

## PROLONGATIONS OF G-STRUCTURES TO TANGENT BUNDLES OF HIGHER ORDER

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*To Professor Katuzi Ono on the occasion of his 60th birthday*

### § Introduction and Notations.

In the previous paper [4] we have studied the prolongations of  $G$ -structures to tangent bundles. The purpose of the present paper is to generalize the previous prolongations and to look at them from a wide view as a special case by considering the tangent bundles of higher order. In fact, in some places, the arguments and calculations in [4] are more or less simplified. Since the usual tangent bundle  $T(M)$  of a manifold  $M$  considers only the first derivatives or first contact elements of  $M$ , the previous paper contains, in most parts, only the calculation of derivatives of first order.

Now, since the tangent bundle  $\overset{r}{T}M$  to a manifold  $M$  of order  $r$  concerns with the derivatives of higher order (up to order  $r$ ), the situations should be much complicated. Nevertheless, the (covariant) functor  $\overset{r}{T}: M \rightarrow \overset{r}{T}M$  from the category of differentiable manifolds and differentiable maps to the same category, fortunately, has many properties similar to the functor  $T: M \rightarrow TM$ . For instance, (i)  $\overset{r}{T}G$  is a Lie group if  $G$  is a Lie group, (ii)  $\overset{r}{T}R^n$  has a natural vector space structure and (iii)  $\overset{r}{T}GL(n)$  can be considered as a Lie subgroup of  $GL(n(r+1))$ . Therefore, we can follow the procedure in [4] by replacing the functor  $T$  with the functor  $\overset{r}{T}$ .

We mention here that Yano and Ishihara [7] study the prolongations of tensor fields to the tangent bundles of order 2 from the viewpoint of tensor analysis.

In §1, we explain the notion of tangent bundles  $\overset{r}{T}M$  of order  $r$  to a manifold  $M$ , tangent bundles of order 1 coinciding with the usual tangent bundle.

In §2, 3, we consider the tangent bundles to a Lie group of order  $r$  and prove that if a Lie group  $G$  operates on a manifold  $M$  effectively then the Lie group  $\overset{r}{T}G$  operates canonically on  $\overset{r}{T}M$  also effectively.

In §4, 5, we consider the vector space  $\overset{r}{T}R^n$  and prove that  $\overset{r}{T}GL(n)$  operates on  $\overset{r}{T}R^n$  as a linear transformation group.

In §6, we consider the tangent bundle of higher order to (principal) fibre bundles.

In §7, we construct a canonical imbedding of  $\overset{r}{T}FM$  into  $F\overset{r}{T}M$ , where  $FM$  denotes the frame bundle of  $M$ . Using the results in §6, 7 we can define in §8 the prolongation  $P^{(r)}$  of order  $r$  of a  $G$ -structure  $P$  to the tangent bundle  $\overset{r}{T}M$  for any  $r$ .

In §9, we prove that a diffeomorphism  $\phi: M \rightarrow M'$  is an isomorphism of  $G$ -structures  $P$  with  $P'$  if and only if  $\overset{r}{T}\phi$  is an isomorphism of  $P^{(r)}$  with  $P'^{(r)}$ .

In §10, we prove that a  $G$ -structure  $P$  is integrable if and only if the prolongation  $P^{(r)}$  is integrable.

In §11, we consider some classical  $G$ -structures and prove, among others, that if a manifold  $M$  has an (resp. an integrable) almost complex structure, symplectic structure, pseudo-Riemannian structure or a (completely integrable) differential system, then  $\overset{r}{T}M$  has canonically the same kind of structures. Moreover, if  $M$  has an almost contact structure, then  $\overset{r}{T}M$  has a canonical almost complex structure for  $r$  odd and has an almost contact structure for  $r$  even.

In this paper, all manifolds and mappings (functions) are assumed to be differentiable of class  $C^\infty$ , unless otherwise stated. If  $\varphi: M \rightarrow N$  is a map of a set  $M$  into a set  $N$  and if  $A$  is a subset of  $M$ , we often denote by  $\varphi$  itself the restriction  $\varphi|_A$  of  $\varphi$  to  $A$ , if there is no confusion. If  $\varphi_i: M_i \rightarrow N_i$  is a map for  $i = 1, 2$ , then the map  $\varphi_1 \times \varphi_2: M_1 \times M_2 \rightarrow N_1 \times N_2$  is defined by  $(\varphi_1 \times \varphi_2)(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2))$  for  $x_i \in M_i$ ,  $i = 1, 2$ . If  $M_1 = M_2 = M$ , the map  $(\varphi_1, \varphi_2): M \rightarrow N_1 \times N_2$  is defined by  $(\varphi_1, \varphi_2)(x) = (\varphi_1(x), \varphi_2(x))$  for  $x \in M$ .

In the following,  $R^n$  denotes always the  $n$ -dimensional real number space. The group of all linear automorphisms of  $R^n$  will be denoted by  $GL(n, R)$  or simply by  $GL(n)$ . If  $a^i_j \in R$  for  $i, j = 1, 2, \dots, n$ , we denote by  $(a^i_j)$  the matrix of degree  $n$  whose  $(i, j)$ -entry is  $a^i_j$ .

**§1. Tangent bundles of order  $r$ .**

Let  $\mathfrak{F}$  be the set of all real valued differentiable functions defined on some neighborhood of  $R$  containing zero. Take two functions  $f$  and  $g$  in  $\mathfrak{F}$ . For a positive integer  $r$  we say  $f$  is  $r$ -equivalent to  $g$  iff  $d^\nu f/dt^\nu = d^\nu g/dt^\nu$  at  $t = 0$  for  $\nu = 0, 1, \dots, r$ , and we will denote it by  $f \underset{r}{\sim} g$ . The relation  $\underset{r}{\sim}$  is clearly an equivalence relation in  $\mathfrak{F}$ . Let  $M$  be an  $n$ -dimensional manifold, and let  $C^\infty(M)$  be the ring of all differentiable functions defined on  $M$ . We denote by  $\tilde{S}(M)$  (resp.  $S(M)$ ) the set of all maps  $\varphi$  of some open interval  $(-\varepsilon, \varepsilon)$  (resp.  $R$ ) into  $M$ ,  $\infty \geq \varepsilon > 0$  depending on  $\varphi$ . Let  $\varphi$  and  $\psi$  be two maps of  $\tilde{S}(M)$ . We say that  $\varphi$  is  $r$ -equivalent to  $\psi$  iff  $f \circ \varphi \underset{r}{\sim} f \circ \psi$  for every  $f \in C^\infty(M)$  and denote it by  $\varphi \underset{r}{\sim} \psi$ . The relation  $\underset{r}{\sim}$  is also an equivalence relation in  $\tilde{S}(M)$ . For  $\varphi \in \tilde{S}(M)$  we denote by  $[\varphi]_r$  the equivalence class in  $\tilde{S}(M)$  containing  $\varphi$ .

**DEFINITION 1. 1.** We call  $[\varphi]_r$  the  $r$ -tangent to  $M$  at  $p \in M$  (or  $r$ -jet) defined by  $\varphi$  iff  $\varphi(0) = p$ .

For any  $r$ -tangent  $[\varphi]_r$  to  $M$  there exists  $\varphi' \in S(M)$  such that  $[\varphi']_r = [\varphi]_r$  by virtue of the following

**LEMMA 1. 2.** Let  $\varphi \in \tilde{S}(M)$ . Then there exist some  $\varepsilon_1 > 0$  and  $\varphi' \in S(M)$  such that  $\varphi$  is defined on  $(-\varepsilon_1, \varepsilon_1)$  and  $\varphi(t) = \varphi'(t)$  for  $|t| < \varepsilon_1$ .

*Proof.* Since  $\varphi \in \tilde{S}(M)$ , there is some  $\varepsilon > 0$  such that  $\varphi$  is defined on  $(-\varepsilon, \varepsilon)$ . We can find a function  $g \in C^\infty(R)$  such that  $g(t) = t$  for  $|t| \leq \varepsilon/2$  and  $g(t) = 0$  for  $|t| \geq 2\varepsilon/3$  and that  $|g(t)| \leq 2\varepsilon/3$  for all  $t \in R$ . Put  $\varepsilon_1 = \varepsilon/2$  and  $\varphi' = \varphi \circ g$ . It is now clear that  $\varphi'$  and  $\varepsilon_1$  satisfy the required conditions.

Q.E.D.

**DEFINITION 1. 3.** Let  $\overset{r}{T}(M)$  (or  $\overset{r}{TM}$ ) be the set of all  $r$ -tangents to  $M$ , and for  $p \in M$  let  $\overset{r}{T}_p(M)$  be the set of all  $r$ -tangents to  $M$  at  $p$ . We define  $\overset{r}{\pi}: \overset{r}{T}(M) \rightarrow M$  by  $\overset{r}{\pi}([\varphi]_r) = \varphi(0)$  for  $[\varphi]_r \in \overset{r}{T}(M)$ .

The notion of 1-tangents to  $M$  at  $p$  coincides with the notion of usual tangent vectors to  $M$  at  $p$ . In order to define the manifold structure in  $\overset{r}{TM}$  we shall prove the following

**LEMMA 1. 4.** Let  $\{x_1, x_2, \dots, x_n\}$  be a local coordinate system on some neighborhood  $U$  of  $p \in M$ . Take two elements  $\varphi$  and  $\psi$  in  $S(M)$  such that

$\varphi(0) = \psi(0) = p$ . Then  $\varphi \underset{r}{\sim} \psi$  if and only if  $x_i \circ \varphi \underset{r}{\sim} x_i \circ \psi$  for  $i = 1, 2, \dots, n$ .

*Proof.* Suppose  $\varphi \underset{r}{\sim} \psi$ . There exist a neighborhood  $V$  of  $p$  contained in  $U$  and a function  $f_i \in C^\infty(M)$  ( $i = 1, 2, \dots, n$ ) such that  $f_i|_V = x_i|_V$ . Since  $f_i \circ \varphi \underset{r}{\sim} f_i \circ \psi$  and since  $x_i \circ \varphi(t) = f_i \circ \varphi(t)$ ,  $x_i \circ \psi(t) = f_i \circ \psi(t)$  for  $|t| < \varepsilon$  with some  $\varepsilon > 0$ , we have  $x_i|_V \varphi \underset{r}{\sim} x_i \circ \psi$  for  $i = 1, 2, \dots, n$ .

Conversely, suppose  $x_i \circ \varphi \underset{r}{\sim} x_i \circ \psi$  for  $i = 1, 2, \dots, n$ . Take  $f \in C^\infty(M)$ . We have to prove  $f \circ \varphi \underset{r}{\sim} f \circ \psi$ , i.e.  $d^\nu(f \circ \varphi)/dt^\nu = d^\nu(f \circ \psi)/dt^\nu$  at  $t = 0$  for  $\nu = 0, 1, 2, \dots, r$ . This holds for  $\nu = 0$ , since  $\varphi(0) = \psi(0)$ . Define  $\Psi: U \rightarrow R^n$  by  $\Psi(q) = (x_1(q), x_2(q), \dots, x_n(q))$  for  $q \in U$ . Then the function  $F = f \circ \Psi^{-1}$  is an element of  $C^\infty(\Psi(U))$  and we have  $f(q) = F(x_1(q), \dots, x_n(q))$  for  $q \in U$ . Since  $f(\varphi(t)) = F(x_1(\varphi(t)), \dots, x_n(\varphi(t)))$ , we have the following

$$(1.1) \quad \frac{d(f \circ \varphi)}{dt} = \sum_{i=1}^n \left[ \frac{\partial F}{\partial x_i} \right]_{x=\Psi(\varphi(t))} \cdot \frac{d(x_i \circ \varphi)}{dt},$$

and hence we get

$$\left[ \frac{d(f \circ \varphi)}{dt} \right]_{t=0} = \sum_{i=1}^n \left[ \frac{\partial F}{\partial x_i} \right]_{x=\Psi(p)} \cdot \left[ \frac{d(x_i \circ \varphi)}{dt} \right]_{t=0}.$$

Similarly, we have

$$\left[ \frac{d(f \circ \psi)}{dt} \right]_{t=0} = \sum_{i=1}^n \left[ \frac{\partial F}{\partial x_i} \right]_{x=\Psi(p)} \cdot \left[ \frac{d(x_i \circ \psi)}{dt} \right]_{t=0}.$$

Hence we obtain  $[d(f \circ \varphi)/dt]_0 = [d(f \circ \psi)/dt]_0$ . Differentiate (1.1) and evaluate at  $t = 0$ , then we get  $[d^2(f \circ \varphi)/dt^2]_0 = [d^2(f \circ \psi)/dt^2]_0$  and so on. Thus we see  $f \circ \varphi \underset{r}{\sim} f \circ \psi$ . Q.E.D.

We define the local coordinate system  $\{x_i^{(\nu)} | i=1, 2, \dots, n; \nu=0, 1, \dots, r\}$  on  $(\pi)^{-1}(U)$  by  $x_i^{(\nu)}([\varphi]_r) = (1/\nu!) [d^\nu(x_i(\varphi(t)))/dt^\nu]_{t=0}$  for  $[\varphi]_r \in (\pi)^{-1}(U)$ .

It is straightforward to see that  $\overset{r}{T}(M)$  has a differentiable manifold structure by these coordinate systems and to see that  $\overset{r}{\pi}$  is a differentiable surjective map of maximal rank. It is also clear that  $\overset{r}{T}_p(M)$  is diffeomorphic to  $R^{rn}$  for any  $p \in M$ .

**DEFINITION 1.5.** The manifold  $TM$  with the projection  $\pi$  is called *the tangent bundle to  $M$  of order  $r$* . If  $U$  is an open subset of  $M$ , then  $(\overset{r}{\pi})^{-1}(U)$  is an open submanifold of  $\overset{r}{T}(M)$  which can be identified with  $\overset{r}{T}(U)$ .

However, it must be noticed that  $\overset{r}{T}(M)(M, \overset{r}{\pi})$  is not a vector bundle over  $M$ .

We define  $\pi_s^r: \overset{r}{T}(M) \rightarrow \overset{s}{T}(M)$  for  $r > s$  by  $\pi_s^r([\varphi]_r) = [\varphi]_s$ , for  $\varphi \in S(M)$ .

On the other hand,  $M$  can be imbedded in  $\overset{r}{T}(M)$  by  $x \rightarrow [\gamma_x]_r$  for  $x \in M$ , where  $\gamma_x \in S(M)$  is defined by  $\gamma_x(t) = x$  for  $t \in R$ .

Let  $N$  be another manifold of dimension  $m$ . For any map  $\Phi: M \rightarrow N$ , we define the induced map  $\overset{r}{T}\Phi: \overset{r}{T}M \rightarrow \overset{r}{T}N$  by  $(\overset{r}{T}\Phi)([\varphi]_r) = [\Phi \circ \varphi]_r$  for  $\varphi \in S(M)$ . It is easy to see that  $\overset{r}{T}\Phi$  is well-defined and that  $\overset{r}{T}\Phi$  is a differentiable map of  $\overset{r}{T}M$  into  $\overset{r}{T}N$ . We shall call  $\overset{r}{T}\Phi$  the tangent to  $\Phi$  of order  $r$  (or simply  $r$ -tangent to  $\Phi$ ).

Let  $\pi_1$  (resp.  $\pi_2$ ) be the projection of  $M \times N$  onto  $M$  (resp.  $N$ ). We can readily see that  $\overset{r}{T}(M \times N)$  can be identified with  $\overset{r}{T}M \times \overset{r}{T}N$  by  $[\varphi]_r \rightarrow ([\pi_1 \circ \varphi]_r, [\pi_2 \circ \varphi]_r)$  for  $\varphi \in S(M \times N)$ .

We can prove the following Propositions 1.6 and 1.7 whose proof will be straightforward.

**PROPOSITION 1.6.** *Let  $M_0, M_1, M_2, M_3$  be manifolds. and let  $\Phi: M_0 \rightarrow M_1, \Phi_1: M_1 \rightarrow M_2, \Phi': M_0 \rightarrow M_2$  and  $\Psi: M_2 \rightarrow M_3$  be maps. Then, we have the following equalities:*

- (i)  $\overset{r}{T}(\Phi_1 \circ \Phi) = (\overset{r}{T}\Phi_1) \circ (\overset{r}{T}\Phi),$
- (ii)  $\overset{r}{T}(\Phi, \Phi') = (\overset{r}{T}\Phi, \overset{r}{T}\Phi'),$
- (iii)  $\overset{r}{T}(\Phi \times \Psi) = \overset{r}{T}\Phi \times \overset{r}{T}\Psi,$
- (iv)  $\overset{r}{T}(1_M) = 1_{\overset{r}{T}M},$

where  $1_M$  stands for the identity map of  $M$ .

**PROPOSITION 1.7.** *Let  $\pi_1$  (resp.  $\pi_2$ ) be the projection of  $M_1 \times M_2$  onto  $M_1$  (resp.  $M_2$ ), and let  $\tilde{\pi}_1$  (resp.  $\tilde{\pi}_2$ ) be the projection of  $\overset{r}{T}M_1 \times \overset{r}{T}M_2$  onto  $\overset{r}{T}M_1$  (resp.  $\overset{r}{T}M_2$ ). Then, we have  $\overset{r}{T}\pi_i = \tilde{\pi}_i$  for  $i = 1, 2$ .*

**PROPOSITION 1.8.** *Let  $M, N$  be manifolds and let  $\Phi$  be a map of  $M$  into  $N$  of maximal rank. Then,  $\overset{r}{T}\Phi$  is a map of  $\overset{r}{T}M$  into  $\overset{r}{T}N$  of maximal rank.*

*Proof.* We shall prove only for the case  $r = 2$ , since the proof for  $r \geq 3$  is similar. Let  $p_0 \in M$  and put  $q_0 = \Phi(p_0)$ . We take a coordinate

neighborhood  $U$  (resp.  $V$ ) of  $p_0$  (resp.  $q_0$ ) with coordinate system  $\{x_1, \dots, x_n\}$  (resp.  $\{y_1, \dots, y_m\}$ ) such that  $\Phi(U) \subset V$ . Then,  $\overset{2}{T}U$  (resp.  $\overset{2}{T}V$ ) has the induced coordinate system  $\{x_i, \dot{x}_i, \ddot{x}_i | i = 1, 2, \dots, n\}$  (resp.  $\{y_j, \dot{y}_j, \ddot{y}_j | j = 1, 2, \dots, m\}$ ). Put  $F_i(x_1, \dots, x_n) = y_i(\Phi(x))$  for  $x \in U$ . Take an element  $[\varphi]_2 \in \overset{2}{T}(U)$  with coordinates  $\{x_i, \dot{x}_i, \ddot{x}_i\}$ , then  $x_i(\varphi(t)) = x_i + \dot{x}_i t + \ddot{x}_i t^2 + \varepsilon_i(t)$ , where  $[d^2\varepsilon_i/dt^2]_0 = 0$ . Hence, we have  $y_i\Phi(x_1(\varphi(t)), \dots, x_n(\varphi(t))) = F_i(x_1, \dots, x_n) + \sum_j \frac{\partial F_i}{\partial x_j} \dot{x}_j t + \frac{1}{2} \left( \sum_{j,k} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_j \dot{x}_k + 2 \sum_j \frac{\partial F_i}{\partial x_j} \ddot{x}_j \right) t^2 + \eta_i(t)$ , where  $[d^2\eta_i/dt^2]_0 = 0$ . Therefore,  $(\overset{2}{T}\Phi)([\varphi]_2) = [\Phi \circ \varphi]_2$  has the following coordinates:

$$(1.2) \quad \begin{cases} y_i = F_i(x), & \dot{y}_i = \sum_j \frac{\partial F_i}{\partial x_j} \dot{x}_j, \\ \ddot{y}_i = \frac{1}{2} \sum_{j,k} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_j \dot{x}_k + \sum_j \frac{\partial F_i}{\partial x_j} \ddot{x}_j. \end{cases}$$

Hence, the map  $\overset{2}{T}\Phi$  has the Jacobian matrix  $J$  with respect to the coordinate systems  $\{x_i^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, 2\}$  and  $\{y_k^{(\nu)} | k = 1, \dots, m; \nu = 0, 1, 2\}$  as follows:

$$(1.3) \quad J = \begin{pmatrix} \left( \frac{\partial F_i}{\partial x_k} \right) & 0 & 0 \\ (J_k^i) & \left( \frac{\partial F_i}{\partial x_k} \right) & 0 \\ (\ddot{J}_k^i) & (\dot{J}_k^i) & \left( \frac{\partial F_i}{\partial x_k} \right) \end{pmatrix}$$

where  $\dot{J}_k^i = \sum_j \frac{\partial^2 F_i}{\partial x_j \partial x_k} \dot{x}_j$  and  $\ddot{J}_k^i = \frac{1}{2} \sum_{j,l} \frac{\partial^3 F_i}{\partial x_j \partial x_k \partial x_l} \dot{x}_j \cdot \dot{x}_l + \sum_j \frac{\partial^2 F_i}{\partial x_j \partial x_k} \ddot{x}_j$ .

Since the Jacobian matrix of  $\Phi$  is  $\left( \frac{\partial F_i}{\partial x_k} \right)$ , which has the maximal rank,  $J$  has also the maximal rank.

**COROLLARY 1.9.** *Let  $\Phi$  be a regular map of  $M$  into  $N$ , namely the differential  $T\Phi$  is an injective map of  $T_p(M)$  into  $T_{\Phi(p)}(N)$  for every point  $p \in M$ . Then,  $\overset{r}{T}\Phi$  is also a regular map of  $\overset{r}{T}M$  into  $\overset{r}{T}N$ .*

*Remark 1.10.* We see that if  $\Phi$  is a regular injective map, then  $\overset{r}{T}\Phi$  is also a regular injective map.

**§2. Tangent groups of order  $r$ .**

Let  $G$  be a Lie group with group multiplication  $\mu: G \times G \rightarrow G$  and with the unit element  $e$ .

**THEOREM 2.1.**  $\overset{r}{T}G$  is a Lie group with group multiplication  $\overset{r}{T}\mu$ . The group  $G$  is a closed subgroup of  $\overset{r}{T}G$  and  $\overset{r}{T}_e(G)$  is a closed normal subgroup of  $\overset{r}{T}G$  such that

$$\overset{r}{T}G = G \cdot \overset{r}{T}_e(G),$$

with  $G \cap \overset{r}{T}_e(G) = \bar{e}$ , where  $\bar{e}$  is the unit element of  $\overset{r}{T}G$ . Moreover the projection  $\overset{r}{\pi}: \overset{r}{T}G \rightarrow G$  is a homomorphism. (cf. [3] for  $r = 1$ )

*Proof.* For any two elements  $\varphi, \psi \in S(G)$  (cf. §1), we define  $\varphi \cdot \psi \in S(G)$  by  $(\varphi \cdot \psi)(t) = \varphi(t) \cdot \psi(t)$  for  $t \in R$ . Then, we have  $(\overset{r}{T}\mu)([\varphi]_r, [\psi]_r) = (\overset{r}{T}\mu)([(\varphi, \psi)]_r) = [\mu \circ (\varphi, \psi)]_r = [\varphi \cdot \psi]_r$  and hence we get

$$(2.1) \quad (\overset{r}{T}\mu)([\varphi]_r, [\psi]_r) = [\varphi \cdot \psi]_r.$$

Since  $(\varphi \cdot \psi) \cdot \eta = \varphi \cdot (\psi \cdot \eta)$  for any  $\varphi, \psi, \eta \in S(G)$ , we see that the multiplication  $\overset{r}{T}\mu$  is associative. Define  $\gamma_e \in S(G)$  by  $\gamma_e(t) = e$  for  $t \in R$  and put  $\bar{e} = [\gamma_e]_r$ . Clearly  $\bar{e}$  is the unit element with respect to  $\overset{r}{T}\mu$ . For  $\varphi \in S(G)$ , we define  $\varphi^{-1} \in S(G)$  by  $\varphi^{-1}(t) = (\varphi(t))^{-1}$  for  $t \in R$ . Then  $\overset{r}{T}\mu([\varphi]_r, [\varphi^{-1}]_r) = [\varphi \cdot \varphi^{-1}]_r = [\gamma_e]_r = \bar{e}$  and hence  $[\varphi^{-1}]_r$  is the inverse element of  $[\varphi]_r$ . Now,  $[\varphi^{-1}]_r = (\overset{r}{T}\iota)[\varphi]_r$ , where  $\iota: G \rightarrow G$  is the map  $x \rightarrow x^{-1}$  for  $x \in G$ . Since  $\overset{r}{T}\iota$  is a differentiable map of  $\overset{r}{T}G$  into itself, we have proved that  $\overset{r}{T}G$  is a Lie group with group multiplication  $\overset{r}{T}\mu$ . Next, since  $G = \{[\gamma_a]_r | a \in G\}$ , where  $\gamma_a(t) = a$  for  $t \in R$ , it follows that  $G$  is a closed subgroup of  $\overset{r}{T}G$ . Similarly we see that  $\overset{r}{T}_e G$  is a closed normal subgroup of  $\overset{r}{T}G$ . Next, any  $[\varphi]_r \in \overset{r}{T}G$  can be written as  $[\varphi]_r = [\gamma_a]_r \cdot [\gamma_{a^{-1}} \cdot \varphi]_r$ , where  $a = \varphi(0)$  and so  $[\gamma_{a^{-1}} \cdot \varphi]_r \in \overset{r}{T}_e G$ . The equality  $G \cap \overset{r}{T}_e G = \bar{e}$  is also clear. Finally the projection  $\overset{r}{\pi}$  is a homomorphism since (2.1) holds. Q.E.D.

**DEFINITION 2.2.** The Lie group  $TG$  with group multiplication  $\overset{r}{T}\mu$  will be called the tangent group to  $G$  of order  $r$ .

PROPOSITION 2. 3. *Let  $\Phi$  be a homomorphism of a Lie group  $G$  into a Lie group  $G'$ . Then  $\overset{r}{T}\Phi$  is also a homomorphism of the tangent group  $\overset{r}{T}G$  of order  $r$  into  $\overset{r}{T}G'$ .*

*Proof.* Let  $\mu'$  be the group multiplication of  $G$ . Since  $\Phi$  is a homomorphism, we have  $\Phi \circ \mu = \mu' \circ (\Phi \times \Phi)$ . By Proposition 1. 6 we have  $\overset{r}{T}\Phi \circ \overset{r}{T}\mu = \overset{r}{T}\mu' \circ (\overset{r}{T}\Phi \times \overset{r}{T}\Phi)$ , which means that  $\overset{r}{T}\Phi$  is a homomorphism of  $\overset{r}{T}G$  into  $\overset{r}{T}G'$ .

PROPOSITION 2. 4. *The projection  $\pi'_s: \overset{r}{T}G \rightarrow \overset{r}{T}G$  for  $r > s$  is a homomorphism of tangent groups.*

*Proof.* Clear from the equality (2. 1).

PROPOSITION 2. 5. *If  $G$  is a Lie subgroup of  $G'$ , then  $\overset{r}{T}(G)$  is also a Lie subgroup of  $\overset{r}{T}(G')$ .*

*Proof.* Let  $\Phi: G \rightarrow G'$  be the injection map. Then  $\Phi$  is a regular map. By Remark 1. 10 and Proposition 2. 3,  $\overset{r}{T}\Phi$  is a regular homomorphism of  $\overset{r}{T}G$  into  $\overset{r}{T}G'$ . Let  $[\varphi]_r$  be an element of  $\overset{r}{T}G$  such that  $(\overset{r}{T}\Phi)([\varphi]_r) = \tilde{e}'$  is the unit element of  $\overset{r}{T}G'$ . Then  $[\Phi \circ \varphi]_r = [\gamma'_e]_r$ , where  $\gamma'_e: R \rightarrow G'$  is defined by  $\gamma'_e(t) = e$  for  $t \in R$ ,  $e$  being the unit element of  $G$ . We see that  $\varphi(0) = e$  and that  $[\varphi]_r = [\gamma_e]_r = \tilde{e}$ . Hence  $\overset{r}{T}\Phi$  is a regular injective homomorphism, which means that  $\overset{r}{T}G$  is a Lie subgroup of  $\overset{r}{T}G'$ . Q.E.D.

### §3. Tangent operations of order $r$ .

Let  $G$  be a Lie group operating on a manifold  $M$  differentiably. We denote by  $\rho: G \times M \rightarrow M$  the operation map of  $G$  on  $M$ .

PROPOSITION 3. 1. *The tangent group  $\overset{r}{T}G$  to  $G$  of order  $r$  operates on the tangent bundle  $\overset{r}{T}M$  of order  $r$  by the operation map  $\overset{r}{T}\rho$  (for the tangent group  $\overset{r}{T}G$ , see [3]).*

*Proof.* Since  $\rho$  is the operation map of  $G$  on  $M$ , we have  $\rho \circ (\mu \times 1_M) = \rho \circ (1_G \times \rho)$ . By Proposition 1. 6 we have  $(\overset{r}{T}\rho) \circ (\overset{r}{T}\mu \times 1_{\overset{r}{T}M}) = \overset{r}{T}\rho \circ (1_{\overset{r}{T}G} \times \overset{r}{T}\rho)$ , which means that  $\tilde{a} \cdot (\tilde{b} \cdot \tilde{x}) = (\tilde{a} \cdot \tilde{b}) \cdot \tilde{x}$  for  $\tilde{a}, \tilde{b} \in \overset{r}{T}(G)$  and  $\tilde{x} \in \overset{r}{T}M$ , where we



have put  $\tilde{a} \cdot \tilde{x} = (\overset{r}{T}\rho)(\tilde{a}, \tilde{x})$ . Let  $\gamma_e: R \rightarrow G$  be the constant map:  $\gamma_e(t) = e$  for  $t \in R$ . Then, for any  $[\varphi]_r \in \overset{r}{T}M$  we have  $\overset{r}{T}\rho([\gamma_e], [\varphi]_r) = \overset{r}{T}\rho([\gamma_e, \varphi]_r) = [\rho \circ (\gamma_e, \varphi)]_r = [\gamma_e, \varphi]_r = [\varphi]_r$ , which means that the unit element  $\tilde{e} = [\gamma_e]_r$  of  $\overset{r}{T}G$  operates on  $\overset{r}{T}M$  as the identity map. Hence we have proved that  $\overset{r}{T}G$  operates on  $\overset{r}{T}M$  by  $\overset{r}{T}\rho$ . Q.E.D.

DEFINITION 3. 2. The operation map  $\overset{r}{T}\rho$  in Proposition 3. 1 will be called the *tangent operation to  $\rho$  of order  $r$* .

PROPOSITION 3. 3. If a Lie group  $G$  operates on  $M$  effectively (i.e.  $a \cdot x = x$  for all  $x \in M$  implies  $a = e$ ), then  $\overset{2}{T}G$  operates on  $\overset{2}{T}M$  effectively by the tangent operation of order 2.

*Proof.* For  $\varphi \in S(G)$  and  $\psi \in S(M)$  we define  $\varphi \cdot \psi \in S(M)$  by  $(\varphi \cdot \psi)(t) = \varphi(t) \cdot \psi(t)$  for  $t \in R$ . Suppose  $\varphi \cdot \psi \underset{2}{\sim} \psi$  for every  $\psi \in S(M)$ . We have to show that  $\varphi \underset{2}{\sim} \gamma_e$ , where  $\gamma_e \in S(G)$  is defined by  $\gamma_e(t) = e$ . First, since  $\varphi(0) \cdot \psi(0) = \psi(0)$  for any  $\psi \in S(M)$ , we see that  $\varphi(0) \cdot x = x$  for any  $x \in M$ , whence  $\varphi(0) = e$  since  $G$  operates effectively on  $M$ . Next take a point  $p_0 \in M$  and fix it. We take a coordinate neighborhood  $U$  (resp.  $V$ ) of  $p_0$  (resp. of  $e$ ) in  $M$  (resp. in  $G$ ) with coordinate system  $\{x_1, \dots, x_n\}$  (resp.  $\{z_1, \dots, z_N\}$ ) such that  $x_i(p_0) = 0$  for  $i = 1, 2, \dots, n$  (resp.  $z_l(e) = 0$  for  $l = 1, 2, \dots, N$ ). Define the functions  $F_i (i = 1, \dots, n)$  by

$$F_i(z_1, \dots, z_N; x_1, \dots, x_n) = x_i(\rho(z, x)).$$

Let  $\{x_i^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, 2\}$  (resp.  $\{z_l^{(\nu)} | l = 1, \dots, N; \nu = 0, 1, 2\}$ ) be the induced coordinate system on  $\overset{r}{T}(U)$  (resp.  $\overset{r}{T}(V)$ ). If  $x_i^{(0)}([\psi]_2) = x_i$ ,  $x_i^{(1)}([\psi]_2) = \dot{x}_i$ ,  $x_i^{(2)}([\psi]_2) = \ddot{x}_i$ , we see that

$$\phi(t) = (\dots, x_i + \dot{x}_i t + \ddot{x}_i t^2 + \varepsilon_i(t), \dots) \in U$$

for small  $|t|$ , where  $[d^2\varepsilon_i/dt^2]_0 = 0$  for  $i = 1, \dots, n$ . Similarly we see that

$$\varphi(t) = (\dots, z_l t + \ddot{z}_l t + \eta_l(t), \dots) \in V$$

for small  $|t|$ , where  $[d^2\eta_l/dt^2]_0 = 0$  for  $l = 1, \dots, N$ . We have the relations  $x_i \circ (\varphi \cdot \psi) \underset{r}{\sim} x_i \circ \psi$  ( $i = 1, 2, \dots, n$ ) for every  $\psi \in S(M)$ . To simplify the notations we define the functions  $f_i(t)$  for  $i = 1, \dots, n$  by

$$f_i(t) = F_i(\dots, \varphi_i(t), \dots; \dots, \psi_i(t), \dots)$$

and we define the variables  $y_\kappa^{(\nu)}$  for  $\kappa = 1, 2, \dots, N+n$ ;  $\nu = 0, 1, \dots, r$  by  $y_\kappa^{(\nu)} = \nu! \cdot z_\kappa^{(\nu)}$  for  $\kappa = 1, 2, \dots, N$  and  $y_\kappa^{(\nu)} = \nu! \cdot x_{\kappa-N}^{(\nu)}$  for  $\kappa = N+1, \dots, N+n$ . By means of these notations we have the following equalities

$$(3.1) \quad \frac{df_i}{dt} = \sum_{\kappa=1}^{N+n} \frac{\partial F_i}{\partial y_\kappa} (\dot{y}_\kappa + \dot{y}_\kappa t + \varepsilon_1^\kappa(t))$$

$$(3.2) \quad \frac{d^2 f_i}{dt^2} = \sum_{\kappa} \frac{\partial F_i}{\partial y_\kappa} (\ddot{y}_\kappa + y_\kappa^{(3)} + \varepsilon_2^\kappa(t)) \\ + \sum_{\kappa, \lambda} \frac{\partial^2 F_i}{\partial y_\kappa \partial y_\lambda} (\dot{y}_\kappa + \dot{y}_\kappa t + \varepsilon_1^\kappa(t)) \cdot (\dot{y}_\lambda + \dot{y}_\lambda t + \varepsilon_1^\lambda(t)),$$

where  $[d\varepsilon_k^\kappa/dt]_0 = 0$  for  $k = 1, 2$ . Since  $f_i(t) = (x_i \circ (\varphi \cdot \psi))(t)$  and since  $x_i \circ (\varphi \cdot \psi) \sim x_i \circ \psi$  we obtain the following relations:

$$(3.3) \quad \sum_{l=1}^N \left[ \frac{\partial F_i}{\partial z_l} \right]_{(0, x)} \dot{z}_l + \sum_{j=1}^n \left[ \frac{\partial F_i}{\partial x_j} \right]_{(0, x)} \dot{x}_j = \dot{x}_i,$$

$$(3.4) \quad 2 \sum_l \left[ \frac{\partial F_i}{\partial z_l} \right]_{(0, x)} \ddot{z}_l + 2 \sum_j \left[ \frac{\partial F_i}{\partial x_j} \right]_{(0, x)} \ddot{x}_j \\ + \sum_{l, m=1}^N \left[ \frac{\partial^2 F_i}{\partial z_l \partial z_m} \right]_{(0, x)} \dot{z}_l \cdot \dot{z}_m + 2 \sum_{l=1}^N \sum_{j=1}^n \left[ \frac{\partial^2 F_i}{\partial z_l \partial x_j} \right]_{(0, x)} \dot{z}_l \dot{x}_j \\ + \sum_{j, k=1}^n \left[ \frac{\partial F_i}{\partial x_j \partial x_k} \right]_{(0, x)} \dot{x}_j \dot{x}_k = \ddot{x}_i$$

for  $i = 1, 2, \dots, n$  and for every  $(x_i, \dot{x}_i, \ddot{x}_i) \in U$ . Now, since  $e \cdot x = x$  for any  $x \in M$ , we have

$$F_i(0, \dots, 0; x_1, \dots, x_n) = x_i$$

for  $i = 1, 2, \dots, n$ . Therefore, we get  $\left[ \frac{\partial F_i}{\partial x_j} \right]_{(0, x)} = \delta_j^i$  for  $i, j = 1, 2, \dots, n$ . Finally, we obtain from (3.3), (3.4) the following relations:

$$(3.5) \quad \sum_{l=1}^N \left[ \frac{\partial F_i}{\partial z_l} \right]_{(0, x)} \dot{z}_l = 0.$$

$$(3.6) \quad 2 \sum_l \left[ \frac{\partial F_i}{\partial z_l} \right]_{(0, x)} \ddot{z}_l + \sum_{l, m} \left[ \frac{\partial^2 F_i}{\partial z_l \partial z_m} \right]_{(0, x)} \dot{z}_l \dot{z}_m \\ + 2 \sum_{l, j} \left[ \frac{\partial^2 F_i}{\partial z_l \partial x_j} \right]_{(0, x)} \dot{z}_l \cdot \dot{x}_j = 0$$

for every  $(x_1, \dots, x_n) \in U$  and  $i = 1, \dots, n$ .

Now, we shall prove the following

LEMMA 3. 4. Let  $a_1, \dots, a_N \in R$ . Suppose  $\sum_{i=1}^N a_i \left[ \frac{\partial F_i}{\partial z_l} \right]_{(0, x)} = 0$  holds for every  $(x_1, \dots, x_n) \in U$  and for  $i = 1, 2, \dots, n$ , where  $U$  is an arbitrary coordinate neighborhood in  $M$ . Then  $a_l = 0$  for  $l = 1, 2, \dots, N$ .

By virtue of this lemma, we see from (3. 5) that  $\dot{z}_l = 0$  for  $l = 1, 2, \dots, N$  and then from (3. 6) it follows that  $\ddot{z}_l = 0$  for  $l = 1, 2, \dots, N$ , which proves that  $\varphi \sim_2 r_e$  and thus the proposition will be proved.

*Proof of Lemma 3. 4.* Suppose  $a_l \neq 0$  for some  $l$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . By taking a linear transformation of the coordinates  $\{z_1, \dots, z_N\}$ , if necessary, we can suppose that  $[\partial F_i / \partial z_1]_{(0, x)} = 0$  for any  $x \in U$  and that  $z_i (\exp \sum_{j=1}^N t_j X_j) = t_i$  for  $i = 1, 2, \dots, N$ , where  $\{X_1, \dots, X_N\}$  is a base of  $\mathfrak{g}$ . Now let  $\tilde{X}_1$  be the vector field on  $M$  induced by the one-parameter group  $\exp tX_1$ . For any point  $x \in U$ , we have  $(\tilde{X}_1)_x = 0$ , since  $(\tilde{X}_1)_x \cdot x_i = [dx_i((\exp tX_1) \cdot x) / dt]_0 = [dF_i(t, 0, \dots, 0; x) / dt]_0 = [\partial F_i / \partial z_1]_{(0, x)} = 0$  for  $i = 1, 2, \dots, n$ . Since  $U$  and  $x$  are arbitrary, we see that  $\tilde{X}_1 = 0$  on  $M$  and that  $\exp tX_1$  operates trivially on  $M$ . It follows that  $\exp tX_1 = e$  for any  $t \in R$  and hence  $X_1 = 0$ , which is a contradiction. Thus Lemma 3. 4 is proved and hence the proof of Proposition 3. 3 is complete. Q.E.D.

More generally, we can prove the following

THEOREM 3. 5. If a Lie group  $G$  operates on  $M$  effectively, then  $\overset{r}{T}G$  operates on  $\overset{r}{T}M$  effectively by tangent operation of order  $r$  for any positive integer  $r$ .

*Proof.* Using the notations of the proof of Proposition 3. 3, especially the notations of (3. 1), we define  $\varphi_\alpha(t)$  by  $\varphi_\alpha(t) = \dot{y}_\alpha + \dot{y}_\alpha t + \epsilon_1^\alpha(t)$  for  $\alpha = 1, 2, \dots, N + n$ . Then the equality (3. 2) can be written as follows:

$$(3. 7) \quad \frac{d^2 f_i}{dt^2} = \sum \frac{\partial F_i}{\partial y_\alpha} \cdot \varphi_\alpha + \sum \frac{\partial^2 F_i}{\partial y_\alpha \partial y_\beta} \varphi_\alpha \varphi_\beta.$$

By differentiating (3. 7), we obtain the following

$$(3. 8) \quad \begin{aligned} \frac{d^3 f_i}{dt^3} &= \sum \frac{\partial^3 F_i}{\partial y_\alpha \partial y_\beta \partial y_\gamma} \varphi_\alpha \varphi_\beta \varphi_\gamma \\ &+ 3 \sum \frac{\partial^2 F_i}{\partial y_\alpha \partial y_\beta} \varphi'_\alpha \varphi_\beta + \sum \frac{\partial F_i}{\partial y_\alpha} \varphi''_\alpha. \end{aligned}$$

In general, by induction on  $\nu = 1, 2, \dots$ , we obtain the following equality

$$\begin{aligned}
(3.9) \quad \frac{d^\nu f_i}{dt^\nu} &= \sum \frac{\partial^\nu F_i}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_\nu}} \varphi_{\alpha_1} \cdots \varphi_{\alpha_\nu} \\
&+ c_1^{(\nu)} \sum \frac{\partial^{\nu-1} F_i}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_{\nu-1}}} \varphi'_{\alpha_1} \varphi_{\alpha_2} \cdots \varphi_{\alpha_{\nu-1}} \\
&+ c_{1,1}^{(\nu)} \sum \frac{\partial^{\nu-2} F_i}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_{\nu-2}}} \varphi'_{\alpha_1} \varphi'_{\alpha_2} \varphi_{\alpha_3} \cdots \varphi_{\alpha_{\nu-2}} \\
&+ c_{\frac{1}{2}}^{(\nu)} \sum \frac{\partial^{\nu-2} F_i}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_{\nu-2}}} \varphi''_{\alpha_1} \varphi_{\alpha_2} \cdots \varphi_{\alpha_{\nu-2}} \\
&+ \cdots + c_{\nu-2}^{(\nu)} \sum \frac{\partial^2 F_i}{\partial y_{\alpha_1} \partial y_{\alpha_2}} \varphi_1^{(\nu-2)} \varphi_{\alpha_2} + \sum \frac{\partial F_i}{\partial y_\alpha} \varphi_\alpha^{(\nu-1)},
\end{aligned}$$

where  $c_{\mu_1 \dots \mu_s}^{(\nu)}$  are some positive integer for  $\sum_{i=1}^s \mu_i = 1, 2, \dots, \nu - 2$  and for any  $\nu = 1, 2, \dots$ .

Suppose  $(\varphi \cdot \psi) \underset{r}{\sim} \psi$  for every  $\psi \in S(M)$  as in the proof of Proposition 3.3. By using (3.9) and Lemma 3.4 repeatedly we can show, by induction on  $\nu$ , that  $z_l = 0$  for any  $l = 1, 2, \dots, N$  and  $\nu = 0, 1, \dots, r$ , which proves that  $\varphi \underset{r}{\sim} \gamma_\sigma$ . Q.E.D.

#### §4. Tangent bundle to $R^n$ of order $r$ .

Let  $R^n$  be the real euclidean space of dimension  $n$ . For any two  $r$ -tangents  $[\varphi]_r, [\psi]_r$  to  $R^n$ , we define their sum by:  $[\varphi]_r + [\psi]_r = [\varphi + \psi]_r$ , where  $(\varphi + \psi)(t) = \varphi(t) + \psi(t)$  for  $t \in R$ . For any  $c \in R$  we define the scalar multiplication of  $[\varphi]_r$  by  $c$  as follows:  $c \cdot [\varphi]_r = [c \cdot \varphi]_r$ , where  $(c \cdot \varphi)(t) = c \cdot \varphi(t)$  for  $t \in R$ . Clearly  $[\varphi]_r + [\psi]_r$  and  $c \cdot [\varphi]_r$  are well-defined.

**THEOREM 4.1.** *By the above sum and scalar multiplication the tangent bundle  $\overset{r}{T}R_n$  to  $R^n$  of order  $r$  is a real vector space of dimension  $n(r+1)$ .*

*Proof.* Straightforward verification. Q.E.D.

**PROPOSITION 4.2.** *Let  $V \oplus W$  be a direct sum of vector subspaces  $V$  and  $W$ , then  $\overset{r}{T}V$  and  $\overset{r}{T}W$  are identified with vector subspaces of  $T(V \oplus W)$  and we have*

$$\overset{r}{T}(V \oplus W) = \overset{r}{T}V \oplus \overset{r}{T}W \quad (\text{direct sum}).$$

**Remark 4.3.** Let  $\{x_1, \dots, x\}$  be the natural coordinate system on  $R^n$  and let  $\{x_i | i = 1, \dots, n; \nu\}$  be the induced coordinate system on  $\overset{r}{T}R^n$ .

Then the sum and scalar multiplication in  $\overset{r}{T}R^n$  in Theorem 4. 1 are as follows:

$$\begin{cases} \binom{(\nu)}{(x_i)} + \binom{(\nu)}{(x'_i)} = \binom{(\nu)}{(x_i + x'_i)}, \\ c \cdot \binom{(\nu)}{(x_i)} = \binom{(\nu)}{(c \cdot x_i)}. \end{cases}$$

**§5. Imbedding of  $\overset{r}{T}GL(n)$  into  $GL(n(r + 1))$ .**

Let  $\rho: GL(n) \times R^n \rightarrow R^n$  be the usual operation of the general linear group  $GL(n)$  on  $R^n$ . By Proposition 3. 1, the tangent group  $\overset{r}{T}GL(n)$  to  $GL(n)$  of order  $r$  operates on  $\overset{r}{T}R^n$  by the tangent operation  $\overset{r}{T}\rho$  to  $\rho$  of order  $r$ . Now, by Theorem 4. 1,  $\overset{r}{T}R^n$  is a vector space of dimension  $n(r+1)$ . We shall prove the following

**THEOREM 5. 1.** *The tangent group  $\overset{r}{T}GL(n)$  to  $GL(n)$  of order  $r$  operates on  $\overset{r}{T}R^n$  effectively as a linear group.*

*Proof.* Since  $\rho$  is effective, we see that  $\overset{r}{T}\rho$  is effective by Theorem 3. 5. For any  $\eta \in S(GL(n))$  and  $\varphi \in S(R^n)$ , we define  $\eta \cdot \varphi \in S(R^n)$  by the equality  $(\eta \cdot \varphi)(t) = \eta(t) \cdot (\varphi(t)) = \rho(\eta(t), \varphi(t))$  for  $t \in R$ . We put  $[\eta]_r \cdot [\varphi]_r = \overset{r}{T}\rho([\eta]_r, [\varphi]_r)$ . Then we have  $[\eta]_r \cdot [\varphi]_r = [\eta \cdot \varphi]_r$ . Take an element  $[\phi]_r$  of  $\overset{r}{T}(R^n)$  and  $c \in R$ . Then we calculate as follows:  $[\eta]_r([\varphi]_r + [\psi]_r) = [\eta]_r \cdot [\varphi + \psi]_r = [\eta \cdot (\varphi + \psi)]_r = [\eta \cdot \varphi + \eta \cdot \psi]_r = [\eta \cdot \varphi]_r + [\eta \cdot \psi]_r = [\eta]_r \cdot [\varphi]_r + [\eta]_r \cdot [\psi]_r$ . Similarly, we have  $[\eta]_r(c \cdot [\varphi]_r) = [\eta]_r[c \cdot \varphi]_r = [\eta \cdot (c\varphi)]_r = [c \cdot (\eta \cdot \varphi)]_r = c[\eta \cdot \varphi]_r = c([\eta]_r \cdot [\varphi]_r)$ . Thus we have proved that  $[\eta]_r$  operates on  $\overset{r}{T}R^n$  as a linear transformation.

Q.E.D.

**DEFINITION 5. 2.** Let  $\{x_1, \dots, x_n\}$  be the natural coordinate system on  $R^n$  and let  $\{\binom{(\nu)}{x_i} | i = 1, \dots, n; \nu = 0, 1, \dots, r\}$  be the induced coordinate system on  $\overset{r}{T}R^n$ . Using these coordinates, Theorem 5. 1 shows that there is a canonical injective homomorphism  $j_n^{(r)}$  of  $\overset{r}{T}GL(n)$  into  $GL(n(r + 1))$ .

Let  $(y^j) \in GL(n)$ . Then  $\overset{r}{T}GL(n)$  has the induced coordinate system  $\{\binom{(\nu)}{y^j} | i, j = 1, \dots, n; \nu = 0, 1, \dots, r\}$ . We denote by  $Y$ , the  $n \times n$ -matrix  $\binom{(\nu)}{y^j}$  for  $\nu = 0, 1, \dots, r$ .

PROPOSITION 5. 3. *The homomorphism  $j_n^{(r)}$  is given by the following equality:*

$$j_n^{(r)}(\dots, \overset{(\nu)}{y_j^i}, \dots) = \begin{pmatrix} Y_0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ Y_1 & Y_0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & Y_1 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & Y_1 & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ Y_r & \dots & \dots & \dots & \dots & \dots & Y_1 & Y_0 \end{pmatrix}$$

*Proof.* We shall prove the proposition only for the case  $r = 2$ , since the proof for the case  $r \geq 3$  is similar. Let  $[\varphi]_2 \in {}^2TGL(n)$  be such that  $[\varphi]_2 = (y_j^i, \dot{y}_j^i, \ddot{y}_j^i)$ . Let  $[\xi]_2 \in {}^2TR^n$  be such that  $[\xi]_2 = (x_i, \dot{x}_i, \ddot{x}_i)$ . Then we can assume that

$$(5. 1) \quad \begin{cases} \varphi(t) = (y_j^i + \dot{y}_j^i t + \ddot{y}_j^i t^2), \\ \xi(t) = (x_i + \dot{x}_i t + \ddot{x}_i t^2) \end{cases}$$

for  $t \in R$ . From (5. 1) it follows that  $(\varphi \cdot \xi)(t) = \varphi(t) \cdot \xi(t) = (\sum_i (y_j^i + \dot{y}_j^i t + \ddot{y}_j^i t^2) (x_i + \dot{x}_i t + \ddot{x}_i t^2)) = (\sum_i y_j^i x_i + \sum_i (\dot{y}_j^i x_i + y_j^i \dot{x}_i) t + \sum_i (y_j^i \ddot{x}_i + \dot{y}_j^i \dot{x}_i + \ddot{y}_j^i x_i) t^2 + \sum_i (\dot{y}_j^i \dot{x}_i + \ddot{y}_j^i \dot{x}_i) t^3 + \sum_i \ddot{y}_j^i \dot{x}_i t^4)$ . Therefore, we get  $[\varphi]_2[\xi]_2 = [\varphi \cdot \xi]_2 = (\sum_i y_j^i x_i, \sum_i (\dot{y}_j^i x_i + y_j^i \dot{x}_i), \sum_i (y_j^i \ddot{x}_i + \dot{y}_j^i \dot{x}_i + \ddot{y}_j^i x_i))$ ,

and hence we obtain

$$j_n^{(2)}([\varphi]_2) = \begin{pmatrix} y_j^i & 0 & 0 \\ \dot{y}_j^i & y_j^i & 0 \\ \ddot{y}_j^i & \dot{y}_j^i & y_j^i \end{pmatrix}$$

which proves the proposition. Q.E.D.

**§6. Tangential fibre bundle of order  $r$ .**

Let  $E(M, \pi, F, G)$  be a fibre bundle with bundle space  $E$ , base  $M$ , projection  $\pi$ , fibre  $F$  and structure group  $G$ . We shall prove the following

PROPOSITION 6. 1.  ${}^rTE({}^rTM, {}^rT\pi, {}^rTF, {}^rTG)$  is a fibre bundle with bundle space  ${}^rTE$ , base  ${}^rTM$ , projection  ${}^rT\pi$ , fibre  ${}^rTF$  and structure group  ${}^rTG$ .

*Proof.* First, since  $G$  operates on  $F$  effectively,  $\overset{\check{r}}{T}G$  operates on  $\overset{\check{r}}{T}F$  effectively by virtue of Theorem 3.5. Let  $\{U_\alpha\}$  be an open covering of  $M$  such that  $E$  is trivial over  $U_\alpha$  with trivialization  $\Psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$  and with transition functions  $g_{\alpha\beta}$ , i.e.  $\Psi_\alpha \circ \Psi_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x) \cdot y)$  for  $x \in U_\alpha \cap U_\beta$  and  $y \in F$ . Clearly  $\{\overset{\check{r}}{T}U_\alpha\}$  is an open covering of  $\overset{\check{r}}{T}M$  and  $\overset{\check{r}}{T}\Psi_\alpha$  is a diffeomorphism of  $(\overset{\check{r}}{T}\pi)^{-1}(\overset{\check{r}}{T}U_\alpha)$  onto  $\overset{\check{r}}{T}U_\alpha \times \overset{\check{r}}{T}F$ . We shall verify the following

$$(6.1) \quad (\overset{\check{r}}{T}\Psi_\alpha) \circ (\overset{\check{r}}{T}\Psi_\beta)^{-1}([\varphi]_r, [\psi]_r) = ([\varphi]_r, ((\overset{\check{r}}{T}g_{\alpha\beta})[\varphi]_r) \cdot [\psi]_r)$$

for  $[\varphi]_r \in \overset{\check{r}}{T}(U_\alpha \cap U_\beta)$  and  $[\psi]_r \in \overset{\check{r}}{T}F$ . We denote by  $\rho: G \times F \rightarrow F$  the operation of  $G$  on  $F$  and by  $\pi_1: U_\alpha \cap U_\beta \times F \rightarrow U_\alpha \cap U_\beta$  (resp.  $\pi_2: U_\alpha \cap U_\beta \times F \rightarrow F$ ) the projection. Similarly we define  $\tilde{\pi}_1: \overset{\check{r}}{T}(U_\alpha \cap U_\beta) \times \overset{\check{r}}{T}F \rightarrow \overset{\check{r}}{T}(U_\alpha \cap U_\beta)$  and  $\tilde{\pi}_2$ . Then, we have the following equalities

$$(6.2) \quad \pi_1 \circ \Psi_\alpha \circ \Psi_\beta^{-1} = \pi_1, \quad \pi_2 \circ \Psi_\alpha \circ \Psi_\beta^{-1} = \rho \circ (g_{\alpha\beta} \times 1_F).$$

Taking the tangent to (6.2) of order  $r$ , we get, by Propositions 1.6 and 1.7, the following

$$(6.3) \quad \begin{cases} \tilde{\pi}_1 \circ \overset{\check{r}}{T}\Psi_\alpha \circ \overset{\check{r}}{T}\Psi_\beta^{-1} = \tilde{\pi}_1, \\ \tilde{\pi}_2 \circ \overset{\check{r}}{T}\Psi_\alpha \circ \overset{\check{r}}{T}\Psi_\beta^{-1} = \overset{\check{r}}{T}\rho \circ (\overset{\check{r}}{T}g_{\alpha\beta} \times 1_{\overset{\check{r}}{T}F}), \end{cases}$$

which proves (6.1). Therefore, we have proved that  $\overset{\check{r}}{T}E$  is a fibre bundle with transition functions  $\{\overset{\check{r}}{T}g_{\alpha\beta}\}$ . Q.E.D.

**DEFINITION 6.2.** We shall call the fibre bundle  $\overset{\check{r}}{T}E(\overset{\check{r}}{T}M, \overset{\check{r}}{T}\pi, \overset{\check{r}}{T}F, \overset{\check{r}}{T}G)$  the *tangential fibre bundle to  $E$  of order  $r$* .

Let  $P(M, \pi, G)$  be a principal fibre bundle with bundle space  $P$ , base  $M$ , projection  $\pi$  and structure group  $G$ , and let  $\{U_\alpha\}$  be an open covering of  $M$  such that  $P$  is trivial over  $U_\alpha$  and let  $\{g_{\alpha\beta}\}$  be the transition function with respect to this covering  $\{U_\alpha\}$ . We denote such a principal fibre bundle by  $P(M, \pi, G) = \{U_\alpha, g_{\alpha\beta}\}$ . (For the general theory of fibre bundles, see [5]). Then, by the proof of Proposition 6.1 we obtain the following

**COROLLARY 6.3.** *From a principal fibre bundle  $P(M, \pi, G) = \{U_\alpha, g_{\alpha\beta}\}$  we get a principal fibre bundle  $\overset{\check{r}}{T}P(\overset{\check{r}}{T}M, \overset{\check{r}}{T}\pi, \overset{\check{r}}{T}G) = \{\overset{\check{r}}{T}U_\alpha, \overset{\check{r}}{T}g_{\alpha\beta}\}$  for any positive integer  $r$ .*

**§7. Imbedding of  $\overset{r}{TFM}$  into  $\overset{r}{FTM}$ .**

Let  $F(M)(M, \pi, GL(n))$  be the frame bundle of an  $n$ -dimensional manifold  $M$  as in [4]. We shall prove the following

**THEOREM 7. 1.** *For any manifold  $M$ , there is a canonical injection  $j_M^{(r)}: \overset{r}{TFM} \rightarrow \overset{r}{FTM}$  of the tangential fibre bundle  $\overset{r}{TFM}$  to  $FM$  of order  $r$  into the frame bundle of  $\overset{r}{TM}$  such that  $j_M^{(r)}(x \cdot g) = j_M^{(r)}(x) \cdot j_n^{(r)}(g)$  for  $x \in \overset{r}{TFM}$ ,  $g \in \overset{r}{TGL}(n)$  and that the following diagram is commutative:*

$$\begin{array}{ccc} \overset{r}{TFM} & \xrightarrow{j_M^{(r)}} & \overset{r}{FTM} \\ \downarrow \overset{r}{T}\pi & \searrow \overset{1}{\overset{r}{TFM}} & \downarrow \tilde{\pi} \\ \overset{r}{TM} & \xrightarrow{\quad} & \overset{r}{TM}, \end{array}$$

where  $\pi: FM \rightarrow M$  (resp.  $\tilde{\pi}: \overset{r}{FTM} \rightarrow \overset{r}{TM}$ ) is the projection.

*Proof.* We shall use the same notations as in the proof of Theorem 2. 4 [4]. We denote by  $J_{\alpha\beta}^{(r)}$  the Jacobian matrix with respect to the coordinate systems  $\{x_{\alpha,i}^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, \dots, r\}$  and  $\{x_{\beta,i}^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, \dots, r\}$ . Using the same arguments as the proof of Theorem 2. 4 [4], in order to prove the Theorem 7. 1, it is sufficient to verify the following relation:

$$(7. 1) \quad J_{\alpha\beta}^{(r)} = j_n^{(r)} \circ \overset{r}{T}J_{\alpha\beta} \text{ on } \overset{r}{T}(U_\alpha) \cap \overset{r}{T}(U_\beta).$$

We shall prove (7. 1) only for  $r = 2$ , since the proof for the case  $r \geq 3$  is similar. Put  $x_i^{(\nu)} = x_{\alpha,i}^{(\nu)}$  and  $y_i^{(\nu)} = x_{\beta,i}^{(\nu)}$  for  $i = 1, 2, \dots, n; \nu = 0, 1, \dots, r$ . By expressing  $y_i$  as a function  $f_i(x_1, \dots, x_n)$  of  $x_1, \dots, x_n$ , we get from (1. 3) the following relation:

$$(7. 2) \quad J_{\alpha\beta}^{(2)} = \begin{pmatrix} J_{\alpha\beta} & 0 & 0 \\ \dot{J}_{\alpha\beta} & J_{\alpha\beta} & 0 \\ \ddot{J}_{\alpha\beta} & \dot{J}_{\alpha\beta} & J_{\alpha\beta} \end{pmatrix}$$

where  $\dot{J}_{\alpha\beta} = (\dot{J}_k^i)$  with  $\dot{J}_k^i = \sum_j \frac{\partial^2 f_i}{\partial x_j \partial x_k} \dot{x}_j$  and  $\ddot{J}_{\alpha\beta} = (\ddot{J}_k^i)$  with  $\ddot{J}_k^i = \frac{1}{2} \sum_{j,l} \frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l} \dot{x}_j \dot{x}_l + \sum_j \frac{\partial^2 f_i}{\partial x_j \partial x_k} \ddot{x}_j$ . Putting  $J_k^i = \frac{\partial f_i}{\partial x_k}$  we get the following



$$(7.3) \quad J_k^i = \sum_j \frac{\partial J_k^i}{\partial x_j} \dot{x}_j, \quad \ddot{J}_k^i = \frac{1}{2} \sum_{j,l} \frac{\partial^2 J_k^i}{\partial x_j \partial x_l} \dot{x}_j \dot{x}_l + \sum_j \frac{\partial J_k^i}{\partial x_j} \ddot{x}_j.$$

Now, consider the map  $J = J_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow GL(n)$ . We can calculate the coordinates  $(y_j^{(\nu)} | i, j = 1, \dots, n; \nu = 0, 1, 2)$  of the image of  $(x_i | i = 1, \dots, n; \nu = 0, 1, 2)$  by the map  $\overset{2}{T}J$  as follows:

$$(7.4) \quad \begin{cases} y_j^{(0)} = J_j^i(x), & y_j^{(1)} = \sum_k \frac{\partial J_j^i}{\partial x_k} \dot{x}_k, \\ y_j^{(2)} = \frac{1}{2} \sum_{k,l} \frac{\partial^2 J_j^i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum_k \frac{\partial J_j^i}{\partial x_k} \ddot{x}_k. \end{cases}$$

By Proposition 5.3 and (7.4), (7.3) we obtain

$$j_n^{(2)} \circ \overset{2}{T}J_{\alpha\beta} = J_{\alpha\beta}^{(2)} \text{ on } \overset{2}{T}(U_\alpha) \cap \overset{2}{T}(U_\beta). \quad \text{Q.E.D.}$$

**§8. Prolongations of G-structures to tangent bundles of order r.**

DEFINITION 8.1. Let  $G$  be a Lie subgroup of  $GL(n)$ . We denote by  $G^{(r)}$  the image of  $\overset{r}{T}G$  by the homomorphism  $j_n^{(r)}$ , i.e.

$$(8.1) \quad G^{(r)} = j_n^{(r)}(\overset{r}{T}G).$$

Clearly,  $G^{(r)}$  is a Lie subgroup of  $GL(n(r+1))$ .

Let  $P(M, \pi, G)$  be a  $G$ -structure on  $M$  (for the general theory of  $G$ -structures see, for instance [1], [2], [4] or [6]). We denote by  $\pi^{(r)}$  the restriction of the projection  $\pi: \overset{r}{F}TM \rightarrow \overset{r}{T}M$  to the subbundle  $P^{(r)} = j_M^{(r)}(\overset{r}{T}P)$ . Then we obtain a  $G^{(r)}$ -structure  $P^{(r)}(\overset{r}{T}M, \pi^{(r)}, G^{(r)})$  on the tangent bundle  $\overset{r}{T}M$  to  $M$  of order  $r$ . We shall call  $P^{(r)}$  the prolongation of order  $r$  of the  $G$ -structure  $P$  to the tangent bundle  $\overset{r}{T}M$  to  $M$  of order  $r$ .

We can easily see the following

PROPOSITION 8.2. *If  $M$  is completely parallelizable, then  $\overset{r}{T}M$  is also completely parallelizable.*

PROPOSITION 8.3. *There is a canonical bundle homomorphism  $\beta_s^r$  of  $P^{(r)}$  into  $P^{(s)}$  for  $r > s$ , i.e. the following diagram*

$$\begin{array}{ccc}
 P^{(\tau)} & \xrightarrow{\beta_s^r} & P^{(s)} \\
 \downarrow \pi^{(\tau)} & & \downarrow \pi^{(s)} \\
 \overset{r}{T}M & \xrightarrow{\pi_s^r} & \overset{s}{T}M
 \end{array}$$

is commutative and there is a canonical homomorphism  $h_s^r: G^{(\tau)} \rightarrow G^{(s)}$  such that

$$\beta_s^r(x \cdot a) = \beta_s^r(x) \cdot h_s^r(a)$$

for  $x \in P^{(\tau)}$  and  $a \in G^{(\tau)}$ .

### §9. Prolongations of isomorphisms of $G$ -structures.

**THEOREM 9.1.** *Let  $M$  and  $M'$  be two manifolds and  $f: M \rightarrow M'$  be a diffeomorphism between them. Then, we have the following commutative diagram:*

$$\begin{array}{ccc}
 \overset{r}{T}FM & \xrightarrow{j_M^{(r)}} & \overset{r}{F}TM \\
 \downarrow \overset{r}{T}Ff & & \downarrow \overset{r}{F}Tf \\
 \overset{r}{T}FM' & \xrightarrow{j_{M'}^{(r)}} & \overset{r}{F}TM'
 \end{array}$$

*Proof.* We use the same notations  $\Phi_\alpha, \Phi'_\alpha, f_\alpha$  as in the proof of Theorem 4.2 [4]. On the other hand, let

$$\begin{aligned}
 \Psi_\alpha &: \overset{r}{T}U_\alpha \times GL(n(r+1)) \rightarrow \overset{r}{F}TU_\alpha \\
 \Psi'_\alpha &: \overset{r}{T}V_\alpha \times GL(n(r+1)) \rightarrow \overset{r}{F}TV_\alpha
 \end{aligned}$$

be the local trivializations of  $\overset{r}{F}TM$  (resp.  $\overset{r}{F}TM'$ ) over  $\overset{r}{T}U_\alpha$  (resp.  $\overset{r}{T}V_\alpha$ ) induced by the coordinate system on  $U_\alpha$  (resp.  $V_\alpha$ ). Define  $f_\alpha^{(r)}: \overset{r}{T}U_\alpha \times GL(n(r+1)) \rightarrow \overset{r}{T}V_\alpha \times GL(n(r+1))$  by the following

$$f_\alpha^{(r)} = \Psi'_\alpha \circ \overset{r}{F}Tf \circ \Psi_\alpha.$$

Let  $j_\alpha^{(r)} = 1_{\overset{r}{T}U_\alpha} \times j_n^{(r)}$  and  $j'_\alpha^{(r)} = 1_{\overset{r}{T}V_\alpha} \times j_n^{(r)}$ . By the same arguments as the proof of Th. 4.2 [4], in order to prove the Theorem 9.1, it is now sufficient to prove the commutativity of the following diagram:

$$(9.1) \quad \begin{array}{ccc} {}^r T U_\alpha \times {}^r T GL(n) & \xrightarrow{j_\alpha^{(r)}} & {}^r T U_\alpha \times GL(n(r+1)) \\ \downarrow {}^r T f_\alpha & & \downarrow f_\alpha^{(r)} \\ {}^r T V_\alpha \times {}^r T GL(n) & \xrightarrow{j'_\alpha^{(r)}} & {}^r T V_\alpha \times GL(n(r+1)). \end{array}$$

We shall prove the commutativity of (9.1) only for the case  $r=2$ , since the case for  $r \geq 3$  is similar. Using the same notations  $y_i, f_i(x), w_i^k, z_i^k$  as in Th. 4.2 [4] (we use  $y_i$  instead of  $y^i$ , etc), we introduce the notations  $f_\kappa(x), x_\kappa, y_\kappa$  for  $\kappa = 1, 2, \dots, 3n$  by the following

$$(9.2) \quad \left\{ \begin{array}{l} f_{i+3n} = \sum \frac{\partial f_i}{\partial x_k} \dot{x}_k, \\ f_{i+2n} = \frac{1}{2} \sum \frac{\partial^2 f_i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum \frac{\partial f_i}{\partial x_k} \ddot{x}_k, \\ x_{i+3n} = \dot{x}_i, \quad x_{i+2n} = \ddot{x}_i, \quad y_{i+3n} = \dot{y}_i, \quad y_{i+2n} = \ddot{y}_i \end{array} \right.$$

for  $i = 1, 2, \dots, n$ . Let  $\{x_\kappa, \bar{w}_\lambda^k | \kappa, \lambda = 1, 2, \dots, 3n\}$  (resp.  $\{y_\kappa, \bar{z}_\lambda^k | \kappa, \lambda = 1, 2, \dots, 3n\}$ ) be the coordinate system on  ${}^2 F T U_\alpha$  (resp.  ${}^2 F T V_\alpha$ ) induced by the coordinate system  $\{x_\kappa\}$  (resp.  $\{y_\kappa\}$ ). Now since the map  $f_\alpha: U_\alpha \times GL(n) \rightarrow V_\alpha \times GL(n)$  is expressed as follows:

$$(9.3) \quad f_\alpha: y_i = f_i(x), \quad z_i^j = \sum w_i^k \frac{\partial f_j}{\partial x_k} \quad (i, j = 1, 2, \dots, n),$$

we obtain the expression of  ${}^2 T f_\alpha$  as follows:

$$(9.4) \quad \left\{ \begin{array}{l} y_i = f_i(x), \quad z_i^j = \sum_k w_i^k \frac{\partial f_j}{\partial x_k}, \\ \dot{y}_i = \sum_k \frac{\partial f_i}{\partial x_k} \dot{x}_k, \\ \dot{z}_i^j = \sum_{k,l} w_i^k \frac{\partial^2 f_j}{\partial x_k \partial x_l} \dot{x}_l + \sum_k \frac{\partial f_j}{\partial x_k} \dot{w}_i^k, \\ \ddot{y}_i = \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum_k \frac{\partial f_i}{\partial x_k} \ddot{x}_k, \\ \ddot{z}_i^j = \frac{1}{2} \left( \sum_{k,l,m} w_i^k \frac{\partial^3 f_j}{\partial x_k \partial x_l \partial x_m} \dot{x}_l \dot{x}_m + \sum_{k,l} \frac{\partial^2 f_j}{\partial x_k \partial x_l} \dot{x}_l \dot{w}_i^k \right) \\ + \sum_{k,l} w_i^k \frac{\partial^2 f_j}{\partial x_k \partial x_l} \ddot{x}_l + \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_j}{\partial x_k \partial x_l} \dot{x}_l \dot{w}_i^k + \sum_k \frac{\partial f_j}{\partial x_k} \ddot{w}_i^k. \end{array} \right.$$

By Proposition 5.3 we get the following

$$(9.5) \quad (j_\alpha^{(2)} \circ T f_\alpha)(x_\kappa, w_\lambda^\kappa) = \left( y_\kappa, \begin{pmatrix} z_i^j & 0 & 0 \\ \dot{z}_i^j & z_i^j & 0 \\ \ddot{z}_i^j & \dot{z}_i^j & z_i^j \end{pmatrix} \right),$$

where  $y_\kappa$  and  $z_\lambda^\kappa$  are given by (9.4).

On the other hand, since  $f: U_\alpha \rightarrow V_\alpha$  is expressed by  $y_i = f_i(x_1, \dots, x_n)$  ( $i = 1, \dots, n$ ), we have the expression of  $T^2 f$  as follows:

$$T^2 f: \begin{cases} y_i = f_i(x), & \dot{y}_i = \sum_k \frac{\partial f_i}{\partial x_k} \dot{x}_k, \\ \ddot{y}_i = \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum_k \frac{\partial f_i}{\partial x_j} \ddot{x}_k. \end{cases}$$

Therefore, we get the expression of  $f_\alpha^{(2)}$  as follows:

$$f_\alpha^{(2)}: \begin{cases} y_i = f_i(x), & \dot{y}_i = \sum_k \frac{\partial f_i}{\partial x_k} \dot{x}_k, \\ \ddot{y}_i = \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum_k \frac{\partial f_i}{\partial x_k} \ddot{x}_k, \\ \ddot{z}_\lambda^\kappa = \sum_{\mu=1}^{3n} \ddot{w}_\lambda^\mu \frac{\partial f_\kappa}{\partial x_\mu} \end{cases}$$

for  $\kappa, \lambda = 1, 2, \dots, 3n$  and  $i = 1, 2, \dots, n$ . Now, we calculate  $\ddot{z}_\lambda^\kappa$  by (9.2) as follows:

$$\begin{aligned} \ddot{z}_\kappa^j &= \sum_k \ddot{w}_\kappa^k \frac{\partial f_j}{\partial x_k}, \\ \ddot{z}_\kappa^{n+j} &= \sum_{k,l} \ddot{w}_\kappa^k \frac{\partial^2 f_j}{\partial x_l \partial x_k} \dot{x}_l + \sum_k \ddot{w}_\kappa^{n+k} \frac{\partial f_j}{\partial x_k}, \\ \ddot{z}_\kappa^{2n+j} &= \sum_k \ddot{w}_\kappa^k \frac{\partial f_{2n+j}}{\partial x_k} - \sum_k \ddot{w}_\kappa^{n+k} \frac{\partial f_{2n+j}}{\partial \dot{x}_k} - \sum_k \ddot{w}_\kappa^{2n+k} \frac{\partial f_{2n+j}}{\partial \ddot{x}_k} \\ &= \sum_k \ddot{w}_\kappa^k \left( \frac{1}{2} \sum_{l,m} \frac{\partial^3 f_j}{\partial x_m \partial x_l \partial x_k} \dot{x}_m \dot{x}_l + \sum_l \frac{\partial^2 f_j}{\partial x_l \partial x_k} \ddot{x}_l \right) \\ &\quad + \sum_{k,l} \ddot{w}_\kappa^{n+k} \frac{\partial^2 f_j}{\partial x_k \partial x_l} \dot{x}_k + \sum_k \ddot{w}_\kappa^{2n+k} \frac{\partial f_j}{\partial x_k}. \end{aligned}$$

for  $\kappa = 1, 2, \dots, 3n$  and  $j = 1, 2, \dots, n$ . By Proposition 5.3 and the above calculations, we have the following equalities:

$$(9.6) \quad f_\alpha^{(2)} \circ j_\alpha^{(2)}(x_\mu; w_i^k, \dot{w}_i^k, \ddot{w}_i^k) = f_\alpha^{(2)} \left( x_\mu; \begin{pmatrix} w_i^k & 0 & 0 \\ \dot{w}_i^k & w_i^k & 0 \\ \ddot{w}_i^k & \dot{w}_i^k & w_i^k \end{pmatrix} \right) \\ = \left( f_i(x), \dot{y}_i, \ddot{y}_i; \begin{pmatrix} \tilde{z}_i^j & 0 & 0 \\ \tilde{z}_i^{n+j} & \tilde{z}_i^j & 0 \\ \tilde{z}_i^{n+j} & \tilde{z}_i^{n+j} & \tilde{z}_i^j \end{pmatrix} \right),$$

where we see that  $\tilde{z}_i^j = \sum_k w_i^k \frac{\partial f_j}{\partial x_k}$ ,  $\tilde{z}_i^{n+j} = \sum_{k,l} w_i^k \frac{\partial^2 f_j}{\partial x_l \partial x_k} \dot{x}_l + \sum_k \dot{w}_i^k \frac{\partial f_j}{\partial x_k}$  and  $\tilde{z}_i^{2n+j} = \sum_k w_i^k \left( \frac{1}{2} \sum_{l,m} \frac{\partial^3 f_j}{\partial x_m \partial x_l \partial x_k} \dot{x}_m \dot{x}_l + \sum_l \frac{\partial^2 f_j}{\partial x_l \partial x_k} \ddot{x}_l \right) + \sum_{k,l} \dot{w}_i^k \frac{\partial^2 f_j}{\partial x_k \partial x_l} \dot{x}_l + \sum_k \ddot{w}_i^k \frac{\partial f_j}{\partial x_k} = \ddot{z}_i^j$ . Therefore, we obtain, by (9.5) and (9.6), the commutativity of (9.1) for  $r = 2$ . Q.E.D.

By the same arguments as the proof of Th. 4.3 [4] we can prove the following

**THEOREM 9.2.** *Let  $\Phi$  be a diffeomorphism of a manifold  $M$  onto a manifold  $M'$ . Let  $P$  (resp.  $P'$ ) be a  $G$ -structure on  $M$  (resp.  $M'$ ). Then  $\Phi$  is an isomorphism of  $P$  with  $P'$  if and only if  $\tilde{T}\Phi$  is an isomorphism of  $P^{(r)}$  with  $P'^{(r)}$ .*

**COROLLARY 9.3.** *Let  $\Phi$  be a diffeomorphism of  $M$  onto itself, and let  $P$  be a  $G$ -structure on  $M$ . Then  $\Phi$  is an automorphism of  $P$  if and only if  $\tilde{T}\Phi$  is an automorphism of the prolongation  $P^{(r)}$  of order  $r$ .*

**§10. Integrability of prolongations of  $G$ -structures.**

In this section, we shall prove that the prolongation of an integrable  $G$ -structure (see Def. 5.1 [4]) of order  $r$  is also integrable and vice versa.

**PROPOSITION 10.1.** *Let  $\{x_1, \dots, x_n\}$  be a local coordinate system on a neighborhood  $U$  in  $M$ , on which we give a  $G$ -structure  $P$ . Let  $\phi$  be a cross section of  $P$  over  $U$ , which is expressed by  $\phi(x) = (\dots, \sum \phi_j^i(x) (\partial/\partial x_i)_{x^*}, \dots)$  for  $x \in U$ . Define  $\phi^{(r)}$  by  $\phi^{(r)} = j_M^{(r)} \circ \tilde{T}\phi$ . Then  $\phi^{(r)}$  is a cross section of the prolongation  $P^{(r)}$  over  $\tilde{T}U$  and is expressed with respect to the induced coordinate system  $\{x_i^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, \dots, r\}$  as follows:*

$$(10.1) \quad \phi^{(r)}(\dots, x_i^{(\nu)}, \dots) = \left( \dots, \sum_{i=1}^n \phi_j^i(x) \left( \frac{\partial}{\partial x_i^{(\nu)}} \right)_X + \sum_{\mu > \nu} \sum_{i=1}^n F_{j,\mu}^{i,\nu}(X) \left( \frac{\partial}{\partial x_i^{(\mu)}} \right)_X, \dots \right),$$

where  $X = (\dots, x_i, \dots) \in {}^r T U$  and  $F_{j,\mu}^{i,\nu}(X)$  is a polynomial of  $x_k^{(\lambda)}$  ( $\lambda \geq \mu$ ;  $k = 1, \dots, n$ ) without constant term and with coefficients, which are partial derivatives of  $\phi_m^l(l, m = 1, \dots, n)$ .

*Proof.* Let  $\pi: F(M) \rightarrow M$  and  $\tilde{\pi}: {}^r F T M \rightarrow {}^r T M$  be the projections. Let  $\Phi_U$  and  $\Psi_U$  be the local trivialization of  $FM$  and  ${}^r F T M$  over  $U$  and  ${}^r T U$ , respectively. We see that

$$j_M^{(r)} \{ {}^r T F M = \Psi_U \circ (1_{\frac{{}^r T U}} \times j_n^{(r)}) \circ ({}^r T \Phi_U)^{-1}.$$

Using Proposition 1.6, we have the following equalities:

$$\tilde{\pi} \circ \phi^{(r)} = \tilde{\pi} \circ j_M^{(r)} {}^r T \phi = {}^r T \pi \circ {}^r T \phi = {}^r T(\pi \circ \phi) = {}^r T 1_U = 1_{\frac{{}^r T U}} \quad .$$

Since  $\phi^{(r)}({}^r T U) = j_M^{(r)} \circ {}^r T \phi(TU) = j_M^{(r)} {}^r T(\phi(U)) \subset j_M^{(r)} {}^r T P = P^{(r)}$ , we see that  $\phi^{(r)}$  is a cross section of  $P^{(r)}$  over  ${}^r T U$ .

We shall prove (10.1) only for the case  $r = 2$ , since the case  $r \geq 3$  is similar. Put  $f(x) = (\phi_j^i(x)) \in GL(n)$  for  $x \in U$ , then we have  $\Phi_U^{-1} \circ \phi = (1_U, f)$ . Hence, we have  $\phi^{(2)} = \Psi_U \circ (1_{\frac{{}^2 T U}} \times j_n^{(2)}) \circ ({}^2 T \Phi)^{-1} \circ {}^2 T \phi = \Psi_U \circ (1_{\frac{{}^2 T U}} \times j_n^{(2)}) \circ {}^2 T(1_U, f) = \Psi_U \circ (1_{\frac{{}^2 T U}} \times j_n^{(2)} \circ {}^2 T f)$ . Therefore, using the expression (1.2) of  ${}^2 T f$  and Proposition 5.3 we get the expression of  $\phi^{(2)}$  as follows:

$$\begin{aligned} \phi^{(2)}(x, \dot{x}, \ddot{x}) &= \Psi_U \left( (x, \dot{x}, \ddot{x}); \begin{pmatrix} \phi_j^i & 0 & 0 \\ \dot{\phi}_j^i & \phi_j^i & 0 \\ \ddot{\phi}_j^i & \dot{\phi}_j^i & \phi_j^i \end{pmatrix} \right) \\ &= \left( \dots, \sum_i \left( \phi_j^i \left( \frac{\partial}{\partial x_i} \right)_x + \dot{\phi}_j^i \left( \frac{\partial}{\partial \dot{x}_i} \right)_x + \ddot{\phi}_j^i \left( \frac{\partial}{\partial \ddot{x}_i} \right)_x \right), \dots, \right. \\ &\quad \left. \sum_i \left( \phi_j^i \left( \frac{\partial}{\partial \dot{x}_i} \right)_x + \dot{\phi}_j^i \left( \frac{\partial}{\partial \ddot{x}_i} \right)_x \right), \dots, \sum_i \phi_j^i \left( \frac{\partial}{\partial \ddot{x}_i} \right)_x, \dots \right), \end{aligned}$$

where 
$$\dot{\phi}_j^i = \sum_k \frac{\partial \phi_j^i}{\partial x_k} \dot{x}_k, \quad \ddot{\phi}_j^i = \frac{1}{2} \sum_{k,l} \frac{\partial^2 \phi_j^i}{\partial x_k \partial x_l} \dot{x}_k \dot{x}_l + \sum_k \frac{\partial \phi_j^i}{\partial x_k} \ddot{x}_k.$$

These functions  $\dot{\phi}_j^i$  and  $\ddot{\phi}_j^i$  have the properties stated in the proposition. Thus the proposition is proved. Q.E.D.

*Remark 10.2.* By the properties of the functions  $F_{j,\mu}^{i,\nu}(X)$ , we see that  $F_{j,\mu}^{i,\nu}$  vanishes if the functions  $\phi_m^l$  are constants for  $l, m = 1, 2, \dots, n$ . The

function  $F_{j,\mu}^{i,\nu}(X)$  also vanishes at  $X = (\dots, x_i, \dots)$  with  $x_k = 0$  for all  $\lambda \geq \mu$  and  $k = 1, \dots, n$ , since  $F_{j,\mu}^{i,\nu}$  is a polynomial of  $x_k^{(\lambda)}$  without constant term.

**THEOREM 10.3.** *Let  $P$  be a  $G$ -structure on a manifold  $M$ . Then,  $P$  is integrable if and only if the prolongation  $P^{(r)}$  of  $P$  order  $r$  is integrable for any  $r$ .*

*Proof.* Suppose  $P$  is integrable. Let  $x_0 \in M$  be any point of  $M$  and let  $\{x_1, \dots, x_n\}$  be a local coordinate system on a neighborhood  $U$  of  $x_0$  such that

$$\phi(x) = \left( \dots, \left( \frac{\partial}{\partial x_i} \right)_x, \dots \right) \in P \text{ for any } x \in U.$$

Then, by Proposition 10.1 and Remark 10.2,  $\phi^{(r)}$  is a cross section of  $P^{(r)}$  and is expressed with respect to the induced coordinate system  $\{x_i^{(\nu)} | i = 1, \dots, n; \nu = 0, 1, \dots, r\}$  as follows:  $\phi^{(r)}(\dots, x_i, \dots) = (\dots, (\partial/\partial x_i)_X, \dots)$  for  $X = (\dots, x_i^{(\nu)}, \dots) \in \overset{r}{T}U$ . Since  $\phi^{(r)}(X) \in P^{(r)}$  and since  $x_0$  is arbitrary, we have proved that  $P^{(r)}$  is integrable.

Conversely, suppose  $P^{(r)}$  is integrable for some  $r$ . To prove that  $P$  is integrable, we use the same arguments as the proof of Prop. 5.5 [4]. Take a point  $p \in M$  and take a coordinate neighborhood  $U$  of  $p$  with coordinate system  $\{x_1, \dots, x_n\}$  such that there is a local cross section  $\phi: U \rightarrow P$  of  $P$  over  $U$ . Then, by Proposition 10.1,  $\phi^{(r)} = j_M^{(r)} \circ \overset{r}{T}\phi$  is a cross section of  $P^{(r)}$  over  $\overset{r}{T}U$ . Now, let  $X_0$  be the element of  $\overset{r}{T}U$  having coordinates  $\{x_i^{(\nu)}\}$  with  $x_i = x_i(p)$  and  $x_i^{(\nu)} = 0$  for all  $\nu \geq 1$  and  $i = 1, \dots, n$ . Since  $P^{(r)}$  is integrable, there can be found a coordinate neighborhood  $\tilde{U}$  of  $X_0$  with coordinate system  $\{y_1, y_2, \dots, y_N\}$  ( $N = n(r+1)$ ) such that  $\tilde{U} \subset \overset{r}{T}U$  and that, if we define  $\tilde{\phi}_0$  by  $\tilde{\phi}_0(X) = ((\partial/\partial y_1)_X, \dots, (\partial/\partial y_N)_X)$ ,  $\tilde{\phi}_0$  is a cross section of  $P^{(r)}$  over  $\tilde{U}$ . Since  $\phi^{(r)}|_{\tilde{U}}$  and  $\tilde{\phi}_0$  are both cross sections of  $P^{(r)}$  over  $\tilde{U}$ , there exists a map  $\tilde{g}: \tilde{U} \rightarrow G^{(r)}$  such that

$$(10.2) \quad \phi^{(r)}(X) = \tilde{\phi}_0(X) \cdot \tilde{g}(X)$$

holds for  $X \in \tilde{U}$ . By Proposition 5.3, there is a map  $g: \tilde{U} \rightarrow G$  such that  $\tilde{g}(X)$  has the following form:

$$(10.3) \quad \tilde{g}(X) = \begin{pmatrix} g(X) & & & 0 \\ & g(X) & & \\ & & \ddots & \\ * & & & g(X) \end{pmatrix}.$$

Since  $\{y_1, \dots, y_N\}$  and  $\{x_i^{(\nu)}\}$  are both coordinate systems on  $\tilde{U}$  we have differentiable functions  $f_\kappa$  such that  $y_\kappa = f_\kappa(\dots, x_i^{(\nu)}, \dots)$  for  $(\dots, x_i^{(\nu)}, \dots) \in \tilde{U}$  and  $\kappa = 1, 2, \dots, N$ . Now if  $\phi(x) = (\dots, \sum_i \phi_j^i(x) (\partial/\partial x_i)_x, \dots)$  for  $x \in U$ , then by Proposition 10.1, (10.2) can be written as follows:

$$(10.4) \quad \sum_i \phi_j^i(x) \left( \frac{\partial}{\partial x_i} \right)_x + \sum_{\mu, i} F_{j\mu}^{i,0}(X) \left( \frac{\partial}{\partial x_i} \right)_x \\ = \sum_i g_j^i(X) \left( \frac{\partial}{\partial y_i} \right)_x + \sum_{\kappa=n+1}^N \bar{g}_j^\kappa(X) \left( \frac{\partial}{\partial y_\kappa} \right)_x$$

for  $j = 1, 2, \dots, n$ , where  $\bar{g}(X) = (g_\lambda^\kappa(X))$  for  $X \in \tilde{U}$ . Since  $(\partial/\partial x_i)_x = \sum (\partial f_\kappa / \partial x_i^{(\nu)}) \cdot (\partial/\partial y_\kappa)_x$ , (10.4) can be written as follows:

$$(10.5) \quad \sum_{i,\kappa} \phi_j^i \cdot \frac{\partial f_\kappa}{\partial x_i} \cdot \left( \frac{\partial}{\partial y_\kappa} \right)_x + \sum_{i,\mu,\kappa} F_{j\mu}^{i,0}(X) \frac{\partial f_\kappa}{\partial x_i} \left( \frac{\partial}{\partial y_\kappa} \right)_x \\ = \sum_i g_j^i(X) \left( \frac{\partial}{\partial y_i} \right)_x + \sum_{\kappa=n+1}^N \bar{g}_j^\kappa(X) \left( \frac{\partial}{\partial y_\kappa} \right)_x.$$

Comparing the coefficients of  $(\partial/\partial y_k)_x$  for  $k \leq n$  in (10.5), we have

$$(10.6) \quad \sum_i \phi_j^i(x) \frac{\partial f_k}{\partial x_i} + \sum_{i,\mu} F_{j\mu}^{i,0}(X) \frac{\partial f_k}{\partial x_i} = g_j^k(X)$$

for  $j, k = 1, 2, \dots, n$ . Now, define maps  $\bar{f}_k: U' \rightarrow R$  and  $\bar{g}: U' \rightarrow G$  by  $\bar{f}_k(x) = f_k(x, 0, \dots, 0)$  and  $(\bar{g}(x)^{-1})_j^i = g_j^i(x, 0, \dots, 0)$  for  $i, j, k = 1, \dots, n$  and  $x \in U' = \pi(\tilde{U})$ .

Putting  $x_k^{(\nu)} = 0$  ( $k = 1, 2, \dots, n$ ;  $\nu = 1, 2, \dots, r$ ) in (10.6) and using Remark 10.2 we obtain

$$(10.7) \quad \sum_i \phi_j^i(x) \frac{\partial \bar{f}_k}{\partial x_i} = (\bar{g}(x)^{-1})_j^k$$

Now, by the same arguments as in the proof of Prop. 5.5 [4, pp. 88-89], we see that there exists a coordinate neighborhood  $U_0$  of  $p$  with coordinate system  $\{\bar{x}_1, \dots, \bar{x}_n\}$  such that the map  $\bar{\phi}$ , defined by  $\bar{\phi}(x) = ((\partial/\partial \bar{x}_1)_x, \dots, (\partial/\partial \bar{x}_n)_x)$  for  $x \in U_0$ , is a cross section of  $P$  over  $U_0$ . Thus  $P$  is integrable.

Q.E.D.



**§11. Prolongations of classical G-structures.**

(I)  $G = GL(n, C)$ .

Let  $J$  be a linear automorphism of  $R^{2n}$  such that  $J^2 = -1_{R^{2n}}$  and let  $GL(n, C; J)$  be the group of all  $a \in GL(2n)$  such that  $a \circ J = J \circ a$ . It is easy to see that  $\overset{r}{T}J$  is a linear automorphism of  $R^{2n(r+1)} = \overset{r}{T}(R^{2n})$  such that  $(\overset{r}{T}J)^2 = -1$ . We shall prove the following

**PROPOSITION 11. 1.** *If  $G = GL(n, C; J)$ , then  $G^{(r)} \subset GL(n(r+1), C; \overset{r}{T}J)$ .*

*Proof.* Take an element  $\bar{a} \in G^{(r)}$ . We have to prove that  $(\bar{a} \circ \overset{r}{T}J)(X) = ((\overset{r}{T}J) \circ \bar{a})(X)$  for every  $X \in \overset{r}{T}(R^{2n})$ . Now, we can find maps  $\varphi \in S(G)$  and  $\psi \in S(R^{2n})$  (cf. Notations in §1) such that  $\bar{a} = [\varphi]_r$  and  $X = [\psi]_r$ . First, it is readily seen that  $\varphi \cdot (J \circ \psi) = J \circ (\varphi \cdot \psi)$  (cf. Notations in Th. 5. 1). Therefore, we have  $\bar{a}(\overset{r}{T}J(X)) = [\varphi]_r([J \circ \psi])_r = [\varphi \cdot [J \circ \psi]]_r = [J \circ (\varphi \cdot \psi)]_r = \overset{r}{T}J([\varphi \cdot \psi]_r) = \overset{r}{T}J([\varphi]_r \cdot [\psi]_r) = \overset{r}{T}J(\bar{a}(X))$ . Q.E.D.

By the same arguments as the proof of Theorem 6. 3 [4], we obtain the following

**THEOREM 11. 2.** (1) *If a manifold  $M$  has an almost complex structure,  $\overset{r}{T}M$  has a canonical almost complex structure for every  $r$ .*

(2) *If a manifold  $M$  has a complex structure, then  $\overset{r}{T}M$  has a canonical complex structure for every  $r$ .*

(II)  $G = S_p(m)$ .

Consider a skew-symmetric non-degenerate bilinear form  $f$  on  $R^{2m}$ . Let  $S_p(m, f)$  be the group of all  $a \in GL(2m)$  which leaves  $f$  invariant. We denote by  $\pi_r$  the projection of  $\overset{r}{T}R = R^{r+1}$  onto  $R$  defined by  $\pi_r([\varphi]_r) = (1/r!) [d^r \varphi / dt^r]_0$  for  $\varphi \in S(R) = C^\infty(R)$ .

**LEMMA 11. 3.** *If  $f$  is a skew-symmetric non-degenerate bilinear form on  $R^{2m}$ , then  $f^{(r)} = \pi_r \circ (\overset{r}{T}f)$  is also a skew-symmetric non-degenerate bilinear form on  $R^{2m(r+1)} = \overset{r}{T}R^{2m}$ .*

*Proof.* We take the skew-symmetric matrix  $(a^i_j) \in GL(2m)$  such that  $f(x, y) = \sum_{(i,j)} a^i_j x_i y_j$  for  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  with  $n = 2m$ . Let  $\{x_i\}$  be the induced coordinate system on  $R^{n(r+1)}$ . Take an element

$[\varphi]_r$  (resp.  $[\psi]_r$ ) of  $\overset{r}{T}R^n$  with coordinates  $\{x_i\}$  (resp.  $\{y_i\}$ ). We can assume that  $\varphi(t) = (\dots, \sum_{\nu=0}^r x_i t^\nu, \dots)$  and  $\psi(t) = (\dots, \sum_{\nu=0}^r y_i t^\nu, \dots)$ . It is now straightforward to see that the following equality holds:

$$(11.1) \quad f^{(r)}([\varphi]_r, [\psi]_r) = \sum_{i,j} \sum_{\nu=0}^r a_{ij}^{(\nu)} x_i^{(r-\nu)} y_j^{(\nu)},$$

which shows that  $f^{(r)}$  is a skew-symmetric non-degenerate bilinear form on  $R^{n(r+1)}$ .

PROPOSITION 11. 4. *If  $G = S_p(m, f)$ , then  $G^{(r)} \subset S_p(m(r+1), f^{(r)})$ .*

*Proof.* Similar to the of Proposition 11. 1.

By the same arguments as the proof of Th. 6. 6 [4] we obtain the following

THEOREM 11. 5. *If a manifold  $M$  has a (resp. an almost) symplectic structure then  $\overset{r}{T}M$  has a canonical (almost) symplectic structure.*

(III)  $G = GL(V, W)$ .

We have the following Proposition whose proof will be omitted.

PROPOSITION 11. 6. *If a manifold  $M$  has a  $k$ -dimensional (completely integrable) differential system, then  $\overset{r}{T}M$  has a canonical  $k(r+1)$ -dimensional (completely integrable) differential system.*

(IV)  $G = O(k, n - k)$ .

Let  $g$  be a symmetric non-degenerate bilinear form on  $R^n$  of signature  $(k, n - k)$  and let  $\pi_r: \overset{r}{T}R \rightarrow R$  be the same projection as in (II) and let  $g^{(r)}$  be the map  $g^{(r)} = \pi_r \circ (\overset{r}{T}g): \overset{r}{T}R^n \times \overset{r}{T}R^n \rightarrow R$ . We denote by  $O(k, n - k, g)$  or simply  $O(g)$  the group of all  $a \in GL(n)$  such that  $a$  leaves  $g$  invariant.

LAMMA 11. 7. *The notations being as above,  $g^{(r)}$  is a symmetric non-degenerate bilinear form on  $R^{n(r+1)}$  of signature  $(n(r+1)/2, n(r+1)/2)$  if  $r$  is odd and of signature  $(k + \frac{rn}{2}, n - k + \frac{rn}{2})$  if  $r$  is even.*

*Proof.* If the bilinear form  $g$  is expressed by a symmetric matrix  $A = (a_{ij}) \in GL(n)$ , then by the same computation as the proof of (11. 1) in Lemma 11. 3, we see that  $g^{(r)}$  is expressed by the following matrix

$$A^{(r)} = \begin{pmatrix} 0 & & & & A \\ & & & \cdot & \\ & & \cdot & & \\ & & & \cdot & \\ A & \cdot & & & 0 \end{pmatrix} \quad (A: (r+1)\text{-times}).$$

Since  $A$  is of signature  $(k, n - k)$ ,  $A^{(r)}$  is of signature  $(n(r + 1)/2, n(r + 1)/2)$  if  $r$  is odd and of signature  $(k + (rn/2), n - k + (rn/2))$  if  $r$  is even.

Q.E.D.

LEMMA 11. 8. *If  $G = O(g)$ , then  $G^{(r)} \subset O(g^{(r)})$ , the signature of  $g^{(r)}$  being given in Lemma 11. 7.*

*Proof.* Omitted.

By the Lemma 11. 8, we obtain the following

THEOREM 11. 9. *If  $M$  has a pseudo-Riemannian metric, then  $\overset{r}{T}M$  has a canonical pseudo-Riemannian metric for every  $r$ .*

(V)  $G = GL(n, C) \times 1 \subset GL(2n + 1)$ .

LEMMA 11. 10. *Let  $G = GL(n, C) \times 1 \subset GL(2n + 1)$ . Then,  $G^{(r)} \subset GL((2n + 1)(r + 1)/2, C)$  if  $r$  is odd and  $G^{(r)} \subset GL((2nr + 2n + r)/2, C) \times 1$  if  $r$  is even.*

*Proof.* We shall omit the proof, which is similar to the proof of Lemma 6. 14 [4].

By Lemma 11. 10. we obtain the following

THEOREM 11. 11. *If  $M$  has an almost contact structure, then (i)  $\overset{r}{T}M$  has a canonical almost complex structure for any odd  $r$  and (ii)  $\overset{r}{T}M$  has a canonical almost contact structure for even  $r$ .*

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