J. Math. Soc. Japan
Vol. 19, No. 4, 1967

Prolongations of tensor fields and connections to tangent bundles III

—Holonomy groups—

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(Received Feb. 24, 1967)

1. Introduction

In our previous paper [3] we introduced the notion of complete lift of an affine connection. Let M be a manifold $T(M)$ its tangent bundle space. Then every affine connection ∇ of M induces in a natural manner an affine connection, called the complete lift ∇^c of ∇, of the manifold $T(M)$. We shall show in this paper that the linear holonomy group $\Phi(\nabla^c)$ of the connection ∇^c coincides with the tangent group $T(\Phi(\nabla))$ of the linear holonomy group $\Phi(\nabla)$ of the connection ∇, i.e.,

$$\Phi(\nabla^c)=T(\Phi(\nabla)).$$

This confirms one of the conjectures we stated at the end of [3].

2. Tangent connection

Let P be a principal fibre bundle over a manifold M with Lie structure group G and projection π. Then $T(P)$ is a principal fibre bundle over $T(M)$ with group $T(G)$ and projection π_*, where π_* denotes the differential of π, (see [1]). (Perhaps the notation $T(\pi)$ instead of π_* would make the whole thing more functorial.) One of the present authors has shown that every connection ∇ in P induces in a natural manner a connection, called the connection tangent to ∇ and denoted by $T(\nabla)$, in the bundle $T(P)$.

We apply these constructions to a subbundle P of the bundle $L(M)$ of linear frames, i.e., a G-structure P on M. The tangent group $T(G)$ is a semidirect product of G with its Lie algebra $\mathfrak{g}$. If we represent an element of G by a matrix $X\in GL(n;R)$, then we may represent also an element of $T(G)$ by a matrix of the form

$$\begin{pmatrix} X & 0 \\ X\xi & X \end{pmatrix}\in GL(2n;R),$$

*) Supported partially by NSF Grant GP-5798.

where ξ is an element of $\mathfrak{gl}(n;R)$. In this way we may consider $T(G)$ as a subgroup of $GL(2n;R)$. In a natural manner we may consider also the bundle $T(P)$ as a $T(G)$-structure on the manifold $T(M)$.

Let ∇ be a connection in P. We view it as an affine connection of M. Similarly, we consider the tangent connection $T(\nabla)$ in the bundle $T(P)$ as an affine connection of the manifold $T(M)$. We assert

$$T(\nabla)=\nabla^c.$$

The verification of this fact is straightforward; see the last formula of § 4 and the last formula of § 6 of Chapter IV in [1].

3. Holonomy theorem

In general, let ∇ be a connection in a principal fibre bundle P over M with group G and let $\Phi(\nabla)$ be its holonomy group. Then the holonomy group $\Phi(T(\nabla))$ of the connection $T(\nabla)$ in $T(P)$ coincides with $T(\Phi(\nabla))$, i.e.,

$$\Phi(T(\nabla))=T(\Phi(\nabla)).$$

This fact was proved in [1] and is essentially equivalent to the so-called holonomy theorem of Ambrose-Singer.

This fact together with the assertion made in § 2 establishes the theorem;

$$\Phi(\nabla^c)=T(\Phi(\nabla)).$$

4. Concluding remarks

It is probably possible to prove the equality $\Phi(\nabla^c)=T(\Phi(\nabla))$ more directly (i.e., without the use of $T(\nabla)$ and equality $T(\nabla)=\nabla^c$) in the frame work of our previous paper [3]. In this respect, the paper of Nijenhuis [2] could be useful. As a matter of fact, in the case of real analytic affine connection, results of Nijenhuis in [2] together with our results in [3] give a simple proof of the theorem above. But it would be more important to find a better definition of $T(\nabla)$ (a definition as simple as that of ∇^c) which yields a simple proof of $T(\nabla)=\nabla^c$.

Finally, the equality $T(\nabla)=\nabla^c$ implies immediately that, if $\Phi(\nabla)$ consists of matrices of the form

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \in GL(n;R),$$

then $\Phi(\nabla^c)$ consists of matrices of the form

$$\begin{pmatrix} X & 0 & 0 & 0 \\ Y & Z & 0 & 0 \\ * & 0 & X & 0 \\ * & * & Y & Z \end{pmatrix}.$$

In particular, the existence of a parallel distribution (i.e., parallel field of tangent subspaces) on M implies the existence of certain parallel distributions on $T(M)$. This fact, of course, can be shown more directly in the frame work of [3].

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Bibliography

[1] S. Kobayashi, Theory of connections, Ann. Mat. Pura Appl., **34** (1957), 119-194.
[2] A. Nijenhuis, On the holonomy groups of linear connections I, II, Indag. Math., **56** (1953), 233-249, **57** (1954), 17-25.
[3] K. Yano and S. Kobayashi, Prolongations of tensor fields and connections to tangent bundles I, J. Math. Soc. Japan, **18** (1966), 194-210.