|s(C)$ is trivial and $f_*(s(C)_r)$ is a trivial vector bundle, $f_*(F \otimes L^j|s(C)_r)$ is an extension of several copies of $\mathscr{O}_C$. For r large, $f_*(F \otimes L^j)$ injects into $f_*(F \otimes L^j|s(C)_r)$. Thus any subbundle of $f_*(F \otimes L^j)$ is a subbundle of $f_*(F \otimes L^j|s(C)_r)$, hence it has nonpositive degree. Therefore any quotient bundle of $(f_*(F \otimes L^j))^*$ has nonnegative degree. This is what we wanted to show.

6.9. Corollary. *$W(k, m)^*$ satisfies condition (Δ') from 3.7.*

Proof. We just showed that up to a degree zero line bundle $W(k, m)|C$ is isomorphic to

$$f_{Y^*}(F_Y \otimes L^{k+m}) \otimes ((f_{Y^*}F_Y \otimes L^k))^{\otimes d-1},$$

where $Y = C \times X_s$. Each of these factors is seminegative and the tensor product is semipositive.

6.10. *Proof of* 6.4. By replacing L with a large tensor power we may assume that $R^i f_*(L^j) = 0$ for $i > 0$ and $j \geq 0$. This is possible since S is of finite type.

Choose k such that

(i) $R^i f_*(F \otimes L^j) = 0$ for $i > 0$ and $j \geq k - n$, and

(ii) $f^*(f_*(F \otimes L^j)) \to F \otimes L^j$ is surjective for $j \geq k$.

By (i) $W(j, m)$ is a vector bundle for $j \geq k$. Now let

$$K = \ker[f^*(f_*(F \otimes L^k)) \to F \otimes L^k],$$

and choose m such that $f^*(f_*(K \otimes L^j)) \to K \otimes L^j$ is surjective for $j \geq m$.

We can consider the multiplication map

$$\text{mult}: W(k, 0) \otimes f_*(L^m) \to W(k, m).$$

At a point $p \in \overline{\overline{\mathscr{Pic}}}{}^H_{X/S}$ where the corresponding sheaf is F_p/X_p this map is (up to a multiplicative constant)

$$[H^0(X_p, F_p \otimes L^k) \otimes H^0(X_p, L^m) \to H^0(X_p, F_p \otimes L^{k+m})] \otimes H^0(X_p, F_p \otimes L^k)^{\otimes d-1}.$$

Note that

$$\ker[H^0(X_p, F_p \otimes L^k) \otimes H^0(X_p, L^m) \to H^0(X_p, F_p \otimes L^{k+m})] = H^0(X_p, \ker[\mathscr{O}_{X_p} \otimes H^0(X_p, F_p \otimes L^k) \to F_p \otimes L^k] \otimes L^m).$$

Therefore by the choice of k and m we can recover

$$\ker[\mathscr{O}_{X_p} \otimes H^0(X_p, F_p \otimes L^k) \to F_p \otimes L^k]$$

from the multiplication map. Thus we can also recover F_p itself.

Now we only need to apply the Ampleness Lemma 3.9. We choose any $\overline{\overline{\mathscr{P}ic}}^H_{X/S} \times_S \{s\}$ and want to show that it is projective. For the semipositive vector bundle W we choose $(W(k, 0) \otimes f_*(L^m))^*$. Since $f_*(L^m)$ is a trivial bundle, W is semipositive by 6.8 and satisfies condition (Δ') by 6.9. For the quotient bundle Q we choose the dual of the kernel of mult. As we just showed, F_p can be recovered from the kernel of the multiplication map, thus the classifying map is finite. Now 3.9 implies that

$$\det(\ker[W(k) \otimes f_*(L^m) \to W(k+m)]^*)$$

is relatively ample on $\overline{\overline{\mathscr{P}ic}}^H_{X/S}/S$. This was to be proved.

6.11. Remark. In the proof we used that S is of finite type only to ensure that for some fixed k the line bundle L^k is relatively very ample (see [1,I,3.4] for the remaining assertions). It is expected that the choice of such a k is always possible. For surfaces this was proved in [20, 2.1.2].

Although not necessary for our considerations, it is frequently very convenient to have a universal family. The following result states that a universal family exists locally in the étale topology on S.

6.12. Theorem. *Assume that X/S is projective, flat, and has geometrically integral fibers, and that there is a section $\sigma: S \to X$ such that X/S is smooth along $\sigma(S)$ (this is always the case if S is a point or the spectrum of a Henselian local ring). Then there exists a universal family over $X \times_S \overline{\overline{\mathscr{P}ic}}_{X/S}$.*

Proof. The idea is the same as in [1, II,1.2]: we rigidify the functor by trivializing along σ. Unfortunately, the universal family will not be locally free along σ, thus it is not clear how to trivialize. This will be taken care of by the following:

6.13. Lemma. *Let $f: X \to S$ be smooth and let F be a sheaf on X, flat over S. Assume that for every $s \in S$ the restriction $F|X_s$ is torsion free of rank one. Then the double dual of F is locally free.*

Proof. The question is local on X so we may assume that X and S are local and complete. Let $0 \in S$ be the closed point. If the fiber has dimension one, then F itself is free. Now assume that the dimension of the fibers is two. We will prove that there is a map $F \to \mathscr{O}_X$ which is an isomorphism in codimension two. This will imply that F^{**} is free.

If I is an ideal of $\mathscr{O}_S$ of finite colength, then let $R_I = \mathscr{O}_X/(f^*I)$ and $X_I = \operatorname{Spec} \mathscr{O}_X/(f^*I)$. Similarly let $F_I = F/[(f^*I) \cdot F]$. If m is the maximal ideal, then F_m is a rank one torsion free sheaf on a smooth surface; thus it has a free resolution of the form

$$0 \to R^n_m \xrightarrow{p_m} R^{n+1}_m \xrightarrow{q_m} F_m \to 0$$

for some integer n. If we have a resolution like this for some I and $I \supset J$, then by flatness of F this resolution has a lifting as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_I^n & \xrightarrow{p_I} & R_I^{n+1} & \xrightarrow{q_I} & F_I & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R_J^n & \xrightarrow{p_J} & R_J^{n+1} & \xrightarrow{q_J} & F_J & \longrightarrow & 0 \end{array}$$

where the vertical maps are the natural quotient maps. The map p_J is given by an $n \times (n+1)$ matrix A_J. We can define a map $r_J \colon R_J^{n+1} \to R_J$ given by the $n \times n$ cofactors of A_J. The sequence

$$R_J^n \xrightarrow{p_J} R_J^{n+1} \xrightarrow{r_J} R_J$$

is exact; thus r_J factors through q_J to get a map $i_J \colon F_J \to R_J$. Furthermore from the construction it is clear that i_J is an extension of i_I, so that we obtain a map $i \colon F \to \mathcal{O}_X$ which is an isomorphism in codimension one. This is what we wanted.

Now assume that the dimension of the fiber is at least three. By the above discussion F^{**} is locally free outside a codimension three subset Z. We claim that this in fact implies that it is locally free. To see this we can localize at a general point of $X_m \cap Z$. Thus we can assume that $F^{**}|X_m$ is locally free outside the closed point x.

For every I, $F^{**} \otimes R_I$ is locally free outside the origin. We can compute the Picard group of $X_I - \{x\}$ as follows. Let J be an ideal of $\mathcal{O}_S$ such that J/I has length one. By flatness we get the following exact sequence:

$$\mathcal{O}_{X_m} \to (\mathcal{O}_X/f^*I)^* \to (\mathcal{O}_X/f^*J)^*,$$

where denotes the group of units. This gives rise to

$$\begin{aligned} H^1(X_I - \{x\}, \mathcal{O}_{X_m}) &\to H^1(X_I - \{x\}, (\mathcal{O}_X/f^*I)^*) \\ &\to H^1(X_J - \{x\}, (\mathcal{O}_X/f^*J)^*). \end{aligned}$$

From the local-global cohomology exact sequence we get that

$$H^1(X_m - \{x\}, \mathcal{O}_{X_m}) \cong H_0^2(X_m, \mathcal{O}_{X_m}) = 0,$$

since X_m is smooth of dimension at least three. Thus

$$H^1(X_I - \{x\}, (\mathcal{O}_X/f^*I)^*) \cong H^1(X_m - \{x\}, \mathcal{O}_{X_m}^*) \cong 0,$$

which shows that F^{**} is locally free on any infinitesimal neighborhood of X_m. Therefore it is locally free.

6.14. *Proof of* 6.11. The section σ induces a section $\overline{\overline{\mathcal{P}ic}}_{X/S} \to X \times_S \overline{\overline{\mathcal{P}ic}}_{X/S}$ which we also denote by σ. By [1,II,1.6] there is an open cover $\{U_i\}$ of $\overline{\overline{\mathcal{P}ic}}_{X/S}$ such that a universal family F_i exists over U_i. If σ_i denotes the restriction of σ to U_i, then let

$$F_i^c = F_i \otimes \pi^*(\sigma_i^*(F_i^{**}))^{-1}.$$

By construction we have a canonical isomorphism $\sigma_i^*((F_i^c)^{**}) \cong \mathcal{O}_{U_i}$. Therefore we can patch together the sheaves F_i^c to a single sheaf F^c. This is the unique universal sheaf such that $\sigma^*((F^c)^{**}) \cong \mathcal{O}_{\overline{\overline{\mathcal{P}ic}}_{X/S}}$.

References

[1] A. Altman & S. Kleiman, *Compactifying the Picard scheme.* I, Advances in Math. **35** (1980) 50–112; II, Amer. J. Math. **101** (1979) 10–41.

[2] S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk SSSR **35** (1971) 1269–1293.

[3] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974) 165–189.

[4] ——, *Algebraisation of formal moduli.* II, Ann. of Math. (2) **91** (1970) 88–135.

[5] M. Artin & G. Winters, *Degenerate fibers and stable reduction of curves*, Topology **10** (1971) 373–384.

[6] C. Barton, *Tensor products of ample vector bundles in characteristic p*, Amer. J. Math. **93** (1971) 429–438.

[7] E. Bombieri & D. Mumford, *Enriques' classification of surfaces in characteristic p.* II, Complex Analysis and Algebraic Geometry, Cambridge Univ. Press, 1977, 23–42.

[8] P. Deligne & D. Mumford, *The irreducibility of space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969) 75–110.

[9] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Inst. Hautes Études Sci. Publ. Math. **67** (1989) 97–144.

[10] T. Fujita, *Kaehler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978) 779–794.

[11] R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. **29** (1966) 63–94.

[12] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. (2) **79** (1964) 109–326.

[13] J. Igusa, *Betti and Picard numbers of abstract algebraic surfaces*, Prac. Nat. Acad. Sci. U.S.A. **46** (1960) 724–726.

[14] Y. Kawamata, *Kodaira dimension of algebraic fiberspaces over curves*, Invent. Math. **66** (1982) 57–71.

[15] ——, *The crepant blowing-up of 3-dimensional canonical singularities and its application to the degeneration of surfaces*, Ann. of Math. (2) **127** (1988) 93–163.

[16] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966) 293–344.

[17] F. Knudsen & D. Mumford, *The projectivity of the moduli space of stable curves.* I, Math. Scand. **39** (1976) 19–55.

[18] F. Knudsen, *The projectivity of the moduli space of stable curves*, II–III, Math. Scand. **52** (1983) 161–212.

[19] D. Knutson, *Algebraic spaces*, Lecture Notes in Math., Vol. 203. Springer, Berlin, 1971.

[20] J. Kollár, *Toward moduli of singular varieties*, Compositio Math. **56** (1985) 369–398.

[21] ——, *Higher direct images of dualizing sheaves*. II, Ann. of Math. (2) **124** (1986) 171–202.

[22] ——, *Subadditivity of the Kodaira dimension: Fibers of general type*, Algebraic Geometry, Sendai 1985, Advanced Studies in Pure Math., Vol. 10, Kinokuniya-North Holland, 1987, 361–398.

[23] J. Kollár & N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988) 299–338.

[24] B. Moishezon, *A criterion for projectivity of complete abstract algebraic varieties*, Amer. Math. Soc. Transl. **63** (1967) 1–50.

[25] L. Moret-Bailly, *Familles de Courbes et de Variétés Abeliennes*, Asterisque **86** (1981) 125–140.

[26] D. Mumford, *Stability of projective varieties*, Enseignement Math. **23** (1977) 39–110.

[27] D. Mumford & J. Fogarthy, *Geometric invariant theory*, 2nd edition, Springer, Berlin, 1982.

[28] Y. Nakai, *A criterion of ample sheaf on a projective scheme*, Amer. J. Math. **85** (1963) 14–26.

[29] M. Reid, *Canonical threefolds*, in Géometrie Algébrique Angers (A. Beauville, ed.), Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, 273–310.

[30] N. I. Shepherd-Barron, *Degenerations with numerically effective canonical divisor*, The Birational Theory of Degenerations, Birkhäuser, Boston, 1983, 33–84.

[31] D. van Straten, *Weakly normal surface singularities and their improvements*, Thesis, Leiden, 1987 (to appear).

[32] L. Szpiro, *Propriétés numerique du faisceau dualisant relatif*, Asterisque **86** (1981) 44–78.

[33] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension*. I, Algebraic Varieties and Analytic Varieties, Advanced Studies in Pure Math., Vol. 1, Kinokuniya-North Holland, 1983, 329–353; II, Classification of Algebraic and Analytic Manifolds, Birkhäuser, Boston, 1983, 567–590.

[34] ——, *Weak positivity and the stability of certain Hilbert points*, Invent. Math. **96** (1989), 639–667.

UNIVERSITY OF UTAH