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Prolongations of tensor fields and connections to tangent bundles II
—Infinitesimal automorphisms—

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1. Introduction.

In our previous paper [3] we defined the notion of complete lift. It is a natural way to prolong tensor fields and connections of a manifold M to the tangent bundle $T(M)$. Referring the reader to our previous paper [3] for necessary notations and terminologies, we state our main result of the present paper.

THEOREM 1. *Let ∇ be a torsionfree affine connection on a manifold M and ∇^c its complete life to $T(M)$. Let X and Y be infinitesimal affine transformations of M and U a parallel tensor field of type* (1, 1) *on M such that*

$$U \circ R(Z, W) = R(UZ, W) = R(Z, UW) = R(Z, W) \circ U \tag{1}$$

for all vector fields Z, W of M,

where R denotes the curvature tensor field of ∇. Then $X^c + Y^v + \iota U$ is an infinitesimal affine transformation of $T(M)$.

Conversely, every infinitesimal affine transformation of $T(M)$ may be uniquely written as $X^c + Y^v + \iota U$, where X, Y and U are as above, if M does not admit a nonzero parallel tensor field A of type (1, 1) *such that*

$$A \circ R(Z, W) = R(AZ, W) = R(Z, AW) = R(Z, W) \circ A = 0 \tag{2}$$

for all vector fields Z, W of M.

The following facts will be also shown.

REMARK 1. In any of the following cases M does not admit a nonzero parallel tensor field A of type (1, 1) satisfying (2):

(a) M is non-flat and the linear holonomy group of M is irreducible;

(b) M is Riemannian and has no Euclidean factor in its de Rham decomposition,

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REMARK 2. If M is pseudo-Riemannian and non-flat and if its linear holonomy group is irreducible, then a parallel tensor field U satisfying (1) is necessarily of the form aI, where $a\in \boldsymbol{R}$ and I is the field of identity linear endomorphisms.

THEOREM 2. *Let g be a pseudo-Riemannian metric on M and g^c its complete lift to $T(M)$. Let X and Y be infinitesimal isometries of M. Then X^c+Y^v is an infinitesimal isometry of $T(M)$.*

Conversely, every infinitesimal isometry of $T(M)$ may be uniquely written as X^c+Y^v, where X and Y are infinitesimal isometries of M, if M does not admit a nonzero parallel tensor field A of type (1, 1) *satisfying* (2) *of Theorem* 1.

REMARK 3. In both Theorems 1 and 2 we assume the non-existence of a tensor field A to insure that every infinitesimal affine transformation of $T(M)$ is necessarily fibre-preserving, i.e., projectable to M. As we shall see in the proofs of Theorems 1 and 2, we can completely determine all fibre-preserving infinitesimal affine transformations or isometries without any assumption on M.

REMARK 4. The following formulas shed light on the structure of the Lie algebra of infinitesimal affine transformations or isometries in Theorems 1 or 2.

$$[X^c, Y^c]=[X, Y]^c; \tag{a}$$

$$[X^v, Y^v]=0; \tag{b}$$

$$[X^c, Y^v]=[X, Y]^v; \tag{c}$$

$$[\iota U, \iota U']=-\iota([U. U']); \tag{d}$$

$$[X^c, \iota U]=\iota(\mathcal{L}_X U); \tag{e}$$

$$[Y^v, \iota U]=(UY)^v. \tag{f}$$

Formulas (a), (b) and (c) are true for any vector fields X and Y of M and have been already proved in our previous paper [3]. Formula (d) holds for any tensor fields U and U' of type (1, 1) on M and can be verified easily. ($[U, U']$ on the right hand side stands for $U\circ U'-U'\circ U$.) Formula (e) holds for any vector field X and any tensor field U of type (1, 1) on M and has been already proved in [3; (4) of Proposition 5.1]. Formula (f) holds for any vector field Y and any tensor field U of type (1, 1) on M and can be verified easily.

REMARK 5. If G is a Lie transformation group acting on a manifold M and if $\mathfrak{g}$ is its Lie algebra (considered as a Lie algebra of vector fields on M), then $\{X^c+Y^v;\ X, Y\in\mathfrak{g}\}$ is the Lie algebra of the tangent group $T(G)$ acting on $T(M)$; this follows from (a), (b) and (c) in Remark 4. In particular, if G denotes the largest connected group of isometries of M in Theorem 2, then

$T(G)$ is a connected group of isometries of $T(M)$ and is in fact the largest one under the assumption stated in Theorem 2.

REMARK 6. For an almost complex structure or an almost symplectic structure, we can prove easily that every fibre-preserving infinitesimal automorphism X of $T(M)$ is necessarily of the form X^c+Y^v, where both X and Y are infinitesimal automorphisms of M. It is however difficult (and probably impossible) to give any useful condition which would imply that every infinitesimal automorphism of $T(M)$ is necessarily fibre-preserving.

The proofs of Theorem 1, Remarks 1 and 2 will be given in § 3 and § 4. The proof of Theorem 2 will be given in § 5. Although our proof of Theorem 2 depends on Theorem 1, an independent proof may be obtained easily using Lemma 2.2.

2. Basic formulas.

Following the notations and the terminologies of [3] we recall some formulas from [3]. Let $x^1, \cdots, x^n$ be a local coordinate system in M and $x^1, \cdots, x^n$, $y^1, \cdots, y^n$ be the induced local coordinate system in $T(M)$. If

$$X=\xi^i\frac{\partial}{\partial x^i},$$

then

$$X^c=\xi^i\frac{\partial}{\partial x^i}+y^j\partial_j\xi^i\frac{\partial}{\partial y^i},$$

$$X^v=\xi^i\frac{\partial}{\partial y^i},$$

where ∂_j denotes the partial differentiation $\frac{\partial}{\partial x^j}$. If

$$C=C^i_j dx^j\otimes\frac{\partial}{\partial x_i},$$

then

$$\iota C=C^i_j y^j\frac{\partial}{\partial y^i}.$$

If Γ^h_{ij} are the coefficients of an affine connection ∇ of M, then the coefficients $\tilde{\Gamma}^\alpha_{\beta\gamma}$ of the affine connection ∇^c of $T(M)$ are given by

$$\tilde{\Gamma}^h_{ij}=\Gamma^h_{ij},\qquad \tilde{\Gamma}^h_{i\bar{j}}=0,\qquad \tilde{\Gamma}^h_{\bar{i}j}=0,\qquad \tilde{\Gamma}^h_{\bar{i}\bar{j}}=0,$$

$$\tilde{\Gamma}^{\bar{h}}_{ij}=y^k\partial_k\Gamma^h_{ij},\qquad \tilde{\Gamma}^{\bar{h}}_{i\bar{j}}=\Gamma^h_{ij},\qquad \tilde{\Gamma}^{\bar{h}}_{\bar{i}j}=\Gamma^h_{ij},\qquad \tilde{\Gamma}^{\bar{h}}_{\bar{i}\bar{j}}=0,$$

where the unbarred indices refer to $x^1, \cdots, x^n$ and the barred indices refer to $y^1, \cdots, y^n$.

A vector field X is said to be an infinitesimal affine transformation of a manifold with affine connection ∇ if

$$\mathcal{L}_X \circ \nabla_Z - \nabla_Z \circ \mathcal{L}_X = \nabla_{[X,Z]} \qquad \text{for all vector fields } Z.$$

In terms of the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ this condition may be formulated as follows (cf. [**2**; p. 8])[1]:

LEMMA 2.1. *A vector field $X=\xi^\alpha \dfrac{\partial}{\partial x^\alpha}$ is an infinitesimal affine transformation if and only if it satisfies*

$$\partial_\gamma \partial_\beta \xi^\alpha + \xi^\lambda \partial_\lambda \Gamma^\alpha_{\beta\gamma} - \Gamma^\lambda_{\beta\gamma} \partial_\lambda \xi^\alpha + \Gamma^\alpha_{\lambda\gamma} \partial_\beta \xi^\lambda + \Gamma^\alpha_{\beta\lambda} \partial_\gamma \xi^\lambda = 0\,.$$

A vector field X is said to be an infinitesimal isometry of a pseudo-Riemannian manifold with metric g if $\mathcal{L}_X g = 0$. In terms of a local coordinate system this condition may be formulated as follows (cf. [**2**; p. 4]).

LEMMA 2.2. *A vector field X with components ξ^α is an infinitesimal isometry of a pseudo-Riemannian manifold with metric $g=(g_{\alpha\beta})$ if and only if it satisfies*

$$\xi^\gamma \partial_\gamma g_{\alpha\beta} + g_{\gamma\beta} \partial_\alpha \xi^\gamma + g_{\alpha\gamma} \partial_\beta \xi^\gamma = 0\,.$$

The left hand side in the equation above represents the components of $\mathcal{L}_X g$.

3. Fibre-preserving infinitesimal affine transformations of $T(M)$.

Let ∇ be an affine connection on a manifold M and ∇^c its complete lift to $T(M)$.

We shall first determine the infinitesimal affine transformations $\tilde{X}$ of $T(M)$ (with respect to ∇^c) which are vertical, i.e., tangent to fibres at each point of $T(M)$. If $\tilde{X}$ is a vertical infinitesimal affine transformation of $T(M)$, then it can be written as follows:

$$\tilde{X} = (C^h_j y^j + D^h) \frac{\partial}{\partial y^h}\,, \tag{1}$$

where C^h_j and D^h depend only on $x^1, \cdots, x^n$. This follows from Lemma 2.1 applied to $\alpha=\bar{h}$, $\beta=\bar{i}$ and $\gamma=\bar{j}$. (Geometrically speaking, each fibre of $T(M)$ is auto-parallel and flat[2] and the restriction of $\tilde{X}$ to each fibre is an ordinary

1) Since we are using notations of [**1**] which are in conformity to those of Eisenhart, *Riemannian Geometry*, the lower indices of the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ in [**2**] are interchanged here.

2) A submanifold M' of a manifold M with affine connection is said to be auto-parallel if every tangent vector of M' remains tangent to M' under a parallel displacement in M along every curve contained in M'. Every auto-parallel submanifold is totally geodesic and, if the connection has no torsion, then the converse is also true. We shall not prove this geometric statement here, since our proofs of Theorems 1 and 2 do not depend on it.

infinitesimal affine transformation acting on a vector space. It then follows immediately that $\tilde{X}$ may be written as (1).) Since $\tilde{X}$ is a vector field on $T(M)$ it follows easily that $C=C^h_j dx^j \otimes \frac{\partial}{\partial x^h}$ is a well defined tensor field on M of type (1,1) and $D=D^h \frac{\partial}{\partial x^h}$ is a well defined vector field on M.

LEMMA 3.1. (a) *D is an infinitesimal affine transformation of M;*

(b) *C is parallel with respect to ∇, i.e., $\nabla C=0$;*

(c) *$C(T(Y,Z))=T(CY,Z)=T(Y,CZ)$ for all vector fields Y and Z on M, where T denotes the torsion tensor of ∇;*

(d) *$C(R(Y,Z)W+(\nabla_Z T)(Y,W))=R(Y,CZ)W+(\nabla_{CZ}T)(Y,W)$ for all vector fields Y, Z and W on M;*

(e) *Conversely, if C and D satisfy* (a), (b), (c) *and* (d), *then the vector field $\tilde{X}$ defined by* (1) *is an infinitesimal affine transformation of $T(M)$.*

PROOF. Setting $\alpha=\bar{h}$, $\beta=i$ and $\gamma=j$ in Lemma 2.1 and applying it to $\tilde{X}$, we obtain

$$(2)\qquad \partial_j\partial_i C^h{}_k+C^l{}_k\partial_l\Gamma^h_{ij}-\Gamma^l_{ij}\partial_l C^h{}_k-\partial_k\Gamma^l_{ij}C^h{}_l+\Gamma^h_{lj}\partial_i C^l{}_k+\Gamma^h_{il}\partial_j C^l{}_k=0$$

and

$$(3)\qquad \partial_j\partial_i D^h+D^l\partial_l\Gamma^h_{ij}-\Gamma^l_{ij}\partial_l D^h+\Gamma^h_{lj}\partial_i D^l+\Gamma^h_{il}\partial_j D^l=0\,.$$

Equation (3) means (cf. Lemma 2.1) that D is an infinitesimal affine transformation of M.

Setting $\alpha=\bar{h}$, $\beta=i$ and $\gamma=\bar{j}$ in Lemma 2.1, we obtain

$$(4)\qquad \partial_i C^h{}_j-\Gamma^l_{ij}C^h{}_l+\Gamma^h_{il}C^l{}_j=0\,.$$

Setting $\alpha=\bar{h}$, $\beta=i$ and $\gamma=\bar{j}$ in Lemma 2.1, we obtain

$$(5)\qquad \partial_j C^h{}_i-\Gamma^l_{ij}C^h{}_l+\Gamma^h_{lj}C^l{}_i=0\,.$$

Equation (5) means that C is parallel. Interchanging i and j in (5) and subtracting the resulting equation from (4), we obtain

$$(6)\qquad C^h{}_l T^l{}_{ij}=T^h{}_{il}C^l{}_j\,,$$

that is,

$$(7)\qquad C(T(Y,Z))=T(Y,CZ)\qquad \text{for all vector fields } Y \text{ and } Z \text{ on } M.$$

From (7) we obtain

$$T(CY,Z)=-T(Z,CY)=-C(T(Z,Y))=C(T(Y,Z))$$

and hence

$$T(CY,Z)=C(T(Y,Z))\,.$$

Using (4) and (5) we eliminate all partial derivatives of C from (2) and obtain

$$(8)\qquad C^h{}_l(R^l{}_{ijk}+\nabla_k T^l{}_{ji})=(R^h{}_{ijl}+\nabla_l T^h{}_{ji})C^l{}_k\,,$$

which is nothing but (d).

Having shown (a), (b), (c) and (d) we see that we obtain (e) by reversing the reasoning above. Q. E. D.

REMARK. $\tilde{X}$ in Lemma 3.1 is therefore of the form $\iota C + D^v$.

LEMMA 3.2. *Let C be as in Lemma 3.1. If X is an infinitesimal affine transformation of M, so is CX.*

PROOF. By Proposition 4.2 of [3] we have

$$(9) \qquad (CX)^v = (\iota_X C)^v = [X^v, \iota C].$$

Since both X^v and ιC are infinitesimal affine transformation of $T(M)$ with respect to ∇^c, so is $(CX)^v$. Hence CX is an infinitesimal affine transformation of M. Q. E. D.

Since $\nabla C = 0$ and $[\nabla_Y, \nabla_Z] - \nabla_{[Y,Z]} = R(Y, Z)$, we obtain

$$(10) \qquad C \circ R(Y, Z) = R(Y, Z) \circ C \quad \text{for all vector fields } Y \text{ and } Z \text{ on } M.$$

This together with (d) of Lemma 3.1 implies

LEMMA 3.3. *If ∇ has no torsion, then the tensor field C in Lemma 3.1 satisfies*

$$C \circ R(Y, Z) = R(CY, Z) = R(Y, CZ) = R(Y, Z) \circ C$$

for all vector fields Y and Z on M.

If $C = aI$ where a is a real number and I is the field of identity endomorphisms of tangent spaces, then C clearly satisfies (b), (c) and (d) of Lemma 3.1. We wish to find a useful condition that guarantees that a tensor field C satisfying (b), (c) and (d) of Lemma 3.1 is necessarily of the form aI. The following lemma gives such a condition.

LEMMA 3.4. *Let M be a non-flat pseudo-Riemannian manifold with irreducible linear holonomy group. Then a tensor field C satisfying* (b), (c) *and* (d) *of Lemma* 3.1 *is necessarily of the form aI.*

PROOF. Since C is parallel and the linear holonomy group is irreducible, it follows that either

$$(11) \qquad C = aI, \quad a \in \boldsymbol{R}$$

or

$$(12) \qquad C = aI + bJ, \quad a, b \in \boldsymbol{R},$$

where J is an (integrable almost) complex structure on M and the metric g is Kaehlerian with respect to J, i. e.,[3)]

$$(13) \qquad g(JY, JZ) = g(Y, Z) \quad \text{for all vector fields } Y \text{ and } Z \text{ on } M,$$

3) Although g need not be positive definite, we shall use the term " Kaehlerian". The term " pseudo-Kaehlerian" has been used in a different sense and hence is not appropriate here.

$$\nabla g=0 \quad \text{and} \quad \nabla J=0.$$

Consider the latter case. From Lemma 3.3 we obtain

$$J\circ R(Y,Z)=R(JY,Z)=R(Y,JZ)=R(Y,Z)\circ J \tag{14}$$
for all vector fields Y and Z on M.

On the other hand, in the Kaehlerian case (whether the metric is definite or not) we have

$$R(JY,JZ)=R(Y,Z) \quad \text{for all vector fields } Y \text{ and } Z \text{ on } M. \tag{15}$$

From (14) and (15) we obtain

$$R(Y,Z)=R(JY,JZ)=J^2R(Y,Z)=-R(Y,Z)$$
for all vector fields Y and Z on M,

and hence $R=0$. This contradicts the assumption that M is non-flat. Q.E.D.

A transformation of $T(M)$ is said to be fibre-preserving if it sends each fibre of $T(M)$ into a fibre. An infinitesimal transformation of $T(M)$ is said to be fibre-preserving if it generates a local 1-parameter group of fibre-preserving transformations. An infinitesimal transformation $\tilde{X}=\xi^i\frac{\partial}{\partial x^i}+\xi^{\bar{i}}\frac{\partial}{\partial y^i}$ is fibre-preserving if and only if ξ^i, $(i=1,\cdots,n)$, depend only on $x^1,\cdots,x^n$. A fibre-preserving infinitesimal transformation X induces an infinitesimal transformation $X=\xi^i\frac{\partial}{\partial x^i}$ of the base manifold M. Assume that $\tilde{X}$ is a fibre-preserving infinitesimal affine transformation of $T(M)$. Then setting $\alpha=h$, $\beta=i$ and $\gamma=j$ in Lemma 2.1, we obtain

LEMMA 3.5. *If $\tilde{X}$ is a fibre-preserving infinitesimal affine transformation of $T(M)$, then the induced infinitesimal transformation X of M is also affine.*

Let $\tilde{X}$ and X be as in Lemma 3.5. Since X^c is an infinitesimal affine transformation of $T(M)$ (cf. Proposition 7.6 of [3]) and $X^c=\xi^i\frac{\partial}{\partial x^i}+y^j\partial_j\xi^i\frac{\partial}{\partial y^i}$ (cf. § 5 of [3]), it follows that $X-X^c$ is a vertical infinitesimal affine transformation. By Lemma 3.1 (see also Remark following Lemma 3.1) we obtain

LEMMA 3.6. *If X is a fibre-preserving infinitesimal affine transformation of $T(M)$, then*

$$\tilde{X}=X^c+D^v+\iota C,$$

where X and D are infinitesimal affine transformations of M and C is a parallel tensor field of type (1,1) *satisfying* (b), (c) *and* (d) *of Lemma* 3.1.

4. General infinitesimal affine transformations of $T(M)$.

In this section we shall show that, under a reasonably weak assumption, every infinitesimal affine transformation of $T(M)$ is fibre-preserving.

First we shall give a geometric argument and then substantiate it later by calculation. Let φ be an affine transformation of $T(M)$. Since each fibre $T_x(M)$ of $T(M)$ is easily seen to be auto-parallel and flat[4], so is $\varphi(T_x(M))$. Since $\pi: T(M)\rightarrow M$ is an affine mapping in the sense of [1; Chapter VI, § 1][5], $\pi_*(\varphi(T_x(M)))$ is also auto-parallel and flat in M. If φ_t is a 1-parameter local group of affine transformations of $T(M)$, then we obtain for each fixed point $x\in M$ a 1-parameter family of auto-parallel flat submanifolds $\pi_*(\varphi_t(T_x(M)))$ of M. We may therefore expect that $\pi_*(\varphi_t(T_x(M)))$ reduces to a point, i. e., φ_t is fibre-preserving for most of affine connections in M.

Let

$$\tilde{X}=\xi^i\frac{\partial}{\partial x^i}+\xi^{\bar{i}}\frac{\partial}{\partial y^i}$$

be an infinitesimal affine transformation of $T(M)$. Setting $\alpha=h$, $\beta=i$ and $\gamma=\bar{j}$ in Lemma 2.1, we obtain

$$\partial_j\partial_i\xi^h=0$$

and hence

$$\xi^h=A^h{}_k y^k+B^h\,, \tag{1}$$

where $A^h{}_k$ and B^h depend only on $x^1,\cdots,x^n$. Since $\tilde{X}$ is a vector field on $T(M)$, it follows easily that $A=A^h{}_k dx^k\otimes\frac{\partial}{\partial x^h}$ is a well defined tensor field of type (1,1) on M and that $B=B^h\frac{\partial}{\partial x^h}$ is a well defined vector field on M.

LEMMA 4.1. (a) *B is an infinitesimal affine transformation of M;*

(b) *A is parallel with respect to ∇, i. e., $\nabla A=0$;*

(c) $A(T(Y,Z))=T(AY,Z)=T(Y,AZ)$ *for all vector fields Y and Z on M;*

(d) $A(R(Y,Z)W+(\nabla_Z T)(Y,W))=R(Y,AZ)W+(\nabla_{AZ}T)(Y,W)$ *for all vector fields Y, Z and W on M.*

PROOF. Setting $\alpha=h$, $\beta=i$ and $\gamma=j$ in Lemma 2.1 and applying it to $\tilde{X}$, we obtain

$$\partial_j\partial_i A^h{}_k+A^l{}_k\partial_l\Gamma^h_{ij}-\Gamma^l_{ij}\partial_l A^h{}_k-\partial_k\Gamma^l_{ij}A^h{}_l+\Gamma^h_{lj}\partial_i A^l{}_k+\Gamma^h_{il}\partial_j A^l{}_k=0\,, \tag{2}$$

and

$$\partial_j\partial_i B^h+B^l\partial_l\Gamma^h_{ij}-\Gamma^l_{ij}\partial_l B^h+\Gamma^h_{lj}\partial_i B^l+\Gamma^h_{il}\partial_j B^l=0\,. \tag{3}$$

Equation (3) means that B is an infinitesimal affine transformaton of M.

Setting $\alpha=h$, $\beta=i$ and $\gamma=\bar{j}$ in Lemma 2.1, we obtain

$$\partial_i A^h{}_j-\Gamma^l_{ij}A^h_l+\Gamma^h_{il}A^l{}_j=0\,. \tag{4}$$

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4) See Footnote 2).

5) We shall not prove this geometric statement here, since our proofs of Theorems 1 and 2 do not depend on it. See our next paper.

Setting $\alpha=h$, $\beta=\bar{i}$ and $\gamma=j$ in Lemma 2.1, we obtain

$$\partial_j A^h{}_i - \Gamma^l_{ij} A^h{}_l + \Gamma^h_{lj} A^l{}_i = 0\,. \tag{5}$$

Calculating in the same way as in the proof of Lemma 3.1, we obtain Lemma 4.1. Q. E. D.

LEMMA 4.2. *$R(Y, AZ) = -R(Y, Z)A$ for all vector fields Y and Z on M.*

PROOF. Setting $\alpha=\bar{h}$, $\beta=\bar{i}$ and $\gamma=\bar{j}$ in Lemma 2.1, we obtain

$$\partial_{\bar{j}}\partial_{\bar{i}}\xi^{\bar{h}} + \Gamma^h_{lj} A^l{}_i + \Gamma^h_{il} A^l{}_j = 0.$$

Hence,

$$\xi^{\bar{h}} = -\frac{1}{2}(\Gamma^h_{lj} A^l{}_i + \Gamma^h_{il} A^l{}_j) y^i y^j + E^h{}_i y^i + F^h\,. \tag{6}$$

Setting $\alpha=\bar{h}$, $\beta=i$ and $\gamma=\bar{j}$ in Lemma 2.1 and using (4) and (6), we obtain (considering only the coefficients of y^k)

$$R^h{}_{jkl} A^l{}_i = -R^h{}_{ikl} A^l{}_j\,.$$ Q. E. D.

Since A is parallel, we have

$$A \circ R(Y, Z) = R(Y, Z) \circ A \qquad \text{for all vector fields } Y \text{ and } Z \text{ on } M.$$

Since $R(Y, Z)$ is skew-symmetric in Y and Z, we obtain

$$R(AY, Z) = -R(Z, AY) = -R(Z, Y) \circ A = R(Y, Z) \circ A\,.$$

By Lemmas 4.1 and 4.2, we have

LEMMA 4.3. *If $T=0$, then*

$$A \circ R(Y, Z) = R(AY, Z) = R(Y, AZ) = R(Y, Z) \circ A = 0$$

for all vector fields Y and Z on M.

Consider the distribution $S = \{S_x\,;\, x \in M\}$ defined by

$$S_x = A(T_x(M))\,.$$

Since A is parallel, so is the distribution.

LEMMA 4.4. *If ∇ is a torsionfree non-flat affine connection with irreducible linear holonomy group on M, then $A=0$ and hence $\tilde{X}$ is fibre-preserving.*

PROOF. Since the linear holonomy group is irreducible, either $S_x=0$ or $S_x=T_x(M)$. In the former case, we have $A=0$. In the latter case, Lemma 4.3 implies $R=0$. Q. E. D.

We consider a more general case. If $T=0$, then every parallel distribution is integrable and totally geodesic. The maximal integral submanifolds defined by $S=\{S_x\}$ are all totally geodesic and flat by Lemma 4.3.

Let M be a Riemannian manifold and M its universal covering space. Let $T_x(M) = \sum_{i=0}^{k} T_x^{(i)}$ be the canonical decomposition or de Rham decomposition in the sense of [1; p. 185] and $\Phi(x)=\Phi_0(x)\times\Phi_1(x)\times\cdots\times\Phi_k(x)$ the corresponding decomposition of the linear holonomy group $\Phi(x)$ of M, where $\Phi_i(x)$ is trivial on $T_x^{(j)}$ if $i\neq j$ and is irreducible on $T_x^{(i)}$ for each $i=1,\cdots,k$ and $\Phi_0(x)$ consists of the identity only. If $\dim T_x^{(0)}=0$, we say that M has no Euclidean factor. If M is complete, this amounts to saying that M has no Euclidean factor in the so-called de Rham decomposition of M. We have

LEMMA 4.5. *If M is a Riemannian manifold without Euclidean factor, then X is fibre-preserving.*

This completes the proofs of Theorem 1 and Remarks 1 and 2 stated in §1.

5. Infinitesimal isometries of $T(M)$.

Let g be a pseudo-Riemannian metric on M and g^c its complete lift to $T(M)$. Let $\tilde{X}$ be an infinitesimal isometry of $T(M)$ with respect to g^c. Since $\tilde{X}$ is an infinitesimal affine transformation of $T(M)$, we may apply our result in §3 and §4. In particular, under a reasonable condition, $\tilde{X}$ is fibre-preserving and may be written as

$$\tilde{X}=X^c+D^v+\iota C\,,$$

where X and D are infinitesimal affine transformations of M and C is a tensor field of type (1,1) on M satisfying (b), (c) and (d) of Lemma 3.1. To obtain more information on X, D and C, we first prove

LEMMA 5.1. *Let C be a tensor field of type* (1,1) *on M. Then*

$$\mathcal{L}_{\iota C}(g^c)=0$$

holds if and only if $C=0$.

PROOF. We recall [3; §6] that

$$g^c=y^k\partial_k g_{ij}dx^i dx^j+g_{ij}dx^i dy^j+g_{ij}dy^i dx^j\,.$$

By a direct calculation using Lemma 2.2 we obtain

$$\mathcal{L}_{\iota C}(g^c)=(\cdots)dx^i dx^j+g_{il}C^l{}_j dx^i dy^j+g_{lj}C^l{}_i dy^i dx^j\,.$$

Hence, if $\mathcal{L}_{\iota C}(g^c)=0$, then $g_{il}C^l{}_j=0$ and consequently $C^l{}_j=0$. Q. E. D.

By Lemma 5.1, every fibre-preserving infinitesimal isometry $\tilde{X}$ of $T(M)$ is of the form X^c+D^v, where X and D are both infinitesimal affine transformations of M. By Proposition 5.1 of [3], we have

$$0=\mathcal{L}_{\tilde{X}}(g^c)=\mathcal{L}_{(X^c+D^v)}(g^c)=\mathcal{L}_{X^c}(g^c)+\mathcal{L}_{D^v}(g^c)=(\mathcal{L}_X g)^c+(\mathcal{L}_D g)^v\,.$$

We assert that $\mathcal{L}_X g=0$ and $\mathcal{L}_D g=0$. In fact, if we set $K=\mathcal{L}_X g$ and $H=\mathcal{L}_D g$ and express them by their components (K_{ij}) and (H_{ij}) in terms of a local coordinate system, then the components of K^c and H^v with respect to the induced local coordinate system are given by [3; §6]

$$\begin{pmatrix} y^k\partial_k K_{ij} & K_{ij} \\ K_{ji} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $K^c+H^v=0$ implies $K=0$ and $H=0$. We have now shown that both X and D are infinitesimal isometries of M. Thus,

LEMMA 5.2. *A fibre-preserving infinitesimal isometry of $T(M)$ is of the form X^c+D^v, where both X and D are infinitesimal isometries of M.*

Lemmas 4.4, 4.5 and 5.2 imply Theorem 2.

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