A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing

ERIK MEIJERING, MEMBER, IEEE

(Encouraged Paper)

This paper presents a chronological overview of the developments in interpolation theory, from the earliest times to the present date. It brings out the connections between the results obtained in different ages, thereby putting the techniques currently used in signal and image processing into historical perspective. A summary of the insights and recommendations that follow from relatively recent theoretical as well as experimental studies concludes the presentation.

Keywords**—***Approximation, convolution-based interpolation, history, image processing, polynomial interpolation, signal processing, splines.*

"It is an extremely useful thing to have knowledge of the true origins of memorable discoveries, especially those that have been found not by accident but by dint of meditation. It is not so much that thereby history may attribute to each man his own discoveries and others should be encouraged to earn like commendation, as that the art of making discoveries should be extended by considering noteworthy examples of it."[1]

I. INTRODUCTION

The problem of constructing a continuously defined function from given discrete data is unavoidable whenever one wishes to manipulate the data in a way that requires information not included explicitly in the data. In this age of ever-increasing digitization in the storage, processing, analysis, and communication of information, it is not difficult to find examples of applications where this problem occurs. The relatively easiest and in many applications often most desired approach to solve the problem is *interpolation*, where an approximating function is constructed in such a way as to agree perfectly with the usually unknown original function at the given measurement points.[2] In view of its increasing relevance, it is only natural that the subject of interpolation is receiving more and more attention these days.[3] However, in times where all efforts are directed toward the future, the past may easily be forgotten. It is no sinecure, scanning the literature, to get a clear picture of the development of the subject through the ages. This is quite unfortunate, since it implies a risk of researchers going over grounds covered earlier by others. History has shown many examples of this and several new examples will be revealed here. The goal of the present paper is to provide a systematic overview of the developments in interpolation theory, from the earliest times to the present date and to put the most well-known techniques currently used in signal and image processing applications into historical perspective. The paper is intended to serve as a tutorial and a useful source of links to the appropriate literature for anyone interested in interpolation, whether it be its history, theory, or applications.

As already suggested by the title, the organization of the paper is largely chronological. Section II presents an

Manuscript received July 19, 2001; revised November 17, 2001. This work was supported by the Netherlands Organization for Scientific Research.

The author is with the Biomedical Imaging Group, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne, Switzerland (e-mail: erik.meijering@epfl.ch).

Publisher Item Identifier S 0018-9219(02)02899-2.

[1]Leibniz, in the opening paragraph of his *Historia et Origo Calculi Differentialis* [1]. The translation given here was taken from a paper by Child [2].

[2]The word "interpolation" originates from the Latin verb *interpolare*, a contraction of "inter," meaning "between," and "polare," meaning "to polish." That is to say, to smooth in between given pieces of information. It seems that the word was introduced in the English literature for the first time around 1612 and was then used in the sense of "to alter or enlarge [texts] by insertion of new matter" [3]. The original Latin word appears [4] to have been used first in a mathematical sense by Wallis in his 1655 book on infinitesimal arithmetic [5].

[3]A search in the multidisciplinary databases of bibliographic information collected by the Institute for Scientific Information in the Web of Science will reveal that the number of publications containing the word "interpolation" in the title, list of keywords, or the abstract has dramatically increased over the past decade, even when taking into account the intrinsic (and likewise dramatic) increase in the number of publications as a function of time.

overview of the earliest known uses of interpolation in antiquity and describes the more sophisticated interpolation methods developed in different parts of the world during the Middle Ages. Next, Section III discusses the origins of the most important techniques developed in Europe during the period of Scientific Revolution, which in the present context lasted from the early 17th until the late 19th century. A discussion of the developments in what could be called the Information and Communication Era, covering roughly the past century, is provided in Section IV. Here, the focus of attention is on the results that have had the largest impact on the advancement of the subject in signal and image processing, in particular on the development of techniques for the manipulation of intensity data defined on uniform grids. Although recently developed alternative methods for specific interpolation tasks in this area will also be mentioned briefly, the discussion in this part of the paper will be restricted mainly to convolution-based methods, which is justified by the fact that these are the most frequently used interpolation methods, probably because of their versatility and relatively low complexity. Finally, summarizing and concluding remarks are made in Section V.

II. Ancient Times and the Middle Ages

In his 1909 book on interpolation [6], Thiele characterized the subject as "the art of reading between the lines in a [numerical] table." Examples of fields in which this problem arises naturally and inevitably are astronomy and, related to this, calendar computation. Because man has been interested in these since day one, it should not surprise us that it is in these fields that the first interpolation methods were conceived. This section discusses the earliest known contributions to interpolation theory.

A. Interpolation in Ancient Babylon and Greece

In antiquity, astronomy was all about time keeping and making predictions concerning astronomical events. This served important practical needs: farmers, e.g., would base their planting strategies on these predictions. To this end, it was of great importance to keep up lists—so-called ephemerides—of the positions of the sun, moon, and the known planets for regular time intervals. Obviously, these lists would contain gaps, due to either atmospherical conditions hampering observation or the fact that celestial bodies may not be visible during certain periods. From his study of ephemerides found on ancient astronomical cuneiform tablets originating from Uruk and Babylon in the Seleucid period (the last three centuries BC), the historian-mathematician Neugebauer [7], [8] concluded that interpolation was used in order to fill these gaps. Apart from linear interpolation, the tablets also revealed the use of more complex interpolation methods. Precise formulations of the latter methods have not survived, however.

An early example of the use of interpolation methods in ancient Greece dates from about the same period. Toomer [9] believes that Hipparchus of Rhodes (190–120 BC) used linear interpolation in the construction of tables of the so-called "chord function" (related to the sine function) for the purpose of computing the positions of celestial bodies. Later examples are found in the *Almagest* ("The Mathematical Compilation," *ca.* 140 AD) of Claudius Ptolemy, the Egypt-born Greek astronomer-mathematician who propounded the geocentric view of the universe which prevailed until the 16th century. Apart from theory, this influential work also contains numerical tables of a wide variety of trigonometric functions defined for astronomical purposes. To avoid the tedious calculations involved in the construction of tables of functions of more than one variable, Ptolemy used an approach that amounts to tabulating the function only for the variable for which the function varies most, given two bounding values of the other variable and to provide a table of coefficients to be used in an "adaptive" linear interpolation scheme for computation of the function for intermediate values of this latter variable [10].

B. Interpolation in Early-Medieval China and India

Analysis of the computational techniques on which early-medieval Chinese ephemerides are based often reveals the use of higher order interpolation formulae.[4] The first person to use second-order interpolation for computing the positions of the sun and the moon in constructing a calendar is said to be the astronomer Liù Zhuó. Around 600 AD, he used this technique in producing the so-called *Huáng jí lì* or "Imperial Standard Calendar." According to Yǎn and Shírán [12], the formula involved in his computations reads in modern notation[5]

$$f(x_0+\xi T) = f(x_0) + \frac{\xi}{2}(\Delta_1+\Delta_2) + \xi(\Delta_1-\Delta_2) - \frac{\xi^2}{2}(\Delta_1-\Delta_2) \quad (1)$$

with $0 \leqslant \xi < 1$, $T > 0$, $\Delta_1 = f(x_0+T) - f(x_0)$ and $\Delta_2 = f(x_0+2T) - f(x_0+T)$ and with $f(x_0)$, $f(x_0+T)$ and $f(x_0+2T)$ the observed results at times x_0, x_0+T and x_0+2T, respectively. This formula is closely related to later Western interpolation formulae, to be discussed in the next section. Methods for second-order interpolation of unequal-interval observations were later used by the astronomer Monk Yì Xíng in producing the so-called "Dà Yǎn Calendar" (727 AD) and by XúÁng in producing the "Xuān Míng Calendar" (822 AD). The latter also used a second-order formula for interpolation of equal-interval observations equivalent to the formula used by Liù Zhuó.

Accurate computation of the motion of celestial bodies, however, requires more sophisticated interpolation techniques than just second order. More complex techniques were later developed by Guō Shōujìng and others. In

[4]The paragraphs on Chinese contributions to interpolation theory are based on the information provided in the books by Martzloff [11] and Yǎn and Shírán [12]. For a more elaborate treatment of the techniques and formulae mentioned here, see the latter. This reference was brought to the author's attention by Phillips in the brief historical notes on interpolation in his recent book [13].

[5]Note that, although supposedly unintended, the actual formula given by Yǎn and Shírán is only valid in cases where the time interval equals one, since the variable is not normalized. The formula given here is identical to theirs, except that we use a normalized variable.

1280 AD, they produced the so-called *Shòu shí lì*, or "Works and Days Calendar" for which they used third-order interpolation. Although they did not write down explicitly third-order interpolation formulae, it follows from their computations recorded in tables that they had grasped the principle.

Important contributions in the area of finite-difference computation were made by the Chinese mathematician Zhū Shìjié. In his book *Sìyuán yùjiàn* ("Jade Mirror of the Four Origins," 1303 AD), he gave the following problem (quoted freely from Martzloff [11]): "Soldiers are recruited in cubes. On the first day, the side of the cube is three. On the following days, it is one more per day. At present, it is 15. Each soldier receives 250 *guan* per day. What is the number of soldiers recruited and what is the total amount paid out?"

In explaining the answer to the first question, Zhū Shìjié gives a sequence of verbal instructions (a "resolutory rule") for finding the solution, which, when cast in modern algebraic notation, reveals the suggestion to use the following formula:

$$\begin{aligned} f(n) = n\Delta_0 &+ \frac{1}{2!}n(n-1)\Delta_0^2 \\ &+ \frac{1}{3!}n(n-1)(n-2)\Delta_0^3 \\ &+ \frac{1}{4!}n(n-1)(n-2)(n-3)\Delta_0^4 \end{aligned} \tag{2}$$

where $f(n)$ is the total number of soldiers recruited in n days and the differences are defined by $\Delta_j = f(j+1) - f(j)$ and $\Delta_j^i = \Delta_{j+1}^{i-1} - \Delta_j^{i-1}$, with $i > 1$ and $j \geqslant 0$ integers. Although the specific problem requires only differences up to fourth order, the proposed formula to solve it can easily be generalized to any arbitrary degree and has close connections with later Western interpolation formulae to be discussed in the next section.

In India, work on higher order interpolation started around the same time as in China.[6] In his work *Dhyānagraha* (*ca.* 625 AD), the astronomer-mathematician Brahmagupta included a passage in which he proposed a method for second-order interpolation of the sine and versed sine functions. Rephrasing the original Sanskrit text in algebraic language, Gupta [15] arrived at the following formula:

$$\begin{aligned} f(x_0 + \xi T) = f(x_0) &+ \frac{\xi}{2}\left\{\Delta f(x_0 - T) + \Delta f(x_0)\right\} \\ &+ \frac{\xi^2}{2}\left\{\Delta f(x_0) - \Delta f(x_0 - T)\right\} \end{aligned} \tag{3}$$

with $\Delta f(x_0) = f(x_0 + T) - f(x_0)$. In a later work, *Khandakhādyaka* (665 AD), Brahmagupta also described a more general method that allowed for interpolation of unequal-interval data. In the case of equal intervals, this method reduces to (3).

Another rule for making second-order interpolations can be found in a commentary on the seventh-century work *Mahābhāskariya* by Bhāskara I, ascribed to Govindasvāmi (*ca.* 800–850 AD). Expressed in algebraic notation, it reads [15]

$$\begin{aligned} f(x_0 + \xi T) &= f(x_0) + \xi \Delta f(x_0) \\ &\quad + \frac{\xi(\xi - 1)}{2}\left\{\Delta f(x_0) - \Delta f(x_0 - T)\right\}. \end{aligned} \tag{4}$$

It is not difficult to see that this formula is equivalent to (3). According to Gupta [15], it is also found in two early 15th-century commentaries by Parameśvara.

C. Late-Medieval Sources on Interpolation

Use of the just described second-order interpolation formulae amounts to fitting a parabola through three consecutive tabular values. Kennedy [16] mentions that parabolic interpolation schemes are also found in several Arabic and Persian sources. Noteworthy are the works *al-Qānūn'l-Mas'ūdi* ("Canon Masudicus," 11th century) by al-Bīrūni and *Zij-i-Khāqāni* (early 15th century) by al-Kāshi. Concerning the parabolic interpolation methods described therein, Gupta [15] and later Rashed [17] have pointed at possible Indian influences, since the important works of Brahmagupta were translated into Arabic as early as the eighth century AD. Not to mention the fact that al-Bīrūni himself travelled through and resided in several parts of India, studied Indian literature in the original, wrote a book about India, and translated several Sanskrit texts into Arabic [18].

III. The Age of Scientific Revolution

Apparently, totally unaware of the important results obtained much earlier in other parts of the world, interpolation theory in Western countries started to develop only after a great revolution in scientific thinking. Especially the new developments in astronomy and physics, initiated by Copernicus, continued by Kepler and Galileo and culminating in the theories of Newton, gave strong impetus to the further advancement of mathematics, including what is now called "classical" interpolation theory.[7] This section highlights the most important contributions to interpolation theory in Western countries until the beginning of the 20th century.

A. General Interpolation Formula for Equidistant Data

Before reviewing the different classical interpolation formulae, we first study one of the better known.[8] Suppose that we are given measurements of some quantity at x_0, $x_0 \pm T$, $x_0 \pm 2T, \ldots$ and that in order to obtain its value at any intermediate point $x_0 + \xi T$, $\xi \in \mathbb{R}$, we locally model it as a poly-

[6] Martzloff [11], referring to Cullen [14], conjectures that this may not be a coincidence, since it was the time when Indian and Chinese astronomers were working together at the court of the Táng.

[7] In constructing the chronology of classical interpolation formulae presented in this section, the interesting—though individually incomplete—accounts given by Fraser [19], Goldstine [20], Joffe [21], and Turnbull [22] have been most helpful.

[8] This section includes explicit formulae only insofar as necessary to demonstrate the link with those in the previous or next section. For a more detailed treatment of these and other formulae, including such aspects as accuracy and implementation, see several early works on interpolation [19], [21], [23]–[26] and the calculus of finite differences [27]–[30], as well as more general books on numerical analysis [20], [31]–[35], most of which also discuss inverse interpolation and the role of interpolation in numerical differentiation and integration.

nomial $f : \mathbb{R} \to \mathbb{R}$ of given degree $n \in \mathbb{N}$, i.e., $f(x_0+\xi T) = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n$. It is easy to show [23] that any such polynomial can be written in terms of factorials $[\xi]^k = \xi(\xi-1)(\xi-2)\cdots(\xi-k+1)$, with $k > 0$ integer, as $f(x_0+\xi T) = c_0 + c_1[\xi] + c_2[\xi]^2 + \cdots + c_n[\xi]^n$. If we now define the first-order difference of any function $\phi : \mathbb{R} \to \mathbb{R}$ at any ξ as $\Delta\phi(\xi) = \phi(\xi+1) - \phi(\xi)$ and similarly the higher order differences as $\Delta^p\phi(\xi) = \Delta^{p-1}\phi(\xi+1) - \Delta^{p-1}\phi(\xi)$, for all $p > 1$ integer, it follows that $\Delta[\xi]^k = k[\xi]^{k-1}$. By repeated application of the difference operator Δ to the factorial representation of $f(x_0+\xi T)$ and taking $\xi = 0$, we find that the coefficients c_k, $k = 0, 1, \ldots, n$ can be expressed as $c_k = \Delta^k f(x_0)/k!$ so that if n could be made arbitrarily large, we would have

$$\begin{aligned} f(x_0+\xi T) =& f(x_0) + \xi\Delta f(x_0) \\ &+ \frac{1}{2!}\xi(\xi-1)\Delta^2 f(x_0) \\ &+ \frac{1}{3!}\xi(\xi-1)(\xi-2)\Delta^3 f(x_0) \\ &+ \frac{1}{4!}\xi(\xi-1)(\xi-2)(\xi-3)\Delta^4 f(x_0) + \cdots. \end{aligned} \tag{5}$$

This general formula[9] was first written down in 1670 by Gregory and can be found in a letter by him to Collins [39]. Particular cases of it, however, had been published several decades earlier by Briggs,[10] the man who brought to fruition the work of Napier on logarithms. In the introductory chapters to his major works [41], [42], he described the precise rules by which he carried out his computations, including interpolations, in constructing the tables contained therein. In the first, e.g., he described a subtabulation rule that, when written in algebraic notation, amounts to (5) for the case when third- and higher order differences are negligible [20], [22]. It is known [20], [43] that still earlier, around 1611, Harriot used a formula equivalent to (5) up to fifth-order differences.[11] However, even he was not the first in the world to write down such rules. It is not difficult to see that the right-hand side of the second-order interpolation formula (1) used by Liù Zhuó can be rewritten so that it equals the first three terms of (5) and, if we replace the integer argument n by the real variable x in Zhū Shìjié's formula (2), we obtain at once (5) for the case when $x_0 = 0$, $f(0) = 0$, and $T = 1$.

[9]It is interesting to note that Taylor [36] obtained his now well-known series as a simple corollary to (5). It follows, indeed, by substituting $\xi = h/T$ and taking $T \to 0$. (It is important to realize here that $f(x_0)$ is in fact $f(x_0+\xi T)|_{\xi=0}$, so that $\Delta f(x_0) = f(x_0+T) - f(x_0)$ and similar for the higher order differences.) The special version for $x_0 = 0$ was later used by Maclaurin [37] as a fundamental tool. See also e.g., Kline [38].

[10]Gregory was also aware of a later method by Mercator, for in his letter he refers to his own method as being "both more easie and universal than either Briggs or Mercator's" [39]. In France, it was Mouton who used a similar method around that time [40].

[11]Goldstine [20] speculates that Briggs was aware of Harriot's work on the subject and is inclined to refer to the formula as the Harriot-Briggs relation. Neither Harriot nor Briggs, however, ever explained how they obtained their respective rules and it has remained unclear up till today.

B. *Newton's General Interpolation Formulae*

Notwithstanding these facts, it is justified to say that "there is no single person who did so much for this field, as for so many others, as Newton" [20]. His enthusiasm becomes clear in a letter he wrote to Oldenburg [44], where he first describes a method by which certain functions may be expressed in series of powers of x and then goes on to say[12]: "But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem. *To describe a geometrical curve which shall pass through any given points*… Although it may seem to be intractable at first sight, it is nevertheless quite the contrary. Perhaps indeed it is one of the prettiest problems that I can ever hope to solve."

The contributions of Newton to the subject are contained in: 1) a letter [45] to Smith in 1675; 2) a manuscript entitled *Methodus Differentialis* [46], published in 1711, although earlier versions were probably written in the middle 1670s; 3) a manuscript entitled *Regula Differentiarum*, written in 1676, but first discovered and published in the 20th century [19], [47]; and 4) Lemma V in Book III of his celebrated *Principia* [48], which appeared in 1687.[13] The latter was published first and contains two formulae. The first deals with equal-interval data and is precisely (5), which Newton seems to have discovered independently of Gregory.[14] The second formula deals with the more general case of arbitrary-interval data and may be derived as follows.

Suppose that the values of the aforementioned quantity are given at $x_0, x_{\pm1}, x_{\pm2}, \ldots$, which may be arbitrary and that in order to obtain its value at intermediate points we model it again as a polynomial function $f : \mathbb{R} \to \mathbb{R}$. If we then define the first-order *divided* difference of any function $\phi : \mathbb{R} \to \mathbb{R}$ for any two $\xi_0 \neq \xi_1$ as $\phi(\xi_0, \xi_1) = \big(\phi(\xi_0) - \phi(\xi_1)\big)/(\xi_0 - \xi_1)$, it follows that the value of f at any $x \in \mathbb{R}$ could be written as $f(x) = f(x_0) + (x - x_0)f(x, x_0)$. If we define the higher order divided differences[15] as $\phi(\xi_0, \ldots, \xi_p) = \big(\phi(\xi_0, \ldots, \xi_{p-1}) - \phi(\xi_1, \ldots, \xi_p)\big)/(\xi_0 - \xi_p)$, for all $p > 1$ integer, we can substitute for $f(x, x_0)$ the expression that follows from the definition of $f(x, x_0, x_1)$ and subsequently for $f(x, x_0, x_1)$ the expression that follows from the definition of $f(x, x_0, x_1, x_2)$, etc., so that if we could go on, we would have

$$\begin{aligned} f(x) =& f(x_0) + (x - x_0)f(x_0, x_1) \\ &+ (x - x_0)(x - x_1)f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + \cdots. \end{aligned}$$

[12]The somewhat free translation from the original Latin is from Fraser [19] and differs, although not fundamentally, from that given by Turnbull [44].

[13]All of these are reproduced (whether or not translated) and discussed in a booklet by Fraser [19].

[14]This is probably why it is nowadays usually referred to as the Gregory–Newton formula. There is reason to suspect, however, that Newton must have been familiar with Briggs' works [19].

[15]Although Newton appears to have been the first to use these for interpolation, he did not call them "divided differences." It has been said [23] that it was De Morgan [49] who first used the term.

(6)

It is this formula that can be considered the most general of all classical interpolation formulae. As we will see in the sequel, all later formula can easily be derived from it.[16]

C. Variations of Newton's General Interpolation Formulae

The presentation of the two interpolation formulae in the *Principia* is heavily condensed and contains no proofs. Newton's *Methodus Differentialis* contains a more elaborate treatment, including proofs and several alternative formulae. Three of those formulae for equal-interval data were discussed a few years later by Stirling [50].[17] These are the Gregory–Newton formula and two central-difference formulae, the first of which is now known as the Newton-Stirling formula[18]

$$\begin{aligned} f(x_0+\xi T) =& f(x_0)+\xi\frac{\Delta f(x_0)+\Delta f(x_0-T)}{2} \\ &+\frac{1}{2!}\xi^2\Delta^2 f(x_0-T) \\ &+\frac{1}{3!}\xi\left(\xi^2-1^2\right)\frac{\Delta^3 f(x_0-T)+\Delta^3 f(x_0-2T)}{2} \\ &+\frac{1}{4!}\xi^2\left(\xi^2-1^2\right)\Delta^4 f(x_0-2T)+\cdots. \end{aligned} \quad (7)$$

It is interesting to note that Brahmagupta's formula (3) is, in fact, the Newton-Stirling formula for the case when the third- and higher order differences are zero.

A very elegant alternative representation of Newton's general formula (6) that does not require the computation of finite or divided differences was published in 1779 by Waring [52]

$$\begin{aligned} f(x) =& f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)\cdots}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots} \\ &+f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)\cdots}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots} \\ &+f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)\cdots}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots}+\cdots. \end{aligned} \quad (8)$$

It is nowadays usually attributed to Lagrange who, in apparent ignorance of Waring's paper, published it 16 years later [53]. The formula may also be obtained from a closely related representation of Newton's formula due to Euler [54]. According to Joffe [21], it was Gauss who first noticed the logical connection and proved the equivalence of the formulae by Newton, Euler, and Waring–Lagrange, as appears from his posthumous works [55], although Gauss did not refer to his predecessors.

In 1812, Gauss delivered a lecture on interpolation, the substance of which was recorded by his then student, Encke, who first published it not until almost two decades later [56]. Apart from other formulae, he also derived the one which is now known as the Newton-Gauss formula

$$\begin{aligned} f(x_0+\xi T) =& f(x_0)+\xi\Delta f(x_0) \\ &+\frac{1}{2!}\xi(\xi-1)\Delta^2 f(x_0-T) \\ &+\frac{1}{3!}(\xi+1)\xi(\xi-1)\Delta^3 f(x_0-T) \\ &+\frac{1}{4!}(\xi+1)\xi(\xi-1)(\xi-2)\Delta^4 f(x_0-2T)+\cdots. \end{aligned} \quad (9)$$

It is this formula[19] that formed the basis for later theories on sampling and reconstruction, as will be discussed in the next section. Note that this formula too had its precursor, in the form of Govindasvāmi's rule (4).

In the course of the 19th century, two more formulae closely related to (9) were developed. The first appeared in a paper by Bessel [57] on computing the motion of the moon and was published by him because, in his own words, he could "not recollect having seen it anywhere." The formula is, however, equivalent to one of Newton's in his *Methodus Differentialis*, which is the second central-difference formula discussed by Stirling [50] and has, therefore, been called the Newton–Bessel formula. The second formula, which has frequently been used by statisticians and actuaries, was developed by Everett [58], [59] around 1900 and reads

$$f(x_0+\xi T) = F(\xi,\delta)f(x_0+T)+F(1-\xi,\delta)f(x_0), \quad (10)$$

where

$$\begin{aligned} F(\xi,\delta) = \xi &+ \frac{1}{3!}\xi(\xi^2-1^2)\delta^2 \\ &+\frac{1}{5!}\xi(\xi^2-1^2)(\xi^2-2^2)\delta^4+\cdots \end{aligned} \quad (11)$$

and use has been made of Sheppard's central-difference operator δ, defined by $\delta\phi(\xi)=\phi(\xi+1/2)-\phi(\xi-1/2)$ and $\delta^p\phi(\xi)=\delta^{p-1}\phi(\xi+1/2)-\delta^{p-1}\phi(\xi-1/2)$, $p>1$ integer, for any function $\phi:\mathbb{R}\to\mathbb{R}$ at any ξ. The elegance of this formula lies in the fact that, in contrast with the earlier mentioned formulae, it involves only the even-order differences of the two table entries between which to interpolate.[20] It was noted later by Joffe [21] and Lidstone [61] that the formulae of Bessel and Everett had alternatively been proven by Laplace by means of his method of *generating functions* [62], [63].

[16]Equation (5), e.g., follows by substituting $x_1=x_0+T$, $x_2=x_0+2T$, $x_3=x_0+3T,\ldots$, and $x=x_0+\xi T$ and rewriting the divided differences $f(x_0,\ldots,x_k)$ in terms of finite differences $\Delta^k f(x_0)$.

[17]Newton's general formula was treated by him in his 1730 booklet [51] on the subject.

[18]The formula may be derived from (6) by substituting $x_1=x_0+T$, $x_2=x_0-T$, $x_3=x_0+2T$, $x_4=x_0-2T,\ldots$, and $x=x_0+\xi T$, rewriting the divided differences $f(x_0,\ldots,x_k)$ in terms of finite differences $\Delta^k f(x_0)$, and rearranging the terms [23].

[19]Even more easily than the Newton-Stirling formula, this formula follows from (6) by proceeding in a similar fashion as in the previous footnote.

[20]This is achieved by expanding the odd-order differences in the Newton-Gauss formula according to their definition and rearranging the terms after simple transformations of the binomial coefficients [23]. Alternatively, we could expand the even-order differences so as to end up with only odd-order differences. The resulting formula appears to have been described first by Steffensen [25] and is, therefore, sometimes referred to as such [31], [60], although he himself calls it Everett's second interpolation formula.

D. Studies on More General Interpolation Problems

By the beginning of the 20th century, the problem of interpolation by finite or divided differences had been studied by astronomers, mathematicians, statisticians, and actuaries,[21] and most of the now well-known variants of Newton's original formulae had been worked out. This is not to say, however, that there are no more advanced developments to report on. Quite to the contrary. Already in 1821, Cauchy [65] studied interpolation by means of a ratio of two polynomials and showed that the solution to this problem is unique, the Waring–Lagrange formula being the special case for the second polynomial equal to one.[22] Generalizations for solving the problem of multivariate interpolation in the case of fairly arbitrary point configurations began to appear in the second half of the 19th century, in the works of Borchardt and Kronecker [68]–[70].

A generalization of a different nature was published in 1878 by Hermite [71], who studied and solved the problem of finding a polynomial of which also the first few derivatives assume prespecified values at given points, where the order of the highest derivative may differ from point to point. In a paper [72] published in 1906, Birkhoff studied the even more general problem: given any set of points, find a polynomial function that satisfies prespecified criteria concerning its value and/or the value of any of its derivatives for each individual point.[23] Hermite and Birkhoff type of interpolation problems—and their multivariate versions, not necessarily on Cartesian grids—have received much attention in the past decades. A more detailed treatment is outside the scope of this paper, however, and the reader is referred to relevant books and reviews [70], [79]–[83].

[21]Many of them introduced their own system of notation and terminology, leading to confusion and researchers reformulating existing results. The point was discussed by Joffe [21], who also made an attempt to standardize yet another system. It is, however, Sheppard's notation [64] for central and mean differences that has survived in most later publications.

[22]It was Cauchy also who, in 1840, found an expression for the error caused by truncating finite-difference interpolation series [66]. The absolute value of this so-called Cauchy remainder term can be minimized by choosing the abscissae as the zeroes of the polynomials introduced later by Tchebychef [67]. See, e.g., Davis [26], Hildebrand [31], or Schwarz [34] for more details.

[23]Birkhoff interpolation, also known as *lacunary interpolation*, initially received little attention, until Schoenberg [73] revived interest in the subject. The problem has since usually been stated in terms of the pair (E, X), where $X = \{x_i\}_{i=0}^m$ is the set of points or nodes and $E = [e_{i,j}]_{i=0,\,j=0}^{m,\,k}$ is the so-called *incidence matrix* or *interpolation matrix*, with $e_{i,j} = 1$ for those i and j for which the interpolating polynomial P is to satisfy a given criterion $P^{(j)}(x_i) = c_{i,j}$ and $e_{i,j} = 0$, otherwise. Several special cases had been studied earlier and carry their own name: if E is an $(m+1) \times 1$ column matrix with $e_{i,0} = 1$ for all $i = 0, \ldots, m$, we have the Waring–Lagrange interpolation problem. If, on the other hand, it is a $1 \times (k+1)$ row matrix with $e_{0,j} = 1$, for all $j = 0, \ldots, k$, we may speak of a Taylor interpolation problem [26]. If E is an $(m+1) \times (k+1)$ matrix with $e_{i,j} = 1$, for all $i = 0, \ldots, m$, and $j = 0, \ldots, k_i$, with $k_i \leqslant k$, we have Hermite's interpolation problem, where the case $k_i = k$ for all $i = 0, \ldots, m$, with usually $k = 1$, is also called *osculatory* or *osculating interpolation* [25], [26], [31]. The problem corresponding to $E = I$, with I the $(m+1) \times (m+1)$ unit matrix, was studied by Abel [74] and later by Gontcharoff and others [26], [75], [76] and, finally, we mention the two-point interpolation problem studied by Lidstone [26], [76]–[78] for which E is a $2 \times (k+1)$ matrix with $e_{i,j} = 1$ for all j even.

E. Approximation Versus Interpolation

Another important development from the late 1800s is the rise of approximation theory. For a long time, one of the main reasons for the use of polynomials had been the fact that they are simply easy to manipulate, e.g., to differentiate or integrate. In 1885, Weierstrass [84] also *justified* their use for approximation by establishing the so-called *approximation theorem*, which states that every continuous function on a closed interval can be approximated uniformly to any prescribed accuracy by a polynomial.[24] The theorem does not provide any means of obtaining such a polynomial, however, and it soon became clear that it does not necessarily apply if the polynomial is forced to agree with the function at given points within the interval, i.e., in the case of an interpolating polynomial.

Examples of meromorphic functions for which the Waring–Lagrange interpolator does not converge uniformly were given by Méray [86], [87] and later Runge [88]—especially the latter has become well known and can be found in most modern books on the topic. A more general result is due to Faber [89], who, in 1914, showed that for any prescribed triangular system of interpolation points there exists a continuous function for which the corresponding Waring–Lagrange interpolation process carried out on these points does not converge uniformly to this function. Although it has later been proven possible to construct interpolating polynomials that *do* converge properly for all continuous functions, e.g., by using the Hermite type of interpolation scheme proposed by Fejér [90] in 1916, these findings clearly revealed the "inflexibility" of algebraic polynomials and their limited applicability to interpolation.

IV. The Information and Communication Era

When Fraser [19], in 1927, summed up the state of affairs in classical interpolation theory, he also expressed his expectations concerning the future and speculated: "The 20th century will no doubt see extensions and developments of the subject of interpolation beyond the boundaries marked by Newton 250 years ago." This section gives an overview of the advances in interpolation theory in the past century, proving that Fraser was right. In fact, he was so right that our rendition of this part of history is necessarily of a more limited nature than the expositions in the previous sections. After having given an overview of the developments that led to the two most important theorems on which modern interpolation theory rests, we focus primarily on their later impact on signal and image processing.

A. From Cardinal Function to Sampling Theory

In his celebrated 1915 paper [91], Whittaker noted that given the values of a function f corresponding to an infinite number of equidistant values of its argument x_0, $x_0 \pm T$, $x_0 \pm 2T$, ... from which we can construct a table of differences for interpolation, there exist many other functions which give rise to exactly the same difference table. He then

[24]For more detailed information on the development of approximation theory, see the recently published historical review by Pinkus [85].

considered the Newton-Gauss formula (9) and set out to answer the question of which one of the cotabular functions is represented by it. The answer, he proved, is that under certain conditions it represents the *cardinal function*

$$C(x) = \sum_{k=-\infty}^{+\infty} f(x_0 + kT) \frac{\sin \frac{\pi}{T}(x - x_0 - kT)}{\frac{\pi}{T}(x - x_0 - kT)} \qquad (12)$$

which he observed to have the remarkable properties that, apart from the fact that it is cotabular with the original function since $C(x_0 + kT) = f(x_0 + kT)$, for all $k \in \mathbb{Z}$, it has no singularities and all "constituents" of period less than $2T$ are absent.

Although Whittaker did not refer to any earlier works, it is now known[25] that the series (12) with $x_0 = 0$ and $T = 1$ had essentially been given as early as 1899 by Borel [94], who had obtained it as the limiting case of the Waring–Lagrange interpolation formula.[26] Borel, however, did not establish the "band-limited" nature of the resulting function nor did Steffensen in a paper [100] published in 1914, in which he gave the same formula as Borel, though he referred to Hadamard [102] as his source. De la Vallée Poussin [103], in 1908, studied the closely related case where the summation is over a finite interval, but the number of known function values in that interval goes to infinity by taking $T = \pi/m$ and $m \to \infty$. In contrast with the Waring–Lagrange polynomial interpolator, which may diverge as we have seen in the previous section, he found that the resulting interpolating function converges to the original function at any point in the interval where that function is continuous and of bounded variation.

The issue of convergence was an important one in subsequent studies of the cardinal function. In establishing the equivalence of the cardinal function and the function obtained by the Newton-Gauss interpolation formula, Whittaker had assumed convergence of both series expansions. Later authors showed, by particular examples, that the former may diverge when the latter converges. The precise relationship was studied by Ferrar [104], who showed that when the series of type (12) converges, (9) also converges and has the same sum. If, on the other hand, (9) is convergent, then (12) is either convergent or has a generalized sum in the sense used by de la Vallée Poussin for Fourier series [105], [106]. Concerning the convergence of the cardinal series, Ferrar [96], [104],[107] and later Whittaker [76], [108], [109] studied several criteria. Perhaps the most important is that of $\sum_{k \neq 0} |s_k/k| < \infty$, where $s_k = f(x_0 + kT)$, being a sufficient condition for having absolute convergence—a criterion that had also been given by Borel [94]. It was Whittaker [76], [109] who gave more refined statements as to the relation between the cardinal series and the truncated Fourier integral representation of a function in the case of convergence—results that also relate to the property called by Ferrar the "consistency" of the series [76], [96], [107],[109], which implies the possibility of reproducing the cardinal function as given in (12) by using its values $C(x_0' + kT')$, with $0 < T' < T$.

It must have been only shortly after publication of Whittaker's works [76], [108], [109] on the cardinal series that Shannon recognized their evident importance to the field of communication. He formulated the now well-known *sampling theorem*, which he first published [110] without proof in 1948 and the subsequent year with full proof in a paper [111] apparently written already in 1940: "If a function f contains no frequencies higher than W cps [cycles per second], it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart." Later on in the paper, he referred to the critical sampling interval $T = 1/2W$ as the *Nyquist interval* corresponding to the band W, in recognition of Nyquist's discovery [112] of the fundamental importance of this interval in connection with telegraphy. In describing the reconstruction process, he pointed out that "There is one and only one function whose spectrum is limited to a band W and which passes through given values at sampling points separated $1/2W$ seconds apart. The function can be simply reconstructed from the samples by using a pulse of the type $\sin(2\pi Wx)/2\pi Wx\ldots$ Mathematically, this process can be described as follows. Let s_k be the kth sample. Then the function f is represented by

$$f(x) = \sum_{k=-\infty}^{\infty} s_k \frac{\sin \pi(2Wx - k)}{\pi(2Wx - k)}." \qquad (13)$$

As pointed out by Higgins [92], the sampling theorem should really be considered in two parts, as done above: the first stating the fact that a bandlimited function is completely determined by its samples, the second describing how to reconstruct the function using its samples. Both parts of the sampling theorem were given in a somewhat different form by Whittaker [76], [108], [109] and before him also by Ogura [113], [114]. They were probably not aware of the fact that the first part of the theorem had been stated as early as 1897 by Borel [115].[27] As we have seen, Borel also used around that time what became known as the cardinal series. However, he appears not to have made the link [92]. In later years, it became known that the sampling theorem had been presented before Shannon to the Russian communication community by Kotel'nikov [118]. In more implicit verbal form, it had also been described in the German literature by Raabe [119]. Several authors [120], [121] have mentioned that Someya [122] introduced the theorem in the Japanese

[25] For more information on the development of sampling theory, the reader is referred to the historical accounts given by Higgins [92] and Butzer and Stens [93].

[26] Since Borel [94], the equivalence of the Waring–Lagrange interpolation formula and (12) in the case of infinitely many known function values between which to interpolate has been pointed out and proven by many authors [76], [92], [93], [95]–[99]. Apart from the Newton-Gauss formula, as shown by Whittaker [91], equivalence also holds for other classical interpolation formulae, such as Newton's divided difference formula [76], [100] or the formulae by Everett *et al.* [101], discussed in the previous section. Given Borel's result, this is an almost trivial observation, since all classical schemes yield the exact same polynomial for a given set of known function values, irrespective of their number.

[27] Several authors, following Black [116], have claimed that this first part of the sampling theorem was stated even earlier by Cauchy in a paper [117] published in 1841. However, the paper of Cauchy does not contain such a statement, as has been pointed out by Higgins [92].

literature parallel to Shannon. In the English literature, Weston [123] introduced it independently of Shannon around the same time.[28]

B. From Osculatory Interpolation Problems to Splines

Having arrived at this point, we go back again more than half a century to follow a parallel development of quite different nature. It is clear that practical application of any of the classical polynomial interpolation formulae discussed in the previous section implies taking into account only the first few of the infinitely many terms. In most situations, it will be computationally prohibitive to consider all or even a large number of known function values when computing an interpolated value. Keeping the number of terms fixed implies fixing the degree of the polynomial curves resulting in each interpolation interval. Irrespective of the degree, however, the composite piecewise polynomial interpolant will generally not be continuously differentiable at the transition points.

The need for smoother interpolants in some applications led in the late 1800s to the development of so-called *osculatory interpolation* techniques, most of which appeared in the actuarial literature [126]–[128]. A well-known example of this is the formula proposed in 1899 by Karup [129] and independently described by King [130] a few years later, which may be obtained from Everett's general formula (10) by taking

$$F(\xi,\delta) = \xi + \frac{1}{2}\xi^2(\xi-1)\delta^2 \tag{14}$$

and results in a piecewise third-degree polynomial interpolant which is continuous and, in contrast with Everett's third-degree interpolant, is also continuously differentiable everywhere.[29] By using this formula, it is possible to reproduce polynomials up to second degree. Another example is the formula proposed in 1906 by Henderson [136], which may be obtained from (10) by substituting

$$F(\xi,\delta) = \xi + \frac{1}{6}\xi(\xi^2-1)\delta^2 - \frac{1}{12}\xi^2(\xi-1)\delta^4 \tag{15}$$

[28]As a consequence of the discovery of the several independent introductions of the sampling theorem, people started to refer to the theorem by including the names of the aforementioned authors, resulting in such catchphrases as "the Whittaker-Kotel'nikov-Shannon sampling theorem" [124] or even "the Whittaker-Kotel'nikov-Raabe-Shannon-Someya sampling theorem" [121]. To avoid confusion, perhaps the best thing to do is to refer to it as the *sampling theorem*, "rather than trying to find a title that does justice to all claimants" [125].

[29]The word "osculatory" originates from the Latin verb *osculari*, which literally means "to kiss" and can be translated here as "joining smoothly." Notice that the meaning of the word in this context is more general than in Footnote 23: Hermite interpolation may be considered that type of osculatory interpolation where the derivatives up to some degree are not only supposed to be continuous everywhere but are also required to assume prespecified values at the sample points. It is especially this latter type of interpolation problem to which the adjective "osculatory" has been attached in later publications [131]–[135]. For a more elaborate discussion of osculatory interpolation in the original sense of the word, see several survey papers [126]–[128].

and also yields a continuously differentiable piecewise third-degree polynomial interpolant, but is capable of reproducing polynomials up to third degree. A third example is the formula published in 1927 by Jenkins [137], obtained from (10) by taking

$$F(\xi,\delta) = \xi + \frac{1}{6}\xi(\xi^2-1)\delta^2 - \frac{1}{36}\xi^3\delta^4. \tag{16}$$

The first and second term of this function are equal to those of Henderson's and Everett's function, but the third term is chosen in such a way that the resulting composite curve is a piecewise third-degree polynomial, which is *twice* continuously differentiable. The price to pay, however, is that this curve is not an interpolant.

The need for practically applicable methods for interpolation or smoothing of empirical data also formed the impetus to Schoenberg's study of the subject. In his 1946 landmark paper [138], [139], he noted that for every osculatory interpolation formula applied to equidistant data, where he assumed the distance to be unity, there exists an even function $\Phi : \mathbb{R} \to \mathbb{R}$ in terms of which the formula may be written as

$$f(x) = \sum_{k=-\infty}^{\infty} y_k\, \Phi(x-k) \tag{17}$$

where Φ, which he termed the *basic function* of the formula, completely determines the properties of the resulting interpolant and reveals itself when applying the initial formula to the impulse sequence defined by $y_0 = 1$ and $y_k = 0$, $\forall k \neq 0$. By analogy with Whittaker's cardinal series (12), Schoenberg referred to the general expression (17) as a formula of the *cardinal type*, but noted that the basic function $\Phi(x) = \sin(\pi x)/\pi x$ is inadequate for numerical purposes due to its excessively low damping rate. The basic functions involved in Waring–Lagrange interpolation, on the other hand, possess the limiting property of being at most continuous, but not continuously differentiable. He then pointed at the smooth curves obtained by the use of a mechanical spline,[30] argued that these are piecewise cubic arcs with a continuous first- and second-order derivative and continued to introduce the notion of the mathematical spline: "A real function f defined for all real x is called a spline curve of order L and denoted by $\mathcal{S}_L$ if it enjoys the following properties: 1) it is composed of polynomial arcs of degree at most $L-1$; 2) it is of class C^{L-2}, i.e., f has $L-2$ continuous derivatives; 3) the only possible function points of the various polynomial arcs are the integer points $x = L$ if L is even, or else the points $x = L + 1/2$ if L is odd." Notice that these requirements are satisfied by the curves resulting from the aforementioned smoothing formula proposed by Jenkins and also studied by Schoenberg [138], which constitutes one of the earliest examples of a spline generating formula.

[30]The word "spline" can be traced back to the 18th century, but by the end of the 19th century was used to refer to "a flexible strip of wood or hard rubber used by draftsmen in laying out broad sweeping curves" [3]. Such mechanical splines were used, e.g., to draw curves needed in the fabrication of cross sections of ships' hulls. Drucks or weights were placed on the strip to force it to go through given points and the free portion of the strip would assume a position in space that minimized the bending energy [140].

After having given the definition of a spline curve, Schoenberg continued to prove that "any spline curve $\mathcal{S}_L$ may be represented in one and only one way in the form

$$\mathcal{S}_L(x) = \sum_{k=-\infty}^{\infty} y_k \, M_L(x-k) \tag{18}$$

for appropriate values of the coefficients y_k. There are no convergence difficulties since $M_L(x)$ vanishes for $|x| > L/2$. Thus, (18) represents an $\mathcal{S}_L$ for arbitrary $\{y_k\}$ and represents the most general one." Here, $M_L : \mathbb{R} \to \mathbb{R}$ denotes the so-called B-spline of degree $n = L-1$, which he had defined earlier in the paper as the inverse Fourier integral

$$M_L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}} \right)^L e^{i\omega x} d\omega \tag{19}$$

and, equivalently, also as[31]

$$M_L(x) = \frac{1}{(L-1)!} \delta^L x_+^{L-1} \tag{20}$$

where δ^p is again the pth-order central difference operator and x_+^n denotes the one-sided power function defined as

$$x_+^n \triangleq \begin{cases} x^n, & \text{if } x \geqslant 0 \\ 0, & \text{if } x < 0. \end{cases} \tag{21}$$

C. Convolution-Based Function Representation

Although there are certainly differences, it is interesting to look at the similarities between the theorems described by Shannon and Schoenberg: both of them involve the definition of a class of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying certain properties and both involve the representation of these functions by a mixed convolution of a set of coefficients c_k with some basic function or *kernel* $\varphi : \mathbb{R} \to \mathbb{R}$, according to the formula

$$f_T(x) = \sum_{k \in \mathbb{Z}} c_k \varphi\left(\frac{x}{T} - k\right) \tag{22}$$

where the subscript T, the sampling interval, is now added to stress the fact that we are dealing with a representation—or, depending on φ, perhaps only an approximation—of any such original function f based on its T-equidistant samples. In the case of Shannon, the f are bandlimited functions, the coefficients c_k are simply the samples $s_k = f(kT)$ and the kernel is the sinc function,[32] defined as $\text{sinc}(x) \triangleq \sin(\pi x)/\pi x$. In Schoenberg's theorem, the functions f are piecewise polynomials of degree n, which join smoothly according to the definition of a spline, the coefficients c_k are computed from the samples s_k and the kernel is the nth degree B-spline.[33]

In the decades to follow, both Shannon's and Schoenberg's paper would prove most fruitful, but largely in different fields. The former had great impact on communication engineering [144]–[147], numerous signal processing and analysis applications [98], [124], [125], [148]–[150] and to some degree also numerical analysis [151]–[154]. Splines, on the other hand and after some two decades of further study by Schoenberg [155]–[157], found their way into approximation theory [158]–[164], mono- and multivariate interpolation [81], [165]–[167], numerical analysis [168], statistics [140], and other branches of mathematics [169]. With the advent of digital computers, splines had a major impact on geometrical modeling and computer-aided geometric design [170]–[174], computer graphics [175], [176], and even font design [177] to mention but a few practical applications. In the remainder of this section, we will focus primarily on the further developments in signal and image processing.

D. Convolution-Based Interpolation in Signal Processing

When using Waring–Lagrange interpolation, the choice for the degree n of the resulting polynomial pieces fixes the number of samples to be used in any interpolation interval to $n+1$. There is still freedom, however, to choose the position of the interpolation intervals, the end points of which constitute the transition points of the polynomial pieces of the interpolant, relative to the sample intervals. It is easy to see that if they are chosen to coincide with the sample intervals, there are n possibilities of choosing the position—in the sequence of samples to be used—of the two samples making up the interpolation interval and that each of these possibilities gives rise to a different impulse response or kernel.

In their 1973 study [178] of these kernels for use in digital signal processing, Schafer and Rabiner concluded that the only ones that are symmetrical and, thus, do not introduce phase distortions are those corresponding to the cases where n is odd and the number of constituent samples on either side of the interpolation interval is the same. It must be pointed out, however, that their conclusion does not hold in general, but is a consequence of letting the interpolation and sample intervals coincide. If the interpolation intervals are chosen according to the parity of n, as in the aforementioned definition of the mathematical spline, then the kernels corresponding to Waring–Lagrange central interpolation for even n will also be symmetrical. Examples of these had already been given by Schoenberg [138]. Schafer and Rabiner also studied the spectral properties of the odd-degree kernels, concluding that the higher order kernels possess considerably better low-pass properties than linear interpolation and

[31] In his original 1946 paper [138], [139], Schoenberg referred to the M_L strictly as "basic functions" or "basic spline curves." The abbreviation "B-splines" was first coined by him twenty years later [141]. It is also interesting here to point at the connection with probability theory: as is clear from (19), $M_L(x)$ can be written as the n-fold convolution of the indicator function $M_1(x)$ with itself from which it follows that a B-spline of degree n represents the probability density function of the sum of $L = n+1$ independent random variables with uniform distribution in the interval $[-1/2, 1/2]$. The explicit formula of this function was known as early as 1820 by Laplace [63] and is essentially (20), as also acknowledged by Schoenberg [138]. For further details on the history of B-splines, see, e.g., Butzer [142].

[32] The term "sinc" is usually held to be short for the Latin *sinus cardinalis* [125]. Although it has become well known in connection with the sampling theorem, it was not used by Shannon in his original papers, but appears to have been introduced first in 1953 by Woodward [143].

[33] Later, in Section IV-I, it will become clear that the kinship between both theorems goes even further, in the sense that the sampling theorem for bandlimited functions is the limiting case of the sampling theorem for splines.

discussed the design of alternative finite impulse response interpolators based on prespecified bandpass and bandstop characteristics. More information on this can also be found in the tutorial review by Crochiere and Rabiner [179].

E. Cubic Convolution Interpolation in Image Processing

The early 1970s was also the time digital image processing really started to develop. One of the first applications reported in the literature was the geometrical rectification of digital images obtained from the first Earth Resources Technology Satellite launched by the United States National Aeronautics and Space Administration in 1972. The need for more accurate interpolations than obtained by standard linear interpolation in this application led to the development of a still very popular technique known as *cubic convolution*, which involves the use of a sinc-like kernel composed of piecewise cubic polynomials. Apart from being interpolating, the kernel was designed to be continuous and to have a continuous first derivative. Cubic convolution was first mentioned by Rifman [180], discussed in some more detail by Simon [181], but the most general form of the kernel appeared first in a paper by Bernstein [182]

$$\psi(x) = \begin{cases} (\alpha+2)|x|^3 - (\alpha+3)|x|^2 + 1, & \text{if } 0 \leqslant |x| < 1 \\ \alpha|x|^3 - 5\alpha|x|^2 + 8\alpha|x| - 4\alpha, & \text{if } 1 \leqslant |x| < 2 \\ 0, & \text{if } 2 \leqslant |x| \end{cases} \tag{23}$$

where α is a free parameter resulting from the fact that the interpolation, continuity and continuous differentiability requirements yield only seven equations, while the two cubic polynomials defining the kernel make up a total of eight unknown coefficients.[34] The explicit cubic convolution kernel given by Rifman and Bernstein in their respective papers is the one corresponding to $\alpha = -1$, which results from forcing the first derivative of ψ to be equal to that of the sinc function at $x = 1$. Two alternative criteria for fixing α were given by Simon [181]. The first consists in requiring the second derivative of the kernel to be continuous at $x = 1$, which results in $\alpha = -3/4$. The second amounts to requiring the kernel to be capable of exact constant slope interpolation, which yields $\alpha = -1/2$. Although not mentioned by Simon, the kernel corresponding to the latter choice for α is not only capable of reproducing linear polynomials, but also quadratic polynomials.

F. Spline Interpolation in Image Processing

The use of splines for digital-image interpolation was first investigated only a little later [183]–[186]. An important paper providing a detailed analysis was published in 1978 by Hou and Andrews [186]. Their approach, mainly centered around the cubic B-spline, was based on matrix inversion techniques for computing the coefficients c_k in (22). Qualitative experiments involving magnification (enlargement) and minification (reduction) of image data demonstrated the superiority of cubic B-spline interpolation over techniques such as nearest neighbor or linear interpolation and even interpolation based on the truncated sinc function as kernel. The results of the magnification experiments also clearly showed the necessity for this type of interpolation to use the coefficients c_k rather than the original samples s_k in (22) in order to preserve resolution and contrast as much as possible.

G. Cubic Convolution Interpolation Revisited

Meanwhile, research on cubic convolution interpolation continued. In 1981, Keys [187] published an important study that provided new approximation-theoretic insights into this technique. He argued that the best choice for α in (23) is that the Taylor series expansion of the interpolant f_T resulting from cubic convolution interpolation of equidistant samples of an original function f agrees in as many terms as possible with that of the original function. By using this criterion, he found that the optimal choice is $\alpha = -1/2$ in which case $f_T(x) - f(x) = \mathcal{O}(T^3)$, for all x. This implies that the interpolation error goes to zero uniformly at a rate proportional to the third power of the sample interval. In other words, for this choice of α, cubic convolution yields a third-order approximation of the original function. For all other choices of α, he found that it yields only a first-order approximation, just like nearest neighbor interpolation. He also pointed at the fact that cubic Lagrange and cubic spline interpolation both yield a fourth-order approximation—the highest possible with piecewise cubics—and continued to derive a cubic convolution kernel with the same property, at the cost of a larger spatial support.

A second complementary study of the cubic convolution kernel (23) was published a little later by Park and Schowengerdt [188]. Rather than studying the properties of the kernel in the spatial domain, they carried out a frequency-domain analysis. The Maclaurin series expansion of the Fourier transform of (23) can be derived as

$$\begin{aligned}\hat{\psi}(\omega) = 1 - \frac{4}{15}(2\alpha+1)\left(\frac{\omega}{2}\right)^2 \\ + \frac{1}{35}(16\alpha+1)\left(\frac{\omega}{2}\right)^4 + \cdots \end{aligned} \tag{24}$$

where ω denotes radial frequency. Based on this fact, they argued that the best choice for the free parameter is $\alpha = -1/2$, since this maximizes the number of terms in which (24) agrees with the Fourier transform of the sinc kernel. That is to say, it provides the best low-frequency approximation to the "ideal" reconstruction filter. Park and Schowengerdt [188], [189] also studied the mean-square error or squared L_2 norm

$$\varepsilon^2 = \int_{-\infty}^{\infty} |f_1(x) - f(x)|^2 \, dx \tag{25}$$

with f_1 the image obtained from interpolating the samples $s_k = f(k)$ of an original image f using any interpolation

[34] Notice here that the application of convolution kernels to two-dimensional (2-D) data defined on Cartesian grids, as digital images are in most practical cases, has traditionally been done simply by extending (22) to $f_T(x,y) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} c_{k,l}\, \varphi(x/T_x - k)\, \varphi(y/T_y - l)$, where T_x and T_y denote the sampling interval in x and y direction, respectively. This approach can, of course, be extended to any number of dimensions and allows for the definition and analysis of kernels in one dimension only.

kernel φ. They showed that if the original image is bandlimited, i.e., $\hat{f}(\omega) = 0$, for all $|\omega|$ larger than some $\omega_{\max} > 0$ in this one-dimensional analysis and if sampling is performed at a rate equal to or higher than the Nyquist rate, this error—which they called the "sampling and reconstruction blur"—is equal to

$$\eta^2 = \frac{1}{2\pi} \int_{-\omega_{\max}}^{\omega_{\max}} \left| \hat{f}(\omega) \right|^2 E(\omega) d\omega \qquad (26)$$

where

$$E(\omega) = |1 - \hat{\varphi}(\omega)|^2 + \sum_{k \in \mathbb{Z}_*} |\hat{\varphi}(\omega + 2\pi k)|^2 \qquad (27)$$

with $\mathbb{Z}_* = \mathbb{Z} \backslash \{0\}$. In the case of undersampling, η^2 represents the average error $\langle \varepsilon^2 \rangle$, where the averaging is over all possible sets of samples $f(k+\tau)$, with $\tau \in [0,1]$. They argued that if the energy spectrum of f is not known, the optimal choice for α is the one that yields the best low-frequency approximation to $E(\omega) = 0$. Substituting ψ for φ and computing the Maclaurin series expansion of the right-hand side of (27), they found that this best approximation is obtained by taking, again, $\alpha = -1/2$. Notwithstanding the achievements of Keys and Park and Schowengerdt, it is interesting to note that cubic convolution interpolation corresponding to $\alpha = -1/2$ had been suggested in the literature at least three times before these authors. Already mentioned is Simon's paper [181], published in 1975. A little earlier, in 1974, Catmull and Rom [190] had studied interpolation by "cardinal blending functions" of the type

$$C_n(x) = \sum_{i=0}^{n} w(x+i) \prod_{\substack{j=i-n \\ j \neq 0}}^{i} \left(\frac{x}{j} + 1 \right) \qquad (28)$$

where n is the degree of polynomials resulting from the product on the right-hand side and w is a weight function or blending function centered around $x = n/2$. Among the examples they gave is the function corresponding to $n = 1$ and w the second-degree B-spline. This function can be shown to be equal to (23) with $\alpha = -1/2$. In the fields of computer graphics and visualization, the third-order cubic convolution kernel is therefore usually referred to as the Catmull–Rom spline. It has also been called the (modified or cardinal) cubic spline [191]–[197]. Finally, this cubic convolution kernel is precisely the kernel implicitly used in the previously mentioned osculatory interpolation scheme proposed around 1900 by Karup and King. More details on this can be found in a recent paper [198], which also demonstrates the equivalence of Keys' fourth-order cubic convolution and Henderson's osculatory interpolation scheme mentioned earlier.

H. Cubic Convolution Versus Spline Interpolation

A comparison of interpolation methods in medical imaging was presented by Parker *et al.* [199] in 1983. Their study included the nearest neighbor kernel, the linear interpolation kernel, the cubic B-spline, and two cubic convolution kernels,[35] viz., the ones corresponding to $\alpha = -1/2$ and $\alpha = -1$. Based on a frequency-domain analysis they concluded that the cubic B-spline yields the most smoothing and that it is therefore better to use a cubic convolution kernel. This conclusion, however, resulted from an incorrect use of the cubic B-spline for interpolation in the sense that the kernel was applied directly to the original samples s_k instead of the appropriate coefficients c_k—an approach that has been suggested (explicitly or implicitly) by many authors over the years [192], [200]–[205]. The point was later discussed by Maeland [206] who derived the true spectrum of the cubic spline interpolator or *cardinal cubic spline* as the product of the spectrum of the required prefilter and that of the cubic B-spline. From a correct comparison of the spectra, he concluded that cubic spline interpolation is superior compared to cubic convolution interpolation—a conclusion that would later be confirmed repeatedly by several evaluation studies (to be discussed in Section IV-M).[36]

I. Spline Interpolation Revisited

In classical interpolation theory, it was already known that it is better or even necessary in some cases to first apply some transformation to the original data before applying a given interpolation formula. The general rule in such cases is to apply transformations that will make the interpolation as simple as possible. The transformations themselves, of course, should preferably also be as simple as possible. Stirling, in his 1730 book [51] on finite differences, wrote: "As in common algebra, the whole art of the analyst does not consist in the resolution of the equations, but in bringing the problems thereto. So likewise in this analysis: there is less dexterity required in the performance of the process of interpolation than in the preliminary determination of the sequences which are best fitted for interpolation."[37] It should be clear from the foregoing discussion that a similar statement applies to convolution-based interpolation using B-splines: the difficulty is not in the convolution, but in the preliminary determination of the coefficients c_k. In order for B-spline interpolation to be a competitive technique, the computational cost of this preprocessing step should be reduced to a minimum—in many situations, the important issue is not just accuracy, but the tradeoff between accuracy and computational cost. Hou and Andrews [186], as many before and after them, solved the problem by setting up a system of equations followed by matrix inversion. Even though there exist optimized techniques [215] for inverting the Toeplitz type of matrices occurring in

[35]Note that Parker *et al.* referred to them consistently as "high-resolution cubic splines." According to Schoenberg's original definition, however, the cubic convolution kernel (23) is not a cubic spline, regardless of the value of α. Some people have called piecewise polynomial functions with less than maximum (nontrivial) smoothness "deficient splines." See also de Boor [133], who adopted the definition of a spline function as a linear combination of B-splines. When using the latter definition, the cubic convolution kernel may indeed be called a spline. We will not do so, however, in this paper.

[36]It is, therefore, surprising that even though there are now textbooks that acknowledge the superiority of spline interpolation [207]–[210], many books since the late 1980s [149], [211]–[214] give the impression that cubic convolution is the state-of-the-art in image interpolation.

[37]The translation from Latin is as given by Whittaker and Robinson [23].

spline interpolation, this approach is unnecessarily complex and computationally expensive.

In the early 1990s, it was shown by Unser *et al.* [216]–[218] that the B-spline interpolation problem can be solved much more efficiently by using a digital-filtering approach. Writing β^n rather than M_{n+1} for a B-spline of degree n, we obtain the following from applying the interpolation requirement to (22):

$$\sum_{l\in\mathbb{Z}} c_l\beta^n(k-l)=s_k, \quad \forall k\in\mathbb{Z}. \tag{29}$$

Recalling that the z transform of a convolution of two discrete sequences is equal to the product of the individual z transforms, the z transform of (29) reads $C(z)B^n(z) = S(z)$. Consequently, the B-spline coefficients can be obtained as

$$C(z)=\left(B^n(z)\right)^{-1}S(z). \tag{30}$$

Since, by definition, $B^n(z)=\sum_{k\in\mathbb{Z}}\beta^n(k)z^{-k}$, it follows from insertion of the explicit form of β^n that $(B^n(z))^{-1}=1$, for $n=0$ and $n=1$, which implies that in these cases $C(z)=S(z)$, that is to say, $c_k=s_k$. For any $n\geqslant 2$, however, $(B^n(z))^{-1}$ is a digital "high-boost" filter that corrects for the blurring effects of the corresponding B-spline convolution kernel. Although this was known to Hou and Andrews [186] and later authors [219]–[221], they did not realize that this filter can be implemented recursively.

Since β^n is even for any $n\in\mathbb{N}$, we have $(B^n(z))^{-1}=(B^n(z^{-1}))^{-1}$, which implies that the poles of the filter come in reciprocal pairs, so that the filter can be factorized as

$$\left(B^n(z)\right)^{-1}=\gamma\prod_{i=1}^{\lfloor n/2\rfloor}H(z;z_i) \tag{31}$$

where $\gamma=1/\beta^n(\lfloor n/2\rfloor)$ is a constant factor and

$$H(z;z_i)=\frac{-z_i}{(1-z_iz^{-1})(1-z_iz)} \tag{32}$$

is the factor corresponding to the pole pair $\{z_i,z_i^{-1}\}$, with $|z_i|<1$. Since the poles of $(B^n(z))^{-1}$ are the zeros of $B^n(z)$, they are obtained by solving $B^n(z)=0$. By a further factorization of (32) into $H(z;z_i)=H^-(z;z_i)H^+(z;z_i)$, with

$$H^+(z;z_i)=\frac{1}{(1-z_iz^{-1})},\ H^-(z;z_i)=\frac{-z_i}{(1-z_iz)} \tag{33}$$

and by using the shift property of the z transform, it is not difficult to show that in the spatial domain, application of $H^+(z;z_i)$ followed by $H^-(z;z_i)$ to given samples s_k, $k=0,1,2,\ldots,K-1$ amounts to applying the recursive filters

$$c_k^+ = s_k+z_ic_{k-1}^+, \quad k=1,2,\ldots,K-1, \tag{34}$$

$$c_k^- = z_i\left(c_{k+1}^- - c_k^+\right), \quad k=K-2,\ldots,1,0 \tag{35}$$

where the c_k^+ are intermediate output samples resulting from the first causal filter and the c_k^- are the output samples resulting from the second anticausal filter. For the initialization of the causal filter, we may use mirror-symmetric boundary conditions, i.e., $s_k=s_l$, for $(k+l)\mathrm{mod}(2K-2)=0$, which results in [222]

$$c_0^+=\frac{1}{1-z_i^{2K-2}}\sum_{l=0}^{2K-3}z_i^ls_l. \tag{36}$$

In most practical cases, K will be sufficiently large to justify taking $1/(1-z_i^{2K-2})=1$ and terminating the summation much earlier. An initial value for the anticausal filter may be obtained from a partial-fraction expansion of (32), resulting in [218]

$$c_{K-1}^-=\frac{-z_i}{(1-z_i^2)}\left(2c_{K-1}^+-s_{K-1}\right). \tag{37}$$

Summarizing, the prefilter $(B^n(z))^{-1}$ corresponding to a B-spline of degree n has $\lfloor n/2\rfloor$ pole pairs $\{z_i,z_i^{-1}\}$, $|z_i|<1$ and (34) and (35), with initial conditions (36) and (37), respectively, need to be applied successively for each z_i, where the input to the next iteration is formed by the output of the previous and the input to the first causal filter by the original samples s_k. The coefficients c_k to be used in (22) are precisely the c_k^- of the final anticausal filter, after scaling by the constant factor γ. Notice, furthermore, that in the subsequent evaluation of the convolution (22) for any x, the polynomial pieces of the kernel are computed most efficiently by using "nested multiplication" by x. In other words, by considering each of the polynomials in the form $x(\cdots(x(xa_n+a_{n-1})+a_{n-2})\cdots)+a_0$ rather than $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. It can easily be seen that the nested form requires only $2n$ floating-point operations (multiplications and additions) compared to $3n-1$ in the case of direct evaluation of the polynomial.[38]

It is interesting to have a look at the interpolation kernel implicitly used in (22) in the case of B-spline interpolation. Writing $(b^n)^{-1}$ for the spatial-domain version of the prefilter $(B^n)^{-1}$ corresponding to an nth degree B-spline, it follows from (30) that the coefficients are given by $c_k=((b^n)^{-1}*s)_k$, where "$*$" denotes convolution. Substituting this expression into (22), together with $\varphi=\beta^n$, we obtain

$$f_T(x)=\sum_{k\in\mathbb{Z}}\left((b^n)^{-1}*s\right)_k\beta^n\left(\frac{x}{T}-k\right) \tag{38}$$

which can be rewritten in cardinal form as

$$f_T(x)=\sum_{k\in\mathbb{Z}}s_k\vartheta^n\left(\frac{x}{T}-k\right) \tag{39}$$

where ϑ^n is the so-called *cardinal spline* of degree n, given by

$$\vartheta^n(x)=\sum_{k\in\mathbb{Z}}(b^n)_k^{-1}\beta^n(x-k). \tag{40}$$

Similar to the sinc function, this kernel satisfies the interpolation property: it vanishes for integer values of its argument, except at the origin, where it assumes unit value. Furthermore, for all $n\geqslant 2$, it has infinite support. And as n goes to

[38]It is precisely this trick that is implemented in the digital filter structure described by Farrow [223] in 1988 and that later authors [224]–[228] have referred to as the "Farrow structure." It was already known, however, to medieval Chinese mathematicians. Jiǎ Xiàn (middle 11th century) appears to have used it for solving cubic equations. A generalization to polynomials of arbitrary degrees was described first by Qín Jiǔsháo [11], [12] (also written as Ch'in Chiu-shao [229]) in his book *Shùshū Jiǔzhāng* ("Mathematical Treatise in Nine Sections," 1247 AD). In the West, it was rediscovered by Horner [230] in 1819 and it can be found under his name in many books on numerical analysis [31], [33], [34], [231]–[233]. Fifteen years earlier, however, it had also been proposed by Ruffini [234] (see also Cajori [235]). And even earlier, around 1669, it was used by Newton [236].

infinity, ϑ^n converges to the sinc function. Although this result was already known to Schoenberg [237], it did not reach the signal and image processing community until recently [238], [239].

In the years to follow, the described digital-filtering approach to spline interpolation would be used in the design of efficient algorithms for such purposes as image rotation [240], the enlargement or reduction of images [241], [242], the construction of multiresolution image pyramids [243], [244], image registration [245], [246], wavelet transformation [247]–[249], texture mapping [250], online signal interpolation [251], and fast spline transformation [252]. For more detailed information, the reader is referred to mentioned papers as well as several reviews [222], [253].

J. Development of Alternative Piecewise Polynomial Kernels

Independent of the just mentioned developments, research on alternative piecewise polynomial interpolation kernels continued. Mitchell and Netravali [192] derived a two-parameter cubic kernel by imposing the requirements of continuity and continuous differentiability, but by replacing the interpolation condition by the requirement of first-order approximation, i.e., the ability to reproduce the constant. By means of an analysis in the spirit of Keys [187], they also obtained a criterion to be satisfied by the two parameters in order to have at least second-order approximation. Special instances of their kernel include the cubic convolution kernel (23) corresponding to $\alpha = -1/2$ and the cubic B-spline. The whole family, sometimes also referred to as BC-splines [196], was later studied by several authors [195], [196], [254] in the fields of visualization and computer graphics.

An extension of the results of Park and Schowengerdt [188] concerning the previously discussed frequency-domain error analysis was presented by Schaum [255]. Instead of the L_2 norm, (25), he studied the performance metric

$$\varepsilon_s^2 = \sum_{k \in \mathbb{Z}} |f_1(k+s) - f(k+s)|^2 \tag{41}$$

which summarizes the total interpolation error at a given shift s with respect to the original sampling grid. He found that in the case of oversampling, this error too is given by (26), where η^2 and E may now be written as η_s^2 and E_s, respectively, with

$$E_s(\omega) = \left| 1 - \sum_{k \in \mathbb{Z}} e^{i2\pi ks} \hat{\varphi}(\omega + 2\pi k) \right|^2. \tag{42}$$

Also, in the case of undersampling, η_s^2 represents the error ε_s^2 averaged over all possible grid placements. He then pointed out that interpolation kernels are optimally designed if as many derivatives as possible of their corresponding error function E_s are zero at $\omega = 0$. By further analyzing E_s, he showed that for L-point interpolation kernels, i.e., kernels that extend over L samples in computing (22) at any x with $c_k = s_k$, this requirement implies that the kernel must be able to reproduce all monomials of degree $n \leqslant L-1$ and that this is the case for the Lagrange central interpolation kernels. Schaum also derived optimal interpolators for specific power spectra $|\hat{f}(\omega)|^2$.

Several authors have developed interpolation kernels defined explicitly as finite-linear combinations of B-splines. Chen *et al.* [256], e.g., described a kernel which they termed the "local interpolatory cardinal spline" and is composed of cubic B-splines only. When applying a scaling factor of two to its argument, this function very closely resembles Keys' fourth-order cubic convolution kernel, except that it is twice rather than once continuously differentiable. Knockaert and Olyslager [257] discovered a class of what they called "modified B-splines." To each integral approximation order $L > 0$ corresponds exactly one kernel of this class. For $L = 1$ and $L = 2$, these are (scaled versions of) the zeroth-degree and first-degree B-spline, respectively. For any $L > 2$, the corresponding kernel is a finite-linear combination of B-splines of different degrees, such that the composite kernel is interpolating and that its degree is as low as possible.

In recent years, the design methodologies originally used by Keys [187] have been employed more than once again in developing alternative interpolation kernels. Dodgson [258], defying the earlier mentioned claim of Schafer and Rabiner concerning even-degree piecewise polynomial interpolators, used them to derive a symmetric second-degree interpolation kernel. In contrast with the quadratic Lagrange interpolator, this kernel is continuous. Its order of approximation, however, is one less. German [259] used Keys' ideas to develop a continuously differentiable quartic interpolator with fifth order of approximation. Also, the present author [260] combined them with Park and Schowengerdt's frequency-domain error analysis to derive a class of odd-degree piecewise polynomial interpolation kernels with increasing regularity. All of these kernels, however, have the same order of approximation as the optimal cubic convolution kernel.

K. Impact of Approximation Theory

The notion of *approximation order*, defined as the rate at which the error of an approximation goes to zero when the distance between the samples goes to zero, has already been used at several points in the previous subsections. In general, an approximating function is computed as a linear combination of basis functions, whose coefficients are based on the samples of the original function. This is the case, e.g., with approximations obtained from (22), where the basis functions are translates of a single kernel φ.

The concept of convolution-based approximation has been studied intensively over the past decades, primarily in approximation theory, but the interesting results that had been obtained in this area of mathematics were not noticed by researchers in signal and image processing until relatively recently [261], [262]. An important example is the theory developed by Strang and Fix [263] in the early 1970s and further studied by many others, which relates the approximation error to properties of the kernel involved. Specifically, it implies that the following conditions are equivalent.

1) The kernel φ has Lth-order zeroes in the Fourier domain. More precisely

$$\begin{cases} \hat{\varphi}(0) \neq 0, \\ \hat{\varphi}^{(n)}(2\pi k) = 0, \quad \forall k \in \mathbb{Z}_*,\ n \in \{0, 1, \ldots, L-1\}. \end{cases} \tag{43}$$

2) The kernel φ is capable of reproducing all monomials of degree $n \leqslant L-1$. That is, for every $n \in \{0, 1, \ldots, L-1\}$, there exist coefficients $c_{k,n} \in \mathbb{R}$ such that

$$\sum_{k \in \mathbb{Z}} c_{k,n}\, \varphi(x-k) = x^n. \tag{44}$$

3) The first L discrete moments of the kernel φ are constants. That is, for every $n \in \{0, 1, \ldots, L-1\}$, there exists a $\mu_n \in \mathbb{R}$ such that

$$\sum_{k \in \mathbb{Z}} (x-k)^n \varphi(x-k) = \mu_n. \tag{45}$$

4) For each sufficiently smooth function f, i.e., a function whose derivatives up to and including order L are in the space L_2, there exists a constant $C \in \mathbb{R}$, which does not depend on f, and a set of coefficients $c_k \in \mathbb{R}$ such that the L_2 norm of the difference between f and its approximation f_T obtained from (22) is bounded as

$$\|f - f_T\|_{L_2} \leqslant C \cdot T^L \cdot \left\| f^{(L)} \right\|_{L_2} \quad \text{as } T \to 0. \tag{46}$$

This result holds for all compactly supported kernels [263], but also extends to noncompactly supported kernels having suitable inverse polynomial decay [264]–[267] or even less stringent properties [268], [269]. Extensions have also been made to L_p norms, $1 \leqslant p \leqslant \infty$ [264], [265], [267], [270], [271]. Furthermore, although the classical Strang–Fix theory applies to the case of an orthogonal projection, alternative approximation methods such as interpolation and quasiinterpolation also yield an $\mathcal{O}(T^L)$ approximation error [270]–[273]. A detailed treatment of the theory is outside the scope of the present paper and the reader is referred to mentioned papers for more information.

An interesting observation that follows from these equivalence conditions is that even though the original function does not at all have to be a polynomial, the capability of a given kernel to let the approximation error go to zero as T^L when $T \to 0$ (fourth condition) is determined by its ability to exactly reproduce all polynomials of maximum degree $L-1$ (second condition). In particular, it follows that in order for the approximation to converge to the original function at all when $T \to 0$, the kernel must have at least approximation order $L = 1$, which implies that it must at least be able to reproduce the constant. If we take $\hat{\varphi}(0) = 1$ (first condition), which yields the usually desirable property of unit gain for $\omega = 0$, we have $\mu_0 = 1$, which implies that the kernel samples must sum to one regardless of the position of the kernel relative to the sampling grid (third condition). This is generally known as the *partition of unity* condition—a condition that is not satisfied by virtually all so-called *windowed sinc functions*, which have frequently been proclaimed as the most appropriate alternative for the "ideal" interpolator.

As can be appreciated from (46), the theoretical notion of approximation order is still rather qualitative and not suitable for precise determination of the approximation error. In many applications, it would be very useful to have a more quantitative way of estimating the error $\varepsilon(T) = \|f - f_T\|_{L_2}$ induced by a given kernel φ and sampling step T. In 1999, it was shown by Blu and Unser [268], [274], [275] that for any convolution-based approximation scheme, this error is given by the relation

$$\varepsilon(T) = \eta(T) + o(T^r) \tag{47}$$

where the first term is a Fourier-domain prediction of the error, given by

$$\eta(T) = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 E(T\omega)\, d\omega} \tag{48}$$

and the second term goes to zero faster than T^r. Here, r denotes the highest derivative of f that still has finite energy. The error function E in (48) is completely determined by the prefiltering and reconstruction kernels involved in the approximation scheme. In the case where the scheme is actually an interpolation scheme, it follows that

$$E(\omega) = \frac{\left| \sum_{k \in \mathbb{Z}_*} \hat{\varphi}(\omega + 2\pi k) \right|^2 + \sum_{k \in \mathbb{Z}_*} \left| \hat{\varphi}(\omega + 2\pi k) \right|^2}{\left| \sum_{k \in \mathbb{Z}} \hat{\varphi}(\omega + 2\pi k) \right|^2}. \tag{49}$$

If the original function f is band-limited or otherwise sufficiently smooth (which means that its intrinsic scale is large with respect to the sampling step T), the prediction (48) is exact, that is to say, $\varepsilon(T) = \eta(T)$. In all other cases, $\eta^2(T)$ represents the average of $\varepsilon^2(T)$ over all possible sets of samples $f(kT+\tau)$, with $\tau \in [0, T]$. Note that if the kernel φ itself is forced to possess the interpolation property $\varphi(k) = \delta_k$, with δ_k the Kronecker symbol, we have from the discrete Fourier transform pair $\varphi(k) = \delta_k \Leftrightarrow \sum_{k \in \mathbb{Z}} \hat{\varphi}(\omega + 2\pi k) = 1$ that the denominator of (49) equals one, so that the error function reduces to the previously mentioned function (27) proposed by Park and Schowengerdt [188], [189].

It will be clear from (48) that the rate at which the approximation error goes to zero as $T \to 0$ is determined by the degree of flatness of the error function (49) near the origin. This latter quantity follows from the Maclaurin series expansion of the function. If all derivatives up to order $2L$ at the origin are zero, this expansion can be written as $E(\omega) = C_\varphi^2\, \omega^{2L} + \mathcal{O}(\omega^{2L+2})$, where $C_\varphi^2 = E^{(2L)}(0)/(2L)!$ and where use has been made of the fact that all odd terms of the expansion are zero because of the symmetry of the function. Substitution into (48) then shows that the last condition in the Strang–Fix theory can be reformulated more quantitatively as follows: for sufficiently smooth functions f, that is to say, functions for which $\|f^{(L)}\|_{L_2}$ is finite, the L_2 norm of the difference between f and the approximation f_T obtained from (22) is given by [253], [274], [276], [277]

$$\|f - f_T\|_{L_2} = C_\varphi \cdot T^L \cdot \left\| f^{(L)} \right\|_{L_2} \quad \text{as } T \to 0. \tag{50}$$

In view of the practical need in many applications to optimize the cost-performance tradeoff of interpolation, this raises the questions of which kernels of given approximation order L have the smallest support and which of the latter kernels have the smallest asymptotic constant C_φ. These questions were recently answered by Blu *et al.* [278]. Concerning the first, they showed that these kernels are piecewise polynomials of degree $L-1$ and that their support is of size L. Moreover, the full class of these so-called maximal-order minimum-support (MOMS) kernels is given by a linear combination of an $(L-1)$th-degree B-spline and its derivatives[39]

$$\varphi(x)=\sum_{n=0}^{L-1}\lambda_n\frac{d^n}{dx^n}\beta^{L-1}(x) \tag{51}$$

where $\lambda_0 = 1$ and the remaining λ_n are free parameters that can be tuned so as to let the kernel satisfy additional criteria. For example, if the kernel is supposed to have maximum smoothness, it turns out that all of the latter λ_n are necessarily zero, so that we are left with the $(L-1)$th-degree B-spline itself. Alternatively, if the kernel is supposed to possess the interpolation property, it follows that the λ_n are such that the kernel boils down to the $(L-1)$th-degree Lagrange central interpolation kernel. A more interesting design goal, however, is to minimize the asymptotic constant C_φ, the general form of which for MOMS kernels is given by [278]

$$C_\varphi=\sqrt{\sum_{n\in\mathbb{Z}_*}\left|\frac{\Lambda_L(i2\pi n)}{(i2\pi n)^L}\right|^2} \tag{52}$$

where $\Lambda_L(x) = \sum_{n=0}^{L-1}\lambda_n x^n$ is a polynomial of degree $L-1$. It was shown by Blu *et al.* [278] that the Λ_L which minimizes (52) for any order L can be obtained from the induction relation

$$\Lambda_{n+1}(x)=\Lambda_n(x)+\frac{x^2}{4\,(4L^2-1)}\Lambda_{n-1}(x) \tag{53}$$

which is initialized by $\Lambda_1(x) = \Lambda_2(x) = 1$. The resulting kernels were coined optimized MOMS (O-MOMS) kernels.

From inspection of (53), it is clear that regardless of the value of L, the corresponding polynomial Λ_L will always consist of even terms only. Hence, if L is even, the degree of Λ_L will be $L-2$ and since an $(L-1)$th-degree B-spline is precisely $L-2$ times continuously differentiable, it follows from (51) that the corresponding kernel is continuous, but that its first derivative is discontinuous. If, on the other hand, L is odd, the degree of the polynomial Λ_L will be $L-1$, so that the corresponding kernel itself will be discontinuous. In order to have at least a continuous derivative, as may be required for some applications, Blu *et al.* [278] also carried out constrained minimizations of (52). The resulting kernels were termed suboptimal MOMS (SO-MOMS).

L. Development of Alternative Interpolation Methods

Although—as announced in the introduction—the main concern in this section is the transition from classical polynomial interpolation approaches to modern convolution-based approaches and the many variants of the latter that have been proposed in the signal and image processing literature, it may be good to extend the perspectives and briefly discuss several alternative methods that have been developed since the 1980s. The goal here is not to be exhaustive, but to give an impression of the more specific interpolation problems and their solutions as studied in mentioned fields of research over the past two decades. Pointers to relevant literature are provided for readers interested in more details.

Deslauriers and Dubuc [280], [281], e.g., studied the problem of extending known function values $f(k)$ at the integers to all integral multiples of $1/b$, where the base b is also integer. In principle, any type of interpolation can be used to compute the $f(k+r/b),\ r=0,1,\ldots,b-1$ and the process can be iterated to find the value $f(x)$ for any rational number x whose denominator is an integral power of b. As pointed out by them, the main properties of this so-called *b-adic interpolation process* are determined by what they called the "fundamental function," i.e., the kernel φ, which reveals itself when feeding the process with a discrete impulse sequence. They showed that when using Waring–Lagrange interpolation[40] of any odd degree $2n-1$, the corresponding kernel has a support limited to $[-2n+1, 2n-1]$ and is capable of reproducing polynomials of maximum degree $2n-1$. Ultimately, it satisfies the b-scale relation

$$\varphi\Big(\frac{x}{b^m}\Big)=\sum_{k\in\mathbb{Z}}c_k\,\varphi(x-k) \tag{54}$$

where $c_k = \varphi(k/b^m)$. Taking $b = 2$, we have a dyadic interpolation process, which has strong links with the multiresolution theory of wavelets. Indeed, for any n, the Deslauriers-Dubuc kernel of order $2n$ is the autocorrelation of the Daubechies n scaling function [284], [285]. More details and further references on interpolating wavelets are given by, e.g., Mallat [285]. See Dai *et al.* [286] for a recent study on dyadic interpolation in the context of image processing.

Another approach that has received quite some attention since the 1980s is to consider the interpolation problem in the Fourier domain. It is well known [287] that multiplication by a phase component in the frequency domain corresponds to a shift in the signal or image domain. Obvious applications of this property are translation and zooming of image data [288]–[292]. Interpolation based on the Fourier shift theorem is equivalent to another Fourier-based method, known as *zero-filled* or *zero-padded interpolation* [293]–[297]. In the spatial domain, both techniques amount to assuming periodicity of the underlying signal or image and convolving the samples with the "periodized sinc," or Dirichlet's kernel [298]–[303]. Because of the assumed periodicity, the infinite convolution sum can be rewritten in finite form. Fourier-based interpolation is especially useful in situations where the data is acquired in Fourier space, such as in magnetic-resonance imaging (MRI), but by use of the fast Fourier transform may in principle be applied to any type of data. Variants of this approach based on the fast Hartley transform [304]–[306] and discrete cosine transform [307]–[309] have also been proposed. More details can be found in mentioned papers and the references therein.

An approach that has hitherto received relatively little attention in the signal and image processing literature is to

[39] Blu *et al.* [278] point out that this fact had been published earlier by Ron [279] in a more mathematically abstract and general context: the theory of exponential B-splines.

[40] Subdivision algorithms for Hermite interpolation were later studied by Merrien and Dubuc [282], [283].

consider sampled data as realizations of a random process at given spatial locations and to make inferences on the unobserved values of the process by means of a statistically optimal predictor. Here, “optimal” refers to minimizing the mean-squared prediction error, which requires a model of the covariance between the data points. This approach to interpolation is generally known as *kriging*—a term coined by the statistician Matheron [310] in honor of D. G. Krige [311], a South-African mining engineer who developed empirical methods for determining true ore-grade distributions from distributions based on sampled ore grades [312], [313]. Kriging has been studied in the context of geostatistics [312], [314], [315], cartography [316], and meteorology [317] and is closely related to interpolation by thin-plate splines or radial basis functions [318], [319]. In medical imaging, the technique seems to have been applied first by Stytz and Parrot [320], [321]. More recent studies related to kriging in signal and image processing include those of Kerwin and Prince [322] and Leung *et al.* [323].

A special type of interpolation problem arises when dealing with binary data, such as, e.g., segmented images. It is not difficult to see that in that case, the use of any of the aforementioned convolution-based interpolation methods followed by requantization practically boils down to nearest-neighbor interpolation. In order to cope with this problem, it is necessary to consider the shape of the objects—that is to say, their contours, rather than their “grey-level” distributions. For the interpolation of slices in a three-dimensional (3-D) data set, e.g., this may be accomplished by extracting the contours of interest and to apply elastic matching to estimate intermediate contours [324], [325]. An alternative is to apply any of the described convolution-based interpolation methods to the distance transform of the binary data. This latter approach to *shape-based interpolation* was originally proposed in the numerical analysis literature [326] and was first adapted to medical image processing by Raya and Udupa [327]. Later authors have proposed variants of their method by using alternative distance transforms [328] and morphological operators [329]–[332]. Extensions to the interpolation of tree-like image structures [333] and even ordinary grey-level images [334], [335] have also been made.

In a way, kriging and shape-based interpolation may be considered the precursors of more recent image interpolation techniques, which attempt to incorporate knowledge about the image content. The idea with most of these techniques is to adapt or choose between existing interpolation techniques, depending on the outcome of an initial image analysis phase. Since edges are often the more prevalent image features, most researchers have focused on gradient-based schemes for the analysis phase, although region-based approaches have also been reported. Examples of specific applications where the potential advantages of adaptive interpolation approaches have been demonstrated are image resolution enhancement or zooming [205], [336]–[342], spatial and temporal coding or compression of images and image sequences [343], [344], texture mapping [250], and volume rendering [345]. Clearly, the results of such adaptive methods depend on the performance of the employed analysis scheme as much as on that of the eventual (often convolution-based) interpolators.

M. Evaluation Studies and Their Conclusions

To return to our main topic: apart from the study by Parker *et al.* [199] discussed in Section IV-H, many more comparative evaluation studies of interpolation methods have been published over the years. Most of these appeared in the medical-imaging-related literature. Perhaps this can be explained from the fact that especially in medical applications, the issues of accuracy, quality, and also speed can be of vital importance. The loss of information and the introduction of distortions and artifacts caused by any manipulation of image data should be minimized in order to minimize their influence on the clinicians’ judgements [276].

Schreiner *et al.* [346] studied the performance of nearest-neighbor, linear and cubic convolution interpolation in generating maximum intensity projections (MIPs) of 3-D magnetic-resonance angiography (MRA) data for the purpose of detection and quantification of vascular anomalies. From the results of experiments involving both a computer-generated vessel model and clinical MRA data, they concluded that the choice for an interpolation method can have a dramatic effect on the information contained in MIPs. However, whereas the improvement of linear over nearest-neighbor interpolation was considerable, the further improvement of cubic convolution interpolation was found to be negligible in this application. Similar observations had been made earlier by Herman *et al.* [191] in the context of image reconstruction from projections.

Ostuni *et al.* [347] analyzed the effects of linear and cubic spline interpolation, as well as truncated and Hann-windowed sinc interpolation on the reslicing of functional MRI (fMRI) data. From the results of forward–backward geometric transformation experiments on clinical fMRI data, they concluded that the interpolation errors caused by cubic spline interpolation are much smaller than those due to linear and truncated-sinc interpolation. In fact, the errors produced by the latter two types of interpolation were found to be similar in magnitude, even with a spatial support of eight sample intervals for the truncated-sinc kernel compared to only two for the linear interpolation kernel. The Hann-windowed sinc kernel with a spatial support extending over six to eight sample intervals performed comparably to cubic spline interpolation in their experiments, but it required much more computation time.

Using similar reorientation experiments, Haddad and Porenta [348] found that the choice for an interpolation technique significantly affects the outcome of quantitative measurements in myocardial perfusion imaging based on single photon emission computed tomography (SPECT). They concluded that cubic convolution is superior to several alternative methods, such as local averaging, linear interpolation and what they called “hybrid” interpolation, which combines in-plane 2-D linear interpolation with through-plane cubic Lagrange interpolation—an approach that had been shown earlier [349] to yield better results than linear interpolation only. Here, we could also mention several studies in the field of remote sensing [193], [350], [351], which also showed the superiority of cubic convolution over linear and nearest-neighbor interpolation.

Grevera and Udupa [197] compared interpolation methods for the very specific task of doubling the number of slices of 3-D medical data sets. Their study included not only convolution-based methods, but also several of the shape-based interpolation methods [334], [352] mentioned earlier. The experiments consisted in subsampling a number of magnetic resonance (MR) and computed tomography (CT) data sets, followed by interpolation to restore the original resolutions. Based on the results they concluded that there is evidence that shape-based interpolation is the most accurate method for this task. Note, however, that concerning the convolution-based methods, the study was limited to nearest-neighbor, linear, and two forms of cubic convolution interpolation (although they referred to the latter as cubic spline interpolation). In a later more task-specific study [353], which led to the same conclusion, shape-based interpolation was compared only to linear interpolation.

A number of independent large-scale evaluations of convolution-based interpolation methods for the purpose of geometrical transformation of medical image data have recently been carried out. Lehmann *et al.* [354], e.g., compared a total of 31 kernels, including the nearest-neighbor, linear, and several quadratic [258] and cubic convolution kernels [187], [188], [192], [355], as well as the cubic B-spline interpolator, various Lagrange- [255] and Gaussian-based [356] interpolators and truncated and Blackman-Harris [357] windowed-sinc kernels of different spatial support. From the results of computational-cost analyses and forward-backward transformation experiments carried out on CCD-photographs, MRI sections, and X-ray images, it followed that, overall, cubic B-spline interpolation provides the best cost-performance tradeoff.

An even more elaborate study was presented by the present author [358], who carried out a cost-performance analysis of a total of 126 kernels with spatial support ranging from two to ten grid intervals. Apart from most of the kernels studied by Lehmann *et al.*, this study also included higher degree generalizations of the cubic convolution kernel [260], cardinal spline, and Lagrange central interpolation kernels up to ninth degree, as well as windowed-sinc kernels using over a dozen different window functions well known from the literature on harmonic analysis of signals [357]. The experiments involved the rotation and subpixel translation of medical data sets from many different modalities, including CT, three different types of MRI, PET, SPECT, as well as 3-D rotational and X-ray angiography. The results revealed that of all mentioned types of interpolation, spline interpolation generally performs statistically significantly better.

Finally, we mention the studies by Thévenaz *et al.* [276], [277], who carried out theoretical as well as experimental comparisons of many different convolution-based interpolation schemes. Concerning the former, they discussed the approximation-theoretical aspects of such schemes and pointed at the importance of having a high approximation order, rather than a high regularity and a small value for the asymptotic constant, as discussed in the previous subsection. Indeed, the results of their experiments, which included all of the aforementioned piecewise polynomial kernels, as well as the quartic convolution kernel by German [259], several types of windowed-sinc kernels and the O-MOMS [278], confirmed the theoretical predictions and clearly showed the superiority of kernels with optimized properties in these terms.

V. Summary and Conclusion

The goal in this paper was to give an overview of the developments in interpolation theory of all ages and to put the important techniques currently used in signal and image processing into historical perspective. We pointed at relatively recent research into the history of science, in particular of mathematical astronomy, which has revealed that rudimentary solutions to the interpolation problem date back to early antiquity. We gave examples of interpolation techniques originally conceived by ancient Babylonian as well as early-medieval Chinese, Indian, and Arabic astronomers and mathematicians and we briefly discussed the links with the classical interpolation techniques developed in Western countries from the 17th until the 19th century.

The available historical material has not yet given reason to suspect that the earliest known contributors to classical interpolation theory were influenced in any way by mentioned ancient and medieval Eastern works. Among these early contributors were Harriot and Briggs who, in the first half of the 17th century, developed higher order interpolation schemes for the purpose of subtabulation. A generalization of their rules for equidistant data was given independently by Gregory and Newton. We saw, however, that it is Newton who deserves the credit for having put classical interpolation theory on a firm foundation. He invented the concept of divided differences, allowing for a general interpolation formula applicable to data at arbitrary intervals and gave several special formulae that follow from it. In the course of the 18th and 19th century, these formulae were further studied by many others, including Stirling, Gauss, Waring, Euler, Lagrange, Bessel, Laplace, and Everett, whose names are nowadays inextricably bound up with formulae that can easily be derived from Newton's *regula generalis*.

Whereas the developments until the end of the 19th century had been impressive, the developments in the past century have been explosive. We briefly discussed early results in approximation theory, which revealed the limitations of interpolation by algebraic polynomials. We then discussed two major extensions of classical interpolation theory introduced in the first half of the 20th century: first, the concept of the cardinal function, mainly due to E. T. Whittaker, but also studied before him by Borel and others and eventually leading to the sampling theorem for bandlimited functions as found in the works of J. M. Whittaker, Kotel'nikov, Shannon, and several others and second, the concept of osculatory interpolation, researched by many and eventually resulting in

Schoenberg's theory of mathematical splines. We pointed at the important consequence of these extensions: a formulation of the interpolation problem in terms of a convolution of a set of coefficients with some fixed kernel.

The remainder of the paper focused on the further development of convolution-based interpolation in signal and image processing. The earliest and probably most important techniques studied in this context were cubic convolution interpolation and spline interpolation and we have discussed in some detail the contributions of various researchers in improving these techniques. Concerning cubic convolution, we highlighted the work of Keys and also Park and Schowengerdt, who presented different techniques for deriving the mathematically most optimal value for the free parameter involved in this scheme. Concerning spline interpolation, we discussed the work of Unser and his coworkers, who invented a fast recursive algorithm for carrying out the prefiltering required for this scheme, thereby making cubic spline interpolation as computationally cheap as cubic convolution interpolation. As a curiosity, we remarked that not only spline interpolation, but cubic convolution interpolation too can be traced back to osculatory interpolation techniques known from the beginning of the 20th century—a fact that, to the author's knowledge, has not been pointed out before.

After a summary of the development of many alternative piecewise polynomial kernels, we discussed some of the interesting results known in approximation theory for some time now, but brought to the attention of signal and image processors only recently. These are, in particular, the equivalence conditions due to Strang and Fix, which link the behavior of the approximation error as a function of the sampling step to specific spatial and Fourier domain properties of the employed convolution kernel. We discussed the extensions by Blu and Unser, who showed how to obtain more precise, quantitative estimates of the approximation error based on an error function which is completely determined by the kernel. We also pointed at their recent efforts toward minimization of the interpolation error, which has resulted in the development of kernels with minimal support and optimal approximation properties.

After a brief discussion of several alternative methods proposed over the past two decades for specific interpolation problems, we finally summarized the results of quite a number of evaluation studies carried out recently, primarily in medical imaging. All of these have clearly shown the dramatic effect the wrong choice for an interpolation method can have on the information content of the data. From this, it can be concluded that the issue of interpolation deserves more attention than it has received so far in some signal and image processing applications. The results of these studies also strongly suggest that in order to reduce the errors caused by convolution-based interpolation, it is more important for a kernel to have good approximation theoretical properties, in terms of approximation order and asymptotic behavior, than to have a high degree of regularity. In applications where the latter is an important issue, it follows that the most suitable kernels are B-splines, since they combine a maximal order of approximation with maximal regularity for a given spatial support.

ACKNOWLEDGMENT

The author would like to thank to Dr. T. Blu (Biomedical Imaging Group, EPFL, Switzerland) for his helpful explanations concerning the theoretical aspects of interpolation and approximation and Prof. Dr. M. Unser (Biomedical Imaging Group, EPFL, Switzerland) for his support and comments on the manuscript. The author would also like to thank Dr. E. Dubbink (Josephine Nefkens Institute, Erasmus University Rotterdam, The Netherlands) and Mr. S. Gheyselinck (Bibliothèque Centrale, EPFL, Switzerland) for providing copies of [129] and [136], respectively, as well as the anonymous reviewers, who brought to his attention some very interesting additional references.

REFERENCES

[1] G. W. Leibniz, "Historia et origo calculi differentialis," in *Mathematische Schriften*, C. I. Gerhardt, Ed. Hildesheim, Germany: Georg Olms Verlag, 1971, vol. 5, pp. 392–410.
[2] J. M. Child, "Newton and the art of discovery," in *Isaac Newton 1642–1727: A Memorial Volume*, W. J. Greenstreet, Ed. London, U.K.: G. Bell, 1927, pp. 117–129.
[3] J. Simpson and E. Weiner, Eds., *The Oxford English Dictionary*, 2nd ed. Oxford, U.K.: Oxford Univ. Press, 1989.
[4] J. Bauschinger, "Interpolation," in *Encyklopädie der Mathematischen Wissenschaften*, W. F. Meyer, Ed. Leipzig, Germany: B. G.Teubner, 1900–1904, pp. 799–820.
[5] J. Wallis, *Arithmetica Infinitorum*. Hildesheim, Germany: Olms Verlag, 1972.
[6] T. N. Thiele, *Interpolationsrechnung*. Leipzig, Germany: B. G. Teubner, 1909.
[7] O. Neugebauer, *Astronomical Cuneiform Texts. Babylonian Ephemerides of the Seleucid Period for the Motion of the Sun, the Moon and the Planets*. London, U.K.: Lund Humphries, 1955.
[8] ——, *A History of Ancient Mathematical Astronomy*. Berlin, Germany: Springer-Verlag, 1975.
[9] G. J. Toomer, "Hipparchus," in *Dictionary of Scientific Biography*, C. C. Gillispie and F. L. Holmes, Eds. New York: Scribner, 1978, vol. XV, pp. 207–224.
[10] G. van Brummelen, "Lunar and planetary interpolation tables in Ptolemy's *Almagest*," *J. Hist. Astron.*, vol. 25, no. 4, pp. 297–311, 1994.
[11] J.-C. Martzloff, *A History of Chinese Mathematics*. Berlin, Germany: Springer-Verlag, 1997.
[12] L. Yǎn and D. Shírán, *Chinese Mathematics: A Concise History*. Oxford, U.K.: Clarendon, 1987.
[13] G. M. Phillips, "Two millennia of mathematics: From Archimedes to Gauss," in *CMS Books in Mathematics*. New York: Springer-Verlag, 2000, vol. 6.
[14] C. Cullen, "An eighth century Chinese table of tangents," *Chin. Sci.*, vol. 5, pp. 1–33, 1982.
[15] R. C. Gupta, "Second order interpolation in Indian mathematics up to the fifteenth century," *Ind. J. Hist. Sci.*, vol. 4, no. 1–2, pp. 86–98, 1969.
[16] E. S. Kennedy, "The Chinese-Uighur calendar as described in the Islamic sources," in *Science and Technology in East Asia*, N. Sivin, Ed. New York: Science History, 1977, pp. 191–199.
[17] R. Rashed, "Analyze combinatoire, analyze numérique, analyze diophantienne et théorie des nombres," in *Histoire Des Sciences Arabes*. Paris, France: Editions du Seuil, 1997, vol. 2, Mathématiques et Physique, pp. 55–91.
[18] E. S. Kennedy, "Al-Bīrūni," in *Dictionary of Scientific Biography*, C. C. Gillispie and F. L. Holmes, Eds. New York: Scribner, 1970, vol. II, pp. 147–158.
[19] D. C. Fraser, *Newton's Interpolation Formulas*. London, U.K.: C. & E. Layton, 1927.

[20] H. H. Goldstine, *A History of Numerical Analysis From the 16th Through the 19th Century*. Berlin, Germany: Springer-Verlag, 1977.
[21] S. A. Joffe, "Interpolation-formulae and central-difference notation," *Trans. Actuar. Soc. Amer.*, vol. 18, pp. 72–98, 1917.
[22] H. W. Turnbull, "James Gregory: A study in the early history of interpolation," in *Proc. Edinburgh Math. Soc.*, vol. 3, 1932, pp. 151–172.
[23] E. T. Whittaker and G. Robinson, *A Short Course in Interpolation*. Glasgow, U.K.: Blackie, 1923.
[24] N. E. Nörlund, *Leçons sur les Séries d'Interpolation*. Paris, France: Gauthier-Villars, 1926.
[25] J. F. Steffensen, *Interpolation*, 2nd ed. New York: Chelsea, 1950.
[26] P. J. Davis, *Interpolation and Approximation*. New York: Blaisdell, 1963.
[27] T. Fort, *Finite Differences and Difference Equations in the Real Domain*. Oxford, U.K.: Clarendon, 1948.
[28] L. M. Milne-Thomson, *The Calculus of Finite Differences*, 3rd ed. New York: Macmillan, 1960.
[29] C. Jordan, *Calculus of Finite Differences*, 3rd ed. New York: Chelsea, 1979.
[30] G. Boole, *Calculus of Finite Differences*, 5th ed. New York: Chelsea, 1970.
[31] F. B. Hildebrand, *Introduction to Numerical Analysis*. New York: McGraw-Hill, 1974.
[32] H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics*, 3rd ed. Cambridge, U.K.: Cambridge Univ. Press, 1966.
[33] F. Szidarovszky and S. Yakowitz, *Principles and Procedures of Numerical Analysis*. New York: Plenum, 1978.
[34] H. R. Schwarz, *Numerical Analysis. A Comprehensive Introduction*. New York: Wiley, 1989.
[35] A. Neumaier, *Introduction to Numerical Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2001.
[36] B. Taylor, *Methodus Incrementorum Directa & Inversa*, London, U.K., 1715.
[37] C. Maclaurin, *A Treatise of Fluxions*, Edinburgh, U.K., 1742.
[38] M. Kline, *Mathematical Thought From Ancient to Modern Times*. New York: Oxford Univ. Press, 1972.
[39] J. Gregory, "Letter to J. Collins (23 november 1670)," in *James Gregory Tercentenary Memorial Volume*, H. W. Turnbull, Ed. London, U.K.: Bell, 1939, pp. 118–137.
[40] G. Mouton, *Observationes Diametrorum Solis et Lunæ Apparentium, Meridianarumque Aliquot Altitudinum Solis & Paucarum Fixarum* Lyons, France, 1670.
[41] H. Briggs, *Arithmetica Logarithmica*. London, U.K.: Guglielmus Iones, 1624.
[42] ——, *Trigonometria Britannica*. Gouda, The Netherlands: Petrus Rammasenius, 1633.
[43] J. A. Lohne, "Thomas Harriot als Mathematiker," *Centaurus*, vol. 11, no. 1, pp. 19–45, 1965.
[44] I. Newton, "Letter to Oldenburg (24 october 1676)," in *The Correspondence of Isaac Newton*, H. W. Turnbull, Ed. Cambridge, U.K.: Cambridge Univ. Press, 1960, vol. II, pp. 110–161.
[45] ——, "Letter to J. Smith (8 may 1675)," in *The Correspondence of Isaac Newton*, H. W. Turnbull, Ed. Cambridge, U.K.: Cambridge Univ. Press, 1959, vol. I, pp. 342–345.
[46] ——, "Methodus differentialis," in *The Mathematical Papers of Isaac Newton*, D. T. Whiteside, Ed. Cambridge, U.K.: Cambridge Univ. Press, 1981, vol. VIII, ch. 4, pp. 236–257.
[47] D. C. Fraser, "Newton and interpolation," in *Isaac Newton 1642–1727: A Memorial Volume*, W. J. Greenstreet, Ed. London, U.K.: Bell, 1927, pp. 45–69.
[48] I. Newton, "Philosophiæ Naturalis Principia Mathematica" (in English), in *Sir Isaac Newton's Mathematical Principles of Natural Philosphy and his System of the World*. Berkeley, CA, 1960. F. Cajori.
[49] A. de Morgan, *The Differential and Integral Calculus*. London, U.K.: Baldwin and Cradock, 1842.
[50] J. Stirling, "Methodus differentialis Newtoniana illustrata," *Philos. Trans.*, vol. 30, no. 362, pp. 1050–1070, 1719.
[51] ——, *Methodus Differentialis sive Tractatus de Summatione et Interpolatione Serierum Infinitarum* London, U.K., 1730.
[52] E. Waring, "Problems concerning interpolations," *Philos. Trans. R. Soc. London*, vol. 69, pp. 59–67, 1779.
[53] J. L. Lagrange, "Leçons élémentaires sur les mathématiques données a l'école normale," in *Œuvres de Lagrange*, J.-A. Serret, Ed. Paris, France: Gauthier-Villars, 1877, vol. 7, pp. 183–287.
[54] L. Euler, "De eximio usu methodi interpolationum in serierum doctrina," in *Opuscula Analytica* Petropoli, 1783, vol. 1, Academia Imperialis Scientiarum, pp. 157–210.
[55] C. F. Gauss, "Theoria interpolationis methodo nova tractata," in *Werke*. Göttingen, Germany: Königlichen Gesellschaft der Wissenschaften, 1866, vol. III, pp. 265–327.
[56] J. F. Encke, "Über interpolation," *Berliner Astronomisches Jahrbuch*, vol. 55, pp. 265–284, 1830.
[57] F. W. Bessel, "Anleitung und Tafeln die stündliche Bewegung des Mondes zu finden," *Astronomische Nachrichten*, vol. II, no. 33, pp. 137–141, 1824.
[58] J. D. Everett, "On a central-difference interpolation formula," *Rep. Br. Assoc. Adv. Sci.*, vol. 70, pp. 648–650, 1900.
[59] ——, "On a new interpolation formula," *J. Inst. Actuar.*, vol. 35, pp. 452–458, 1901.
[60] M. K. Samarin, "Steffensen interpolation formula," in *Encyclopaedia of Mathematics*. Norwell, MA: Kluwer, 1992, vol. 8, p. 521.
[61] G. J. Lidstone, "Notes on Everett's interpolation formula," in *Proc. Edinburgh Math. Soc.*, vol. 40, 1922, pp. 21–26.
[62] P. S. de Laplace, "Mémoire sur les suites (1779)," in *Œuvres Complètes de Laplace*. Paris, France: Gauthier-Villars et Fils, 1894, vol. 10, pp. 1–89.
[63] ——, *Théorie Analytique des Probabilités*, 3rd ed. Paris, France: Ve. Courcier, 1820, vol. 7.
[64] W. F. Sheppard, "Central-difference formulæ," in *Proceedings of the London Mathematical Society*, vol. 31, 1899, pp. 449–488.
[65] A.-L. Cauchy, *Cours d'Analyze de l'École Royale Polytechnique: Part I: Analyze Algébrique*. Paris, France: Imprimerie Royale, 1821, pt. I.
[66] A. Cauchy, "Sur les fonctions interpolaires," *Comptes Rendus des Séances de l'Académie des Sciences*, vol. 11, no. 20, pp. 775–789, 1841.
[67] P. L. Tchebychef, "Sur les quadratures," *Journal de Mathématiques Pures et Appliquées*, ser. II, vol. 19, pp. 19–34, 1874.
[68] C. W. Borchardt, "Über eine Interpolationsformel für eine art symmetrischer Functionen und über deren Anwendung," *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin*, pp. 1–20, 1860.
[69] L. Kronecker, "Über einige Interpolationsformeln für ganze Functionen mehrer Variabeln," *Monatsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pp. 686–691, 1865.
[70] M. Gasca and T. Sauer, "On the history of multivariate polynomial interpolation," *J. Comput. Appl. Math.*, vol. 122, no. 1–2, pp. 23–35, 2000.
[71] C. Hermite, "Sur la formule d'interpolation de Lagrange," *Journal für die Reine und Angewandte Mathematik*, vol. 84, no. 1, pp. 70–79, 1878.
[72] G. D. Birkhoff, "General mean value and remainder theorems with applications to mechanical differentiation and quadrature," *Trans. Amer. Math. Soc.*, vol. 7, no. 1, pp. 107–136, 1906.
[73] I. J. Schoenberg, "On Hermite-Birkhoff interpolation," *J. Math. Anal. Applicat.*, vol. 16, no. 3, pp. 538–543, 1966.
[74] N. H. Abel, "Sur les fonctions génératrices et leurs déterminantes," in *Œuvres Complètes de Niels Henrik Abel*, 2nd ed. Christiania: Grøndahl & Søn, 1881, vol. 2, ch. XI, pp. 67–81.
[75] W. Gontcharoff, "Recherches sur les dérivées successives des fonctions analytiques. Généralization de la série d'Abel," *Annales Scientifiques de l'École Normale Supérieure*, ser. 3, vol. 47, pp. 1–78, 1930.
[76] J. M. Whittaker, "Interpolatory function theory," in *Cambridge Tracts in Mathematics and Mathematical Physics*. Cambridge, U.K.: Cambridge Univ. Press, 1935.
[77] G. J. Lidstone, "Notes on the extension of Aitken's theorem (for polynomial Interpolation) to the Everett types," in *Proc. Edinburgh Math. Soc.*, vol. 2, 1930, pp. 16–19.
[78] J. M. Whittaker, "On Lidstone's series and two-point expansions of analytic functions," in *Proc. London Math. Soc.*, vol. 36, 1934, pp. 451–469.
[79] G. G. Lorentz, K. Jetter, and S. D. Riemenschneider, *Birkhoff Interpolation*. Reading, MA: Addison-Wesley, 1983, vol. 19, Encyclopedia of Mathematics and Its Applications.
[80] R. A. Lorentz, *Multivariate Birkhoff Interpolation*, Germany: Springer-Verlag, 1992, vol. 1516, Lecture Notes in Mathematics.
[81] B. D. Bojanov, H. A. Hakopian, and A. A. Sahakian, *Spline Functions and Multivariate Interpolations*. Norwell, MA: Kluwer, 1993.

[82] M. Gasca and T. Sauer, "Polynomial interpolation in several variables," *Adv. Comput. Math.*, vol. 12, no. 4, pp. 377–410, 2000.
[83] R. A. Lorentz, "Multivariate Hermite interpolation by algebraic polynomials: A survey," *J. Comput. Appl. Math.*, vol. 122, no. 1–2, pp. 167–201, 2000.
[84] K. Weierstrass, "Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen reeller Argumente," in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pp. 633/789–639/805.
[85] A. Pinkus, "Weierstrass and approximation theory," *J. Approx. Theory*, vol. 107, no. 1, pp. 1–66, 2000.
[86] Ch. Méray, "Observations sur la légitimité de l'interpolation," *Annales Scientifiques de l'École Normale Supérieure*, ser. 3, vol. 1, pp. 165–176, 1884.
[87] ——, "Nouveaux exemples d'interpolations illusoires," *Bulletin des Sciences Mathématiques.*, ser. 2, vol. 20, pp. 266–270, 1896.
[88] C. Runge, "Über emperische Functionen und die Interpolation zwischen äquidistanten Ordinaten," *Zeitschrift für Mathematik und Physik*, vol. 46, pp. 224–243, 1901.
[89] G. Faber, "Über die interpolatorische Darstellung stetiger Functionen," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23, pp. 192–210, 1914.
[90] L. Fejér, "Ueber interpolation," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse*, pp. 66–91, 1916.
[91] E. T. Whittaker, "On the functions which are represented by the expansions of interpolation-theory," in *Proc. R. Soc. Edinburgh*, vol. 35, 1915, pp. 181–194.
[92] J. R. Higgins, "Five short stories about the cardinal series," *Bull. Amer. Math. Soc. New Ser.*, vol. 12, no. 1, pp. 45–89, 1985.
[93] P. L. Butzer and R. L. Stens, "Sampling theory for not necessarily band-limited functions: A historical overview," *SIAM Rev.*, vol. 34, no. 1, pp. 40–53, 1992.
[94] E. Borel, "Mémoire sur les séries divergentes," *Annales Scientifiques de l'École Normale Supérieure*, ser. 2, vol. 16, pp. 9–131, 1899.
[95] T. A. Brown, "Fourier's integral," in *Proc. Edinburgh Math. Soc.*, vol. 34, 1915–1916, pp. 3–10.
[96] W. L. Ferrar, "On the cardinal function of interpolation-theory," in *Proc. R. Soc. Edinburgh*, vol. 45, 1925, pp. 269–282.
[97] D. L. Jagerman and L. J. Fogel, "Some general aspects of the sampling theorem," *IRE Trans. Inform. Theory*, vol. 2, pp. 139–146, 1956.
[98] R. J. Marks II, *Introduction to Shannon Sampling and Interpolation Theory*. Berlin, Germany: Springer-Verlag, 1991.
[99] G. Hinsen and D. Klösters, "The sampling series as a limiting case of Lagrange interpolation," *Applicable Anal.*, vol. 49, no. 1–2, pp. 49–60, 1993.
[100] J. F. Steffensen, "Über eine Klasse von ganzen Funktionen und ihre Anwending auf die Zahlentheorie," *Acta Mathematica*, vol. 37, pp. 75–112, 1914.
[101] J. McNamee, F. Stenger, and E. L. Whitney, "Whittaker's cardinal function in retrospect," *Math. Comput.*, vol. 25, no. 113, pp. 141–154, 1971.
[102] J. Hadamard, "La série de Taylor et son prolongement analytique," *Scientia: Physico-Mathématique*, 1901.
[103] Ch.-J. de la Vallée Poussin, "Sur la convergence des formules d'interpolation entre ordonnées équidistantes," *Bulletin de la Classe des Sciences Académie Royale de Belgique*, pp. 319–410, 1908.
[104] W. L. Ferrar, "On the cardinal function of interpolation-theory," in *Proc. R. Soc. Edinburgh*, vol. 46, 1926, pp. 323–333.
[105] C.-J. de la Vallée Poussin, "Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par des polyn mes et de suites limitées de Fourier," *Bulletin de la Classe des Sciences Académie Royale de Belgique*, pp. 193–254, 1908.
[106] A. A. Zakharov, "De la Vallée-Poussin summation method," in *Encyclopaedia of Mathematics*. Norwell, MA: Kluwer, 1989, vol. 3, pp. 13–14.
[107] W. L. Ferrar, "On the consistency of cardinal function interpolation," in *Proc. R. Soc. Edinburgh*, vol. 47, 1927, pp. 230–242.
[108] J. M. Whittaker, "On the cardinal function of interpolation theory," in *Proc. Edinburgh Math. Soc.*, vol. 1, 1927–1929, pp. 41–46.
[109] ——, "The "Fourier" theory of the cardinal function," in *Proc. Edinburgh Math. Soc.*, vol. 1, 1927–1929, pp. 169–176.
[110] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, pp. 379–423, 1948.
[111] ——, "Communication in the presence of noise," *Proc. Inst. Radio Eng.*, vol. 37, no. 1, pp. 10–21, 1949.
[112] H. Nyquist, "Certain topics in telegraph transmission theory," *Trans. Amer. Inst. Electr. Eng.*, vol. 47, pp. 617–644, 1928.
[113] K. Ogura, "On a certain transcendental integral function in the theory of interpolation," *Thoku Math. J.*, vol. 17, pp. 64–72, 1920.
[114] ——, "On some central difference formulas of interpolation," *Thoku Math. J.*, vol. 17, pp. 232–241, 1920.
[115] E. Borel, "Sur l' interpolation," *Comptes Rendus des Séances de l'Académie des Sciences*, vol. 124, no. 13, pp. 673–676, 1897.
[116] H. S. Black, *Modulation Theory*. New York: Van Nostrand, 1953.
[117] A. Cauchy, "Mémoire sur diverses formules d'analyze," *Comptes Rendus des Séances de l'Académie des Sciences*, vol. 12, no. 6, pp. 283–298, 1841.
[118] V. A. Kotel'nikov, "On the transmission capacity of the "ether" and wire in electrocommunications," in *Modern Sampling Theory: Mathematics and Applications*, J. J. Benedetto and P. J. S. G. Ferreira, Eds. Boston, MA: Birkhäuser, 2000, ch. 2, pp. 27–45.
[119] H. Raabe, "Untersuchungen an der wechselzeitigen Mehrfachübertragung (Multiplexübertragung)," *Elektr. Nach.tech.*, vol. 16, no. 8, pp. 213–228, 1939.
[120] H. D. Lüke, "Zur Entstehung des Abtasttheorems," *Nach.tech. Zeit.*, vol. 31, no. 4, pp. 271–274, 1978.
[121] P. L. Butzer, "A survey of the Whittaker-Shannon sampling theorem and some of its extensions," *J. Math. Res. Expos.*, vol. 3, no. 1, pp. 185–212, 1983.
[122] I. Someya, *Waveform Transmission* (in Japanese). Tokyo, Japan: Shyukyoo, 1949.
[123] J. D. Weston, "A note on the theory of communication," *London, Edinburgh, Dublin Philos. Mag. J. Sci.*, ser. 7, vol. 40, no. 303, pp. 449–453, 1949.
[124] A. J. Jerri, "The Shannon sampling theorem—its various extensions and applications: A tutorial review," *Proc. IEEE*, vol. 65, pp. 1565–1596, Nov. 1977.
[125] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*. Oxford, U.K.: Clarendon, 1996.
[126] T. N. E. Greville, "The general theory of osculatory interpolation," *Trans. Actuar. Soc. Amer.*, vol. 45, pp. 202–265, 1944.
[127] H. L. Seal, "Graduation by piecewise cubic polynomials: A historical review," *Blätter Deutsch. Gesellschaft Versich.math.*, vol. 15, pp. 89–114, 1981.
[128] E. S. W. Shiu, "A survey of graduation theory," in *Actuarial Mathematics*, H. H. Panjer, Ed. Providence, RI: Amer. Math. Soc., 1986, vol. 35, Proceedings of Symposia in Applied Mathematics.
[129] J. Karup, "Über eine neue mechanische Ausgleichungsmethode," in *Transactions of the Second International Actuarial Congress*, G. King, Ed. London, U.K.: Layton, 1899, pp. 31–77.
[130] G. King, "Notes on summation formulas of graduation, with certain new formulas for consideration," *J. Inst. Actuar.*, vol. 41, pp. 530–565, 1907.
[131] S. W. Kahng, "Osculatory interpolation," *Math. Comput.*, vol. 23, no. 107, pp. 621–629, 1969.
[132] H. E. Salzer, "Formulas for bivariate hyperosculatory interpolation," *Math. Comput.*, vol. 25, no. 113, pp. 119–133, 1971.
[133] C. de Boor, "A practical guide to splines," in *Applied Mathematical Sciences*. Berlin, Germany: Springer-Verlag, 1978, vol. 27.
[134] J. R. Busch, "Osculatory interpolation in $\mathbb{R}^n$," *SIAM J. Numer. Anal.*, vol. 22, no. 1, pp. 107–113, 1985.
[135] X.-Z. Liang and L.-Q. Li, "On bivariate osculatory interpolation," *J. Comput. Appl. Math.*, vol. 38, pp. 271–282, 1991.
[136] R. Henderson, "A practical interpolation formula. With a theoretical introduction," *Trans. Actuar. Soc. Amer.*, vol. 9, no. 35, pp. 211–224, 1906.
[137] W. A. Jenkins, "Graduation based on a modification of osculatory interpolation," *Trans. Actuar. Soc. Amer.*, vol. 28, pp. 198–215, 1927.
[138] I. J. Schoenberg, "Contributions to the problem of approximation of equidistant data by analytic functions. Part A—on the problem of smoothing or graduation. A first class of analytic approximation formulae," *Quart. Appl. Math.*, vol. IV, no. 1, pp. 45–99, 1946.
[139] ——, "Contributions to the problem of approximation of equidistant data by analytic functions. Part B—on the problem of osculatory interpolation. A second class of analytic approximation formulae," *Quart. Appl. Math.*, vol. IV, no. 2, pp. 112–141, 1946.
[140] G. Wahba, *Spline Models for Observational Data*. Philadelphia, PA: SIAM, 1990.
[141] I. J. Schoenberg, "On spline functions," in *Inequalities*, O. Shisha, Ed. New York: Academic, 1967.

[142] P. L. Butzer, M. Schmidt, and E. L. Stark, "Observations on the history of central B-splines," *Archive for History of Exact Sciences*, vol. 39, pp. 137–156, 1988.

[143] P. M. Woodward, *Probability and Information Theory, with Applications to Radar*. London, U.K.: Pergamon, 1953.

[144] K. S. Shanmugam, *Digital and Analog Communication Systems*. New York: Wiley, 1979.

[145] R. H. Jones and N. C. Steele, *Mathematics in Communication Theory*. Chichester, U.K.: Ellis Horwood, 1989.

[146] R. E. Blahut, *Digital Transmission of Information*. Reading, MA: Addison-Wesley, 1990.

[147] L. W. Couch, *Digital and Analog Communication Systems*, 4th ed. New York: Macmillan, 1993.

[148] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[149] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*. Reading, MA: Addison-Wesley, 1993.

[150] K. R. Castleman, *Digital Image Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[151] R. W. Hamming, *Numerical Methods for Scientists and Engineers*, 2nd ed. New York: McGraw-Hill, 1973.

[152] F. Stenger, "Numerical methods based on Whittaker cardinal, or sinc functions," *SIAM Rev.*, vol. 23, no. 2, pp. 165–224, 1981.

[153] J. Lund and K. L. Bowers, *Sinc Methods for Quadrature and Differential Equations*. Philadelphia, PA: SIAM, 1992.

[154] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*. Berlin, Germany: Springer-Verlag, 1993.

[155] I. J. Schoenberg, "Cardinal interpolation and spline functions," *J. Approx. Theory*, vol. 2, no. 2, pp. 167–206, 1969.

[156] ——, *Cardinal Spline Interpolation*. Philadelphia, PA: SIAM, 1973.

[157] ——, *Selected Papers*. Boston, MA: Birkhäuser, 1988.

[158] C. de Boor, "On calculating with B-splines," *J. Approx. Theory*, vol. 6, no. 1, pp. 50–62, 1972.

[159] A. S. B. Holland and B. N. Sahney, *The General Problem of Approximation and Spline Functions*. Melbourne, FL: Krieger, 1979.

[160] L. L. Schumaker, *Spline Functions: Basic Theory*. New York: Wiley, 1981.

[161] G. Nürnberger, *Approximation by Spline Functions*. Berlin, Germany: Springer-Verlag, 1989.

[162] C. de Boor, K. Höllig, and S. Riemenschneider, *Box Splines*. Berlin, Germany: Springer-Verlag, 1993, vol. 98, Applied Mathematical Sciences.

[163] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*. New York: Springer-Verlag, 1993, vol. 303, Grundlehren der Mathematischen Wissenschaften.

[164] N. Dyn, D. Leviatan, D. Levin, and A. Pinkus, *Multivariate Approximation and Applications*, N. Dyn, D. Leviatan, D. Levin, and A. Pinkus, Eds. Cambridge, U.K.: Cambridge Univ. Press, 2001.

[165] C. K. Chui, *Multivariate Splines*. Philadelphia, PA: SIAM, 1993.

[166] H. Späth, *One Dimensional Spline Interpolation Algorithms*. Wellesley, MA: A. K. Peters, 1995.

[167] ——, *Two Dimensional Spline Interpolation Algorithms*. Wellesley, MA: A. K. Peters, 1995.

[168] P. M. Prenter, *Splines and Variational Methods*. New York: Wiley, 1975.

[169] J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, *The Theory of Splines and Their Applications*. New York: Academic, 1967, vol. 38, Mathematics in Science and Engineering.

[170] P. Lancaster and K. Salkauskas, *Curve and Surface Fitting: An Introduction*. New York: Academic, 1986.

[171] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Boston, MA: Academic, 1988.

[172] P. Dierckx, *Curve and Surface Fitting With Splines*. Oxford, U.K.: Clarendon, 1993.

[173] D. Marsh, *Applied Geometry for Computer Graphics and CAD*. Berlin, Germany: Springer-Verlag, 1999.

[174] C. A. Micchelli, *Mathematical Aspects of Geometrical Modeling*. Philadelphia, PA: SIAM, 1995.

[175] T. Pavlidis, *Algorithms for Graphics and Image Processing*. Berlin, Germany: Springer-Verlag, 1982.

[176] R. H. Bartels, J. C. Beatty, and B. A. Barsky, *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*. San Mateo, CA: Morgan Kaufmann, 1987.

[177] D. E. Knuth, *TEX and METAFONT: New Directions in Typesetting*. Bedford, MA: American Mathematical Soc., 1979.

[178] R. W. Schafer and L. R. Rabiner, "A digital signal processing approach to interpolation," *Proc. IEEE*, vol. 61, pp. 692–702, June 1973.

[179] R. E. Crochiere and L. R. Rabiner, "Interpolation and decimation of digital signals—A tutorial review," *Proc. IEEE*, vol. 69, pp. 300–331, Mar. 1981.

[180] S. S. Rifman, "Digital rectification of ERTS multispectral imagery," in *Proc. Symp. Significant Results Obtained From the Earth Resources Technology Satellite-1*, vol. 1, sec. B, 1973, NASA SP-327, pp. 1131–1142.

[181] K. W. Simon, "Digital image reconstruction and resampling for geometric manipulation," in *Symposium on Machine Processing of Remotely Sensed Data*, C. D. McGillem and D. B. Morrison, Eds. Piscataway, NJ: IEEE Press, 1975, pp. 3A-1–3A-11.

[182] R. Bernstein, "Digital image processing of earth observation sensor data," *IBM J. Res. Devel.*, vol. 20, no. 1, pp. 40–57, 1976.

[183] H. C. Andrews and C. L. Patterson, "Digital interpolation of discrete images," *IEEE Trans. Comput.*, vol. C-25, pp. 196–202, Feb. 1976.

[184] T. Pavlidis, *Structural Pattern Recognition*. Berlin, Germany: Springer-Verlag, 1977.

[185] H. S. Hou and H. C. Andrews, "Least squares image restoration using spline basis functions," *IEEE Trans. Comput.*, vol. 26, pp. 856–873, Sept. 1977.

[186] ——, "Cubic splines for image interpolation and digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-26, pp. 508–517, Dec. 1978.

[187] R. G. Keys, "Cubic convolution interpolation for digital image processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1153–1160, Dec. 1981.

[188] S. K. Park and R. A. Schowengerdt, "Image reconstruction by parametric cubic convolution," *Comput. Vis., Graphics, Image Processing*, vol. 23, no. 3, pp. 258–272, 1983.

[189] ——, "Image sampling, reconstruction and the effect of sample-scene phasing," *Appl. Opt.*, vol. 21, no. 17, pp. 3142–3151, 1982.

[190] E. Catmull and R. Rom, "A class of local interpolating splines," in *Computer Aided Geometric Design*, R. E. Barnhill and R. F. Riesenfeld, Eds. New York: Academic, 1974, pp. 317–326.

[191] G. T. Herman, S. W. Rowland, and M. Yau, "A comparitive study of the use of linear and modified cubic spline interpolation for image reconstruction," *IEEE Trans. Nucl. Sci.*, vol. NS-26, pp. 2879–2894, Feb. 1979.

[192] D. P. Mitchell and A. N. Netravali, "Reconstruction filters in computer graphics," *Comput. Graphics*, vol. 22, no. 4, pp. 221–228, 1988.

[193] B. Khan, L. W. B. Hayes, and A. P. Cracknell, "The effects of higher order resampling on AVHRR data," *Int. J. Remote Sensing*, vol. 16, no. 1, pp. 147–163, 1995.

[194] M. J. Bentum, B. B. A. Lichtenbelt, and T. Malzbender, "Frequency analysis of gradient estimators in volume rendering," *IEEE Trans. Visual. Comput. Graphics*, vol. 2, pp. 242–254, Sept. 1996.

[195] R. Machiraju and R. Yagel, "Reconstruction error characterization and control: A sampling theory approach," *IEEE Trans. Visual. Comput. Graphics*, vol. 2, pp. 364–378, Dec. 1996.

[196] T. Möller, R. Machiraju, K. Mueller, and R. Yagel, "Evaluation and design of filters using a Taylor series expansion," *IEEE Trans. Visual. Comput. Graphics*, vol. 3, pp. 184–199, Apr.–June 1997.

[197] G. J. Grevara and J. K. Udupa, "An objective comparison of 3-D image interpolation methods," *IEEE Trans. Med. Imag.*, vol. 17, pp. 642–652, Aug. 1998.

[198] E. Meijering and M. Unser, "A note on cubic convolution interpolation,", 2001, submitted for publication.

[199] J. A. Parker, R. V. Kenyon, and D. E. Troxel, "Comparison of interpolating methods for image resampling," *IEEE Trans. Med. Imag.*, vol. MI-2, pp. 31–39, Mar. 1983.

[200] C. E. Floyd, R. J. Jaszczak, and R. E. Coleman, "Image resampling on a cylindrical sector grid," *IEEE Trans. Med. Imag.*, vol. MI-5, pp. 128–131, Sept. 1986.

[201] P. V. Sankar and L. A. Ferrari, "Simple algorithms and architectures for *B*-spline interpolation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 10, pp. 271–276, 1988.

[202] W. K. Pratt, *Digital Image Processing*, 2nd ed. New York: Wiley, 1991.

[203] M. Moshfeghi, "Directional interpolation for magnetic resonance angiography data," *IEEE Trans. Med. Imag.*, vol. 12, pp. 366–379, June 1993.

[204] L. R. Watkins and J. Le Bihan, "Fast, accurate interpolation with B-spline scaling functions," *Electron. Lett.*, vol. 30, no. 13, pp. 1024–1025, 1994.
[205] A. M. Darwish, M. S. Bedair, and S. I. Shaheen, "Adaptive resampling algorithm for image zooming," *Proc. Inst. Electr. Eng. Vision, Image, Signal Processing*, vol. 144, no. 4 , pp. 207–212, 1997.
[206] E. Maeland, "On the comparison of interpolation methods," *IEEE Trans. Med. Imag.*, vol. 7, pp. 213–217, Sept. 1988.
[207] G. Wolberg, *Digital Image Warping*. Los Alamitos, CA: IEEE Comput. So. Press, 1990.
[208] B. Jähne, *Digital Image Processing: Concepts, Algorithms and Scientific Applications*, 4th ed. Berlin, Germany: Springer-Verlag, 1997.
[209] ——, *Practical Handbook on Image Processing for Scientific Applications*. Boca Raton, FL: CRC Press, 1997.
[210] A. Bovik, *Handbook of Image and Video Processing*. San Diego, CA: Academic, 2000.
[211] G. A. Baxes, *Digital Image Processing: Principles and Applications*. New York: Wiley, 1994.
[212] R. A. Schowengerdt, *Remote Sensing: Models and Methods for Image Processing*, 2nd ed. San Diego, CA: Academic, 1997.
[213] J.-L. Starck, F. Murtagh, and A. Bijaoui, *Image Processing and Data Analysis: The Multiscale Approach*. Cambridge, U.K.: Cambridge Univ. Press, 1998.
[214] M. Sonka, V. Hlavac, and R. Boyle, *Image Processing, Analysis and Machine Vision*, 2nd ed. Boston, MA: PWS-Kent, 1999.
[215] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd ed. Baltimore, MD: Johns Hopkins Univ. Press, 1989.
[216] M. Unser, A. Aldroubi, and M. Eden, "Fast B-spline transforms for continuous image representation and interpolation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 13, pp. 277–285, Mar. 1991.
[217] ——, "B-spline signal processing: Part I—Theory," *IEEE Trans. Signal Processing*, vol. 41, pp. 821–833, Feb. 1993.
[218] ——, "B-spline signal processing: Part II—Efficient design and applications," *IEEE Trans. Signal Processing*, vol. 41, pp. 834–848, Feb. 1993.
[219] C.-H. Lee, "Restoring spline interpolation of CT images," *IEEE Trans. Med. Imag.*, vol. 2, pp. 142–149, Sept. 1983.
[220] K. Toraichi, S. Yang, M. Kamada, and R. Mori, "Two-dimensional spline interpolation for image reconstruction," *Pattern Recognit.*, vol. 21, no. 3, pp. 275–284, 1988.
[221] A. Goshtasby, F. Cheng, and B. A. Barsky, "B-spline curves and surfaces viewed as digital filters," *Comput. Vis., Graphics, Image Processing*, vol. 52, no. 2, pp. 264–275, 1990.
[222] M. Unser, "Splines: A perfect fit for signal and image processing," *IEEE Signal Processing Mag.*, vol. 16, pp. 22–38, Nov. 1999.
[223] C. W. Farrow, "A continuously variable digital delay element," in *Proceedings of the IEEE International Symposium on Circuits and Systems*. Piscataway, NJ: IEEE Press, 1988, pp. 2641–2645.
[224] L. Erup, F. M. Gardner, and R. A. Harris, "Interpolation in digital modems—Part II: Implementation and performance," *IEEE Trans. Commun.*, vol. 41, pp. 998–1008, June 1993.
[225] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol. 13, pp. 30–60, Jan. 1996.
[226] S. R. Dooley, R. W. Stewart, and T. S. Durrani, "Fast on-line B-spline interpolation," *Electron. Lett.*, vol. 35, no. 14, pp. 1130–1131, 1999.
[227] K. Rajamani, Y.-S. Lai, and C. W. Farrow, "An efficient algorithm for sample rate conversion from CD to DAT," *IEEE Signal Processing Lett.*, vol. 7, pp. 288–290, Oct. 2000.
[228] J. Vesma, "A frequency-domain approach to polynomial-based interpolation and the Farrow structure," *IEEE Trans. Circuits Syst. II*, vol. 47, pp. 206–209, Mar. 2000.
[229] H. Peng-Yoke, "Ch'in Chiu-shao," in *Dictionary of Scientific Biography*, C. C. Gellispie and F. L. Holmes, Eds. New York: Scribner, 1970, vol. III, pp. 249–256.
[230] W. G. Horner, "A new method of solving numerical equations of all orders by continuous approximation," *Philos. Trans. R. Soc. London*, vol. 109, pp. 308–335, 1819.
[231] P. Henrici, *Elements of Numerical Analysis*. New York: Wiley, 1964.
[232] D. E. Knuth, "Seminumerical algorithms," in *The Art of Computer Programming*. Reading, MA: Addison-Wesley, 1969, vol. 2.
[233] J. von zur Gathen and J. Gerhard, *Modern Computer Algebra*. Cambridge, U.K.: Cambridge Univ. Press, 1999.
[234] P. Ruffini, *Sopra la Determinazione delle Radici nelle Equazioni Numeriche di Qualunque Grado*. Modena, Italy: La Società Italiana delle Scienze, 1804.
[235] F. Cajori, "Horner's method of approximation anticipated by Ruffini," *Bull. Amer. Math. Soc.*, vol. 17, no. 8, pp. 409–414, 1911.
[236] I. Newton, "De Analysi per Æquationes Numero Terminorum Infinitas," in *Analysis per Quantitates, Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis*, W. Jones, Ed. London, U.K., 1711, pp. 1–21.
[237] I. J. Schoenberg, "Notes on spline functions III: On the convergence of the interpolating cardinal splines as their degree tends to infinity," *Isr. J. Math.*, vol. 16, pp. 87–93, 1973.
[238] A. Aldroubi, M. Unser, and M. Eden, "Cardinal spline filters: Stability and convergence to the ideal sinc interpolator," *Signal Process.*, vol. 28, no. 2, pp. 127–138, 1992.
[239] M. Unser, A. Aldroubi, and M. Eden, "Polynomial spline signal approximations: Filter design and asymptotic equivalence with Shannon's sampling theorem," *IEEE Trans. Inform. Theory*, vol. 38, pp. 95–103, Jan. 1992.
[240] M. Unser, P. Thévenaz, and L. Yaroslavsky, "Convolution-based interpolation for fast, high-quality rotation of images," *IEEE Trans. Image Processing*, vol. 4, pp. 1371–1381, Oct. 1995.
[241] M. Unser, A. Aldroubi, and M. Eden, "Enlargement or reduction of digital images with minimum loss of information," *IEEE Trans. Image Processing*, vol. 4, pp. 247–258, Mar. 1995.
[242] C. Lee, M. Eden, and M. Unser, "High-quality image resizing using oblique projection operators," *IEEE Trans. Image Processing*, vol. 7, pp. 679–692, May 1998.
[243] M. Unser, A. Aldroubi, and M. Eden, "The L_2 polynomial spline pyramid," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 15, pp. 364–379, Apr. 1993.
[244] Y.-P. Wang and S. L. Lee, "Scale-space derived from B-splines," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 20, pp. 1040–1055, Oct. 1998.
[245] P. Thévenaz, U. E. Ruttimann, and M. Unser, "A pyramid approach to subpixel registration based on intensity," *IEEE Trans. Image Processing*, vol. 7, pp. 27–41, Jan. 1998.
[246] J. Kybic, P. Thévenaz, A. Nirkko, and M. Unser, "Unwarping of unidirectionally distorted EPI images," *IEEE Trans. Med. Imag.*, vol. 19, pp. 80–93, Feb. 2000.
[247] M. Unser, A. Aldroubi, and S. J. Schiff, "Fast implementation of the continuous wavelet transform with integer scales," *IEEE Trans. Signal Processing*, vol. 42, pp. 3519–3523, Dec. 1994.
[248] M. J. Vrhel, C. Lee, and M. Unser, "Fast continuous wavelet transform: A least-squares formulation," *Signal Process.*, vol. 57, no. 2, pp. 103–119, 1997.
[249] ——, "Rapid computation of the continuous wavelet transform by oblique projections," *IEEE Trans. Signal Processing*, vol. 45, pp. 891–900, Apr. 1997.
[250] C.-L. Huang and K.-C. Chen, "Directional moving averaging interpolation for texture mapping," *Graph. Models Image Process.*, vol. 58, no. 4, pp. 301–313, 1996.
[251] T. J. Lim and M. D. Macleod, "On-line interpolation using spline functions," *IEEE Signal Processing Mag.*, vol. 3, pp. 144–146, Sept. 1996.
[252] L. A. Ferrari, J. H. Park, A. Healey, and S. Leeman, "Interpolation using a fast spline transform (FST)," *IEEE Trans. Circuits Syst. I*, vol. 46, pp. 891–906, Aug. 1999.
[253] M. Unser, "Sampling—50 years after Shannon," *Proc. IEEE*, vol. 88, pp. 569–587, Apr. 2000.
[254] S. R. Marschner and R. J. Lobb, "An evaluation of reconstruction filters for volume rendering," in *Proceedings of the IEEE Conference on Visualization*, R. D. Bergerson and A. E. Kaufman, Eds. Los Alamitos, CA: IEEE Comput. Soc. Press, 1994, pp. 100–107.
[255] A. Schaum, "Theory and design of local interpolators," *Graph. Models Image Process.*, vol. 55, no. 6, pp. 464–481, 1993.
[256] J. J. Chen, A. K. Chan, and C. K. Chui, "A local interpolatory cardinal spline method for the determination of eigenstates in quantum-well structures with arbitrary potential profiles," *IEEE J. Quantum Electron.*, vol. 30, pp. 269–274, Feb. 1994.
[257] L. Knockaert and F. Olyslager, "Modified B-splines for the sampling of bandlimited functions," *IEEE Trans. Signal Processing*, vol. 47, pp. 2328–2332, Aug. 1999.
[258] N. A. Dodgson, "Quadratic interpolation for image resampling," *IEEE Trans. Image Processing*, vol. 6, pp. 1322–1326, Sept. 1997.

[259] I. German, "Short kernel fifth-order interpolation," *IEEE Trans. Signal Processing*, vol. 45, pp. 1355–1359, May 1997.
[260] E. H. W. Meijering, K. J. Zuiderveld, and M. A. Viergever, "Image reconstruction by convolution with symmetrical piecewise nth order polynomial kernels," *IEEE Trans. Image Processing*, vol. 8, pp. 192–201, Feb. 1999.
[261] M. Unser, "Approximation power of biorthogonal wavelet expansions," *IEEE Trans. Signal Processing*, vol. 44, pp. 519–527, Mar. 1996.
[262] M. Unser and I. Daubechies, "On the approximation power of convolution-based least squares versus interpolation," *IEEE Trans. Signal Processing*, vol. 45, pp. 1697–1711, July 1997.
[263] G. Strang and G. Fix, "A Fourier analysis of the finite element variational method," in *Constructive Aspects of Functional Analysis*. Rome, Italy: Edizioni Cremonese, 1973, Centro Internazionale Matematico Estivo, pp. 793–840.
[264] C. de Boor and R.-Q. Jia, "Controlled approximation and a characterization of the local approximation order," *Proc. Amer. Math. Soc.*, vol. 95, no. 4, pp. 547–553, 1985.
[265] W. A. Light, "Recent developments in the Strang-Fix theory for approximation orders," in *Curves and Surfaces*, P. J. Laurent, A. le Méhauté, and L. L. Schumaker, Eds. New York: Academic, 1991, pp. 285–292.
[266] W. A. Light and E. W. Cheney, "Quasiinterpolation with translates of a function having noncompact support," *Constructive Approx.*, vol. 8, no. 1, pp. 35–48, 1992.
[267] R.-Q. Jia and J. Lei, "Approximation by multiinteger translates of functions having global support," *J. Approx. Theory*, vol. 72, no. 1, pp. 2–23, 1993.
[268] T. Blu and M. Unser, "Approximation error for quasiinterpolators and (multi-)wavelet expansions," *Appl. Comput. Harmonic Anal.*, vol. 6, no. 2, pp. 219–251, 1999.
[269] C. de Boor, "Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$," *Trans. Amer. Math. Soc.*, vol. 341, no. 2, pp. 787–806, 1994.
[270] ——, "Quasiinterpolants and approximation power of multivariate splines," in *Computation of Curves and Surfaces*, W. Dahmen, M. Gasca, and C. A. Micchelli, Eds. Norwell, MA: Kluwer, 1990, pp. 313–345.
[271] J. Lei, "L_p-approximation by certain projection operators," *J. Math. Anal. Applicat.*, vol. 185, no. 1, pp. 1–14, 1994.
[272] C. de Boor, "The polynomials in the linear span of integer translates of a compactly supported function," *Constructive Approx.*, vol. 3, no. 2, pp. 199–208, 1987.
[273] C. K. Chui and H. Diamond, "A characterization of multivariate quasiinterpolation formulas and its applications," *Numerische Mathematik*, vol. 57, no. 2, pp. 105–121, 1990.
[274] T. Blu and M. Unser, "Quantitative Fourier analysis of approximation techniques: Part I—Interpolators and projectors," *IEEE Trans. Signal Processing*, vol. 47, pp. 2783–2795, Oct. 1999.
[275] ——, "Quantitative Fourier analysis of approximation techniques: Part II—Wavelets," *IEEE Trans. Signal Processing*, vol. 47, pp. 2796–2806, Oct. 1999.
[276] P. Thévenaz, T. Blu, and M. Unser, "Interpolation revisited," *IEEE Trans. Med. Imag.*, vol. 19, pp. 739–758, July 2000.
[277] ——, "Image interpolation and resampling," in *Handbook of Medical Imaging, Processing and Analysis*, I. N. Bankman, Ed. San Diego, CA: Academic, 2000, ch. Ch. 25, pp. 393–420.
[278] T. Blu, P. Thévenaz, and M. Unser, "MOMS: Maximal-order interpolation of minimal support," *IEEE Trans. Image Processing*, vol. 10, pp. 1069–1080, July 2001.
[279] A. Ron, "Factorization theorems for univariate splines on regular grids," *Isr. J. Math.*, vol. 70, no. 1, pp. 48–68, 1990.
[280] S. Dubuc, "Interpolation through an iterative scheme," *J. Math. Anal. Applicat.*, vol. 114, no. 1, pp. 185–204, 1986.
[281] G. Deslauriers and S. Dubuc, "Symmetric iterative interpolation processes," *Constructive Approx.*, vol. 5, no. 1, pp. 49–68, 1989.
[282] J.-L. Merrien, "A family of Hermite interpolants by bisection algorithms," *Numer. Algor.*, vol. 2, no. 2, pp. 187–200, 1992.
[283] S. Dubuc and J.-L. Merrien, "Dyadic Hermite interpolation on a rectangular mesh," *Adv. Comput. Math.*, vol. 10, no. 3/4, pp. 343–365, 1999.
[284] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992.
[285] S. Mallat, *A Wavelet Tour of Signal Processing*. San Diego, CA: Academic, 1998.
[286] D.-Q. Dai, T.-M. Shih, and F.-T. Chau, "Polynomial preserving algorithm for digital image interpolation," *Signal Process.*, vol. 67, no. 1, pp. 109–121, 1998.
[287] R. N. Bracewell, *The Fourier Transform and Its Applications*, 2nd ed. New York: McGraw-Hill, 1978.
[288] D. Kramer, A. Li, I. Simovsky, C. Hawryszko, J. Hale, and L. Kaufman, "Applications of voxel shifting in magnetic resonance imaging," *Invest. Radiol.*, vol. 25, no. 12, pp. 1305–1310, 1990.
[289] N. M. Hylton, I. Simovsky, A. J. Li, and J. D. Hale, "Impact of section doubling on MR angiography," *Radiol.*, vol. 185, no. 3, pp. 899–902, 1992.
[290] Y. P. Du, D. L. Parker, W. L. Davis, and G. Cao, "Reduction of partial-volume artifacts with zero-filled interpolation in three-dimensional MR angiography," *J. Magn. Reson. Imag.*, vol. 4, no. 5, pp. 733–741, 1994.
[291] W. F. Eddy, M. Fitzgerald, and D. C. Noll, "Improved image registration by using Fourier interpolation," *Magn. Reson. Med.*, vol. 36, no. 6, pp. 923–931, 1996.
[292] Q.-S. Chen, R. Crownover, and M. S. Weinhous, "Subunity coordinate translation with Fourier transform to achieve efficient and quality three-dimensional medical image interpolation," *Med. Phys.*, vol. 26, no. 9, pp. 1776–1782, 1999.
[293] K. P. Prasad and P. Satyanarayana, "Fast interpolation algorithm using FFT," *Electron. Lett.*, vol. 22, no. 4, pp. 185–187, 1986.
[294] S. Holm, "FFT pruning applied to time domain interpolation and peak localization," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1776–1778, Dec. 1987.
[295] M. R. Smith and S. T. Nichols, "Efficient algorithms for generating interpolated (zoomed) MR images," *Magn. Reson. Med.*, vol. 7, no. 2, pp. 156–171, 1988.
[296] D. Fraser, "Interpolation by the FFT revisited—An experimental investigation," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 37, pp. 665–675, May 1989.
[297] T. Smit, M. R. Smith, and S. T. Nichols, "Efficient sinc function interpolation technique for center padded data," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1512–1517, Sept. 1990.
[298] T. J. Cavicchi, "DFT time-domain interpolation," *Proc. Inst. Electr. Eng. F*, vol. 139, no. 3, pp. 207–211, 1992.
[299] T. Schanze, "Sinc interpolation of discrete periodic signals," *IEEE Trans. Signal Processing*, vol. 43, pp. 1502–1503, June 1995.
[300] L. P. Yaroslavsky, "Signal sinc-interpolation: A fast computer algorithm," *Bioimaging*, vol. 4, pp. 225–231, 1996.
[301] ——, "Efficient algorithm for discrete sinc interpolation," *Appl. Opt.*, vol. 36, no. 2, pp. 460–463, 1997.
[302] F. Candocia and J. C. Principe, "Sinc interpolation of discrete periodic signals," *IEEE Trans. Signal Processing*, vol. 46, pp. 2044–2047, July 1998.
[303] S. R. Dooley and A. K. Nandi, "Notes on the interpolation of discrete periodic signals using sinc function related approaches," *IEEE Trans. Signal Processing*, vol. 48, pp. 1201–1203, Apr. 2000.
[304] J. I. Agbinya, "Fast interpolation algorithm using fast Hartley transform," *Proc. IEEE*, vol. 75, pp. 523–524, Apr. 1987.
[305] C.-Y. Hsu and T.-P. Lin, "Novel approach to discrete interpolaion using the subsequence FHT," *Electron. Lett.*, vol. 24, no. 4, pp. 223–224, 1988.
[306] D. Liu, D. Wang, and Z. Wang, "Interpolation using Hartley transform," *Electron. Lett.*, vol. 28, no. 2, pp. 209–210, 1992.
[307] Z. Wang, "Interpolation using type I discrete cosine transform," *Electron. Lett.*, vol. 26, no. 15, pp. 1170–1172, 1990.
[308] J. I. Agbinya, "Interpolation using the discrete cosine transform," *Electron. Lett.*, vol. 28, no. 20, pp. 1927–1928, 1992.
[309] ——, "Two dimensional interpolation of real sequences using the DCT," *Electron. Lett.*, vol. 29, no. 2, pp. 204–205, 1993.
[310] G. Matheron, "Principles of geostatistics," *Econom. Geology*, vol. 58, pp. 1246–1266, 1963.
[311] D. G. Krige, "A statistical approach to some basic mine valuation problems on the Witwatersrand," *J. Chem., Metalll, Mining Soc. S. Africa*, vol. 52, no. 6, pp. 119–139, 1951.
[312] N. A. C. Cressie, *Statistics for Spatial Data*. New York: Wiley, 1993.
[313] N. Cressie, "The origins of kriging," *Math. Geology*, vol. 22, no. 3, pp. 239–252, 1990.
[314] J.-P. Chilès and P. Delfiner, *Geostatistics: Modeling Spatial Uncertainty*. New York: Wiley, 1999.
[315] M. L. Stein, *Interpolation of Spatial Data: Some Theory for Kriging*. Berlin, Germany: Springer-Verlag, 1999.

[316] N. S.-N. Lam, "Spatial interpolation methods: A review," *Amer. Cartograph.*, vol. 10, no. 2, pp. 129–149, 1983.
[317] L. S. Gandin, *Objective Analysis of Meteorological Fields*. Jerusalem, Israel: Israel Program Scientific Translations, 1965.
[318] O. Dubrule, "Comparing splines and kriging," *Comput. Geosci.*, vol. 10, no. 2/3, pp. 327–338, 1984.
[319] D. E. Myers, "Kriging, cokriging, radial basis functions and the role of positive definiteness," *Comput. Math. Applicat.*, vol. 24, no. 12, pp. 139–148, 1992.
[320] M. R. Stytz and R. W. Parrott, "Using kriging for 3-D medical imaging," *Comput. Med. Imaging Graph.*, vol. 17, no. 6, pp. 421–442, 1993.
[321] R. W. Parrott, M. R. Stytz, P. Amburn, and D. Robinson, "Toward statistically optimal interpolation for 3-D medical imaging," *IEEE Eng. Med. Biol.*, vol. 12, pp. 49–59, 1993.
[322] W. S. Kerwin and J. L. Prince, "The kriging update model and recursive space-time function estimation," *IEEE Trans. Signal Processing*, vol. 47, pp. 2942–2952, Nov. 1999.
[323] W.-Y. V. Leung, P. J. Bones, and R. G. Lane, "Statistical interpolation of sampled images," *Opt. Eng.*, vol. 40, no. 4, pp. 547–553, 2001.
[324] W.-C. Lin, C.-C. Liang, and C.-T. Chen, "Dynamic elastic interpolation for 3-D medical image reconstruction from serial cross sections," *IEEE Trans. Med. Imag.*, vol. 7, pp. 225–232, Sept. 1988.
[325] S.-Y. Chen, W.-C. Lin, C.-C. Liang, and C.-T. Chen, "Improvement on dynamic elastic interpolation technique for reconstructing 3-D objects from serial cross sections," *IEEE Trans. Med. Imag.*, vol. 9, pp. 71–83, Mar. 1990.
[326] D. Levin, "Multidimensional reconstruction by set-valued approximation," *IMA J. Numer. Anal.*, vol. 6, no. 2, pp. 173–184, 1986.
[327] S. P. Raya and J. K. Udupa, "Shape-based interpolation of multidimensional objects," *IEEE Trans. Med. Imag.*, vol. 9, pp. 32–42, Mar. 1990.
[328] G. T. Herman, J. Zheng, and C. A. Bucholtz, "Shape-based interpolation," *IEEE Comput. Graph. Applicat.*, vol. 12, pp. 69–79, May 1992.
[329] M. Joliot and B. M. Mazoyer, "Three-dimensional segmentation and interpolation of magnetic resonance brain images," *IEEE Trans. Med. Imag.*, vol. 12, pp. 269–277, June 1993.
[330] J.-F. Guo, Y.-L. Cai, and Y.-P. Wang, "Morphology-based interpolation for 3D medical image reconstruction," *Comput. Med. Imaging Graph.*, vol. 19, no. 3, pp. 267–279, 1995.
[331] T.-Y. Lee and W.-H. Wang, "Morphology-based three-dimensional interpolation," *IEEE Trans. Med. Imag.*, vol. 19, pp. 711–721, July 2000.
[332] V. Chatzis and I. Pitas, "Interpolation of 3-D binary images based on morphological skeletonization," *IEEE Trans. Med. Imag.*, vol. 19, pp. 699–710, July 2000.
[333] W. E. Higgins, C. Morice, and E. L. Ritman, "Shape-based interpolation of tree-like structures in three-dimensional images," *IEEE Trans. Med. Imag.*, vol. 12, pp. 439–450, Sept. 1993.
[334] G. J. Grevera and J. K. Udupa, "Shape-based interpolation of multidimensional grey-level images," *IEEE Trans. Med. Imag.*, vol. 15, pp. 881–892, Dec. 1996.
[335] K.-S. Chuang, C.-Y. Chen, L.-J. Yuan, and C.-K. Yeh, "Shape-based grey-level image interpolation," *Phys. Med. Biol.*, vol. 44, no. 6, pp. 1565–1577, 1999.
[336] K. Jensen and D. Anastassiou, "Subpixel edge localization and the interpolation of still images," *IEEE Trans. Image Processing*, vol. 4, pp. 285–295, Mar. 1995.
[337] S. Thurnhofer and S. K. Mitra, "Edge-enhanced image zooming," *Opt. Eng.*, vol. 35, no. 7, pp. 1862–1870, 1996.
[338] K. P. Hong, J. K. Paik, H. J. Kim, and C. H. Lee, "An edge-preserving image interpolation system for a digital camcorder," *IEEE Trans. Consumer Electron.*, vol. 42, pp. 279–284, Aug. 1996.
[339] G. Ramponi, "Warped distance for space-variant linear image interpolation," *IEEE Trans. Image Processing*, vol. 8, pp. 629–639, May 1999.
[340] W. K. Carey, D. B. Chuang, and S. S. Hemami, "Regularity-preserving image interpolation," *IEEE Trans. Image Processing*, vol. 8, pp. 1293–1297, Sept. 1999.
[341] Q. Gao and F.-F. Yin, "Two-dimensional direction-based interpolation with local centered moments," *Graph. Models Image Process.*, vol. 61, no. 6, pp. 323–339, 1999.
[342] X. Li and M. T. Orchard, "New edge-directed interpolation," *IEEE Trans. Image Processing*, vol. 10, pp. 1521–1527, Oct. 2001.
[343] P. W. M. Tsang and W. T. Lee, "An adaptive decimation and interpolation scheme for low complexity image compression," *Signal Process.*, vol. 65, no. 3, pp. 391–401, 1998.
[344] ——, "Two-dimensional adaptive decimation and interpolation scheme for low bit-rate image coding," *J. Vis. Commun. Image Represent.*, vol. 11, no. 4, pp. 343–359, 2000.
[345] R. Sánchez and M. Carvajal, "Wavelet based adaptive interpolation for volume rendering," in *Proceedings of the 1998 Symposium on Volume Visualization*. Los Alamitos, CA: IEEE Comput. Soc. Press, 1998, pp. 127–134.
[346] S. Schreiner, C. B. Paschal, and R. L. Galloway, "Comparison of projection algorithms used for the construction of maximum intensity projection images," *J. Comput.-Assisted Tomography*, vol. 20, no. 1, pp. 56–67, 1996.
[347] J. L. Ostuni, A. K. S. Santha, V. S. Mattay, D. R. Weinberger, R. L. Levin, and J. A. Frank, "Analysis of interpolation effects in the reslicing of functional MR images," *J. Comput.-Assisted Tomography*, vol. 21, no. 5, pp. 803–810, 1997.
[348] M. Haddad and G. Porenta, "Impact of reorientation algorithms on quantitative myocardial spect perfusion imaging," *J. Nucl. Med.*, vol. 39, no. 11, pp. 1864–1869, 1998.
[349] W. G. Kuhle, G. Porenta, S. C. Huang, M. E. Phelps, and H. R. Schelbert, "Issues in the quantitation of reoriented cardiac PET images," *J. Nucl. Med.*, vol. 33, no. 6, pp. 1235–1242, 1992.
[350] D. P. Roy and O. Dikshit, "Investigation of image resampling effects upon the textural information content of a high spatial resolution remotely sensed image," *Int. J. Remote Sens.*, vol. 15, no. 5, pp. 1123–1130, 1994.
[351] R. Hanssen and R. Bamler, "Evaluation of interpolation kernels for SAR interferometry," *IEEE Trans. Geosci. Remote Sens.*, vol. 37, pp. 318–321, Jan. 1999.
[352] A. Goshtasby, D. A. Turner, and L. V. Ackerman, "Matching of tomographic slices for interpolation," *IEEE Trans. Med. Imag.*, vol. 11, pp. 507–516, Dec. 1992.
[353] G. J. Grevera, J. K. Udupa, and Y. Miki, "A task-specific evaluation of three-dimensional image interpolation techniques," *IEEE Trans. Med. Imag.*, vol. 18, pp. 137–143, Feb. 1999.
[354] T. M. Lehmann, C. Gönner, and K. Spitzer, "Survey: Interpolation methods in medical image processing," *IEEE Trans. Med. Imag.*, vol. 18, pp. 1049–1075, Nov. 1999.
[355] P.-E. Danielsson and M. Hammerin, "High-accuracy rotation of images," *Graph. Models Image Process.*, vol. 54, no. 4, pp. 340–344, 1992.
[356] C. R. Appledorn, "A new approach to the interpolation of sampled data," *IEEE Trans. Med. Imag.*, vol. 15, pp. 369–376, June 1996.
[357] F. J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proc. IEEE*, vol. 66, pp. 51–83, Jan. 1978.
[358] E. H. W. Meijering, W. J. Niessen, and M. A. Viergever, "Quantitative evaluation of convolution-based methods for medical image interpolation," *Med. Image Anal.*, vol. 5, no. 2, pp. 111–126, 2001.

Erik Meijering (Member, IEEE) was born in The Netherlands, on July 31, 1971. He received the M.Sc. degree (*cum laude*) in electrical engineering from Delft University of Technology, Delft, The Netherlands, in 1996 and the Ph.D. degree from Utrecht University, Utrecht, The Netherlands, in 2000.

He is currently with the Biomedical Imaging Group of the Swiss Federal Institute of Technology, Lausanne, Switzerland. His current research interests include many aspects of digital image processing and analysis, such as reconstruction, registration, enhancement, segmentation, visualization, and interpolation, as well as their applications in biomedical imaging.

Dr. Meijering received a Fellowship by the Netherlands Organization for Scientific Research (NWO) for studying the theoretical and implementational aspects of adaptive interpolation techniques.