Promotion and Evacuation

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Dedicated to Anders Björner on the occasion of his sixtieth birthday.

Abstract

Promotion and evacuation are bijections on the set of linear extensions of a finite poset first defined by Schützenberger. This paper surveys the basic properties of these two operations and discusses some generalizations. Linear extensions of a finite poset P may be regarded as maximal chains in the lattice $J(P)$ of order ideals of P. The generalizations concern permutations of the maximal chains of a wider class of posets, or more generally bijective linear transformations on the vector space with basis consisting of the maximal chains of *any* poset. When the poset is the lattice of subspaces of $\mathbb{F}_q^n$, then the results can be stated in terms of the expansion of certain Hecke algebra products.

1 Introduction.

Promotion and evacuation are bijections on the set of linear extensions of a finite poset. Evacuation first arose in the theory of the RSK algorithm, which associates a permutation in the symmetric group $\mathfrak{S}_n$ with a pair of standard Young tableaux of the same shape [31, pp. 320–321]. Evacuation was described by M.-P. Schützenberger [25] in a direct way not involving the RSK algorithm. In two follow-up papers [26][27] Schützenberger extended the definition of evacuation to linear extensions of any finite poset. Evacuation is described in terms of a simpler operation called promotion. Schützenberger established many fundamental properties of promotion and evacuation, including the result that evacuation is an involution. Schützenberger's work was simplified by Haiman [15] and

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Malvenuto and Reutenauer [19], and further work on evacuation was undertaken by a number of researchers (discussed in more detail below).

In this paper we will survey the basic properties of promotion and evacuation. We will then discuss some generalizations. In particular, the linear extensions of a finite poset P correspond to the maximal chains of the distributive lattice $J(P)$ of order ideals of P. We will extend promotion and evacuation to bijections on the vector space whose basis consists of all maximal chains of a finite graded poset Q. The case $Q = B_n(q)$, the lattice of subspaces of the vector space $\mathbb{F}_q^n$, leads to some results on expanding a certain product in the Hecke algebra $\mathcal{H}_n(q)$ of $\mathfrak{S}_n$ in terms of the standard basis $\{T_w : w \in \mathfrak{S}_n\}$.

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2 Basic results.

We begin with the original definitions of promotion and evacuation due to Schützenberger. Let P be a p-element poset. We write $s \lessdot t$ if t *covers* s in P, i.e., $s < t$ and no $u \in P$ satisfies $s < u < t$. The set of all linear extensions of P is denoted $\mathcal{L}(P)$. Schützenberger regards a linear extension as a bijection $f\colon P \to [p] = \{1, 2, \dots, p\}$ such that if $s < t$ in P, then $f(s) < f(t)$. (Actually, Schützenberger considers bijections $f\colon P \to \{k+1, k+2, \dots, k+p\}$ for some $k \in \mathbb{Z}$, but we slightly modify his approach by always ensuring that $k = 0$.) Think of the element $t \in P$ as being labelled by $f(t)$. We now define a bijection $\partial\colon \mathcal{L}(P) \to \mathcal{L}(P)$, called *promotion*, as follows. Let $t_1 \in P$ satisfy $f(t_1) = 1$. Remove the label 1 from t_1. Among the elements of P covering t_1, let t_2 be the one with the smallest label $f(t_2)$. Remove this label from t_2 and place it at t_1. (Think of "sliding" the label $f(t_2)$ down from t_2 to t_1.) Now among the elements of P covering t_2, let t_3 be the one with the smallest label $f(t_3)$. Slide this label from t_3 to t_2. Continue this process until eventually reaching a maximal element t_k of P. After we slide $f(t_k)$ to t_{k-1}, label t_k with $p+1$. Now subtract 1 from every label. We obtain a new linear extension $f\partial \in \mathcal{L}(P)$. Note that we let ∂ operate on the *right*. Note also that $t_1 \lessdot t_2 \lessdot \cdots \lessdot t_k$ is a maximal chain of P, called the *promotion chain* of f. Figure 1(a) shows a poset P and a linear extension f. The promotion chain is indicated by circled dots and arrows. Figure 1(b) shows the labeling after the sliding operations and the labeling of the last element of the promotion chain by $p + 1 = 10$. Figure 1(c) shows the linear extension $f\partial$ obtained by subtracting 1 from the labels in Figure 1(b).

It should be obvious that $\partial\colon \mathcal{L}(P) \to \mathcal{L}(P)$ is a bijection. In fact, let ∂^* denote *dual promotion*, i.e., we remove the *largest* label p from some element $u_1 \in P$, then slide the *largest* label of an element covered by u_1 up to u_1, etc. After reaching a minimal element u_k, we label it by 0 and then add 1 to each label, obtaining $f\partial^*$. It is easy to check that

$$\partial^{-1} = \partial^*.$$

We next define a variant of promotion called *evacuation*. The evacuation of a linear extension $f \in \mathcal{L}(P)$ is denoted $f\epsilon$ and is another linear extension of P. First compute $f\partial$.

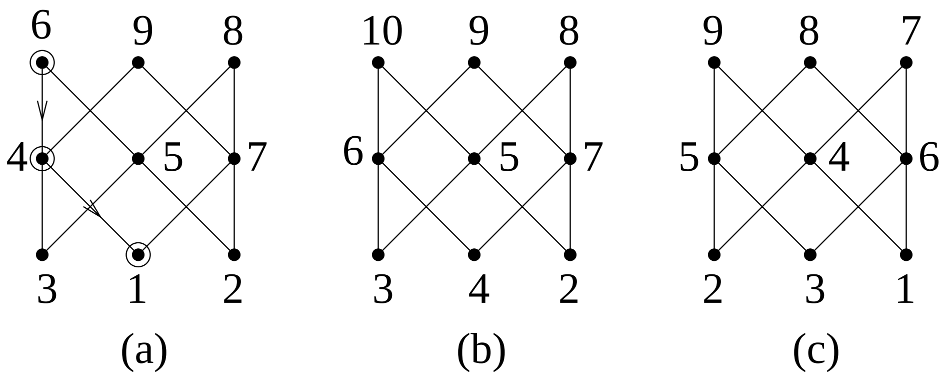


Figure 1: The promotion operator ∂ applied to a linear extension

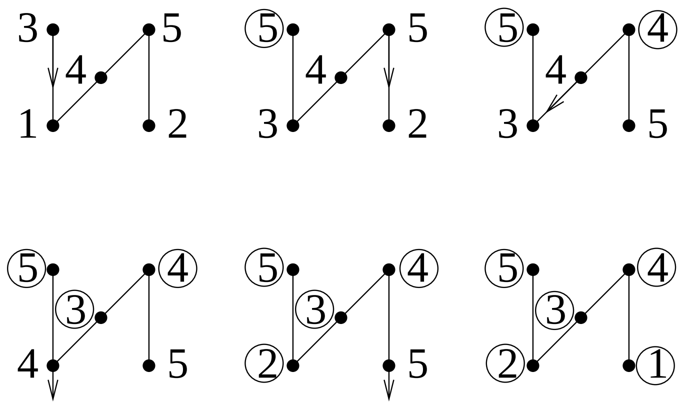


Figure 2: The evacuation of a linear extension f

Then "freeze" the label p into place and apply ∂ to what remains. In other words, let P_1 consist of those elements of P labelled $1, 2, \ldots, p-1$ by $f\partial$, and apply ∂ to the restriction of ∂f to P_1. Then freeze the label $p-1$ and apply ∂ to the $p-2$ elements that remain. Continue in this way until every element has been frozen. Let $f\epsilon$ be the linear extension, called the *evacuation* of f, defined by the frozen labels.

NOTE. A standard Young tableau of shape λ can be identified in an obvious way with a linear extension of a certain poset P_λ. Evacuation of standard Young tableaux has a nice geometric interpretation connected with the nilpotent flag variety. See van Leeuwen [18, §3] and Tesler [36, Thm. 5.14].

Figure 2 illustrates the evacuation of a linear extension f. The promotion paths are shown by arrows, and the frozen elements are circled. For ease of understanding we don't subtract 1 from the unfrozen labels since they all eventually disappear. The labels are always frozen in descending order $p, p-1, \ldots, 1$. Figure 3 shows the evacuation of $f\epsilon$, where f is the linear extension of Figure 2. Note that (seemingly) miraculously we have $f\epsilon^2 = f$. This example illustrates a fundamental property of evacuation given by Theorem 2.1(a) below.

We can define *dual evacuation* analogously to dual promotion. In symbols, if $f \in \mathcal{L}(P)$

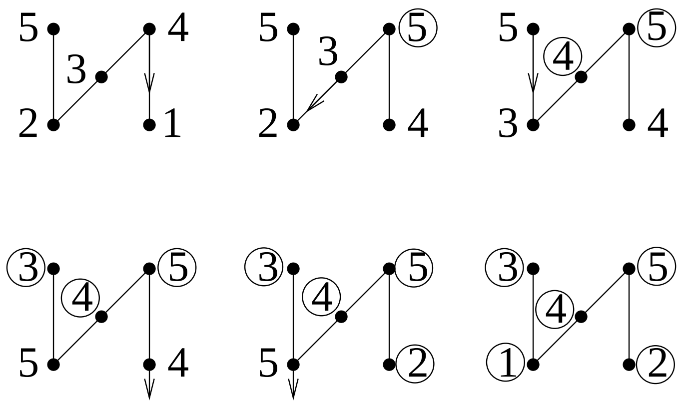


Figure 3: The linear extension evac(evac(f))

then define $f^* \in \mathcal{L}(P^*)$ by $f^*(t) = p + 1 - f(t)$. Thus

$$f\epsilon^* = (f^*\epsilon)^*.$$

We can now state three of the four main results obtained by Schützenberger.

Theorem 2.1. *Let P be a p-element poset. Then the operators ϵ, ϵ^*, and ∂ satisfy the following properties.*

- *(a) Evacuation is an involution, i.e., $\epsilon^2 = 1$ (the identity operator).*
- *(b) $\partial^p = \epsilon\epsilon^*$*
- *(c) $\partial\epsilon = \epsilon\partial^{-1}$*

Theorem 2.1 can be interpreted algebraically as follows. The bijections ϵ and ϵ^* generate a subgroup D_P of the symmetric group $\mathfrak{S}_{\mathcal{L}(P)}$ on all the linear extensions of P. Since ϵ and (by duality) ϵ^* are involutions, the group they generate is a dihedral group D_P (possibly degenerate, i.e., isomorphic to $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) of order 1 or $2m$ for some $m \geq 1$. If ϵ and ϵ^* are not both trivial (which can only happen when P is a chain), so they generate a group of order $2m$, then m is the order of ∂^p. In general the value of m, or more generally the cycle structure of ∂^p, is mysterious. For a few cases in which more can be said, see Section 4.

The main idea of Haiman [15, Lemma 2.7, and page 91] (further developed by Malvenuto and Reutenauer [19]) for proving Theorem 2.1 is to write linear extensions as *words* rather than functions and then to describe the actions of ∂ and ϵ on these words. The proof then becomes a routine algebraic computation. Let us first develop the necessary algebra in a more general context.

Let G be the group with generators $\tau_1, \ldots, \tau_{p-1}$ and relations

$$\begin{aligned} \tau_i^2 &= 1, \quad 1 \leq i \leq p-1 \\ \tau_i\tau_j &= \tau_j\tau_i, \quad \text{if } |i-j| > 1. \end{aligned} \tag{1}$$

Some readers will recognize that G is an infinite Coxeter group ($p \geq 3$) with the symmetric group $\mathfrak{S}_p$ as a quotient. Define the following elements of G:

$$\begin{aligned} \delta &= \tau_1\tau_2\cdots\tau_{p-1} \\ \gamma = \gamma_p &= \tau_1\tau_2\cdots\tau_{p-1}\cdot\tau_1\tau_2\cdots\tau_{p-2}\cdots\tau_1\tau_2\cdot\tau_1 \\ \gamma^* &= \tau_{p-1}\tau_{p-2}\cdots\tau_1\cdot\tau_{p-1}\tau_{p-2}\cdots\tau_2\cdots\tau_{p-1}\tau_{p-2}\cdot\tau_{p-1}. \end{aligned}$$

Lemma 2.2. *In the group G we have the following identities:*

(a) $\gamma^2 = (\gamma^*)^2 = 1$

(b) $\delta^p = \gamma\gamma^*$

(c) $\delta\gamma = \gamma\delta^{-1}$.

Proof. (a) Induction on p. For $p = 2$, we need to show that $\tau_1^2 = 1$, which is given. Now assume for $p-1$. Then

$$\begin{aligned} \gamma_p^2 = {} & \tau_1\tau_2\cdots\tau_{p-1}\cdot\tau_1\cdots\tau_{p-2}\cdots\tau_1\tau_2\tau_3\cdot\tau_1\tau_2\cdot\tau_1 \\ & \cdot\tau_1\tau_2\cdots\tau_{p-1}\cdot\tau_1\cdots\tau_{p-2}\cdots\tau_1\tau_2\tau_3\cdot\tau_1\tau_2\cdot\tau_1. \end{aligned}$$

We can cancel the two middle τ_1's since they appear consecutively. We can then cancel the two middle τ_2's since they are now consecutive. We can then move one of the middle τ_3's past a τ_1 so that the two middle τ_3's are consecutive and can be cancelled. Now the two middle τ_4's can be moved to be consecutive and then cancelled. Continuing in this way, we can cancel the two middle τ_i's for all $1 \leq i \leq p-1$. When this cancellation is done, what remains is the element γ_{p-1}^2, which is 1 by induction.

(b,c) Analogous to (a). Details are omitted. □

Proof of Theorem 2.1. A glance at Theorem 2.1 and Lemma 2.2 makes it obvious that they should be connected. To see this connection, regard the linear extension $f \in \mathcal{L}(P)$ as the word (or permutation of P) $f^{-1}(1), \ldots, f^{-1}(p)$. For $1 \leq i \leq p-1$ define operators $\tau_i \colon \mathcal{L}(P) \to \mathcal{L}(P)$ by

$$\tau_i(u_1u_2\cdots u_p) = \begin{cases} u_1u_2\cdots u_p, & \text{if } u_i \text{ and } u_{i+1} \text{ are comparable in } P \\ u_1u_2\cdots u_{i+1}u_i\cdots u_p, & \text{otherwise.} \end{cases} \qquad (2)$$

Clearly τ_i is a bijection, and the τ_i's satisfy the relations (1). By Lemma 2.2, the proof of Theorem 2.1 follows from showing that

$$\partial = \delta := \tau_1\tau_2\cdots\tau_{p-1}.$$

Note that if $f = u_1u_2\cdots u_p$, then $f\delta$ is obtained as follows. Let j be the least integer such that $j > 1$ and $u_1 < u_j$. Since f is a linear extension, the elements $u_2, u_3, \ldots, u_{j-1}$ are incomparable with u_1. Move u_1 so it is between u_{j-1} and u_j. (Equivalently, cyclically shift the sequence $u_1u_2\cdots u_{j-1}$ one unit to the left.) Now let k be the least integer such

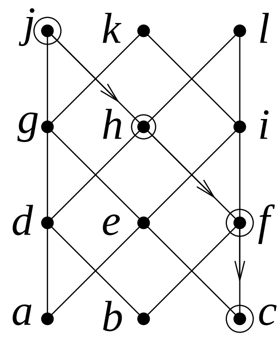


Figure 4: The promotion chain of the linear extension $cabdfeghjilk$

that $k > j$ and $u_j < u_k$. Move u_j so it is between u_{k-1} and u_k. Continue in this way reaching the end. For example, let z be the linear extension $cabdfeghjilk$ of the poset in Figure 4 (which also shows the promotion chain for this linear extension). We factor z from left-to-right into the longest factors for which the first element of each factor is incomparable with the other elements of the factor:

$$z = cabd \cdot feg \cdot h \cdot jilk.$$

Cyclically shift each factor one unit to the left to obtain $z\delta$:

$$z\delta = abdc \cdot egf \cdot h \cdot ilkj = abdcegfhkilj.$$

Now consider the process of promoting the linear extension f of the previous paragraph, given as a function by $f(u_i) = i$ and as a word by $u_1u_2 \cdots u_p$. The elements $u_2, \dots, u_{j-1}$ are incomparable with u_1 and thus will have their labels reduced by 1 after promotion. The label j of u_j (the least element in the linear extension f greater than u_1) will slide down to u_1 and be reduced to $j - 1$. Hence $f\partial = u_2u_3 \cdots u_{j-1}u_1 \cdots$. Exactly analogous reasoning applies to the next step of the promotion process, when we slide the label k of u_k down to u_j. Continuing in this manner shows that $z\delta = z\partial$, completing the proof of Theorem 2.1. □

Note. The operators $\tau_i \colon \mathcal{L}(P) \to \mathcal{L}(P)$ have the additional property that $(\tau_i\tau_{i+1})^6 = 1$, but we see no way to exploit this fact.

Theorem 2.1 states three of the four main results of Schützenberger. We now discuss the fourth result. Let $f \colon P \to [p]$ be a linear extension, and apply ∂ p times, using Schützenberger's original description of ∂ given at the beginning of this section. Say $f(t_1) = p$. After applying sufficiently many ∂'s, the label of t_1 will slide down to a new element t_2 and then be decreased by 1. Continuing to apply ∂, the label of t_2 will eventually slide down to t_3, etc. Eventually we will reach a minimal element t_j of P. We call the chain $\{t_1, t_2, \dots, t_j\}$ the *principal chain* of f (equivalent to Schützenberger's definition of "orbit"), denoted $\rho(f)$. For instance, let f be the linear extension of Figure 5(b) of the poset of Figure 5(a). After applying ∂, the label 5 of e slides down to d and becomes 4. Two more applications of ∂ cause the label 3 to d to slide down to a. Thus $\rho(f) = \{a, d, e\}$.

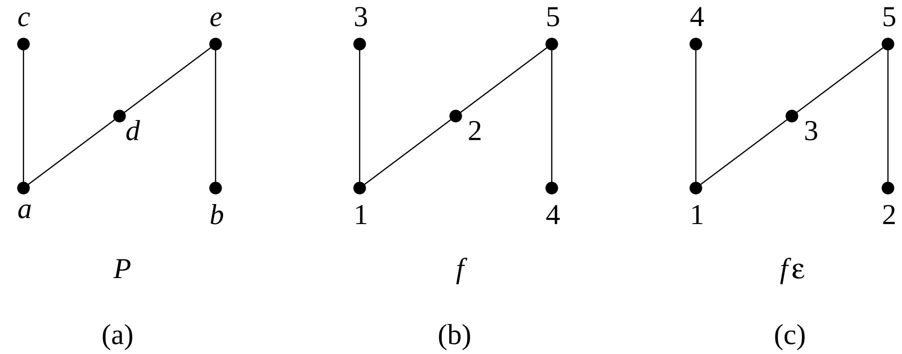


Figure 5: A poset P with a linear extension and its evacuation

Now apply ∂ to the evacuation $f\epsilon$. Let $\sigma(f\epsilon)$ be the chain of elements of P along which labels slide, called the *trajectory* of f. For instance, Figure 5(c) shows $f\epsilon$, where f is given by Figure 5(b). When we apply ∂ to $f\epsilon$, the label 1 of a is removed, the label 3 of d slides to a, and the label 5 of e slides to d. Schützenberger's fourth result is the following.

Theorem 2.3. *For any finite poset P and $f \in \mathcal{L}(P)$ we have $\rho(f) = \sigma(f\epsilon)$.*

Proof (sketch). Regard the linear extension $f\partial^i$ of P as the word $u_{i1}u_{i2}\cdots u_{ip}$. It is clear that

$$\rho(f) = \{u_{0p}, u_{1,p-1}, u_{2,p-2}, \dots, u_{p-1,1}\}$$

(where multiple elements are counted only once). On the other hand, let $\psi_j = \tau_1\tau_2\cdots\tau_{p-j}$, and regard the linear extension $f\psi_1\psi_2\cdots\psi_i$ as the word $v_{i1}v_{i2}\cdots v_{ip}$. It is clear that $v_{ij} = u_{ij}$ if $i+j \leq p$. In particular, $u_{i,p-i} = v_{i,p-i}$. Moreover, $f\epsilon = v_{2,p}, v_{3,p-1}, \dots, v_{p+1,1}$. We leave to the reader to check that the elements of $\rho(f)$ written in increasing order, say $z_1 < z_2 < \cdots < z_k$, form a subsequence of $f\epsilon$, since $u_{i,p-i} = v_{i,p-i}$. Moreover, the elements of $f\epsilon$ between z_j and z_{j+1} are incomparable with z_j. Hence when we apply ∂ to $f\epsilon$, the element z_1 moves to the right until reaching z_2, then z_2 moves to the right until reaching z_3, etc. This is just what it means for $\sigma(f\epsilon) = \{z_1, \dots, z_k\}$, completing the proof. □

Promotion and evacuation can be applied to other properties of linear extensions. We mention three such results here. For the first, let $e(P)$ denote the number of linear extensions of the finite poset P. If A is the set of minimal (or maximal) elements of P, then it is obvious that

$$e(P) = \sum_{t \in A} e(P - t). \tag{3}$$

An *antichain* of P is a set of pairwise incomparable elements of P. Edelman, Hibi, and Stanley [9] use promotion to obtain the following generalization of equation (3) (a special case of an even more general theorem).

Theorem 2.4. *Let A be an antichain of P that intersects every maximal chain. Then*

$$e(P) = \sum_{t \in A} e(P - t).$$

The second application of promotion and evacuation is to the theory of sign balance. Fix an ordering $t_1, \ldots, t_p$ of the elements of P, and regard a linear extension of $f\colon P \to [p]$ as the permutation w of P given by $w(t_i) = f^{-1}(i)$. A finite poset P is *sign balanced* if it has the same number of even linear extensions as odd linear extensions. It is easy to see that the property of being sign balanced does not depend on the ordering $t_1, \ldots, t_p$. While it is difficult in general to understand the cycle structure of the operator ∂ (regarded as a permutation of the set of all linear extensions f of P), there are situations when we can analyze its effect on the parity of f. Moreover, Theorem 3.1 determines the cycle structure of ϵ. This idea leads to the following result of Stanley [32, Cor. 2.2 and 2.4].

Theorem 2.5. *(a) Let $\#P = p$, and suppose that the length ℓ of every maximal chain of P satisfies $p \equiv \ell \pmod 2$. Then P is sign-balanced.*

(b) Suppose that for all $t \in P$, the lengths of all maximal chains of the principal order ideal $\Lambda_t := \{s \in P : s \leq t\}$ have the same parity. Let $\nu(t)$ denote the length of the longest chain of Λ_t, and set $\Gamma(P) = \sum_{t \in P} \nu(t)$. If $\binom{p}{2} \equiv \Gamma(P) \pmod 2$ then P is sign-balanced.

Our final application is related to an operation ψ on antichains A of a finite poset P. Let

$$I_A = \{s \in P : s \leq t \text{ for some } t \in A\},$$

the order ideal *generated* by A. Define $A\psi$ to be the set of minimal elements of $P - I_A$. The operation ψ is a bijection on the set $\mathcal{A}(P)$ of antichains of P, and there is considerable interest in determining the cycle structure of ψ (see, e.g., Cameron [7] and Panyushev [20]). Here we will show a connection with the case $P = \boldsymbol{m} \times \boldsymbol{n}$ (a product of chains of sizes m and n) and promotion on $\boldsymbol{m} + \boldsymbol{n}$ (where $+$ denotes disjoint union). We first define a bijection $\Phi\colon \mathcal{L}(\boldsymbol{m}+\boldsymbol{n}) \to \mathcal{A}(\boldsymbol{m} \times \boldsymbol{n})$. We can write $w \in \mathcal{L}(\boldsymbol{m}+\boldsymbol{n})$ as a sequence $(a_m, a_{m-1}, \ldots, a_1, b_n, b_{n-1}, \ldots, b_1)$ of m 1's and n 2's in some order. The position of the 1's indicate when we choose in w (regarded as a word in the elements of $\boldsymbol{m}+\boldsymbol{n}$) an element from the first summand $\boldsymbol{m}$. Let $m \geq i_1 > i_2 > \cdots > i_r \geq 1$ be those indices i for which $a_i = 2$. Let $j_1 < j_2 < \cdots < j_r$ be those indices j for which $b_j = 1$. Regard the elements of $\boldsymbol{m} \times \boldsymbol{n}$ as pairs (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$, ordered coordinatewise. Define

$$\Phi(w) = \{(i_1, j_1), \ldots, (i_r, j_r)\} \in \mathcal{A}(\boldsymbol{m} \times \boldsymbol{n}).$$

For instance (writing a bar to show the space between a_1 and b_6), $\Phi(1211221|212211) = \{(6,1), (3,2), (2,5)\}$. It can be checked that $\Phi(w\partial) = \Phi(w)\psi$. Hence ψ on $\boldsymbol{m} \times \boldsymbol{n}$ has the same cycle type as ∂ on $\boldsymbol{m}+\boldsymbol{n}$, which is relatively easy to analyze. We omit the details here.

3 Self-evacuation and P-domino tableaux

In this section we consider self-evacuating linear extensions of a finite poset P, i.e., linear extensions f such that $f\epsilon = f$. The main result asserts that the number of self-evacuating $f \in \mathcal{L}(P)$ is equal to two other quantities associated with P. We begin by defining these two other quantities.

An *order ideal* of P is a subset I such that if $t \in I$ and $s < t$, then $s \in I$. A *P-domino tableau* is a chain $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_r = P$ of order ideals of P such that $I_i - I_{i-1}$ is a two-element chain for $2 \leq i \leq r$, while I_1 is either a two-element or one-element chain (depending on whether p is even or odd). In particular, $r = \lceil p/2 \rceil$.

Note. In [32, §4] domino tableaux were defined so that $I_r - I_{r-1}$, rather than I_1, could have one element. The definition given in the present paper is more consistent with previously defined special cases.

Now assume that the vertex set of P is $[p]$ and that P is a *natural partial order*, i.e., if $i < j$ in P then $i < j$ in $\mathbb{Z}$. A linear extension of P is thus a permutation $w = a_1 \cdots a_p \in \mathfrak{S}_p$. The *descent set* $D(w)$ of w is defined by

$$D(w) = \{1 \leq i \leq p-1 \,:\, a_i > a_{i+1}\},$$

and the *comajor index* $\mathrm{comaj}(w)$ is defined by

$$\mathrm{comaj}(w) = \sum_{i \in D(w)} (p-i). \tag{4}$$

(Note. Sometimes the comajor index is defined by $\mathrm{comaj}(w) = \sum_{i \in [p-1]-D(w)} i$, but we will use equation (4) here.) Set

$$W'_P(x) = \sum_{w \in \mathcal{L}(P)} x^{\mathrm{comaj}(w)}.$$

It is known from the theory of P-partitions (e.g., [30, §4.5]) that $W'_P(x)$ depends only on P up to isomorphism.

Note. Usually in the theory of P-partitions one works with the *major index* $\mathrm{maj}(w) = \sum_{i \in D(w)} i$ and with the polynomial $W_P(x) = \sum_{w \in \mathcal{L}(P)} x^{\mathrm{maj}(w)}$. Note that if p is even then $\mathrm{comaj}(w) \equiv \mathrm{maj}(w) \pmod 2$, so $W_P(-1) = W'_P(-1)$.

Theorem 3.1. *Let P be a finite natural partial order. Then the following three quantities are equal.*

(i) $W'_P(-1)$.

(ii) The number of P-domino tableaux.

(iii) The number of self-evacuating linear extensions of P.

In order to prove Theorem 3.1, we need one further result about the elements τ_i of equation (1).

Lemma 3.2. *Let G be the group of Lemma 2.2. Write*

$$\begin{aligned} \delta_i &= \tau_1 \tau_2 \cdots \tau_i \\ \delta_i^* &= \tau_i \tau_{i-1} \cdots \tau_1. \end{aligned}$$

Let $u, v \in G$. The following two conditions are equivalent.

(i) $u\delta_1^*\delta_3^*\cdots\delta_{2j-1}^* = v\delta_1^*\delta_3^*\cdots\delta_{2j-1}^*\cdot\delta_{2j-1}\delta_{2j-2}\cdots\delta_2\delta_1$.

(ii) $u\tau_1\tau_3\cdots\tau_{2j-1} = v$.

Proof of Lemma 3.2. The proof is a straightforward extension of an argument due to van Leeuwen [17, §2.3] (but not expressed in terms of the group G) and more explicitly to Berenstein and Kirillov [2]. (About the same time as van Leeuwen, a special case was proved by Stembridge [35] using representation theory. Both Stembridge and Berenstein-Kirillov deal with *semistandard tableaux*, while here we consider only the special case of standard tableaux. While standard tableaux have a natural generalization to linear extensions of any finite poset, it is unclear how to generalize semistandard tableaux analogously so that the results of Stembridge and Berenstein-Kirillov continue to hold.) Induction on j. The case $j=1$ asserts that $u\tau_1 = v\tau_1\tau_1$ if and only if $u\tau_1 = v$, which is immediate from $\tau_1^2 = 1$. Now assume for $j-1$, and suppose that (i) holds. First cancel $\delta_{2j-1}^*\delta_{2j-1}$ from the right-hand side. Now take the last factor τ_i from each factor δ_i ($1\leq i\leq 2j-2$) on the right-hand side and move it as far to the right as possible. The right-hand side will then end in $\tau_{2j-2}\tau_{2j-3}\cdots\tau_1 = \delta_{2j-2}^*$. The left-hand side ends in $\delta_{2j-1}^* = \tau_{2j-1}\delta_{2j-2}^*$. Hence we can cancel the suffix δ_{2j-2}^* from both sides, obtaining

$$u\delta_1^*\delta_3^*\cdots\delta_{2j-3}^*\tau_{2j-1} = v\delta_1^*\delta_3^*\cdots\delta_{2j-3}^*\cdot\delta_{2j-3}\delta_{2j-4}\cdots\delta_2\delta_1. \tag{5}$$

We can now move the rightmost factor τ_{2j-1} on the left-hand side of equation (5) directly to the right of u. Applying the induction hypothesis with u replaced by $u\tau_{2j-1}$ yields (ii). The steps are reversible, so (ii) implies (i). □

Proof of Theorem 3.1. The equivalence of (i) and (ii) appears (in dual form) in [32, Theorem 5.1(a)]. Namely, let $w = a_1\cdots a_p \in \mathcal{L}(P)$. Let i be the least nonnegative integer (if it exists) for which

$$w' := a_1\cdots a_{p-2i-2}a_{p-2i}a_{p-2i-1}a_{p-2i+1}\cdots a_p \in \mathcal{L}(P).$$

Note that $(w')' = w$. Now exactly one of w and w' has the descent $p-2i-1$. The only other differences in the descent sets of w and w' occur (possibly) for the numbers $p-2i-2$ and $p-2i$. Hence $(-1)^{\mathrm{comaj}(w)} + (-1)^{\mathrm{comaj}(w')} = 0$. The surviving permutations $w = b_1\cdots b_p$ in $\mathcal{L}(P)$ are exactly those for which the chain of order ideals

$$\emptyset \subset \cdots \subset \{b_1, b_2, \ldots, b_{p-4}\} \subset \{b_1, b_2, \ldots, b_{p-2}\} \subset \{b_1, b_2, \ldots, b_p\} = P$$

is a P-domino tableau. We call w a *domino linear extension*; they are in bijection with domino tableaux. Such permutations w can only have descents in positions $p-j$ where j is even, so $(-1)^{\mathrm{comaj}(w)} = 1$. Hence (i) and (ii) are equal.

To prove that (ii) and (iii) are equal, let τ_i be the operator on $\mathcal{L}(P)$ defined by equation (2). Thus w is self-evacuating if and only if

$$w = w\tau_1\tau_2\cdots\tau_{p-1}\cdot\tau_1\cdots\tau_{p-2}\cdots\tau_1\tau_2\tau_3\cdot\tau_1\tau_2\cdot\tau_1.$$

On the other hand, note that w is a domino linear extension if and only if

$$w\tau_{p-1}\tau_{p-3}\tau_{p-5}\cdots\tau_h = w,$$

where $h = 1$ if p is even, and $h = 2$ if p is odd. It follows from Lemma 3.2 (letting $u = v = w$) that w is a domino linear extension if and only if

$$\widetilde{w} := w\tau_1 \cdot \tau_3\tau_2\tau_1 \cdot \tau_5\tau_4\tau_3\tau_2\tau_1 \cdots \tau_m\tau_{m-1}\cdots\tau_1$$

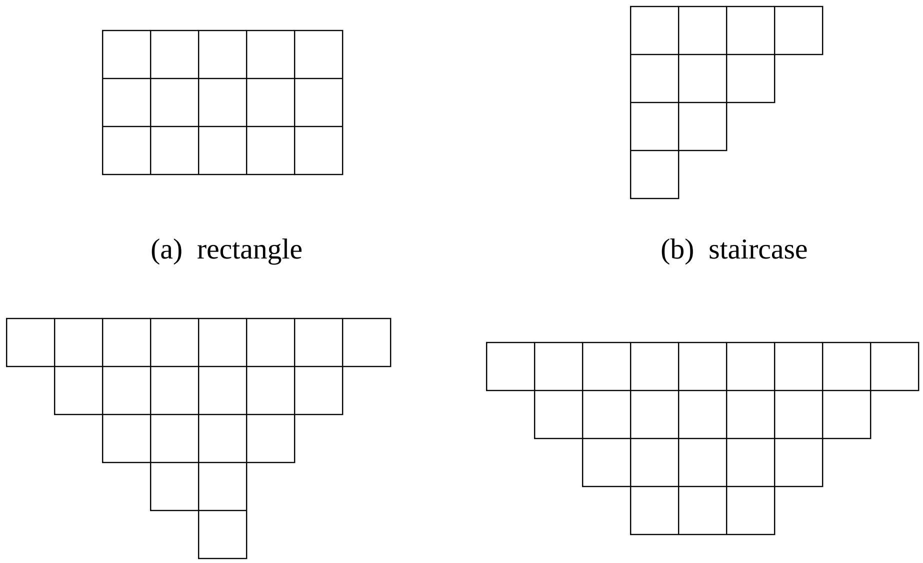
is self-evacuating, where $m = p - 1$ if p is even, and $m = p - 2$ if p is odd. The proof follows since the map $w \mapsto \widetilde{w}$ is then a bijection between domino linear extensions and self-evacuating linear extensions of P. □

The equivalence of (i) and (iii) above is an instance of Stembridge's "$q = -1$ phenomenon." Namely, suppose that an involution ι acts on a finite set S. Let $f : S \to \mathbb{Z}$. (Usually f will be a "natural" combinatorial or algebraic statistic on S.) Then we say that the triple (S, ι, f) exhibits the $q = -1$ phenomenon if the number of fixed points of ι is given by $\sum_{t\in S}(-1)^{f(t)}$. See Stembridge [33][34][35]. The $q = -1$ phenomenon has been generalized to the action of cyclic groups by V. Reiner, D. Stanton, and D. White [23], where it is called the "cyclic sieving phenomenon." For further examples of the cyclic sieving phenomenon, see C. Bessis and V. Reiner [3], H. Barcelo, D. Stanton, and V. Reiner [1], and B. Rhoades [24]. In the next section we state a deep example of the cyclic sieving phenomenon, due to Rhoades, applied to the operator ∂ when P is the product of two chains.

4 Special cases.

There are a few "nontrivial" classes of posets P known for which the operation $\partial^p = \epsilon\epsilon^*$ can be described in a simple explicit way, so in particular the order of the dihedral group D_P generated by ϵ and ϵ^* can be determined. There are also some "trivial" classes, such as hook shapes (a disjoint union of two chains with a $\hat{0}$ adjoined), where it is straightforward to compute the order of ∂ and D_P. The nontrivial classes of posets are all connected with the theory of standard Young tableaux or shifted tableaux, whose definition we assume is known to the reader. A standard Young tableau of shape λ corresponds to a linear extension of a certain poset P_λ in an obvious way, and similarly for a standard shifted tableau. (As mentioned in the introduction, Schützenberger originally defined evacuation for standard Young tableaux before extending it to linear extensions of any finite poset.) We will simply state the known results here. The posets will be defined by examples which should make the general definition clear. In these examples, the elements increase as we move down or to the right, so that the upper-left square is always the unique minimal element of P_λ.

Theorem 4.1. *For the following shapes and shifted shapes P with a total of $p = \#P$ squares, we have the indicated properties of ∂^p and D_P.*



(a) rectangle (b) staircase (c) shifted double staircase (d) shifted trapezoid

Figure 6: Some shapes and shifted shapes

(a) Rectangles (Figure 6(a)). Then $f\partial^p = f$ *and* $D_P \cong \mathbb{Z}/2\mathbb{Z}$ *(if* $m, n > 1$*). Moreover, if* $f = (a_{ij})$ *(where we are regarding a linear extension of the rectangle* P *as a labeling of the squares of* P*), then* $f\epsilon = (p + 1 - a_{m+1-i,n+1-j})$*.*

(b) Staircases (Figure 6(b)). Then $f\partial^p = f^t$ *(the transpose of* f*) and* $D \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$*.*

(c) Shifted double staircases (Figure 6(c)). Then $f\partial^p = f$ *and* $D_P \cong \mathbb{Z}/2\mathbb{Z}$*.*

(d) Shifted trapezoids (Figure 6(d)). Then $f\partial^p = f$ *and* $D_P \cong \mathbb{Z}/2\mathbb{Z}$*.*

Theorem 4.1(a) follows easily from basic properties of *jeu de taquin* due to Schützenberger [28] (see also [31, Ch. 7, Appendix 1]) and is often attributed to Schützenberger. We are unaware, however, of an explicit statement in the work of Schützenberger. Part (b) is due to Edelman and Greene [8, Cor. 7.23]. Parts (c) and (d) are due to Haiman [15, Thm. 4.4], who gives a unified approach also including (a) and (b).

The equivalence of (i) and (iii) in Theorem 3.1 was given a deep generalization by Rhoades [24] when P is an $m \times n$ rectangular shape (so $p = mn$), as mentioned in the previous section. By Theorem 4.1(a) we have $f\partial^p = f$ when P is a rectangular shape of size p. Thus every cycle of ∂, regarded as a permutation of the set $\mathcal{L}(P)$, has length d dividing p. We can ask more generally for the precise cycle structure of ∂, i.e., the number of cycles of each length $d|p$. Equivalently, for any $d \in \mathbb{Z}$ (or just any $d|p$) we can ask for the quantity

$$e_d(P) = \#\{f \in \mathcal{L}(P) \,:\, f = f\partial^d\}.$$

To answer this question, define the *major index* of the linear extension $f \in \mathcal{L}(P)$ by

$$\mathrm{maj}(f) = \sum_i i,$$

where i ranges over all entries of P for which $i+1$ appears in a lower row than i [31, p. 363]. For instance, if f is given by

$$f = \begin{array}{cccc} 1 & 3 & 4 & 8 \\ 2 & 5 & 6 & 11 \\ 7 & 9 & 10 & 12 \end{array},$$

then $\mathrm{maj}(f) = 1+4+6+8+11 = 30$. Let

$$F(q) = \sum_{f \in \mathcal{L}(P)} q^{\mathrm{maj}(f)}.$$

It is well known [31, Cor. 7.21.5] that

$$F(q) = \frac{q^{n\binom{m}{2}}(1-q)(1-q^2)\cdots(1-q^p)}{\prod_{t\in P}(1-q^{h(t)})},$$

where $h(t)$ is the hook length of t. If say $m \leq n$, then we have more explicitly

$$\prod_{t\in P}(1-q^{h(t)})$$

$$= [1][2]^2[3]^3\cdots[m]^m[m+1]^m\cdots[n]^m[n+1]^{m-1}[n+2]^{m-2}\cdots[n+m-1],$$

where $[i] = 1-q^i$. The beautiful result of Rhoades is the following.

Theorem 4.2. *Let P be an $m \times n$ rectangular shape. Set $p = mn$ and $\zeta = e^{2\pi i/p}$. Then for any $d \in \mathbb{Z}$ we have*

$$e_d(P) = F(\zeta^d).$$

Rhoades' proof of this theorem uses Kazhdan-Lusztig theory and a characterization of the dual canonical basis of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ due to Skandera [29]. Several questions are suggested by Theorems 4.1 and 4.2.

1. Is there a more elementary proof of Theorem 4.2? For the special case of $2 \times n$ and $3 \times n$ rectangles, see [21]. The authors of [21] are currently hoping to extend their proof to general rectangles.

2. Can Theorem 4.2 be extended to more general posets, in particular, the posets of Theorem 4.1(b,c,d)?

3. Can Theorem 4.1 itself be extended to other classes of posets? A possible place to look is among the d-complete posets of Proctor [22]. Some work along these lines is being done by Kevin Dilks (in progress at the time of this writing).

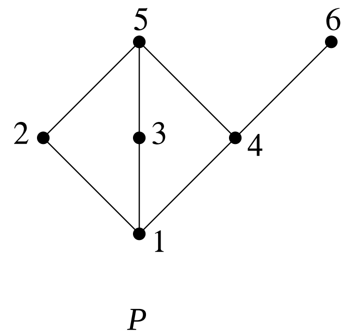


Figure 7: A linear extension of a poset P

5 Growth diagrams

There is an alternative approach to promotion and evacuation, kindly explained by an anonymous referee. This approach is based on the growth diagrams developed by S. Fomin in a series of papers [10][11][12][13]. In [31, pp. 424–429] Fomin uses growth diagrams to develop Schützenberger's work on evacuation related to the RSK algorithm. This approach can be extended to arbitrary posets by replacing Young diagrams with order ideals of P.

Let $f\colon P \to [p]$ be a linear extension of the p-element poset P. For simplicity we will denote the element $t \in P$ satisfying $f(t) = i$ by i. Figure 7 shows an example that we will use throughout this discussion.

We now define the *growth diagram* $\mathcal{D}(P, f)$ of the pair (P, f). Begin with the points $(a, b) \in \mathbb{Z}^2$ satisfying $a, b \geq 0$ and $a + b \leq p$. We want to label each of these points (a, b) with an order ideal $I(a, b)$ of P. In general we will have $\#I(a, b) = a + b$. We first label all the points satisfying $a + b = p$ with the elements $\{1, 2, \ldots, p\}$ of the entire poset P, and the points $(0, b)$ with the order ideal $\{1, 2, \ldots, b\}$. See Figure 8.

We now inductively label the remaining points according to the following *local rule*: suppose that we have labelled all the corners except the bottom-right corner of a unit square. The bottom-left corner (a, b) will be labelled with an order ideal $I = I(a, b)$; the top-left corner $(a, b+1)$ will be labeled $I \cup \{i\}$ for some $1 \leq i \leq p$; and the top-right corner $(a + 1, b + 1)$ will be labelled $I \cup \{i, j\}$. We then define the labelling of the bottom-right corner $(a + 1, b)$ by

$$I(a+1, b) = \begin{cases} I(a, b) \cup \{i\}, & \text{if } i < j \text{ in } P \\ I(a, b) \cup \{j\}, & \text{if } i \parallel j \text{ in } P, \end{cases}$$

where $i \parallel j$ denotes that i and j are incomparable. The labelling begins at $(1, p - 2)$ and works its way down and to the right. See Figure 9 for a diagram of the local rule and Figure 10 for the completed growth diagram of our example.

The bottom row of the growth diagram $\mathcal{D}(P, f)$ lists a chain $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = P$ of order ideals of P with $\#I_i = i$. This chain corresponds to the linear extension g of P given by $g(t) = i$ if $t \in I_i - I_{i-1}$. Now every lattice path from $(0, 0)$ to a point

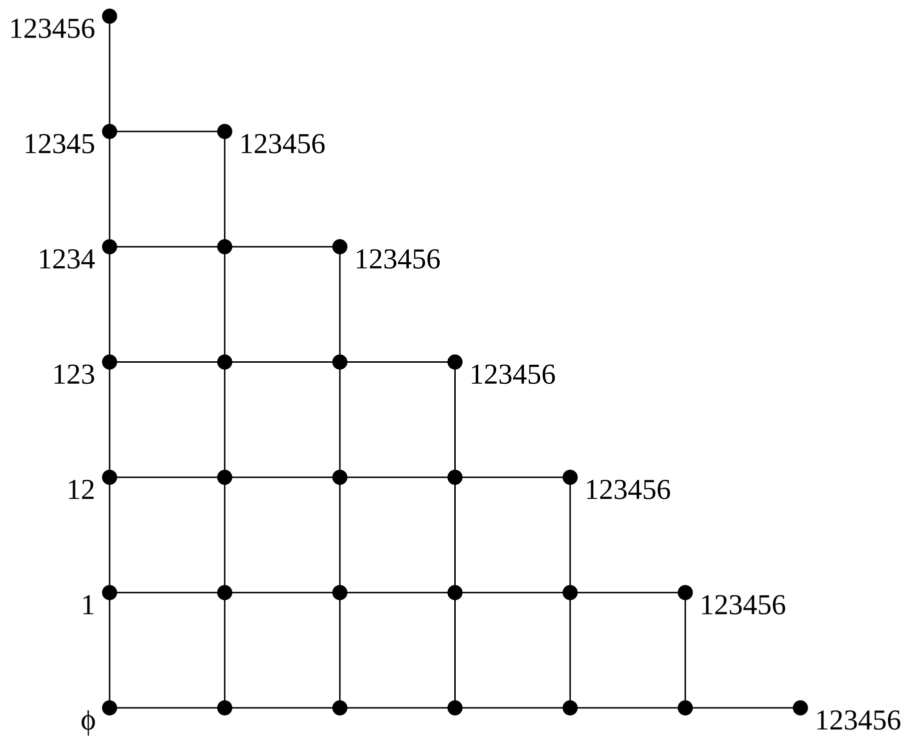


Figure 8: Initialization of the growth process

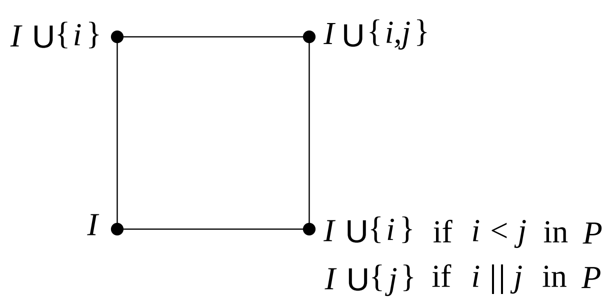


Figure 9: The local growth rule

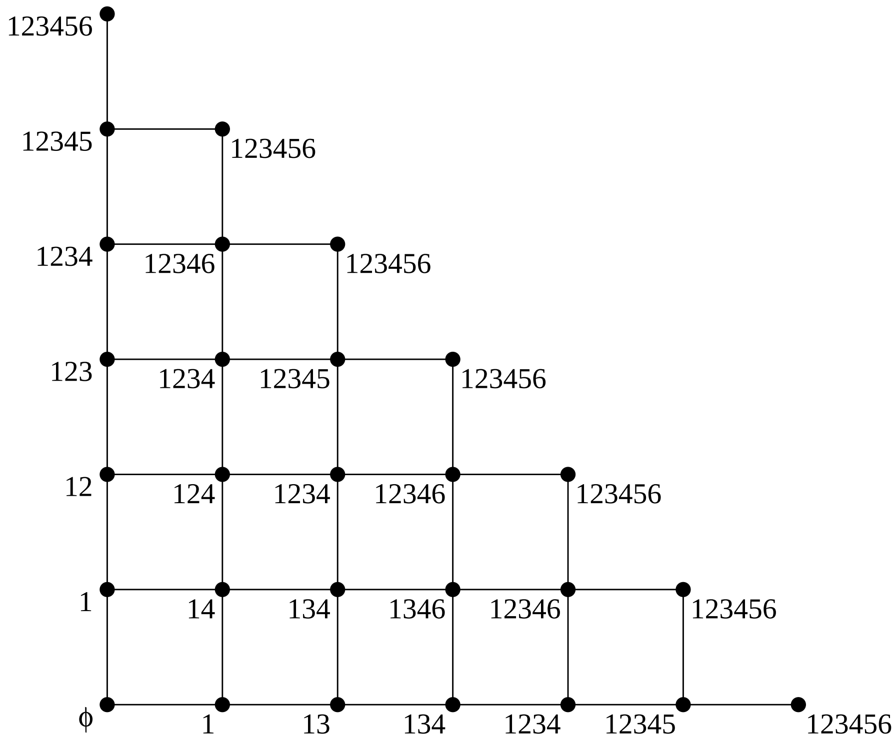


Figure 10: A growth diagram

$(a.b)$ with $a + b = p$ with steps $(1, 0)$ and $(0, 1)$ defines a linear extension of P, just as we have done for the linear extension g. By analyzing how these linear extensions change as we alter the lattice path by changing two consecutive steps $(0, 1), (1, 0)$ to $(1, 0), (0, 1)$, we can deduce that $g = f\epsilon^*$, the dual evacuation of f. If we reflect $\mathcal{D}(P, f)$ about the main diagonal then we obtain $\mathcal{D}(P, g) = \mathcal{D}(P, \epsilon^*)$. Hence it is geometrically obvious that $(\epsilon^*)^2 = 1$. In a similar manner we can obtain the other parts of Theorem 2.1 and (with a little more work) Lemma 3.2.

6 Generalizations.

The basic properties of evacuation given in Sections 2 and 3 depend only on the formal properties of the group G defined by equation (1). It is easy to find other examples of operators satisfying these conditions that are more general than the operators τ_i operating on linear extensions of posets. Hence the theory of promotion and evacuation extends to these more general situations.

Let $J(P)$ denote the set of all order ideals of the finite poset P, ordered by inclusion. By a well-known theorem of Birkhoff (see [30, Thm. 3.41]), the posets $J(P)$ coincide with the finite distributive lattices. There is a simple bijection [30, §3.5] between maximal chains $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = P$ of $J(P)$ and linear extensions of P, viz., associate with this chain the linear extension $f\colon P \to [p]$ defined by $f(t) = i$ if $t \in I_i - I_{i-1}$. In terms of the maximal chain $\mathfrak{m}\colon \emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = P$ of $J(P)$, the operator τ_i on linear extensions of P can be defined as follows. The interval $[I_{i-1}, I_{i+1}]$ contains either three or four elements, i.e., either I_i is the unique element satisfying $I_{i-1} \subset I_i \subset I_{i+1}$ or there is exactly one other such element I'. In the former case define $\tau_i(\mathfrak{m}) = \mathfrak{m}$; in the latter case, $\tau_i(\mathfrak{m})$ is obtained from $\mathfrak{m}$ by replacing I_i with I'.

The exact same definition of τ_i can be made for any finite graded poset, say for convenience with a unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$, for which every interval of rank 2 contains either three or four elements. Let us call such posets *slender*. Clearly the τ_i's satisfy the conditions (1). Thus Lemma 2.2 applies to the operators γ, γ^*, and δ. (These observations seem first to have been made by van Leeuwen [17, §2], after similar results by Malvenuto and Reutenauer [19] in the context of graphs rather than posets.) We also have an analogue for slender posets Q of the equivalence of (ii) and (iii) in Lemma 3.2. The role of P-domino tableau is played by *domino chains* of Q, i.e., chains $\hat{0} = t_0 < t_1 < \cdots < t_r = \hat{1}$ in P for which the interval $[t_{i-1}, t_i]$ is a two-element chain for $2 \le i \le r$, while $[t_0, t_1]$ is either a two-element or one-element chain (depending on whether the rank of Q is even or odd). We then have that the number of self-evacuating maximal chains of Q is equal to the number of domino chains of Q.

Some example of slender posets are Eulerian posets [30, §3.14], which include face posets of regular CW-spheres [4] and intervals in the Bruhat order of Coxeter groups W (including the full Bruhat order of W when W is finite). Eulerian posets Q have the property that every interval of rank 2 contains four elements. Hence there are no domino chains when $\mathrm{rank}(Q) > 1$, and therefore also no self-evacuating maximal chains. Non-Eulerian slender posets include the weak order of a finite Coxeter group [5][6, Ch. 3] and face posets of regular CW-balls. We have not systematically investigated whether there are examples for which more can be said, e.g., an explicit description of evacuation or the determination of the order of the dihedral group generated by γ and γ^*.

There is a simple example that can be made more explicit, namely, the face lattice L_n of an n-dimensional cross-polytope $\mathcal{C}_n$ (the dual to an n-cube). The vertices of $\mathcal{C}_n$ can be labelled $1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}$ so that vertices i and $\bar{i}$ are antipodal for all i. A maximal chain $\hat{0} = t_0 < t_1 < \cdots < t_{n+1} = \hat{1}$ of L_n can then be encoded as a *signed permutation* $a_1 \cdots a_n$, i.e., take a permutation $b_1 \cdots b_n$ and place bars above some subset of the b_i's. Thus a_i is the unique vertex of the face t_i that does not lie in t_{i-1}. Write $'$ for the reversal of the bar, i.e., $i' = \bar{i}$ and $\bar{i}' = i$. Let $w = a_1 \cdots a_n$ be a signed permutation of $1, 2, \ldots, n$. Then it is easy to compute that

$$\begin{aligned} w\delta &= a_2\, a_3 \cdots a_n\, a_1' \\ w\gamma &= a_1'\, a_n\, a_{n-1} \cdots a_2 \\ w\gamma^* &= a_n'\, a_{n-1}' \cdots a_1' \\ w\delta^{n+1} = w\gamma\gamma^* &= a_2'\, a_3' \ldots a_n'\, a_1. \end{aligned}$$

Thus $\gamma\gamma^*$ has order n if n is odd and $2n$ if n is even. The dihedral group generated by γ and γ^* has order $2n$ if n is odd and $4n$ if n is even.

Can the concepts of promotion and evacuation be extended to posets that are not slender? We discuss one way to do this. Let P be a graded poset of rank n with $\hat{0}$ and $\hat{1}$.

If $\mathfrak{m}\colon \hat{0} = t_0 < t_1 < \cdots < t_n = \hat{1}$ is a maximal chain of P, then we would like to define $\mathfrak{m}\tau_i$ so that (1) $\tau_i^2 = 1$, and (2) the action of τ_i is "local" at rank i, i.e., $\mathfrak{m}\tau_i$ should only involve maximal chains that agree with $\mathfrak{m}$ except possibly at t_i. There is no "natural" choice of a single chain $\mathfrak{m}' = \mathfrak{m}\tau_i$, so we should be unbiased and choose a linear combination of chains. Thus let K be a field of characteristic 0. Write $\mathcal{M}(P)$ for the set of maximal chains of P and $K\mathcal{M}(P)$ for the K-vector space with basis $\mathcal{M}(P)$. For $1 \le i \le n-1$ define a linear operator $\tau_i\colon K\mathcal{M}(P) \to K\mathcal{M}(P)$ as follows. Let $N_i(\mathfrak{m})$ be the set of maximal chains $\mathfrak{m}'$ of P that differ from $\mathfrak{m}$ exactly at t_i, i.e., $\mathfrak{m}'$ has the form

$$\mathfrak{m}'\colon \hat{0} = t_0 < t_1 < \cdots < t_{i-1} < t_i' < t_{i+1} < \cdots < t_n = \hat{1},$$

where $t_i' \neq t_i$. Suppose that $\#N_i(\mathfrak{m}) = q \ge 1$. Then set

$$\tau_i(\mathfrak{m}) = \frac{1}{q+1}\left((q-1)\mathfrak{m} - 2\sum_{\mathfrak{m}' \in N_i(\mathfrak{m})} \mathfrak{m}'\right). \tag{6}$$

When $q = 0$ we set $\mathfrak{m}\tau_i = \mathfrak{m}$, though it would make no difference to set $\mathfrak{m}\tau_i = -\mathfrak{m}$ to remain consistent with equation (6). It is easy to check that $\tau_i^2 = 1$. In fact, $\pm\tau_i$ are the unique involutions of the form $a\mathfrak{m} + b\sum_{\mathfrak{m}' \in N_i(\mathfrak{m})} \mathfrak{m}'$ for some $a, b \in K$ with $b \neq 0$ when $q \ge 1$. It is clear that also $\tau_i\tau_j = \tau_j\tau_i$ if $|j - i| \ge 2$, so the τ_i's satisfy (1). Hence we can define promotion and evacuation on the maximal chains of any finite graded poset so that Lemma 2.2 holds, as well as an evident analogue of the equivalence of (ii) and (iii) in Theorem 3.1.

The obvious question then arises: are there interesting examples? We will discuss one example here, namely, the lattice $B_n(q)$ of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ (ordered by inclusion). This lattice is the "q-analogue" of the boolean algebra B_n of all subsets of the set $\{1, 2, \ldots, n\}$, ordered by inclusion. The boolean algebra B_n is the lattice of order ideals of an n-element antichain A. Hence promotion and evacuation on the maximal chains of B_n are equivalent to "classical" promotion and evacuation on A. The linear extensions of A are just *all* the permutations w of $\{1, \ldots, n\}$, and the evacuation $w\epsilon$ of $w = a_1 a_2 \cdots a_n$ is just the reversal $a_n \cdots a_2 a_1$. Thus we are asking for a kind of q-analogue of reversing a permutation.

This problem can be reduced to a computation in the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group $\mathfrak{S}_n$ over the field K (of characteristic 0). Recall (e.g., [16, §7.4]) that $\mathcal{H}_n(q)$ has generators $T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} (T_i + 1)(T_i - q) &= 0 \\ T_i T_j &= T_j T_i, \quad |i - j| \ge 2 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}. \end{aligned}$$

If $q = 1$ then we have $T_i^2 = 1$, and the above relations are just the Coxeter relations for the group algebra $K\mathfrak{S}_n$.

For $1 \le i \le n-1$ let s_i denote the adjacent transposition $(i, i+1) \in \mathfrak{S}_n$. A *reduced decomposition* of an element $w \in \mathfrak{S}_n$ is a sequence $(a_1, \ldots, a_r)$ of integers $1 \le a_i \le n-1$

such that $w = s_{a_1} \cdots s_{a_r}$ and r is as small as possible, namely, r is the number of inversions of w. Define $T_w = T_{a_1} \cdots T_{a_r}$. In particular, $T_{\mathrm{id}} = 1$ and $T_{s_k} = T_k$. A standard fact about $\mathcal{H}_n(q)$ is that T_w is independent of the choice of reduced decomposition of w, and the T_w's for $w \in \mathfrak{S}_n$ form a K-basis for $\mathcal{H}_n(q)$. We also have the multiplication rule

$$T_u T_k = \begin{cases} T_{us_k}, & \text{if } l(us_k) = l(u) + 1, \\ qT_{us_k} + (q-1)T_u, & \text{if } l(us_k) = l(u) - 1, \end{cases} \tag{7}$$

for any $u \in \mathfrak{S}_n$.

Let $\mathrm{End}(K\mathcal{M}(B_n(q)))$ be the set of all linear transformations

$$K\mathcal{M}(B_n(q)) \to K\mathcal{M}(B_n(q)).$$

Let

$$t_i = -\frac{q+1}{2}\tau_i + \frac{q-1}{2}I,$$

the endomorphism sending a maximal chain $\mathfrak{m}$ to $\sum_{\mathfrak{m}' \in N_i(\mathfrak{m})} \mathfrak{m}'$. It is easy to check that the map $T_i \mapsto t_i$ extends to an algebra homomorphism (i.e., a representation of $\mathcal{H}_n(q)$) $\varphi\colon \mathcal{H}_n(q) \to \mathrm{End}(K\mathcal{M}(B_n(q)))$. Moreover, φ is injective. If we fix a maximal chain $\mathfrak{m}_0$, then the set $\mathcal{M}(B_n(q))$ has a *Bruhat decomposition* [14, §23.4]

$$\mathcal{M}(B_n(q)) = \bigsqcup_{w \in \mathfrak{S}_n} \Omega_w,$$

where $\bigsqcup$ denotes disjoint union and $\Omega_{\mathrm{id}} = \{\mathfrak{m}_0\}$. Defining $t_w = \varphi(T_w)$, we then have

$$t_w(\mathfrak{m}_0) = \sum_{\mathfrak{m} \in \Omega_w} \mathfrak{m}.$$

(In fact, this equation could be used to define Ω_w.) Let $E_i = \frac{1}{q+1}(q-1-2T_i) \in \mathcal{H}_n(q)$, so $E_i^2 = 1$. It follows that

$$\mathfrak{m}_0 \epsilon = \sum_{w \in \mathfrak{S}_n} c_w(q) \sum_{\mathfrak{m} \in \Omega_w} w,$$

where $c_w(q)$ is defined by the Hecke algebra expansion

$$E_1 E_2 \cdots E_{n-1} E_1 E_2 \cdots E_{n-2} \cdots E_1 E_2 E_1 = \sum_{w \in \mathfrak{S}_n} c_w(q) T_w. \tag{8}$$

Note that by Lemma 2.2(a) the right-hand side of equation (8) remains invariant if we reverse the order of the factors on the left-hand side. In general, however, the expression $E_{a_1} \cdots E_{a_r}$ is *not* the same for all reduced decompositions $(a_1, \ldots, a_r)$ $(r = \binom{n}{2})$ of $w_0 = n, n-1, \ldots, 1$.

When $w \in \mathfrak{S}_4$ the values of $c_w(q)$ are given by

$$\begin{aligned}
c_{1234}(q) &= (q-1)^2/(q+1)^2 \\
c_{1243}(q) &= -2(q-1)^3/(q+1)^4 \\
c_{1324}(q) &= -16q(q-1)(q^2+1)/(q+1)^6 \\
c_{1342}(q) &= 4(q-1)^2/(q+1)^4 \\
c_{1423}(q) &= 4(q-1)^2/(q+1)^4 \\
c_{1432}(q) &= -8(q-1)^3/(q+1)^6 \\
c_{2134}(q) &= -2(q-1)^3/(q+1)^4 \\
c_{2143}(q) &= 4(q-1)^2/(q+1)^4 \\
c_{2314}(q) &= -4(q-1)^4/(q+1)^6 \\
c_{2341}(q) &= -8(q-1)/(q+1)^4 \\
c_{2413}(q) &= 0 \\
c_{2431}(q) &= 16(q-1)^2/(q+1)^6 \\
c_{3124}(q) &= -4(q-1)^4/(q+1)^6 \\
c_{3142}(q) &= 0 \\
c_{3214}(q) &= 8(q-1)^3/(q+1)^6 \\
c_{3241}(q) &= 0 \\
c_{3412}(q) &= 16(q-1)^2/(q+1)^6 \\
c_{3421}(q) &= -32(q-1)/(q+1)^6 \\
c_{4123}(q) &= -8(q-1)/(q+1)^4 \\
c_{4132}(q) &= 16(q-1)^2/(q+1)^6 \\
c_{4213}(q) &= 0 \\
c_{4231}(q) &= -32(q-1)/(q+1)^6 \\
c_{4312}(q) &= -32(q-1)/(q+1)^6 \\
c_{4321}(q) &= 64/(q+1)^6.
\end{aligned}$$

Although many values of $c_w(q)$ appear to be "nice," not all are as nice as the above data suggests. For instance,

$$\begin{aligned}
c_{12453}(q) &= 4(q^2+6q+1)(q-1)^4/(q+1)^8 \\
c_{13245}(q) &= -2(q^4-8q^3-2q^2-8q+1)(q-1)^5/(q+1)^{10} \\
c_{13425}(q) &= -4(q^6-6q^5-33q^4+12q^3-33q^2-6q+1)(q-1)^2/ \\
&\qquad (q+1)^{10}.
\end{aligned}$$

We will prove two results about the $c_w(q)$'s.

Theorem 6.1. *Let* id *denote the identity permutation in* $\mathfrak{S}_n$*. Then*

$$c_{\mathrm{id}}(q) = \left(\frac{q-1}{q+1}\right)^{\lfloor n/2 \rfloor}.$$

Proof (sketch). I am grateful to Monica Vazirani for assistance with the following proof. Define a scalar product on $\mathcal{H}_n(q)$ by

$$\langle T_u, T_v \rangle = q^{\ell(u)}\delta_{uv},$$

where $\ell(u)$ denotes the number of inversions of u (i.e., the length of u as an element of the Coxeter group $\mathfrak{S}_n$). Then one can check that for any $g, h \in \mathcal{H}_n(q)$ we have

$$\langle T_i g, h \rangle = \langle g, T_i h \rangle$$

and

$$\langle g T_i, h \rangle = \langle g, h T_i \rangle.$$

Since $E_i^2 = 1$ it follows that

$$\langle E_i g E_i, 1 \rangle = \langle g, 1 \rangle. \tag{9}$$

Now

$$c_{\mathrm{id}}(q) = \langle E_1 E_2 \cdots E_{n-1} E_1 E_2 \cdots E_{n-2} \cdots E_1 E_2 E_1, 1 \rangle.$$

Using equation (9) and the commutation relation $E_i E_j = E_j E_i$ if $|i-j| \geq 2$, we obtain

$$c_{\mathrm{id}}(q) = \langle E_{n-1} E_{n-3} \cdots E_r, 1 \rangle,$$

where $r = 1$ if n is even, and $r = 2$ if n is odd. For any subset S of $\{n-1, n-3, \ldots, r\}$ we have

$$\prod_{i \in S} T_i = T_{\prod_{i \in S} s_i}.$$

(The T_i's and s_i's for $i \in S$ commute, so the above products are well-defined.) Hence we obtain the scalar product $\langle E_{n-1} E_{n-3} \cdots E_r, 1 \rangle$ by setting $T_i = 0$ in each factor of the product $E_{n-1} E_{n-2} \cdots E_r$, so we get

$$\langle E_{n-1} E_{n-3} \cdots E_r, 1 \rangle = \left(\frac{q-1}{q+1} \right)^{\lfloor n/2 \rfloor},$$

completing the proof. ☐

If $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$, then write $\widehat{w}$ for the reversal $a_n \cdots a_2 a_1$. Equivalently, $\widehat{w} = w_0 w$, where $w_0 = n, n-1, \ldots, 1$ (the longest permutation in $\mathfrak{S}_n$). Our second result on the polynomials $c_w(q)$ is the following.

Theorem 6.2. *Let $w \in \mathfrak{S}_n$, and let $\kappa(w)$ denote the number of cycles of w. Then $c_w(q)$, regarded as a rational function of q, has numerator divisible by $(q-1)^{n-\kappa(\widehat{w})}$.*

Proof. Consider the coefficient of T_w in the expansion of the product on the left-hand side of (8). For each factor $E_i = \frac{1}{q+1}(q-1-2T_i)$ we must choose a term $(q-1)/(q+1)$ or $-2T_i/(q+1)$. If we choose $(q-1)/(q+1)$ then we have introduced a factor of $q-1$. If we choose $-2T_i/(q+1)$ and multiply some T_u by it, then a T_v so obtained satisfies either $v = us_i$ or $v = u$; in the latter case a factor of $q-1$ is introduced. It follows that every contribution to the coefficient of T_w arises from choosing a subsequence $(b_1,\ldots,b_j)$ of the reduced decomposition $(1,2,\ldots,p-1,1,2,\ldots,p-2,\ldots,1,2,1)$ of w_0 such that

$$w = s_{b_1}\cdots s_{b_j}, \tag{10}$$

in which case we will obtain a factor $(q-1)^{\binom{n}{2}-j}$. The b_i's correspond to the terms that do *not* introduce a factor of $q-1$.

Now let $\boldsymbol{a} = (a_1,\ldots,a_{\binom{n}{2}})$ be a reduced decomposition of w_0. It is a well-known and simple consequence of the strong exchange property for reduced decompositions (e.g. [6, Thm. 1.4.3]) that if k is the length of the longest subsequence $(b_1,\ldots,b_k)$ of $\boldsymbol{a}$ such that $s_{b_1}\cdots s_{b_k} = w$, then $\binom{n}{2}-k$ is the minimum number of transpositions $t_1,\ldots,t_k$ for which $w = w_0t_1\cdots t_k$. This number is just $n-\kappa(w_0^{-1}w) = n-\kappa(w_0w) = n-\kappa(\widehat{w})$, so $k = \binom{n}{2}-n+\kappa(\widehat{w})$.

It follows that the largest possible value of j in equation (10) is $\binom{n}{2}-n+\kappa(\widehat{w})$. Thus $\binom{n}{2}-j \geq n-\kappa(\widehat{w})$, completing the proof. $\square$

Theorem 6.2 need not be best possible. For instance, some values of $c_w(1)$ can be 0, such as $c_{2413}(q)$. For a nonzero example, we have that $(q-1)^4$ divides $c_{2314}(q)$, but $4-\kappa(4132)=2$.

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