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PROLONGATIONS AND STABILITY IN DYNAMICAL SYSTEMS

by J. AUSLANDER and P. SEIBERT (1)

Introduction.

In this paper we present a unified theory of stability and boundedness in dynamical systems by means of prolongations. The notion of prolongation was first used, in a very special sense, by Poincaré and, subsequently, by Bendixson in their studies of the asymptotic behavior of trajectories in the plane. In a much more general sense, prolongations were considered by Ura [12, 13], who recognized their close relation to the concept of stability in the sense of Liapunov. Consider the map which associates to every point in the state space the positive semi-orbit issuing from it. The first prolongation is obtained by extending this map to one which is closed, (considered as a subset of the product space). By alternating extensions to maps which are transitive and closed respectively, we obtain a sequence of more and more extensive prolongations and, following Ura, associate to each of these a concept of stability: A compact invariant set is called Q-stable if it is invariant under the prolongation Q.

In particular, there exists a smallest prolongation which is both closed and transitive. The corresponding concept of

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stability is called absolute stability. This notion turns out to play a key role in another context, namely in connection with the « generalized Liapunov function », introduced by Zubov [15]. While Liapunov stability of a compact invariant set can be characterized by the existence of a generalized (not necessarily continuous) Liapunov function, it has been known that there exist cases of stable sets for which no *continuous* Liapunov function can be found (*e.g.*, certain critical points in the plane of the « center-focus » type). We prove (Theorem 6) that the existence of a continuous Liapunov function is necessary and sufficient for absolute stability.

It has been observed [14], that between the concepts of stability (in the sense of Liapunov) and boundedness (Lagrange stability), a kind of duality exists. In Chapter VI we formalize this duality by compactifying the phase space. In this way we obtain from every stability theorem a corresponding boundedness theorem.

In Chapter VII, some aspects of asymptotic stability are discussed. It is shown that asymptotic stability implies absolute stability. On the other hand, the dual concept, namely ultimate boundedness, implies the existence of a compact invariant set which is asymptotically stable in the large. While asymptotic stability cannot be characterized in terms of invariance under a prolongation, it is proved that it can indeed be characterized by the property of being the image of one of its neighborhoods under a map obtained from a prolongation by deleting the positive semi-orbit.

In the concluding chapter we study stability under persistent perturbations or, as we call it more briefly, « strict stability ». The dynamical system here is assumed to be given by a differential system in euclidean n-space. It is shown that strict stability can be characterized in terms of invariance under a closed, transitive map which has essential properties in common with the prolongations. Thus some results concerning absolute stability can be carried over. Moreover, strict stability occupies an intermediate place between asymptotic and absolute stability. The complete analysis of the relation between strict and asymptotic stability, however, requires the development of some additional methods and will therefore be published separately.

1. Definitions and notations.

1. In this section, we establish our notations, and also recall the basic notions in the theory of dynamical systems. X will denote a locally compact metric space with metric d. (In Chapter VI we shall assume in addition that X is second countable.) If $A \subset X$, $\bar{A}$ will denote the closure of A, A^0 the interior of A. R and R^+ denote the reals and the non-negative reals, respectively. If $\varepsilon > 0, S_\varepsilon(A) = [y \in X | d(y, A) < \varepsilon]$. A set will be called *relatively compact* if its closure is compact. The boundary of a set A we denote by ∂A.

By a *dynamical system* or *continuous flow* $\mathcal{F}$ on X, we mean a continuous map $\pi : X \times R \to X$ satisfying

$$(a) \qquad \pi(x, 0) = x \quad (x \in X),$$
$$(b) \quad \pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2) \qquad (x \in X; t_1, t_2 \in R).$$

Typically, dynamical systems arise from the solution curves of autonomous systems of differential equations, $\dot{x} = f(x)$, if f satisfies suitable hypotheses [10, p. 17ff]. However, except for Chapter VIII, we shall consider dynamical systems abstractly without explicit reference to a system of differential equations. As general references, consult [2], [10], and [15].

If $x \in X$, the set $\{\pi(x, t) | t \in R\}$ is called the *orbit* or *trajectory* through x, and will be denoted by $\gamma(x)$. The positive *semi-orbit*, denoted by $\gamma^+(x)$, is the set $\{\pi(x, t) | t \geqslant 0\}$. The negative *semi-orbit* $\gamma^-(x)$ is defined analogously. The *omega limit* set of x, $\Omega(x)$, is the set $\bigcap \{\overline{\gamma^+(\pi(x, t))} | t \geqq 0\}$; clearly $\Omega(x)$ is the set of points y for which there exists a sequence $\{t_n\}$ of real numbers with $t_n \to +\infty$ and $\pi(x, t_n) \to y$. Similarly, the alpha limit set of x, $A(x)$, is defined to be $\bigcap \{\overline{\gamma^-(\pi(x, t))} | t \leqq 0\}$.

A subset A of X is called *invariant* if $\pi(x, t) \in A$ whenever $x \in A$ and t is real. If $x \in A$ and $t \geqq 0$ imply $\pi(x, t) \in A$, we say that A is *positively invariant.* We remark that the alpha and omega limit sets of a point are invariant.

In conformity with current pratice, we shall suppress the map π notationally; if $x \in X$ and $t \in R$, we write xt in place of $\pi(x, t)$.

2. In this paper we shall frequently be concerned with maps from X to 2^X (the set of all subsets of X). If $Q : X \to 2^X$, and $A \subset X$, then $Q(A) = \bigcup \{Q(x) | x \in A\}$. If a family *of maps* $Q : X \to 2^X$ ($\alpha \in \mathfrak{A}$, *some index set*) is given, by $\bigcup \{Q_\alpha | \alpha \in \mathfrak{A}\}$ we mean the map $Q : X \to 2^X$ defined by $Q(x) = \bigcup \{Q_\alpha(x) | \alpha \in \mathfrak{A}\}$. Finally, if m is a positive integer, the map: $X \to 2^X$ is defined inductively by $Q^1 = Q$, and $Q^m = Q \circ Q^{m-1}$.

2. The first prolongation.

3. Let $x \in X$, and let $\mathfrak{N}(x)$ denote the neighborhood filter of x. Following Ura ([12], [13]), we define the first prolongation of x, denoted by $D_1(x)$, by $\bigcap \{\overline{\gamma^+(W)} | W \in \mathfrak{N}(x)\}$.

It is easy to see that $y \in D_1(x)$ if and only if there exist sequences $x_n \in X$ and $t_n \geqslant 0$ such that $x_n \to x$ and $x_n t_n \to y$. The first prolongation may be regarded as an extension of the orbit closure of x. Indeed, it is an immediate consequence of the definition that $\overline{\gamma^+(x)} \subset D_1(x)$.

A simple example of a non-trivial prolongation (that is, $D_1(x) \neq \overline{\gamma^+(x)}$) is provided by the dynamical system in the plane defined by the differential equations $\dot{x}_1 = x_1$, $\dot{x}_2 = -x_2$; this is a system with a saddle point at the origin. Let $x = (0, -1)$ Then $\overline{\gamma^+(x)}$ consists of the points $(0, x_2)$, with $-1 \leqq x_2 \leqq 0$, whereas $D_1(x)$ contains, in addition to $\overline{\gamma^+(x)}$, all points of the x_1-axis.

A second example is furnished by the differential equation (in polar coordinates) $\dot{r} = r(r-1)^2$, $\dot{\theta} = 1$, which has an unstable critical point at the origin and a limit cycle, stable from the inside and unstable from the outside, at $r = 1$. In this case, the first prolongation of the origin is the closed unit disc.

4. We observe the following elementary properties of the first prolongation:

a) If A is compact, and $x \in A$, then $D_1(x) \subset A$, or $D_1(x)$ meets the boundary of A.

b) If $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $y_n \in D_1(x_n)$, and if $x_n \to x$, $y_n \to y$, then $y \in D_1(x)$.

c) If A is compact, $D_1(A)$ is closed. (This is a consequence of *b*).)

It is not difficult to verify these properties directly. However, they will follow immediately from developments in Chapter III.

5. The first prolongation is intimately connected with the notion of Liapunov stability. We recall that the compact positively invariant set M is said to be *Liapunov stable* (or simply *stable*) if for every neighborhood U of M, there is a neighborhood W of M with $\gamma^+(W) \subset U$. It is not difficult to show that M is stable if and only if $D_1(M) = M$ ([12, p. 341]). In the general case (M not necessarily stable), $D_1(M)$ may be regarded as a « measure of instability » of M. In this connection, we have :

THEOREM 1. — *Let* M *be a compact positively invariant set. Then* $D_1(M)$ *is the intersection of all closed positively invariant neighborhoods of* M.

Proof. — Let $y \in D_1(M)$, and let W be a closed positively invariant neighborhood of M. Choose $x \in M$ such that $y \in D_1(x)$. Then $y \in \overline{\gamma^+(W)} = W$. Thus $D_1(M) \subset W$.

Now suppose $y \notin D_1(M)$. Then $y \notin D_1(x)$, for each $x \in M$. Therefore, if $x \in M$, there is *a* $W(x) \in \mathcal{N}(x)$ with $y \notin \overline{\gamma^+(W(x))}$. By compactness of M, there exist $x_1, \ldots, x_k \in M$ such that $M \subset W = \cup \{W(x_i) | i = 1, \ldots, k\}$. Let $W^* = \overline{\gamma^+(W)}$. W^* is a closed positively invariant neighborhood of M and $y \notin W^*$. The proof is completed.

COROLLARY 1. — *The compact positively invariant set* M *is Liapunov stable if and only if every neighborhood of* M *contains a positively invariant neighborhood of* M.

3. Abstract prolongations and semi-prolongations.

6. We now wish to generalize the notion of prolongation. Toward this end, we define two operators, $\mathcal{D}$ and $\mathcal{S}$, on the class of maps from X to 2^X. If $Q : X \to 2^X$, we define $\mathcal{D}Q$ by

$$\mathcal{D}Q(x) = \bigcap_{W \in \mathcal{N}(x)} \overline{Q(W)}.$$

is defined by

$$\mathcal{J}Q(x) = \bigcup_{n=1,\dots} Q^n(x).$$

We note that $y \in \mathcal{D}Q(x)$ if and only if there are sequences $\{x_n\}$ and $\{y_n\}$ with $y_n \in Q(x_n)$ such that $y_n \to y$ and $x_n \to x$. Also $y \in \mathcal{J}Q(x)$ if and only if there are points $x_1, \dots, x_n$ in X with $x = x_1$, $y = x_n$, and $x_{j+1} \in Q(x_j)(j = 1, \dots, n-1)$.'

The operator $\mathcal{D}$ may be considered a closure operator, in the following sense. Let S denote the relation in X defined by : $(x, y) \in S$ if and only if $y \in Q(x)$. Then it is readily verified that $y \in \mathcal{D}Q(x)$ if and only if $(x, y) \in \overline{S}$.

The following statements follow easily from the definition of $\mathcal{D}$ and $\mathcal{J}$, and from the above remarks.

(*a*) $\mathcal{D}^2 = \mathcal{D}$, and $\mathcal{J}^2 = \mathcal{J}$; that is, $\mathcal{D}$ and $\mathcal{J}$ are idempotent operators.

(*b*) If A is compact, $\mathcal{D}Q(A)$ is closed.

(*c*) Suppose V is a continuous real valued function on X, such that $y \in Q(x)$ implies $V(y) \leqslant V(x)$. Then $y \in \mathcal{D}Q(x) \cup \mathcal{J}Q(x)$ implies $V(y) \leqslant V(x)$.

Definition. — An *abstract prolongation* (or simply *prolongation*) is a map $Q : X \to 2^X$ satisfying

(α) If $x \in X$, then $\gamma^+(x) \subset Q(x)$.

(β) $\mathcal{D}Q = Q$.

(γ) If A is a compact subset of X, and $x \in A$, then either $Q(x) \subset A$, or $Q(x)$ meets the boundary of A.

If the map $Q : X \to 2^X$ satisfies (α) and (γ) above, but not necessarily (β), it will be called a *semi-prolongation.*

If Q is a semi-prolongation, and $\mathcal{J}Q = Q$, then Q is said to be *transitive.*

7. The following lemma indicates how, given a collection of semi-prolongations, new prolongations and semi-prolongations can be formed.

Lemma 1. — (i). *If* $\{Q_\beta\}$, $(\beta \in \mathcal{B})$, *is a collection of semi-prolongations, and* $Q = \bigcup\{Q_\beta | \beta \in \mathcal{B}\}$, *then* Q *is a semi-prolongation.*

(ii) *If* Q_1 *and* Q_2 *are semi-prolongations and* $Q = Q_1 \circ Q_2$, *then* Q *is a semi-prolongation.*

(iii) *If* Q *is a semi-prolongation,* $\mathcal{J}Q$ *is a semi-prolongation and* $\mathcal{D}Q$ *is a prolongation.*

Proof. — (i) That Q satisfies property (α) is obvious. Suppose A is compact, $x \in A$, and $Q(x) \not\subset A$. Then for some $\beta \in \mathcal{B}$, $Q_\beta(x) \not\subset A$, and therefore $Q_\beta(x) \cap \partial A \neq \emptyset$. Therefore, $Q(x) \cap \partial A \neq \emptyset$, and property (γ) is verified.

(ii) If $x \in X$, $\gamma^+(x) \subset Q_1(x) \subset Q_1(Q_2(x)) = Q(x)$, so (α) holds. Let A be compact, and suppose $x \in A$ with $Q(x) \not\subset A$. If there is a $z \in Q_2(x) \cap \partial A$, then $z \in Q_1(z) \subset Q_1(Q_2(x)) = Q(x)$, so $z \in Q(x) \cap \partial A$. If $Q_2(x) \subset A$, then, since $Q_1(Q_2(x)) \not\subset A$, there exists $z \in Q_2(x)$ with $Q_1(z) \not\subset A$. Then, there is a $y \in Q_1(z) \cap \partial A$. It follows that $y \in Q(x) \cap \partial A$.

(iii) If Q is a semi-prolongation, it follows from (i) and (ii) that $\mathcal{J}Q$ is a semi-prolongation. We show that $\mathcal{D}Q$ is a prolongation. Since $\gamma^+(x) \subset Q(x) \subset \mathcal{D}Q(x)$, property (α) holds, and since $\mathcal{D}^2 = \mathcal{D}$, (β) is satisfied. We show that (γ) holds. Let $x \in A$, a compact subset of X. It is clear that we need only consider the case in which $Q(x) \subset A$, but $\mathcal{D}Q(x) \not\subset A$. If $x \in \partial A$, then $x \in \mathcal{D}Q(x) \cap \partial A$, and there is nothing to prove. Therefore, suppose $x \in A^0$ and let $y \in \mathcal{D}Q(x)$ with $y \notin A$. Then there are sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$, $y_n \in Q(x_n)$, and $y_n \to y$. We may assume $x_n \in A$, and $y_n \notin A$ (since A is closed). Now, since Q is a semi-prolongation, there exist $y'_n \in Q(x_n) \cap \partial A$, and since ∂A is compact, we may assume $y'_n \to y' \in \partial A$. Then $y' \in \mathcal{D}Q(x) \cap \partial A$, and the proof of (γ) is completed.

THEOREM 2. — *Let* M *be a compact subset of* X, *and let* Q *be an abstract prolongation. Then* $Q(M) = M$ *if and only if, whenever* W *is a neighborhood of* M, *there is a neighborhood* U *of* M *such that* $Q(U) \subset W$.

Proof. — Suppose $Q(M) = M$, and suppose there is a neighborhood W of M, such that for every neighborhood U of M, $Q(U) \not\subset W$. It is no loss of generality to assume that W is compact.

Then there exist sequences $\{x_n\}$ and $\{y_n\}$, with $y_n \in Q(x_n)$, $x_n \to M$, and $y_n \notin W$. Since M is compact, we may assume that

$x_n \to x \in M$. By property (γ) in the definition of prolongation, there exist $y'_n \in Q(x_n) \cap \partial W$, and since ∂W is compact, we may assume that $y'_n \to y' \in \partial W$. Then $y' \in \mathfrak{D}Q(x) = Q(x) \subset Q(M) = M$, which is a contradiction.

To prove the converse statement, suppose that $y \notin M$. Let W be a neighborhood of M such that $y \notin W$, and let U be a neighborhood of M with $Q(U) \subset W$. Then, $Q(M) \subset Q(U) \subset W$, so $y \notin Q(M)$. Therefore $Q(M) \subset M$, and since $M \subset Q(M)$, the proof is completed.

Corollary 2. — *Let* M *be a compact subset of* X *and let* Q *be a transitive prolongation. Then* $Q(M) = M$ *if and only if* M *possesses a fundamental system of compact neighborhoods* $\{U_n\}$ *such that* $Q(U_n) = U_n$.

Indeed, if $\{W_n\}$ is any fundamental system of neighborhoods, choose N_n compact and such that $Q(N_n) \subset W_n$, and define $U_n = Q(N_n)$.

4. The higher prolongations and stability of order α.

9. If $x \in X$, let $E_0(x)$ be $\gamma^+(x)$, the positive semi-orbit of x. Clearly E_0 is a transitive semi-prolongation. Then, by lemma 1, $\mathfrak{D}E_0 = \mathfrak{D}\mathcal{I}E_0$ is a prolongation, and indeed it is equal to D_1, as defined in Chapter II. We define $E_1 = \mathcal{I}D_1$, and $D_2 = \mathfrak{D}E_1$.

Now, let α be any ordinal number. We define the prolongation D_α inductively. Suppose for every ordinal $\beta < \alpha$, the prolongation D_β has been defined. Let $E_\beta = \mathcal{I}D_\beta$, and let $E^*_\alpha = \bigcup\{E_\beta | \beta < \alpha\}$. Define $D_\alpha = \mathfrak{D}E^*_\alpha$.

Observe that $y \in D_\alpha(x)$ if and only if there are sequences $\{x_n\}$ and $\{y_n\}$ in X with $x_n \to x$, $y_n \to y$, and $y_n \in D^{k_n}_{\beta_n}(x_n)$, where β_n are ordinal numbers less than α, and k_n are positive integers.

By Lemma 1, E_β and E^*_α are semi-prolongations, and therefore D_α is a prolongation. Observe that if $\beta < \alpha$, we have $D_\beta \subset E_\beta \subset E^*_\alpha \subset D_\alpha$. If α is a successor ordinal, $E^*_\alpha = E_{\alpha-1}$, and $D_\alpha = \mathfrak{D}E_{\alpha-1}$. ([2]).

([2]) These are essentially the same as the prolongations D^+_β of Ura [13]. However Ura includes the semi-prolongations E_β, as well as the D_α among his transfinite sequence $\{D^+_\beta\}$; therefore our system of numbering differs from his.

THEOREM 3. — *Let* γ *denote the first uncountable ordinal number. Then:*

(i) $D_\gamma = \bigcup \{D_\beta | \beta < \gamma\}$;

(ii) If $\gamma' > \gamma$, *then* $D_{\gamma'} = D_\gamma$.

(Therefore D_γ *is a transitive prolongation.)*

Proof. — (i) Let $y \in D_\gamma(x)$. Then there are sequences $\{x_n\}$, $\{y_n\}$ in X, and a sequence $\{\beta_n\}$ of ordinal numbers, such that $\beta_n < \gamma$, $y_n \in E_{\beta_n}(x_n)$, $x_n \to x$, and $y_n \to y$. Let β be an ordinal number such that $\beta_n < \beta < \beta + 1 < \gamma$. Such ordinals exist, [3, p. 30]. Then $y_n \in E_\beta(x_n)$, and $y \in D_{\beta+1}(x)$.

(ii) We first show that D_γ is transitive, or, what is the same thing, that $D_\gamma^2 = D_\gamma$. Suppose that $y \in D_\gamma(x)$, and $z \in D_\gamma(y)$. Let $\beta < \gamma$ such that $y \in D_\beta(x)$ and $z \in D_\beta(y)$. Then

$$z \in D_\beta^2(x) \subset E_\beta(x) \subset D_\gamma(x).$$

Hence we have $E_\gamma = \mathcal{J} D_\gamma = D_\gamma$. Then $D_{\gamma+1} = \mathfrak{D} E_\gamma = \mathfrak{D} D_\gamma = D_\gamma$. A simple induction show that if $\gamma' > \gamma$, $D_{\gamma'} = D_\gamma$.

We shall frequently write D instead of D_γ. By Theorem 3, $\mathcal{J} D = \mathfrak{D} D = D$; therefore D *is the smallest closed transitive map containing the positive semi-orbit.*

10. *Definition.* — Let M be a compact positively invariant set, and let α be an ordinal number. M is said to be *stable of order* α, or α*-stable*, if $D_\alpha(M) = M$.

If M is α-stable for every ordinal number α, then M is said to be *absolutely stable.*

Theorem 3 tells us that M is absolutely stable if and only if M is stable of order γ, where γ denotes the first uncountable ordinal.

Note that stability of order 1 is the same thing as Liapunov stability.

THEOREM 4. ([13]). — *Let* M *be a compact positively invariant set. Then the following statements are equivalent:*

(i) M *is stable of order* α.

(ii) *If* W *is a neighborhood of* M, *there exists a neighborhood* U *of* M *such that* $D_\alpha(U) \subset W$.

Proof. — This is an immediate consequence of Theorem 2.

11. We conclude this chapter with some examples which will illustrate the notions of prolongation and stability of order α; (cf. also [13], pp 191-194 [3]). The examples are all special cases of the equation

$$\ddot{x} + f(x^2 + \dot{x}^2)\dot{x} + x = 0.$$

To every zero of the function $f(r^2) = f(x^2 + \dot{x}^2)$ there corresponds a limit cycle $x^2 + \dot{x}^2 = r^2$. The orbits between two neighboring limit cycles are spirals with decreasing or increasing distance from the origin, depending upon the sign of f. The compact invariant set M under consideration is the origin.

a) Let $f(r^2) = -r \sin^2 \frac{\pi}{r^2}$ for $0 < r$, and let $f(0) = 0$, (figure 1). Then $D_1(0) = \{0\}$, so the origin is stable of order 1.

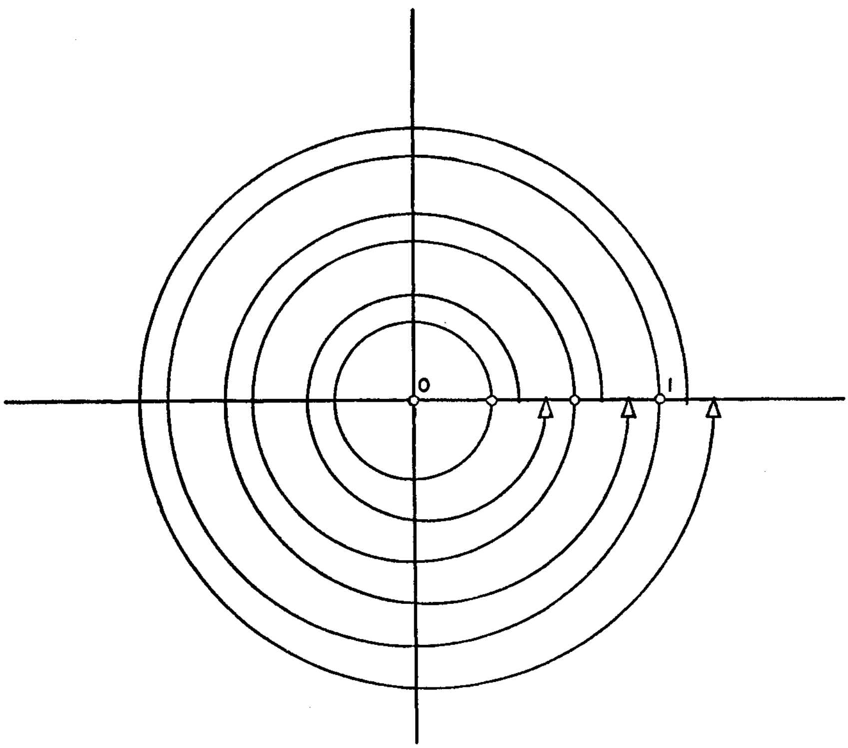

FIG. 1.

[3] Added in proof: This example was also considered by N. N. Krasovskii [Stability of motion, Stanford 1963 (Russian original: Moscow 1959), pp. 46 f]. He also pointed out that the construction of a continuons Liapunov function of the form $V(x)$ is not possible in this case.

For n a positive integer, let C_n denote the circle $r = \dfrac{1}{\sqrt{n}}$; C_n is an invariant set. If $p \in C_n$ $(n > 2)$, then $D_1(p)$ consists of the closed annulus A_n bounded by C_n and C_{n-1}, and therefore $E_1(p) = \bigcup \{A_k | k \leq n\}$. Now let $p_n \in C_n$. Then $p_n \to 0$, as $n \to \infty$, and it follows that $D_2(0)$ consists of the entire unit disc. Hence the origin is not stable of order 2.

b) Let

$$f(r^2) = \begin{cases} 0, & \text{for } r = \dfrac{1}{n + \dfrac{1}{m}} \quad (n = 0, 1, 2, \ldots; m = 1, 2, \ldots) \\ < 0, & \text{elsewhere.} \end{cases}$$

An analysis similar to that given in the preceding example shows that the origin is stable of order 2, but not of order 3. If n is any positive integer, it is clear that we may define a function f_n, similar to f above, so that in the dynamical system $\mathcal{F}_n$ determined by the equation

$$\ddot{x} + f_n(x^2 + \dot{x}^2)\dot{x} + x = 0,$$

the origin is stable of order n, but not of order $n + 1$.

By appropriately combining the $\mathcal{F}_n$, we may define a dynamical system $\mathcal{F}$ which is m-stable, for every positive integer m, but not stable of order ω (where ω denotes the first infinite ordinal). We may suppose that $f_n\left(\dfrac{1}{k}\right) = 0$, $\begin{cases} (n = 1, 2, \ldots; \\ k = n - 1, n). \end{cases}$

Now define $g(r^2) = f_n(r^2)$ for $\dfrac{1}{\sqrt{n}} \leqslant r \leqslant \dfrac{1}{\sqrt{n-1}}$, and $g(0)=0$.

Then $\mathcal{F}$, the dynamical system determined by

$$\ddot{x} + g(x^2 + \dot{x}^2)\dot{x} + x = 0,$$

coincides with $\mathcal{F}_n$ on the annulus $B_n = \left\{ r \middle| \dfrac{1}{\sqrt{n}} \leqslant r \leqslant \dfrac{1}{\sqrt{n-1}} \right\}$.

Now let m be a fixed positive integer, and let $\{x_k\}$ be a sequence tending to 0. Then, for k sufficiently large, $x_k \in \bigcup \{B_n | n \geqslant m\}$, and if $y_k \in E_{m-1}(x_k)$, then $y_k \to 0$. Hence $\{0\}$ is stable of order m. On the other hand, let $x_n \to 0$, with $x_n \in B_n$. Then $y = (1, 0) \in E_{n+1}(x_n) \subset E^*_\omega(x_n)$, and $(1, 0) \in D_\omega(0)$.

c) Let $f(r^2) = r \sin \frac{\pi}{r^2}$ (figure 2). Here we have stable limit cycles alternating with unstable ones. The origin is absolutely stable.

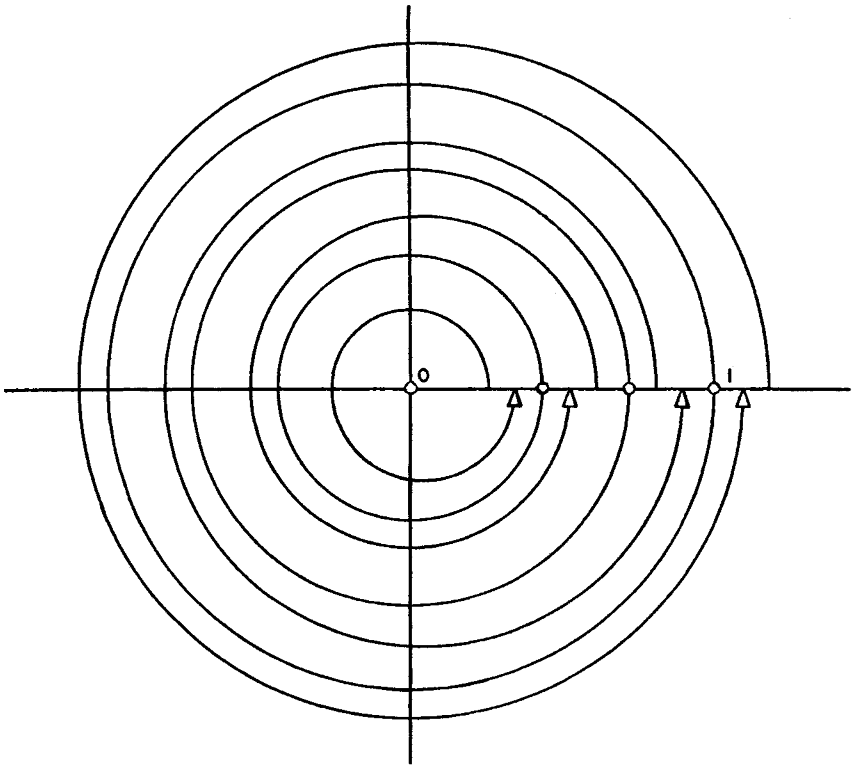

FIG. 2.

5. Liapunov functions.

12. The study of stability in dynamical systems has been facilitated by the use of generalized Liapunov functions [6], [15].

DEFINITION. — Let M be a compact positively invariant set. A *generalized Liapunov function* for M is a non-negative function V defined in a positively invariant neighborhood W of M, and satisfying:

a) If $\varepsilon > 0$, then there exists $\lambda > 0$ such that $V(x) > \lambda$, for x not in $S_\varepsilon(M)$.

b) If $\lambda > 0$, there exists $\eta > 0$ such that $V(x) < \lambda$, for $x \in S_\eta(M)$.

c) If $x \in W$, and $t \geq 0$, then $V(xt) \leq V(x)$.

Conditions *a*) and *b*) may be succinctly summarized by the condition:

If $\{x_n\}$ is a sequence in W, then $V(x_n) \to 0$ if and only if $x_n \to M$. (In particular, $V(x) = 0$ if and only if $x \in M$.)

We shall usually omit the adjective « generalized » and speak simply of Liapunov functions.

The following theorem is a purely topological version of one of Liapunov's stability theorems. It may be found in [6] and [15].

THEOREM 5. — *The compact set* M *is Liapunov stable if and only if there exists a generalized Liapunov functions for* M.

LEMMA 2. *Let* V *be a generalized Liapunov function for the compact positively invariant set* M. *Let* $W_\lambda = \{x \in X | V(x) \leq \lambda\}$. *Then the sets* $\{W_\lambda | \lambda > 0\}$ *constitue a fundamental systems of neighborhoods of* M.

This is an easy consequence of the definition.

13. A generalized Liapunov function is not necessarily continuous (although it is always possible, in the case of Liapunov stability, to find a Liapunov function which is continuous on every orbit). The role of continuity of the genralized Liapunov function is demonstrated by the following theorem.

THEOREM 6. — *Let* M *be a compact subset of* X. *Then the following are equivalent:*

(*a*) *There is a generalized Liapunov function* V *for* M *which is continuous in some neighborhood* W *of* M.

(*b*) M *possesses a fundamental system of absolutely stable compact neighborhoods.*

(*c*) M *is absolutely stable.*

Proof. — We show $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. $(a) \Rightarrow (b)$: Since X is locally compact, we may assume that W is compact. Using the notation of Lemma 2, let $\eta > 0$ such that $W_\eta \subset W$. Then $\{W_\lambda | 0 < \lambda < \eta\}$ is a fundamental system of compact neighborhoods of M. We show that each W_λ is absolutely stable. Let $0 < \lambda < \eta$. Let $x \in W_\lambda$, and let $y \in D_1(x)$. Then there exist $x_n \to x$, $t_n \geq 0$, such that $x_n t_n \to y$. Now, $x \in W_\lambda$, so

$x_n \in W_\eta \subset W$, for $n \geqq n_0$. Then $V(x_n t_n) \leqq V(x_n)$, and by continuity of V on W, it follows that $V(y) \leqq V(x)$. That is, if $\lambda < \eta$, $D_1(W_\lambda) \subset W_\lambda$. Now, let α be an ordinal number, and suppose that $D_\beta(W_\lambda) \subset W_\lambda$, for all $\lambda < \eta$, and all ordinals $\beta < \alpha$. Then, $E_\beta(W_\lambda) \subset W_\lambda$ and therefore $E^*_\alpha(W_\lambda) \subset W_\lambda$. By an argument similar to that above for D_1, we obtain

$$D_\alpha(W_\lambda) \subset W_\lambda.$$

Since α is arbitrary, the sets W_λ are absolutely stable.

$(b) \Longrightarrow (c)$. Let γ denote the first uncountable ordinal number. Since D_γ is a transitive prolongation, (c) follows immediately from Corollary 2.

$(c) \Longrightarrow (a)$. Suppose that M is absolutely stable. Then, for each dyadic rational number $\lambda = j/2^n (n = 0, 1, 2, \ldots; j$ an integer such that $1 \leqq j \leqq 2^n)$, we construct a set W_λ such that (1) W_λ is a compact neighborhood of M, (2). If $\lambda < \lambda'$, $W_\lambda \subset$ interior $W_{\lambda'}$, (3) W_λ is absolutely stable, and (4) $\bigcap \{W_\lambda | \lambda$ a dyadic rational$\} = M$.

To see that such a construction is possible, first obtain a fundamental system of compact absolutely stable neighborhoods $W_{1/2^n}$, $n = 0, 1, \ldots$, such that $W_{1/2^{n+1}} \subset$ interior $W_{1/2^n}$, $n = 0, 1, \ldots$. This is possible by virtue of Corollary 2. Now, to define, for example $W_{3/4}$, observe that W_1 is a compact neighborhood of the absolutely stable set $W_{1/2}$. Then, again by Corollary 2, we may find an absolutely stable compact neighborhood $W_{3/4}$ of $W_{1/2}$ such that $W_{3/4} \subset$ interior of W_1. Proceeding in this manner, we can define the sets W_λ (λ a dyadic rational) with the required properties.

If $x \in W$, define $V(x) = \inf\ [\lambda | x \in W_\lambda]$. Clearly $V(x) = 0$ if and only if $x \in M$. Let $t > 0$. We show $V(xt) \leqq V(x)$. Suppose $V(xt) > V(x)$. Then there is a dyadic rational λ with $V(xt) > \lambda > V(x)$. Then $x \in W_\lambda$, and, for any ordinal number α, $xt \in D_\alpha(x) \subset D_\alpha(W_\lambda) \subset W_\lambda$. That is, $V(xt) \leqq \lambda$, which is a contradiction.

Finally we show that V is continuous on W_1^0. If not, then for some $x \in W_1^0$, (say $V(x) = \tau$), there exists a sequence $x_n \to x$ such that (i) $V(x_n) \to \tau' < \tau$ or (ii) $V(x_n) \to \tau' > \tau$. In case (i), let λ, λ' be dyadic rationals such that

$$\tau' < \lambda' < \lambda < \tau.$$

Then $x \notin W_\lambda$, and $x_n \in W_{\lambda'}$, for n sufficiently large. Since $W_{\lambda'} \subset W_\lambda$, we have $x_n \subset W_\lambda$, and since W_λ is closed, $x \in W_\lambda$, which is a contradiction. In case (ii) let λ, λ' be dyadic rationals with $\tau < \lambda < \lambda' < \tau'$. Then $V(x_n) > \lambda'$ for n sufficiently large, and $x_n \notin W_{\lambda'}$. Now, $x \in W_\lambda \subset W^0_{\lambda'}$. But $x_n \to x$, and since $x_n \notin W_{\lambda'}$, $x_n \notin W^0_{\lambda'}$. Again we have reached a contradiction, and the proof is completed.

14. In conclusion, we remark that the developments in this and the preceding chapter could just as well have been applied to *any* semi-prolongation Q_0, and the successive prolongations Q_α obtained by alternate applications of $\mathcal{I}$ and $\mathcal{D}$. Then we would have a notion of « Q_α-stability » defined by $Q_\alpha(M) = M$. In particular, Theorem 6 may be formulated in terms of a semi-prolongation Q_0, the smallest transitive prolongation Q containing Q_0, and a continuous non-negative function V with the property that $V(y) \leqq V(x)$ if $y \in Q_0(x)$, (see Lemma 1 in [1]). The proof is an exact paraphrase of the proof of Theorem 6. We shall make use of these remarks in Chapter VIII.

6. The duality between boundedness and stability.

15. The dynamical system $\mathcal{F}$ is said to be *bounded* or *Lagrange stable* if $\overline{\gamma^+(x)}$ is compact for every $x \in X$. It is natural to generalize this notion as follows. If α is an ordinal number, we say that $\mathcal{F}$ is *bounded of order* α, or α-*bounded*, if $D_\alpha(x)$ is compact, for every $x \in X$. This is easily seen to be equivalent to the assertion that $D_\alpha(A)$ is compact whenever A is compact. $\mathcal{F}$ is said to be *absolutely bounded* if it is α-bounded for every ordinal number α.

In this section, we assume that X is second countable (as well as locally compact metric). Under this assumption, it turns out that there is a kind of duality between boundedness and stability. This duality may be established by means of the following device. For $x \in X$, let $E_0^-(x) = \gamma^-(x)$, the negative semi-orbit of x. We may define negative prolongations D_α^- in a manner completely analogous to the definitions of D_α, that is, we let $D_1^- = \mathcal{D}E_0^-$, $E_1^- = \mathcal{I}D_1^-$, and so on. Then, for any ordinal number α, $D_\alpha^- = \mathfrak{D}\Big(\bigcup_{\beta<\alpha} E_\beta^-\Big)$, where $E_\beta^- = \mathcal{G}D_\beta^-$. We note that $x \in D^-(y)$ if and only if $y \in D_\alpha(x)$. We may then define *negative* stability of order α (for a compact *negatively* invariant set M), by $D_\alpha^-(M) = M$. Clearly all the theorems of the preceding sections may be phrased so as to apply to negative stability. [In particular, condition *c*) in the definition of generalized Liapunov function would read :

$$V(xt) \geqq V(x),$$

for $x \in X$, and $t \geqq 0$.]

Let $\tilde{X}$ denote the one-point-compactification of X. Then $\tilde{X} = X \cup \{\omega\}$, where ω denotes the point at infinity. The assumption that X is second countable guarantees that $\tilde{X}$ is metrizable, ([3], p. 125). We extend the dynamical system $\mathcal{F}$ to a dynamical system $\tilde{\mathcal{F}}$ on $\tilde{X}$ by defining $\omega t = \omega$, for all real t. Then $\{\omega\}$ is a compact (positively and negatively) invariant set in $\tilde{X}$. The prolongations pertaining to $\tilde{\mathcal{F}}$, we also distinguish notationally by a tilde.

The duality between boundedness and stability is embodied in the next theorem.

THEOREM 7. — *$\mathcal{F}$ is α-bounded if and only if $\{\omega\}$ is negatively α-stable.*

Proof. — If $\mathcal{F}$ is not α-bounded, then $D_\alpha(x)$ is not compact, for some $x \in X$. That is, $\omega \in \tilde{D}_\alpha(x)$, and $x \in \tilde{D}_\alpha^-(\omega)$, so $\{\omega\}$ is negatively α-unstable.

Suppose, conversely, that $\mathcal{F}$ is α-bounded. If $\{\omega\}$ is negatively α-unstable, then there are sequences $\{x_n\}$ and $\{y_n\}$ in X, with $x_n \to \omega$, $y_n \in \tilde{E}_{\beta_n}^-(x_n)$ $(\beta_n < \alpha)$, and $y_n \to y \in X$. Then, $\omega \in \tilde{D}_\alpha(y)$. Now, let K be a compact subset of X such that $D_\alpha(y) \subset K^0$. Since $\omega \in \tilde{D}_\alpha(y)$, it follows from the defining properties of a prolongation that there is a $z \in \tilde{D}_\alpha(y) \cap \partial K$. Then there are sequences $y_n \to y$, $z_n \to z$ such that $z_n \in \tilde{E}_{\beta_n}(y_n)$, $(\beta_n < \alpha)$. Now, it follows easily that $\tilde{E}_\beta(y') = E_\beta(y')$, for all $\beta < \alpha$ and all $y' \in X$. Then

$$z \in D_\alpha(y) \cap \partial K.$$

This is a contradiction.

16. It follows from Theorem 7 that every stability theorem has a boundedness theorem as its counterpart.

Continuing in this vein, we define a (*generalized*) *Liapunov function at infinity* to be a positive real-valued function V defined in the complement of a compact subset K of X satisfying

a) V is bounded on every compact set.

b) The set $\{x | V(x) \leqq \lambda\}$ is a relatively compact subset of X.

c) If $x \in X - K$, and $t \geqq 0$, then $V(xt) \leqq V(x)$.

Equivalently, we may consider an extended real valued function $\tilde{V}$, defined in a neighborhood of ω in $\tilde{X}$, such that $\tilde{V}(\omega) = +\infty$, and such that $\tilde{V}(x_n) \to +\infty$ if and only if $x_n \to \omega$.

It follows that $\tilde{V}$ is a generalized Liapunov function at infinity if and only if $V = \dfrac{1}{\tilde{V}}$ is a «negative» (non-decreasing) Liapunov Function for the set $\{\omega\}$.

Using this observation, it is easy to prove the following theorems, which are the duals of Theorem 5 and 6.

THEOREM 8. — *The dynamical system $\mathcal{F}$ is bounded of order* 1 *if and only if there exists a generalized Liapunov function at infinity* (4).

THEOREM 9. — *The following statements are equivalent.*

a) *The dynamical system $\mathcal{F}$ is absolutely bounded.*

b) *Every compact set is contained in an absolutely stable compact set.*

c) *There exists a continuous (generalized) Liapunov function at infinity.*

7. Asymptotic stability and ultimate boundedness.

17. The compact set M is said to be *asymptotically stable* if it is Liapunov stable, and if there exists a neighborhood W of M such that $d(xt, M) \to 0$ as $t \to \infty$, for all $x \in W$. The

(4) This theorem is a generalization of Theorem 4 in [5].

last condition is equivalent to the statement that $\Omega(x)$ is a non-empty subset of M, for each $x \in W$. If M is asymptotically stable, the largest neighborhood W of M such that $\Omega(W) \subset M$ is called the *region of attraction* or *domain of asymptotic stability*; W is an open subset of X. If $W = X$, then M is said to be *asymptotically stable in the large*, or *completely stable*, [5].

If M is asymptotically stable, then (since X is locally compact), it is known that M is uniformly asymptotically stable, [6, p. 38]. That is, if A is a compact subset of W, and U is a neighborhood of M, then there is a $T > 0$ such that $At \subset U$, for all $t \geq T$.

THEOREM 10. — *If the compact set* M *is asymptotically stable, it is absolutely stable. Indeed,* M *is asymptotically stable if and only if there exists a continuous Liapunov function* V *for* M *such that* $V(xt) < V(x)$, *whenever* $x \notin M$, *and* $t > 0$.

Proof. — Suppose that M is asymptotically stable. Let W be the domain of asymptotic stability of M, and let U be a relatively compact neighborhood of M, with $\overline{U} \subset W$. If $x \in W$, define $Y(x) = \sup_{t \geq 0} d(xt, M)$.

It is known that Y is a Liapunov function for M, [6, p. 36(5)]. We show that Y is continuous on U. Let $x \in U - M$, and suppose $d(x, M) = 2\varepsilon > 0$. Let $T > 0$ be such that $Ut \subset S_\varepsilon(M)$, for all $t \geq T$. Suppose $\{x_n\}$ is a sequence in U such that $x_n \to x$. We show that $Y(x_n) \to Y(x)$. Let $\tau_n, \tau \geq 0$ be such that $Y(x_n) = d(x_n\tau_n, M)$, and $Y(x) = d(x\tau, M)$. Since

$$0 \leq \tau_n < T,$$

we may suppose $\tau_n \to \tau' \geq 0$. Then

$$Y(x_n) = d(x_n\tau_n, M) \to d(x\tau', M).$$

We show that $Y(x) = d(x\tau', M)$. Obviously $Y(x) \geq d(x\tau', M)$. Suppose that $Y(x) = d(x\tau, M) > d(x\tau', M)$. Let $\lambda > 0$ such that $d(x\tau', M) + 2\lambda < d(x\tau, M)$. For n sufficiently large, $d(x_n\tau, x\tau) < \lambda$, and $d(x_n\tau_n, x\tau') < \lambda$. Then it follows that

$$d(x_n\tau, M) > d(x\tau, M) - \lambda > d(x\tau', M) + \lambda > d(x_n\tau_n, M),$$

(5) This function, which is due to Okamura and Yoshizawa, is a Liapunov function for M whenever M is 1-stable.

that is $d(x_n\tau, M) > d(x_n\tau_n, M) = Y(x_n)$, which is a contradiction. Since Y is always continuous on M, it is continuous on U, and by Theorem 6, M is absolutely stable.

The function Y is not, in general, strictly decreasing on orbits of points outside M, as we require in the statement of the theorem. However, it is easily verified that the function V defined, for $x \in U$, by

$$V(x) = \int_0^\infty \alpha(t) Y(xt)\, dt$$

where α is any positive, non-increasing, summable function, is a continuous Liapunov function for M satisfying $V(xt) < V(x)$, for $x \in U - M$, and $t > 0$.

Suppose conversely, that V is a continuous Liapunov function satisfying $V(xt) < V(x)$, for $x \in U - M$, and $t > 0$, where U is a neighborhood of M. Let $\eta > 0$ be chosen so that $W_\eta = \{x | V(x) \leq \eta\}$ is a compact neighborhood of M. We show that $\Omega(W_\eta) \subset M$. By the defining properties of generalized Liapunov functions, it is sufficient to show that

$$\lim_{t \to +\infty} V(xt) = 0,$$

for all $x \in W_\eta$. Suppose the contrary. Then, for some $x \in W_\eta$, $\lim_{t \to +\infty} V(xt) = \lambda > 0$. Let $z \in \Omega(x)$. Then $V(z) = \lambda$. But, if $\tau > 0$, $z\tau \in \Omega(x)$, and $V(z\tau) = \lambda$. This is a contradiction. The last proof is due to La Salle ([5]).

18. The dynamical system $\mathcal{F}$ is called *ultimately bounded* [14] if $\Omega(X)$ is a non-empty, relatively compact subset of X.

LEMMA 3. — *Suppose that $\mathcal{F}$ is ultimately bounded. Let* A *be a compact subset of* X. *Then $\gamma^+(A)$ is relatively compact.*

Proof. — We may suppose $\Omega(X) \subset A^0$. If the conclusion of the lemma is false, there are point $x_n \in \partial A$ and $t_n > 0$ such that $x_n t_n$ contains no convergent subsequence; clearly $t_n \to +\infty$. We may also suppose (by replacing x_n by $x_n\tau_n$, for some $\tau_n > 0$, if necessary) that $x_n t \notin A$, for $0 < t \leq t_n$. Now, suppose $x_n \to x \in \partial A$, Let N be a neighborhood of x and $t > 0$ such that $Nt \subset A^0$. Then, for all n sufficiently large, $x_n t \in A^0$, and $t < t_n$. This is a contradiction.

The next theorem shows that asymptotic stability and ultimate boundedness are dual notions, in the sense of Chapter VI (provided that the space X is second countable). As in Chapter VI, ω denotes the point at infinity in $\tilde{X}$.

THEOREM 11. — *$\mathcal{F}$ is ultimately bounded if and only if the point $\{\omega\}$ is negatively asymptotically stable (in the dynamical system $\tilde{\mathcal{F}}$).*

Proof. — Suppose that $\mathcal{F}$ is ultimately bounded. In order to show that $\{\omega\}$ is negatively asymptotically stable, we apply Theorems (14.1) and (14.3) of [6] (modified so as to apply to negative stability). Then, we must show:

a) There is a neighborhood W′ of $\{\omega\}$, such that if $x \in W'$, and $x \neq \{\omega\}$, then there is a $t \in R$ with $xt \notin W'$.

b) If N′ is a neighborhood of $\{\omega\}$, then there is a neighborhood W′ of $\{\omega\}$ such that $\gamma^+(X - N') \subset X - W'$.

To prove *a*), let W be any compact neighborhood of $\Omega(X)$, and let $W' = X - W$.

Since $X - N'$ is a compact subset of X, *b*) is a consequence of Lemma 3.

Now, suppose $\{\omega\}$ is negatively asymptotically stable. Let W be a compact set in X such that $W' = \tilde{X} - W$ is the domain of asymptotic stability of $\{\omega\}$, We show $\Omega(X) \subset W$. If not, there is an $x \in X$, and $t_n \to +\infty$ such that $xt_n \to z \in W'$. We may suppose all $xt_n \in W'$. Let $U' = \tilde{X} - \{x\}$; U′ is a neighborhood of ω. Choose a compact subset K of W′ such that $xt_n \in K$, for all n. As we observed in § 17, asymptotic stability in a locally compact space is uniform; hence there is a $T > 0$ such that $Kt \subset U'$, for $t < -T$. But $t_n > T$, for n sufficiently large, and therefore $x = (xt_n)(-t_n) \in K(-t_n) \subset U'$. This a contradiction.

19. Similar to Theorem 10 we have:

THEOREM 12. — *Let X be second countable, and suppose that $\mathcal{F}$ is ultimately bounded. Then:*

(i) *$\mathcal{F}$ is absolutely bounded.*

(ii) *There is a continuous generalized Liapunov function at infinity V, defined on the complement of a compact set K, such that if $x \in X - K$, and $t > 0$, $V(xt) < V(x)$.*

(iii) *There exists a compact set M which is completely stable.*

Proof. — Statements (i) and (ii) follow immediately by dualizing Theorem 10. To prove (iii), let K be a compact subset of X with $\Omega(X) \subset K$, and let $M = D(K)$. Since $\mathcal{F}$ is absolutely bounded, closedness of D implies that M is compact, We show that M is completely stable [6].

Recalling that $D^2 = D$, we have

$$D_1(M) \subset D(M) = D(D(K)) = D(K) = M.$$

Hence M is 1-stable. Since $\Omega(X) \subset K \subset D(K) = M$, M is completely stable.

20. Lemma 4. — *Let* M *be asymptotically stable, and let* W *be the domain of asymptotic stability of* M. *Let* N *be a compact positively invariant set with* $M \subset N \subset W$. *Then* N *is asymptotically stable.*

Proof. — Since $\Omega(W) \subset M \subset N$, it is only necessary to show that N is Liapunov stable. Let U be a relatively compact open set with $N \subset U \subset \overline{U} \subset W$. Since U is a relatively compact neighborhood of M, there exists $T > 0$ such that $Ut \subset U$, for all $t \geqq T$. Now, let $y \in D_1(N)$. Then $y \in D_1(x)$, for some $x \in N$. Hence there exist sequences $\{x_n\}$ in X, and $t_n \geqq 0$ such that $x_n \to x$, and $x_n t_n \to y$. If the sequence $\{t_n\}$ is bounded, then $y \in N$, since N is positively invariant. If not, $t_n \geqq T$, for n sufficiently large, and $x_n t_n \in U t_n \subset U$, so $y \in \overline{U}$. Since U is an arbitrary relatively compact neighborhood of M, we have $D_1(N) \subset N$. The proof is completed.

Now, suppose that M is completely stable. Then Lemma 4 tells us that any compact positively invariant superset of M is also completely stable. Therefore, it is reasonable to ask for a smallest or « minimal » set which is completely stable (that is, one which contains no non-empty proper subset with the same property). We will show that such a set exists. First, we state a lemma, the proof of which is left to the reader.

Lemma 5. — (i) *Let* M *be completely stable and let* $t > 0$. *Then* Mt *is completely stable.*

[6] Actually, it can be shown that $D_1(K)$ is completely stable. However, it is not in general true that K is asymptotically stable. See [9] for an interesting example of this phenomenon.

(ii) *Let* $\{M_\alpha | \alpha \in \mathfrak{A}\}$ *be a family of completely stable sets. Then* $M^* = \bigcap\{M_\alpha | \alpha \in \mathfrak{A}\}$ *is not vacuous, and is completely stable.*

THEOREM 13. — *Assume the compact positively invariant set* M *to be completely stable. Then* $M^* = \bigcap\{Mt | t \geq 0\}$ *is the minimal completely stable set.*

Proof. — Since the collection of sets $\{Mt | t \geq 0\}$ constitutes a decreasing family of non-empty compact sets, M^* is non-empty, and, by Lemma 5, M^* is completely stable. It is easily shown that M^* is positively and negatively invariant. Suppose that N is a completely stable proper subset of M^*. Let $y \in M^* - N$, and let U be a neighborhood of N with $y \notin U$. Let $t_0 > 0$ such that $M^* t_0 \subset U$. Then $z = y(-t_0) \in M^*$ and $y = zt_0 \in M^* t_0 \subset U$, which is a contradiction.

21. Asymptotic stability cannot be described in terms of invariance under an abstract prolongation, as the following considerations indicate. Suppose there were an abstract prolongation Q such that $Q(M) = M$ if and only if M is asymptotically stable. Now, consider a compact invariant set M, such that $M = \bigcap\{M_n | n = 1, 2, \ldots\}$, where each M_n is asymptotically stable, but M is not asymptotically stable. Such exist; see example *c*) in Chapter IV. Then $Q(M) \subset Q(M_n) = M_n$, so $Q(M) \subset \bigcap\{M_n | n = 1, 2, \ldots\} = M$, and M would have to be asymptotically stable.

Nevertheless the prolongations D_α do throw some light on the notion of asymptotic stability. For $x \in X$, and α an ordinal number, define $D'_\alpha(x) = D_\alpha(x) - \gamma^+(x)$. Then $D'_\alpha(x)$ is, so to speak, the « non-trivial part » of $D_\alpha(x)$. Of course $D'_\alpha(x)$ may be empty.

THEOREM 14. — *If* M *is asymptotically stable, and* W *is the domain of asymptotic stability, then* $D'_\alpha(W) \subset M$, *for every ordinal number* α.

Conversely, if there is a neighborhood W *of* M *such that, for every* $x \in W$, *there exists an ordinal number* $\alpha \geq 1$ *for which* $D'_\alpha(x)$ *is a non-empty subset of* M, *then* M *is asymptotically stable.*

Proof. — Suppose M is asymptotically stable. Let W denote the domain of asymptotic stability. Then, using the property of uniform asymptotic stability, it is easy to show that, for $x \in W$, $D_1(x) \subset \gamma^+(x) \cup M$. That is, $D_1'(x) \subset M$. Now, let α be any ordinal number. Suppose for all $\beta < \alpha$, and all $x \in W$, $D_\beta'(x) \subset M$. It follows immediately that if $y \in E_\alpha^*(x)$, then either $y \in M$, or $y \in \gamma^+(x)$.

Now, suppose $x \in W$, and $y \in D_\alpha(x)$. Then, there are sequences $x_n \to x$, $y_n \to y$ with $y_n \in E_\alpha^*(x_n)$. If infinitely many y_n are in M, then $y \in M$. If not, then $y_n \in \gamma^+(x_n)$, and

$$y \in D_1(x) = \gamma^+(x) \cup D_1'(x) \subset \gamma^+(x) \cup M.$$

That is, $D_\alpha(x) \subset \gamma^+(x) \cup M$, and therefore $D_\alpha'(x) \subset M$, for all $x \in W$.

To prove the second part of the theorem, observe that if $z \in M$, then $D_1(z) = \gamma^+(z) \cup D_1'(z) \subset M \cup M = M$, so that M is Liapunov stable. Let $x \in W - M$. We show that $\Omega(x) \subset M$. Let α be an ordinal number for which $D_\alpha'(x) \neq \emptyset$ and $D_\alpha'(x) \subset M$. Then $D_\alpha(x) = \gamma^+(x) \cup D_\alpha'(x) \subset \gamma^+(x) \cup M$. Therefore, since

$$\Omega(x) \subset D_\alpha(x)$$

it is enough to show that $\Omega(x) \cap \gamma^+(x) = \emptyset$ or, what is the same thing, $x \notin \Omega(x)$. First, we show that x is not periodic. If it is, let $A = \gamma^+(x)$, and let N be a compact neighborhood of A with $N \cap M = \emptyset$. Then since $\emptyset \neq D_\alpha'(x) \subset M$, it follows that $D_\alpha(x) \not\subset N$. Therefore, there is a point $z \in D_\alpha(x) \cap \partial N$. Since $z \notin \gamma^+(x)$, $z \in M$. This contradicts $N \cap M = \emptyset$.

Finally, suppose $x \in \Omega(x)$, and that x is not periodic. Let N be a neighborhood of x with $N \cap M = \emptyset$. Then, (by [10], Theorem 4.10, p. 348) there is a point $z \in N \cap \Omega(x)$, with $z \notin \gamma^+(x)$. Then $z \in D_\alpha'(x)$. But this contradicts $D_\alpha'(x) \subset M$.

8. Strict stability and boundedness.

22. In this chapter, X denotes a region of euclidean n-dimensional space R^n. Moreover, the dynamical system $\mathcal{F}$ under consideration is assumed to consist of the solution curves of the autonomous system of differential equations

$$\dot{x} = f(x) \tag{1}$$

where x and f are n-vectors. We assume that f is defined and continuous in X, that $f(x) = 0(\|x\|)$, and satisfies a local Lipschitz condition in X. Under these assumptions, the solutions of (1) depend continuously on the right side of (1), and form a dynamical system in X [10, Chapter I]. Then, if $x \in X$ and $t \in R$, by xt we mean $\pi(x, t)$ where π is the unique solution of (1) satisfying $\pi(x, 0) = x$.

Let $\delta > 0$. By a *δ-solution* of (1) we mean an absolutely continuous curve ψ in X satisfying

$$(2) \qquad \|\dot{\psi}(t) - f(\psi)(t))\| < \delta$$

for all $t \in R$ for which $\dot{\psi}(t)$ is defined.

If $x \in X$, let $\Psi_\delta(x)$ be the set of δ-solutions ψ of (1) satisfying $\psi(0) = x$.

Next, we introduce the following subsets of X:

$$P_\delta(x, t) = \{\psi(t) | \psi \in \Psi_\delta(x)\};$$
$$P_\delta(x) = \bigcup \{P_\delta(x, t) | t \geqq 0\},$$

and

$$P(x) = \bigcap \{P_\delta(x) | \delta > 0\}.$$

The set $P(x)$ consists of the points y for which, for any $\delta > 0$, there is a δ-solution ψ such that $\psi(0) = x$ and $\psi(t) = y$, for some $t \geqq 0$.

Note also that $y \in P_\delta(x, t)$ if and only if $x \in P_\delta(y, -t)$, and that $P_\delta(x, t + t') = P_\delta(P_\delta(x, t), t')$.

As an example, consider a parallel flow in the plane, defined by the equations $\dot{x}_1 = 1$, $\dot{x}_2 = 0$. If $x \in R^2$, then $P(x)$ coincides with the positive semi-orbit $\gamma^+(x)$.

A second example is furnished by the equations of a harmonic oscillator with damping $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - \zeta x_2$, $(\zeta > 0)$. Here $P(0) = \{0\}$, where $\{0\}$ denotes the origin, which is a stable focus.

Consider next a center, given by $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$. If $x \in R^2$, and $\delta > 0$, any point $y \in R^2$ may be joined with x by a δ-solution. Then $P_\delta(x) = R^2$, and consequently $P(x) = R^2$.

23. The following lemma is an easy consequence of the continuous dependence of the solutions of (1) on the function f(7).

(7) It follows, for example, from the « fundamental inequality », [10, p. 13].

LEMMA 6. — *Let $x \in \mathrm{X}$ and let ε and τ be positive real numbers. Then there exists $\delta > 0$ such that if $d(x, x') < \delta$, and $y' \in \mathrm{P}_\delta(x', t)$, where $|t| \leqq \tau$, then $d(xt, y') < \varepsilon$.*

The next lemma plays a central role in all subsequent considerations.

LEMMA 7. — *Let $\{x_n\}$ and $\{y_n\}$ be sequences in X with $x_n \to x$ and $y_n \to y$. Let δ_n be a sequence of positive real numbers with $\delta_n \to 0$, and suppose $y_n \in \mathrm{P}_{\delta_n}(x_n)$. Then $y \in \mathrm{P}(x)$.*

Proof. — If $y = x$, there is nothing to prove, so suppose $y \neq x$. Let $\delta > 0$. We show $y \in \mathrm{P}_\delta(x)$. By hypothesis, there exists for every n, a δ_n-solution ψ_n of (1) satisfying $\psi_n(0) = x_n$, $\psi_n(t_n) = y_n$ with $t_n > 0$. Let U and V be relatively compact disjoint neighborhoods of x and y respectively, and suppose that $x_n \in \mathrm{U}$, $y_n \in \mathrm{V}$, and $\delta_n < \delta/3$. Let $t' = \inf t_n$. It can be shown, using lemma 6, that $t' > 0$. Let A be a compact neighborhood of $\overline{\mathrm{U}} \cup \overline{\mathrm{V}}$,and let $0 < t_0 \leqq t'$ such that $\mathrm{U}t \subset \mathrm{A}$ and $\mathrm{V}(-t) \subset \mathrm{A}$ for $0 \leqq t \leqq \frac{t_0}{2}$.

Now we define the following sequence of functions:

$$\varphi_n(t) = \begin{cases} \psi_n(t) + \dfrac{x_n - x}{t_0}(2t - t_0) & \text{for } 0 \leqq t \leqq \dfrac{t_0}{2}, \\ \psi_n(t) & \text{for } \dfrac{t_0}{2} \leqq t \leqq t_n - \dfrac{t_0}{2}, \\ \psi_n(t) + \dfrac{y - y_n}{t_0}(2t + t_0 - 2t_n) & \text{for } t_n - \dfrac{t_0}{2} \leqq t \leqq t_n. \end{cases}$$

Clearly, $\varphi_n(0) = x$ and $\varphi_n(t_n) = y$. We show that φ_n is a δ-solution of (1), for n sufficiently large. Differentiating, we obtain

$$\dot\varphi_n(t) = \begin{cases} \dot\psi_n(t) + 2\dfrac{x_n - x}{t_0}, & 0 \leqq t \leqq \dfrac{t_0}{2}, \\ \dot\psi_n(t), & \dfrac{t_0}{2} \leqq t \leqq t_n - \dfrac{t_0}{2}, \\ \dot\psi_n(t) + 2\dfrac{y - y_n}{t_0}(2t + t_0 - 2t_n), & t_n - \dfrac{t_0}{2} \leqq t \leqq t_n. \end{cases}$$

This holds almost everywhere in $[0, t_n]$. Now

$$\begin{aligned}\|\dot\varphi_n(t) - f(\varphi_n(t))\| &\leq \|\dot\varphi_n(t) - \dot\psi_n(t)\| + \|\dot\psi_n(t)) - f(\psi_n(t))\| \\ &\qquad + \|f(\psi_n(t)) - f(\varphi_n(t))\| \\ &< \frac{2}{t_0}\max\{\|x_n - x\|, \|y_n - y\|\} + \frac{\delta}{3} + \|f(\psi_n(t)) - f(\varphi_n(t))\|.\end{aligned}$$

In order to show that φ_n is a δ-solution of (1) for large n, it is therefore only necessary to show that $\|f(\psi_n(t)) - f(\varphi_n(t))\| \to 0$, as $n \to \infty$.

Now, $\varphi_n(t) = \psi_n(t)$, for $\frac{t_0}{2} \leq t \leq t_n - \frac{t_0}{2}$, so we need only consider the intervals $I_0 = \left[0, \frac{t_0}{2}\right]$, and $I_n = \left[t_n - \frac{t_0}{2}, t_n\right]$. Let N be a relatively compact neighborhood of A.

Since $Ut \subset A$ and $V(-t) \subset A$, for $0 \leq t \leq \frac{t_0}{2}$, we may apply Lemma 6, and obtain $\psi_n(t) \in N$, for $t \in I_0 \cup I_n$. Since $\|\varphi_n(t) - \psi_n(t)\| \leq \max\{\|x_n - x\|, \|y_n - y\|\}$, we may find a compact set $K \supset N$ such that $\varphi_n(t) \in K$, for $t \in I_0 \cup I_n$, and n large. Since f is uniformly continuous on K, we obtain $\|f(\psi_n(t)) - f(\varphi_n(t))\| \to 0$ as $n \to \infty$. The proof is completed.

24. Lemma 8. — P *is a transitive prolongation.*

Proof. — P obviously satisfies condition (α), and (β) follows immediately from Lemma 7 by putting $\delta_n = 0$. To show that (γ) holds, let A be a compact subset of X, let $x \in A^0$ and let $y \in P(x) - A$. Then there exists a sequence of numbers $\delta_n > 0$, $\delta_n \to 0$, and δ_n-solutions ψ_n of (1) with $\psi_n(0) = x$ and $\psi_n(t_n) = y$, where $t_n > 0$. Let $0 < \tau_n \leq t_n$ such that $y_n = \psi_n(\tau_n) \in \partial A$. By choosing a subsequence if necessary we may suppose $y_n \to y \in \partial A$. By Lemma 7, $y \in P(x)$, and (γ) is proved.

To prove that P is transitive, let $y \in P(x)$ and $z \in P(y)$. Then, if $\delta > 0$, there are δ-solutions ψ_1 and ψ_2 with $\psi_1(0) = x$, $\psi_1(t_1) = y$, $\psi_2(0) = y$, $\psi_2(t_2) = z$, where $t_1, t_2 > 0$. Then

$$\psi(t) = \begin{cases} \psi_1(t) & \text{for} \quad 0 \leq t \leq t_1, \\ \psi_2(t - t_1) & \text{for} \quad t_1 \leq t \leq t_1 + t_2 \end{cases}$$

is a δ-solution with $\psi(0) = x$ and $\psi(t_1 + t_2) = z$.

Corollary 3. — *If* M *is compact, then* P(M) *is closed and positively invariant.*

The compact set M in X is called *strictly stable* if, for any $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that

$$P_\delta(S_\eta(M)) \subset S_\varepsilon(M).$$

The dynamical system $\mathcal{F}$ determined by (1) is called *strictly bounded* if, for any $\sigma > 0$, there exist $\delta > 0$ and $\tau > 0$ such that $\|x\| < \sigma$ implies $\|P_\delta(x)\| < \tau$ [8].

Theorem 15. — *Let* M *be a compact subset of* X. *Then* M *is strictly stable if and only if* P(M) = M.

Proof. — By Theorem 2 and Lemma 8, P(M) = M if and only if, for every neighborhood W of M, there is a neighborhood U of M such that $P(U) \subset W^o$. We may suppose that $\overline{U}$ and $\overline{W}$ are compact. Then an application of Lemma 7 tells us that $P_\delta(U) \subset W$, for some $\delta > 0$. That is, M is strictly stable. The converse is obvious.

Theorem 15 immediately yields:

Corollary 4. — *Let* $M_n (n = 1, 2, \ldots)$ *be a decreasing family of compact invariant sets, each of which is strictly stable. Then* $M = \bigcap_{n=1,2,\ldots,} M_n$ *is strictly stable.*

Applying the duality principle between stability and boundedness, discussed in Chapter VI, we obtain:

Theorem 16. — *The dynamical system* $\mathcal{F}$ *is strictly bounded if and only if* P(B) *is compact whenever* B *is compact.*

The last theorem in this section is the strict stability analogue of Theorem 6. The proof is similar to that of Theorem 6, and is therefore omitted. (See the concluding remarks in Chapter V.)

Theorem 17. — *Let* M *be a compact positively invariant set. Then the following are equivalent.*

(8) Strict stability is also called ***stability under persistent perturbations*, *total stability*** and ***weak stability under perturbations***. Similarly, strict boundedness is called ***boundeness under persistent perturbations*** or ***total boundedness*** [14].

(a) M *is strictly stable.*

(b) *There is a fundamental sequence of strictly stable neighborhoods of* M.

(c) *There exists a non-negative continuous function* V *defined on a neighborhood* W *of* M *such that* $V(x) = 0$ *if and only if* $x \in M$ *and such that* $V(y) \leq V(x)$ *whenever* $y \in P(x)$.

25. Now, we shall indicate the relationship between strict stability and boundedness with some of the stability and boundedness notions studied in earlier chapters. Actually, (i) is known ([7], [8], and [11]) but the proof is included here for completeness.

THEOREM 18. — (i) *If the compact invariant set* M *is asymptotically stable, then it is strictly stable.*

(ii) *If* M *is strictly stable, it is absolutely stable.*

(iii) *If the dynamical system* $\mathcal{F}$ *is strictly bounded, it is absolutely bounded.*

Proof. — (i) We show: if $\varepsilon > 0$, there exist $\zeta > 0$ and $\delta > 0$ such that $x \in S(M, \delta)$ implies $P_\zeta(x, t) \subset S(M, \varepsilon)$ for all $t \geq 0$. We may suppose that $S(M, \varepsilon)$ is contained in the domain of asymptotic stability of M. Choose δ and t so that $0 < \delta < \frac{\varepsilon}{2}$, $S(M, \delta)R^+ \subset S\left(M, \frac{\varepsilon}{2}\right)$, and $S(M, \varepsilon)t \subset S\left(M, \frac{\delta}{2}\right)$, for $t \geq \tau$. By Lemma 6 we may choose $\zeta > 0$ so that $x \in S(M, \delta)$ and $y \in P_\zeta(x, t)$, $0 \leq t \leq \tau$ implies $d(xt, y) < \delta/2$. Then, for

$$x \in S(M, \delta),\ 0 \leq t \leq \tau,$$

we have $P_\zeta(x, t) \subset S\left(M, \frac{\varepsilon}{2} + \frac{\delta}{2}\right) \subset S(M, \varepsilon)$. Moreover, since $x\tau \in S(M, \delta/2)$, $P_\zeta(x, \tau) \subset S(M, \delta)$. Now since

$$P_\zeta(x, t + \tau) = P_\zeta(P_\zeta(x, \tau), t),$$

we obtain immediately $P_\zeta(x, t) \subset S(M, \varepsilon)$, for $0 \leq t \leq 2\tau$, and $P_\zeta(x, 2\tau) \subset S(M, \delta)$, whenever $x \in S(M, \delta)$. The conclusion follows by an easy induction.

(ii) If $x \in X$, and γ is the first uncountable ordinal, then $D_\gamma(x)$ is the smallest transitive prolongation containing $\gamma^+(x)$. Since $\gamma^+(x) \subset P(x)$, and P is a transitive prolongation,

$D_\gamma(x) \subset P(x)$. Therefore, if $P(M) \subset M$, $D_\gamma(M) \subset M$ and the assertion follows from theorem 15.

(iii) This is an immediate consequence of Theorem 16.

None of the converses to the statements in Theorem 18 is valid. Let $M = \bigcap M_n$, where each M_n is asymptotically stable, but M is not. [Example *c*) in Chapter IV.] Then each M_n is strictly stable, and Corollary 4 tells us that M is strictly stable. A center in the plane provides an example of an absolutely bounded dynamical system is not strictly bounded; here the origin is absolutely stable and not strictly stable.

26. Actually, using the method of proof of Theorem 18, it may be shown [11], that asymptotic stability implies *strict asymptotic stability*, that is, Mis strictly stable and there is a neighborhood W of, M such that if $\varepsilon > 0$, then there exist δ and $\tau > 0$ such that $x \in W$ implies $P_\delta(xt) \subset S_\varepsilon(M)$, for $t \geqq \tau$. Conversely, strict asymptotic stability implies asymptotic stability.

Strict asymptotic stability (and therefore asymptotic stability) can be conveniently characterized in terms of a set which is formed in an analogous manner as the omega limit set. If $x \in X$, we define $\Omega_P(x) = \bigcap_{t \in R} P(xt)$. Since $P(xt') \subset P(xt)$ whenever $t' > t$, it follows that $\Omega_P(x) = \bigcap_{t \geqq t_0} P(xt)$.

Our final theorem characterizes strict asymptotic stability by means of Ω_P.

THEOREM 19. — *The compact invariant set* M *is (strictly) asymptotically stable if and only if there is a neighborhood* W *of* M *such that* $\Omega_P(W) \subset M$.

Proof. — Suppose M is strictly asymptotically stable. We will find a neighborhood W of M such that $\Omega_P(W) \subset S_\varepsilon(M)$ for every $\varepsilon > 0$. Choose W as in the definition of strict asymptotic stability, and let $\varepsilon > 0$. Choose δ and τ such that $P_\delta(x, t) \subset S_\varepsilon(M)$, for $t \geqq \tau$, and $x \in W$. Then

$$P_\delta(x, t + \tau) \subset S_\varepsilon(M),$$

for $x \in W$ and $t \geqq 0$. Now

$$P_\delta(x\tau, t) \subset P_\delta(P_\delta(x, \tau), t) = P_\delta(x, \tau + t) \subset S_\varepsilon(M)$$

for $t \geq 0$. Then $P_\delta(x\tau) \subset S_\varepsilon(M)$; it follows that $P(x\tau) \subset S_\varepsilon(M)$, and then $\Omega_P(x) \subset S_\varepsilon(M)$.

Conversely, suppose that W is a compact neighborhood of M for which $\Omega_P(W) \subset M$. We show that if $\varepsilon > 0$, there exist τ and $\delta > 0$ such that $P_\delta(x, t) \subset S_\varepsilon(M)$, for $t \geq \tau$, and $x \in W$. If not, there are sequences $x_n \in W$, $\delta_n \to 0$, $t_n \to +\infty$, and $y_n \in P_{\delta_n}(x_n, t_n)$ such that $d(y_n, M) = \varepsilon > 0$. Without loss of generality, suppose $y_n \to y$ and $x_n \to x \in W$. Let $\tau > 0$, and suppose all $t_n > \tau$. Now, $y_n \in P_{\delta_n}(x_n, t_n) = P_{\delta_n}(P_{\delta_n}(x_n, \tau), t_n - \tau)$. Let $z_n \in P_{\delta_n}(x_n, \tau)$ such that $y_n \in P_{\delta_n}(z_n, t_n - \tau)$. By Lemma 6, $z_n \to x\tau$, and by Lemma 7, $y \in P(x\tau)$. That is, $y \in \bigcap_{\tau \geq 0} P(x\tau) \subset \Omega_P(x) \subset \Omega_P(W)$. But $d(y, M) = \varepsilon$, and this contradicts $\Omega_P(W) \subset M$.

27. In the two following diagrams we summarize the relations between the various types of stability and boundedness discussed in this paper. Except where indicated, the converses of the implications are not true.

$$\begin{tikzcd}
\text{asymptotic stability} \arrow[r, leftrightarrow] \arrow[d] & \text{strict asymptotic stability} \arrow[d] \\
\text{absolute stability} \arrow[d] & \text{strict stability (total stability)} \arrow[l] \\
\text{stability of order } \alpha (\alpha > 1) \arrow[d] & \\
\text{Liapunov stability} &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{ultimate boundedness} \arrow[r, leftrightarrow] \arrow[d] & \text{strict ultimate boundedness} \arrow[d] \\
\text{absolute boundedness} \arrow[d] & \text{strict boundedness (total boundedness)} \arrow[l] \\
\text{boundedness of order } \alpha (\alpha > 1) \arrow[d] & \\
\text{boundedness} &
\end{tikzcd}$$

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