

Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2

J.W.S. Cassels
University of Cambridge

E.V. Flynn
University of Liverpool



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Chapters are divided into sections. Some topics occupy several sections. We give a name to a section and list it here only when it starts a new topic.

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