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Prolongations of Golden Structure to Tangent Bundle of Order 2

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ABSTRACT

In this paper, we study $2nd$ lift of golden structure to tangent bundle of order 2. We investigate integrability and parallelism of golden structures in $T_2(M)$. Moreover, we define golden semi-Riemannian metric in $T_2(M)$.

Key words: Golden structure, semi-Riemannian manifold, prolongations, tangent bundle of order two, lift, integrability.

1. INTRODUCTION

Lifts of structures on any differentiable manifold M to its tangent bundle of order 2 were introduced and studied by several authors, see [3, 11, 14, 15].

In 2007 Hretcanu [7] introduced the golden structure on manifold M. In the recent paper of Crasmareanu and Hretcanu [2], the geometry of the golden structure on a manifold was studied. They investigated the geometry of the golden structure on a manifold by using corresponding almost product structure. Recently, the geometry of the golden structure has been studied in [5, 8, 9, 12, 13].

In the previous paper [13] we have studied the prolongations of golden structures to tangent bundles. The purpose of the present paper is to generalize the previous prolongations to the tangent bundles of order 2. In particular, we follow the spirit of [13].

The paper is organized as follows. In Section 2, we establish $2nd$ lift of golden structures in tangent bundle of order 2. Section 3 deal with integrability and parallelism of golden structures in tangent bundle of order 2. In the last section we study golden semi-Riemannian metric in $T_2(M)$.

2. PROLONGATIONS OF GOLDEN STRUCTURE TO TANGENT BUNDLE OF ORDER 2

Let M be an $n-$dimensional differentiable manifold of class C^∞. The set of all $2-$jets of M is called the tangent bundle of order 2 and denoted by $T_2(M)$, and $\pi_2: T_2(M) \to M$ is the bundle projection of $T_2(M)$ to M. Then $T_2(M)$ is also a differentiable manifold of class C^∞ and its dimension

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is $3n$. The tangent bundle $T_1(M)$ of order 1 is the tangent bundle TM over M and $\pi_1: TM \to M$ is the bundle projection of TM to M. If π_{12} is the mapping $\pi_{12}: T_2(M) \to TM$ then $T_2(M)$ has a bundle structure over TM with projection π_{12} [14,15].

Let $\Im_l^k(M)$ be the set of tensor fields of class C^∞ and of the type (k,l) in M. Let $\Phi \in \Im_1^1(M)$ and Φ have local components Φ_i^h in a coordinate neighborhood U of M. Then the second lift Φ^{II} of Φ in $T_2(M)$ will have the components of the form [15]

$$\Phi^{II} = \begin{bmatrix} \Phi_i^h & 0 & 0 \\ y^s \partial_s \Phi_i^h & \Phi_i^h & 0 \\ z^s \partial_s \Phi_i^h + y^s y^t \partial_s \partial_t \Phi_i^h & 2y^s \partial_s \Phi_i^h & \Phi_i^h \end{bmatrix}$$

with respect to the induced coordinates in $T_2(M)$.

For any $\Phi, G \in \Im_1^1(M)$, we have [15]

$$(\Phi G)^{II} = \Phi^{II} G^{II}. \tag{1}$$

Replacing G by Φ in (1), we obtain

$$(\Phi^2)^{II} = (\Phi^{II})^2. \tag{2}$$

Definition 1 ([2,7]). A non null tensor field Φ type $(1,1)$ and of class C^∞ satisfying

$$\Phi^2 - \Phi - I = 0 \tag{3}$$

is called a golden structure on M.

Taking the second lift on both sides of equation (3), we get $(\Phi^2 - \Phi - I)^{II} = 0$. Using (2) and $I^{II} = I$, we obtain

$$(\Phi^{II})^2 - \Phi^{II} - I = 0. \tag{4}$$

Thus we have the following proposition.

Proposition 1. Let Φ be an element of $\Im_1^1(M)$. Then the second lift Φ^{II} of Φ is an golden structure in $T_2(M)$ if and only if so is Φ.

Remark 1. If Φ (respectively, P) be a golden structure (respectively, an almost product structure) on M then Φ^{II} (respectively, P^{II}) is a golden structure (respectively, an almost product structure) and $\widetilde{\Phi}^{II} = I - \Phi^{II}$ (respectively, $-P^{II}$) is also a golden structure (respectively, an almost product structure) in $T_2(M)$.

Theorem 1 ([2]). An almost product structure P induces a golden structure as follows:

$$\Phi = \tfrac{1}{2}\left(I + \sqrt{5}P\right) \tag{5}$$

Conversely, any golden structure Φ yields an almost product structure

$$P = \tfrac{1}{\sqrt{5}}(2\Phi - I). \tag{6}$$

Taking the second lift on both sides of equation (5), (6) and taking account of Remark 1, we have the following theorem.

Theorem 2. Let P be an almost product structure on M. An almost product structure P^{II} induces a golden structure in $T_2(M)$ as follows:

$$\Phi^{II} = \tfrac{1}{2}\left(I + \sqrt{5}P^{II}\right). \tag{7}$$

Conversely, let Φ be a golden structure on M. Any golden structure Φ^{II} yields an almost product structure in $T_2(M)$

$$P^{II} = \frac{1}{\sqrt{5}}(2\Phi^{II} - I).$$

From ([2], Example 2.4), we give following example.

Example 1 (Triple structures in terms of golden structures on $\mathbf{T_2(M)}$).

From (7), we get

$$\Phi_{F^{II}} = \frac{1}{2}\left(I + \sqrt{5}F^{II}\right),\ \Phi_{P^{II}} \frac{1}{2}\left(I + \sqrt{5}P^{II}\right),$$
$$\Phi_{J^{II}} \frac{1}{2}\left(I + \sqrt{5}J^{II}\right)$$

where $F, P \in \Im_1^1(M)$ and $J = P \circ F$. Hence we obtain

$$\sqrt{5}\Phi_{J^{II}} = 2\Phi_{P^{II}}\Phi_{F^{II}} - \Phi_{P^{II}} - \Phi_{F^{II}} + \phi I$$

where $\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the golden ratio and the $\left(\Phi_{F^{II}}, \Phi_{P^{II}}, \Phi_{J^{II}}\right)$ is

1) An (ahp)-structure in $T_2(M)$ if and only if $\left(\Phi_F, \Phi_P, \Phi_J\right)$ is (ahp)-structure on M.

2) An (abpc)-structure in $T_2(M)$ if and only if $\left(\Phi_F, \Phi_P, \Phi_J\right)$ is (abpc)-structure on M.

3) An (apbc)-structure in $T_2(M)$ if and only if $\left(\Phi_F, \Phi_P, \Phi_J\right)$ is (apbc)-structure on M.

4) An (ahc)-structure in $T_2(M)$ if and only if $\left(\Phi_F, \Phi_P, \Phi_J\right)$ is (ahc)-structure on M.

3. INTEGRABILITY AND PARALLELISM OF GOLDEN STRUCTURES IN TANGENT BUNDLE OF ORDER 2

Let P, Φ are almost product and golden structures on M, respectively. Then the Nijenhuis tensor N_P of P and N_Φ of Φ are tensor fields of type $(1,2)$ given by [2,15]

$$N_P(X,Y) = [PX,PY] - P[PX,Y] - P[X,PY] + P^2[X,Y],$$
$$N_\Phi(X,Y) = [\Phi X,\Phi Y] - \Phi[\Phi X,Y] - \Phi[X,\Phi Y] + \Phi^2[X,Y].$$

for any $X,Y \in \mathfrak{I}_0^1(M)$, respectively.

For any $X,Y \in \mathfrak{I}_0^1(M)$ and $\Phi = \frac{1}{2}\left(I + \sqrt{5}P\right)$, the following relations are satisfied [2]

$$N_P(X,Y) = \frac{4}{5}N_\Phi(X,Y). \tag{8}$$

For any $X,Y \in \mathfrak{I}_0^1(M)$ and $\Phi \in \mathfrak{I}_1^1(M)$, we have [15]

$$\begin{aligned}&(X+Y)^{II} = X^{II} + Y^{II},\\ &[X^{II},Y^{II}] = [X,Y]^{II},\\ &\Phi^{II}X^{II} = (\Phi X)^{II}.\end{aligned} \tag{9}$$

From ([2], (4.2), (4.3) and (4.4)), (1) and (2), we obtain

$$\begin{aligned}r^{II} &= \frac{1}{\sqrt{5}}\Phi^{II} - \frac{1-\phi}{\sqrt{5}}I,\\ s^{II} &= -\frac{1}{\sqrt{5}}\Phi^{II} + \frac{\phi}{\sqrt{5}}I,\end{aligned} \tag{10}$$

$$\begin{aligned}&r^{II} + s^{II} = I,\ r^{II}s^{II} = s^{II}r^{II} = 0,\\ &(r^{II})^2 = r^{II},\ (s^{II})^2 = s^{II},\end{aligned} \tag{11}$$

$$\begin{aligned}&\Phi^{II}r^{II} = r^{II}\Phi^{II} = \phi r^{II},\\ &\Phi^{II}s^{II} = s^{II}\Phi^{II} = (1-\phi)s^{II}.\end{aligned} \tag{12}$$

R and S are complementary distributions corresponding to the projection operators r and s in M, respectively. Let Φ be a golden structure in M. Then the second lift r^{II} of r and s^{II} of s are complementary projection tensors in $T_2(M)$. Thus there exist in $T_2(M)$ two complementary distributions R^{II} and S^{II} determined by r^{II} and s^{II}, respectively.

Let $N_{P^{II}}$, $N_{\Phi^{II}}$ be the Nijenhuis tensor of P^{II} and of Φ^{II} in $T_2(M)$, respectively. Then in view of (2), we have

$$\begin{aligned}N_{P^{II}}(X^{II},Y^{II}) &= [P^{II}X^{II},P^{II}Y^{II}] - P^{II}[P^{II}X^{II},Y^{II}]\\ &\quad - P^{II}[X^{II},P^{II}Y^{II}] + (P^2)^{II}[X^{II},Y^{II}],\end{aligned} \tag{13}$$

$$\begin{aligned}N_{\Phi^{II}}(X^{II},Y^{II}) &= [\Phi^{II}X^{II},\Phi^{II}Y^{II}] - \Phi^{II}[\Phi^{II}X^{II},Y^{II}]\\ &\quad - \Phi^{II}[X^{II},\Phi^{II}Y^{II}] + (\Phi^2)^{II}[X^{II},Y^{II}]\end{aligned} \tag{14}$$

where $X,Y \in \mathfrak{I}_0^1(M)$.

Proposition 2. The second lift S^{II} of a distribution S in $T_2(M)$ is integrable if and only if S is integrable in M.

Proof. The distribution S is integrable if and only if [2]

$$r[sX,sY] = 0 \tag{15}$$

for any $X,Y \in \mathfrak{I}_0^1(M)$.

Taking second lift on both sides of equation (15) and using (9), we get

$$r^{II}[s^{II}X^{II},s^{II}Y^{II}] = 0 \tag{16}$$

where $r^{II} = (I-s)^{II} = I - s^{II}$, is the projection tensor complementary to s^{II}. Thus the conditions (15) and (16) are equivalent. Hence the theorem is proved. □

Therefore, we have the following result.

Proposition 3. For any $X,Y \in \mathfrak{I}_0^1(M)$, let the distribution S be integrable in M, that is $rN_\Phi(sX,sY) = 0$ [2]. Then the distribution S^{II} is integrable in $T_2(M)$ if and only if

$$r^{II}N_{\Phi^{II}}(s^{II}X^{II},s^{II}Y^{II}) = 0.$$

Proof. Let $N_{\Phi^{II}}$ be the Nijenhuis tensor of Φ^{II} in $T_2(M)$. Then in view of (14), we have

$$\begin{aligned}N_{\Phi^{II}}(s^{II}X^{II},s^{II}Y^{II}) &= [\Phi^{II}s^{II}X^{II},\Phi^{II}s^{II}Y^{II}]\\ &\quad - \Phi^{II}[\Phi^{II}s^{II}X^{II},s^{II}Y^{II}]\\ &\quad - \Phi^{II}[s^{II}X^{II},\Phi^{II}s^{II}Y^{II}]\\ &\quad + (\Phi^2)^{II}[s^{II}X^{II},s^{II}Y^{II}].\end{aligned} \tag{17}$$

Equation (17), with the help of (4) and (12) gives

$$\begin{aligned}N_{\Phi^{II}}(s^{II}X^{II},s^{II}Y^{II}) &= (2\phi-1)\Phi^{II}[s^{II}X^{II},s^{II}Y^{II}]\\ &\quad + (3-\phi)[s^{II}X^{II},s^{II}Y^{II}].\end{aligned}$$

Multiplying throughout by $\frac{1}{5}r^{II}$ and from (12), we get

$$\begin{aligned}\frac{1}{5}r^{II}N_{\Phi^{II}}(s^{II}X^{II},s^{II}Y^{II}) &= r^{II}[s^{II}X^{II},s^{II}Y^{II}]\\ &= \left(rN_\Phi(sX,sY)\right)^{II}.\end{aligned}$$

Using (16) (or $rN_\Phi(sX,sY) = 0$), we have

$$r^{II}N_{\Phi^{II}}(s^{II}X^{II},s^{II}Y^{II}) = 0.$$ □

Proposition 4. The second lift R^{II} of a distribution R in $T_2(M)$ is integrable if and only if R is integrable in M.

Proof. The distribution R is integrable if and only if [2]

$$s[rX,rY] = 0 \tag{18}$$

for any $X,Y \in \mathfrak{I}_0^1(M)$.

Taking second lift on both sides of equation (18) and using (9), we get

$$s^{II}[r^{II}X^{II},r^{II}Y^{II}] = 0 \tag{19}$$

where $s^{II} = (I-r)^{II} = I - r^{II}$, is the projection tensor complementary to r^{II}. Thus the conditions (18) and (19) are equivalent. Hence the theorem is proved. □

Proposition 5. For any $X,Y \in \mathfrak{I}_0^1(M)$, let the distribution R be integrable in M, that is $sN_\Phi(rX,rY) = 0$ [2]. Then the distribution R^{II} is integrable in $T_2(M)$, if and only if

$$s^{II}N_{\Phi^{II}}(r^{II}X^{II},r^{II}Y^{II})=0.$$

Proof. Taking account of the Nijenhuis tensor of Φ^{II}, we obtain

$$\begin{aligned}N_{\Phi^{II}}(r^{II}X^{II},r^{II}Y^{II})&=[\Phi^{II}r^{II}X^{II},\Phi^{II}r^{II}Y^{II}]\\&-\Phi^{II}[\Phi^{II}r^{II}X^{II},r^{II}Y^{II}]\\&-\Phi^{II}[r^{II}X^{II},\Phi^{II}r^{II}Y^{II}]\\&+(\Phi^2)^{II}[r^{II}X^{II},r^{II}Y^{II}].\end{aligned}\qquad(20)$$

Equation (20), with the help of (4) and (12) gives

$$\begin{aligned}N_{\Phi^{II}}(r^{II}X^{II},r^{II}Y^{II})&=(1-2\phi)\Phi^{II}[r^{II}X^{II},r^{II}Y^{II}]\\&+(2+\phi)[r^{II}X^{II},r^{II}Y^{II}].\end{aligned}$$

Multiplying throughout by $\frac{1}{5}s^{II}$ and from (12), we get

$$\begin{aligned}\frac{1}{5}s^{II}N_{\Phi^{II}}(r^{II}X^{II},r^{II}Y^{II})&=s^{II}[r^{II}X^{II},r^{II}Y^{II}]\\&=\big(sN_\Phi(rX,rY)\big)^{II}.\end{aligned}$$

Using (19) (or $sN_\Phi(rX,rY)=0$), we have

$$s^{II}N_{\Phi^{II}}(r^{II}X^{II},r^{II}Y^{II})=0.$$ □

Proposition 6. For any $X,Y\in\mathfrak{I}_0^1(M)$ and $\Phi^{II}=\frac{1}{2}\big(I+\sqrt{5}P^{II}\big)$, the following relation between $N_{P^{II}}$ and $N_{\Phi^{II}}$ is satisfying

$$N_{P^{II}}(X^{II},Y^{II})=\frac{4}{5}N_{\Phi^{II}}(X^{II},Y^{II}).$$

Proof. In view of (8), (9) and (13) we have

$$\begin{aligned}N_{P^{II}}(X^{II},Y^{II})&=\big(N_P(X,Y)\big)^{II}=\left(\frac{4}{5}N_\Phi(X,Y)\right)^{II}\\&=\frac{4}{5}N_{\Phi^{II}}(X^{II},Y^{II}).\end{aligned}$$ □

Proposition 7. Let P almost product structure on M and the second lifts Φ^{II} of Φ is golden structures in $T_2(M)$. Then Φ^{II} is integrable in $T_2(M)$ if and only if P is integrable in M.

Proposition 8. For any vector fields X and Y on M, let the golden structure Φ be integrable in M, that is $N_\Phi(X,Y)=0$ [2]. Then the golden structure Φ^{II} is integrable in $T_2(M)$ if and only if

$$N_{\Phi^{II}}(X^{II},Y^{II})=0.$$

Proof. From the equation (14), we have

$$\begin{aligned}N_{\Phi^{II}}(X^{II},Y^{II})&=[\Phi^{II}X^{II},\Phi^{II}Y^{II}]-\Phi^{II}[\Phi^{II}X^{II},Y^{II}]\\&-\Phi^{II}[X^{II},\Phi^{II}Y^{II}]+(\Phi^2)^{II}[X^{II},Y^{II}].\end{aligned}$$

In view of equations (9), we have

$$N_{\Phi^{II}}(X^{II},Y^{II})=\big(N_\Phi(X,Y)\big)^{II}=0$$

since the golden structure Φ is integrable in M. □

Recall [2] that if the golden structure Φ is integrable then both the distributions R and S are integrable. Therefore we get following proposition.

Proposition 9. If the second lift Φ^{II} of Φ is integrable in $T_2(M)$ then both of the distributions R^{II} and S^{II} are integrable on $T_2(M)$.

Let ∇ be a linear connection on M. To the pair (Φ,∇) we associate two other linear connections [1, 2, 10]:

i) The Schouten connection

$$\tilde{\nabla}_XY=r(\nabla_XrY)+s(\nabla_XsY).$$

ii) The Vrănceanu connection

$$\breve{\nabla}_XY=r(\nabla_{rX}rY)+s(\nabla_{sX}sY)+r[sX,rY]+s[rX,sY].$$

Let ∇ be a linear connection on M. Then there exists a unique linear connection ∇^{II} in $T_2(M)$ which satisfies

$$\nabla^{II}_{X^{II}}Y^{II}=(\nabla_XY)^{II}$$

for any $X,Y\in\mathfrak{I}_0^1(M)$ [15]. Thus, to the pair (Φ^{II},∇^{II}) we have two other linear connections in $T_2(M)$:

i') The Schouten connection

$$\tilde{\nabla}^{II}_{X^{II}}Y^{II}=r^{II}\big(\nabla^{II}_{X^{II}}r^{II}Y^{II}\big)+s^{II}\big(\nabla^{II}_{X^{II}}s^{II}Y^{II}\big).$$

ii') The Vrănceanu connection

$$\begin{aligned}\breve{\nabla}^{II}_{X^{II}}Y^{II}&=r^{II}\big(\nabla^{II}_{r^{II}X^{II}}r^{II}Y^{II}\big)+s^{II}\big(\nabla^{II}_{s^{II}X^{II}}s^{II}Y^{II}\big)\\&+r^{II}[s^{II}X^{II},r^{II}Y^{II}]+s^{II}[r^{II}X^{II},s^{II}Y^{II}].\end{aligned}$$

From [2, 4], we get the following two propositions.

Proposition 10. The projectors r^{II}, s^{II} are parallels with respect to Schouten and Vrănceanu connections for every linear connection ∇^{II} on $T_2(M)$ and Φ^{II} is parallel with respect to Schouten and Vrănceanu connections.

Proof. From (11), for $X,Y\in\mathfrak{I}_0^1(M)$, we have

$$\begin{aligned}\big(\tilde{\nabla}^{II}_{X^{II}}r^{II}\big)Y^{II}&=\tilde{\nabla}^{II}_{X^{II}}r^{II}Y^{II}-r^{II}\big(\tilde{\nabla}^{II}_{X^{II}}Y^{II}\big)\\&=r^{II}\big(\nabla^{II}_{X^{II}}r^{II}Y^{II}\big)-r^{II}\big(\nabla^{II}_{X^{II}}r^{II}Y^{II}\big)\\&=0,\end{aligned}$$

$$\begin{aligned}\big(\breve{\nabla}^{II}_{X^{II}}r^{II}\big)Y^{II}&=\breve{\nabla}^{II}_{X^{II}}r^{II}Y^{II}-r^{II}\big(\breve{\nabla}^{II}_{X^{II}}Y^{II}\big)\\&=r^{II}\big(\nabla^{II}_{r^{II}X^{II}}r^{II}Y^{II}\big)+r^{II}[s^{II}X^{II},r^{II}Y^{II}]\\&-r^{II}\big(\nabla^{II}_{r^{II}X^{II}}r^{II}Y^{II}\big)-r^{II}[s^{II}X^{II},r^{II}Y^{II}]\end{aligned}$$

$= 0$

Similar relations hold for s^{II}.
From (10) and (12) it results that Φ^{II} is parallel with respect to Schouten and Vrănceanu connections. □

We know that, a distribution D^{II} on $T_2(M)$ is called parallel with respect to the linear connection ∇^{II} if $X^{II} \in \Im_0^1(T_2(M))$ and $Y^{II} \in D^{II}$ implies $\nabla^{II}_{X^{II}} Y^{II} \in D^{II}$.

Proposition 11. The distributions R^{II}, S^{II} are parallel with respect to Schouten and Vrănceanu connections for the linear connection ∇^{II} in $T_2(M)$.

Proof. Let $X \in \Im_0^1(M)$ and $Y \in R$. Thus, $X^{II} \in \Im_0^1(T_2(M))$ and $Y^{II} \in R^{II}$. Since $s^{II}Y^{II} = (sY)^{II} = 0$, $r^{II}Y^{II} = (rY)^{II} = Y^{II}$, we have

$$\tilde{\nabla}^{II}_{X^{II}} Y^{II} = r^{II}\left(\nabla^{II}_{X^{II}} Y^{II}\right) \in R^{II},$$

$$\check{\nabla}^{II}_{X^{II}} Y^{II} = r^{II}\left(\nabla^{II}_{r^{II}X^{II}} Y^{II}\right) + r^{II}[s^{II}X^{II}, Y^{II}] \in R^{II}.$$

Similar relations hold for S^{II}. □

4. GOLDEN SEMI-RIEMANNIAN METRICS IN TANGENT BUNDLE OF ORDER 2

Definition 2 ([6]). A semi-Riemannian almost product structure is a pair (g, P) with g a semi-Riemannian metric on M and an almost product structure P is a g −symmetric endomorphism

$$g(PX, Y) = g(X, PY)$$

for every $X, Y \in \Im_0^1(M)$.

Proposition 12 ([15]). Let g be a semi-Riemannian metric in M. Then g^{II} is a semi-Riemannian metric in $T_2(M)$.

Let g a semi-Riemannian metric and P an almost product structure on M, then the pair (g^{II}, P^{II}) is a semi-Riemannian almost product structure on $T_2(M)$ if and only if so is (g, P). Thus, we have

$$g^{II}(P^{II}X^{II}, Y^{II}) = g^{II}(X^{II}, P^{II}Y^{II}).$$

From equations (5) and (7), we obtain following proposition.

Proposition 13. The almost product structure P is a g −symmetric endomorphism if and only if golden structure Φ^{II} is a g^{II} −symmetric endomorphism.

Definition 3 ([2]). A golden Riemannian structure on M is a pair (g, Φ) with

$$g(\Phi X, Y) = g(X, \Phi Y).$$

The triple (M, g, Φ) is a golden Riemannian manifold.

Definition 4 ([13]). A golden semi-Riemannian structure on M is a pair (g, Φ) with
$g(\Phi X, Y) = g(X, \Phi Y)$.

The triple (M, g, Φ) is a golden semi-Riemannian manifold.

Proposition 14. If Φ is a golden semi-Riemannian structure in M, then the second lift Φ^{II} of Φ is a golden semi-Riemannian structure in $T_2(M)$.

Corollary. Let (M, g, Φ) be a golden semi-Riemannian manifold, then on a golden semi-Riemannian manifold $(T_2(M), g^{II}, \Phi^{II})$:

i) The projectors r^{II}, s^{II} are $g^{II}-$ symmetric endomorphism if and only if r, s are g −symmetric.

ii) The distribution R^{II}, S^{II} are g^{II} −orthogonal if and only if R, S are g −orthogonal.

iii) The golden structure Φ^{II} is $N_{\Phi^{II}}$ −symmetric if and only if the golden structure Φ is N_Φ −symmetric.

From ([2], Proposition 5.2), we have following proposition.

Proposition 15. A semi-Riemannian almost product structure is a locally product structure if P^{II} is parallel with respect to the Levi-Civita connection $\overset{g^{II}}{\nabla^{II}}$ of g^{II}, i.e. $\overset{g^{II}}{\nabla^{II}} P^{II} = 0$ and if ∇^{II} is a symmetric linear connection then the Nijenhuis tensor of P^{II} verifies

$$N_{P^{II}}(X^{II}, Y^{II}) = \left(\nabla^{II}_{P^{II}X^{II}} P^{II}\right)Y^{II} - \left(\nabla^{II}_{P^{II}Y^{II}} P^{II}\right)X^{II} - P^{II}\left(\nabla^{II}_{X^{II}} P^{II}\right)Y^{II} + P^{II}\left(\nabla^{II}_{Y^{II}} P^{II}\right)X^{II}.$$

Proposition 16. On a locally product golden semi-Riemannian manifold the golden structure Φ^{II} is integrable.

By this result and from ([2], Theorem 5.1) , we have following theorem.

Theorem 3. If linear connection

$$\nabla^{II}_{X^{II}} Y^{II} = \frac{1}{5}[3\bar{\nabla}^{II}_{X^{II}} Y^{II} + \Phi^{II}\left(\bar{\nabla}^{II}_{X^{II}} \Phi^{II} Y^{II}\right) - \Phi^{II}\left(\bar{\nabla}^{II}_{X^{II}} Y^{II}\right) - \bar{\nabla}^{II}_{X^{II}} \Phi^{II} Y^{II}] + O_{P^{II}} Q^{II}(X^{II}, Y^{II})$$

where $\bar{\nabla}^{II}$ is second lift of a linear connection $\bar{\nabla}$ and Q^{II} is second lift of an $(1,2)$ −tensor field Q for which $O_P Q$ is an associated Obata operator

$$O_P Q(X, Y) = \frac{1}{2}[Q(X, Y) + PQ(X, PY)]$$

for the corresponding almost product structure (6) then Φ^{II} is parallel with respect to ∇^{II} linear connection, i.e. $\nabla^{II}\Phi^{II} = 0$.

From ([2], Example 5.6), we give following example.

Example 2.

$$\begin{cases} R^{II} = Span\left\{x^1\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}+y^1\frac{\partial}{\partial y^1}+z^1\frac{\partial}{\partial z^1}\right\} \\ S^{II} = Span\left\{\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^2}-y^1\frac{\partial}{\partial y^2}-z^1\frac{\partial}{\partial z^2}\right\} \end{cases}$$

R^{II} and S^{II} are defined complementary distributions orthogonal with respect to second lift of the Euclidean metric of $\mathbb{R}^2$. These distributions are associated to the golden structure

$$\begin{cases} \Phi^{II}\left(\left(\frac{\partial}{\partial x^1}\right)^{II}\right) = \frac{\phi(x^1)^2+(1-\phi)}{(x^1)^2+1}\frac{\partial}{\partial x^1} + \frac{\sqrt{5}x^1}{(x^1)^2+1}\frac{\partial}{\partial x^2} + \frac{\phi(y^1)^2+(1-\phi)}{(y^1)^2+1}\frac{\partial}{\partial y^1} + \frac{\sqrt{5}y^1}{(y^1)^2+1}\frac{\partial}{\partial y^2} + \frac{\phi(z^1)^2+(1-\phi)}{(z^1)^2+1}\frac{\partial}{\partial z^1} + \frac{\sqrt{5}z^1}{(z^1)^2+1}\frac{\partial}{\partial z^2} \\ \Phi^{II}\left(\left(\frac{\partial}{\partial x^2}\right)^{II}\right) = \frac{\sqrt{5}x^1}{(x^1)^2+1}\frac{\partial}{\partial x^1} + \frac{(1-\phi)(x^1)^2+\phi}{(x^1)^2+1}\frac{\partial}{\partial x^2} + \frac{\sqrt{5}y^1}{(y^1)^2+1}\frac{\partial}{\partial y^1} + \frac{(1-\phi)(y^1)^2+\phi}{(y^1)^2+1}\frac{\partial}{\partial y^2} + \frac{\sqrt{5}z^1}{(z^1)^2+1}\frac{\partial}{\partial z^1} + \frac{(1-\phi)(z^1)^2+\phi}{(z^1)^2+1}\frac{\partial}{\partial z^2} \end{cases}$$

which is integrable since

$$N_{\Phi^{II}}\left(\left(\frac{\partial}{\partial x^1}\right)^{II},\left(\frac{\partial}{\partial x^2}\right)^{II}\right) = 0.$$

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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