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Proof by contradiction: Teaching and learning considerations in the secondary mathematics classroom

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Proof by contradiction: Teaching and learning considerations in the secondary mathematics classroom

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Gregory Hine has written a very interesting paper on the application of proofs using the method of 'proof by contradiction'. Proof by contradiction first takes the assumption that the thing we are proving is not true, and secondly then shows that any consequences from it are not possible. Greg presented a version of this paper at the recent AAMT conference in Brisbane.

This professional practice paper is underpinned philosophically by the indisputable centrality of proof to the discipline of mathematics. Proof offers students the opportunity to deepen their own understanding of mathematical ideas, to construct and defend logical arguments, and to think critically about the veracity of mathematical statements. Such opportunities afford students key skills required for further study, and arguably for a myriad of careers. Proof by contradiction is a particular mathematical technique taught in Australian senior secondary classrooms (ACMSM025, ACMSM063) which will be explored in this paper. In particular, several worked examples will be outlined alongside implications for best instructional practice within the context of the secondary mathematics classroom.

Introduction

A considerable amount of literature highlights the centrality of proof to the discipline of mathematics (Hanna & de Villiers, 2008; Hine, Lesseig & Boardman, 2018; Stylianou, Blanton, & Knuth, 2009). This centrality is reflected in policy documents and national curricula which govern the teaching and learning of mathematics (e.g. *ACARA*, 2017; *Common Core State Standards Initiative*, 2010). Such importance is supported by the notion that engaging in proof activity helps students reason about mathematical ideas as they critique arguments or construct their own logically sound explanations or justifications (Lesseig, Hine, Na & Boardman, 2019). Working with proofs enables students to explore the axiomatic structure of the discipline and the infallible nature of mathematical truths (Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012), and through this exploration, students can develop all mathematical skills (Milbou, Deprez & Laenens, 2013). Additionally, constructing mathematical arguments to convince oneself or others of a statement's truth (or falsehood) provides opportunities for students to deepen their understanding of underlying mathematical ideas (Lesseig et al., 2019).

Despite these recognised affordances, the extent to which proof plays a significant role in the teaching and learning of mathematics across the grades is subject to variation and debate (Hanna & de Villiers, 2008; Stylianou et al., 2009). For instance, research has revealed that secondary mathematics teachers often hold a limited view on the purpose of proof instruction and its appropriateness for all students (Bergqvist, 2005; Knuth, 2002). With such a narrowed view, teachers can relegate proof to verifying formulas in secondary school geometry lessons, neglecting the explanatory role proof can play in the learning of mathematics at all levels (Hanna, 2000; Knuth, 2002). Furthermore, there is a tendency for teachers to focus on the structure of a proof rather than its substance (Dickerson & Doerr, 2014); this focus of proof as a formalistic mechanism has been reported as a common and most recent experience for prospective secondary mathematics teachers (Boyle, Bleiler, Yee, & Ko, 2015; Varghese, 2009).

This paper will explore proof by contradiction, which is a form of proof that establishes the truth or validity of a proposition. It achieves this by showing that the proposition being false would imply a contradiction (Hine & McNab, 2014). Within the *Australian Curriculum: Mathematics* (ACMSM025, ACMSM063), proof by contradiction is taught in the senior secondary course Specialist Mathematics (ACARA, 2019). After outlining the proof by contradiction method, various worked examples will be presented with a running commentary to assist teachers wishing to use these examples in the classroom. Then, several difficulties associated with teaching and learning this proof method—as reported by researchers and scholars alike—will be offered to the readership. The final section of the paper contains a number of examples for teachers and students to practice using the proof by contradiction method.

Proof in mathematics education

One popular view of a mathematical proof has been described as a sequence of steps, written almost exclusively in symbols, where each step follows logically from an earlier part of the proof and where the last line is the statement being proved (Garnier & Taylor, 2010). Lawson (2016) suggested that in order to understand how proofs work, three simple assumptions are needed:

1. Mathematics only deals in statements that are capable of either being true or false
2. If a statement is true then its negation is false, and if a statement is false then its negation is true
3. Mathematics is free from contradictions (p. 12).

According to Otani (2015), mathematical proof can be classified into two types. The first type is direct proof, which claims that a statement Q is true based on a premise P that is supposed to be true (this rule of logic is referred to as *modus ponens*). The other type of proof is where an indirect claim is made that a conclusion Q is true. Mathematical proof by contradiction is one of the latter type of proof methods, which is logically constructed, formally valid, and devoid of uncertainty and probability (Otani, 2015).

Proof by contradiction

Proving by contradiction is common practice amongst mathematicians (Amit & Portnov-Neeman, 2017) who cannot derive, or find it difficult to derive, that a conclusion Q is true from a premise P directly (Otani, 2015). Polya (1957) claimed that using indirect proof is the height of intellectual achievement, and that it promotes students' thinking to higher levels. This method of indirect proof is frequently referred to as *reductio ad absurdum* (reduction to absurdity), and it claims indirectly the truth about a conclusion Q. According to (Antonini & Mariotti, 2008), we suppose both the premise P and the negation of the conclusion Q ($\neg Q$) are true. Using a logically constructed series of mathematical statements, deriving a contradiction (i.e., the falsehood of ($\neg Q$) implies the truth of conclusion Q (Otani, 2015)). In a real-world example, suppose that it was alleged that a person had committed an offence in a certain city. Using evidence that this person was not in the city at the time of the offence contradicts the allegation and hence establishes the person's innocence. The steps to follow using the proof by contradiction method with mathematical examples are:

1. Commence with claim P
2. Assume that P is an incorrect claim and develop an oppositional claim $\neg P$
3. Work logically to determine an inconsistency between P and $\neg P$, and prove that $\neg P$ is false
4. Conclude that if claim $\neg P$ is false then by contradiction the premise P is true.

The steps of this method will be demonstrated in three worked examples which now follow.

Worked example 1

Prove by contradiction that the difference of any rational number and any irrational number is irrational.

To get started, we develop a rational number x and an irrational number y such that their difference $(x-y)$ is rational (the negation of the original premise).

By definition of a rational number, we have $x=\frac{a}{b}$ for some integers a and b with $b \neq 0$ and $x-y=\frac{c}{d}$ for some integers c and d with $d \neq 0$.

Starting with $x-y=\frac{c}{d}$ (*We need to prove this result will give a rational number*)

$$\frac{a}{b}-y=\frac{c}{d} \quad \textit{Substitute for } x$$

$$y=\frac{a}{b}-\frac{c}{d}=\frac{(ad-bc)}{bd} \quad \textit{Add fractions and simplify}$$

Now the result $(ad-bc)$ is an integer (because a, b, c, d are all integers and products and differences of integers are integers), and $bd \neq 0$ (by zero product property). Therefore, by definition of a rational number, y is rational. However, this finding contradicts our original supposition that y is irrational. Hence, the supposition is false and the theorem is true.

Worked example 2

Prove by contradiction that for every prime integer, p, $\sqrt{p}$ is irrational.

Let's start by asserting that the negation of this premise is true, viz. $\sqrt{p}$ is a rational number.

This means that $\sqrt{p}$ can be written as the ratio of two integers, a and b such that:

$$\sqrt{p}=\frac{a}{b},\ b \neq 0 \qquad \text{Equation 1}$$

From this statement we can assume that a and b have no common factors (if there were any common factors, these could be cancelled in both numerator and denominator).

If we square both sides of Equation 1 we obtain:

$$p=\frac{a^2}{b^2} \quad \text{Equation 2}$$

And rearranging gives:

$$pb^2=a^2 \quad \text{Equation 3}$$

which implies that a^2 is a multiple of p.

Furthermore, we can deduce that if a^2 is a multiple of p, then a is also multiple of p (see Question 1 in Exercises). We can therefore write $a=pw$ for some natural number w.

Substituting this value of a into Equation 3 $\left(pb^2=a^2\right)$, we can obtain:

$$pb^2=p^2w^2$$

Dividing both sides of this equation by p gives us

$$b^2=pw^2$$

Now since the RHS of this equation is a multiple of p, so must the LHS. Thus, b^2 is a multiple of p.

Furthermore, it follows from an earlier deduction that if b^2 is a multiple of p, then b is a multiple of p.

It has been shown that that both a and b are multiples of p as they have the common factor of p.

This contradicts the original assumption that $\frac{a}{b}$ was fully cancelled down (or in other words, the only natural number to divide both a and b is 1).

In summary, $\sqrt{p}$ cannot be written as a fraction and hence $\sqrt{p}$ is irrational.

Worked example 3

If n is an integer, then n^2+2 is not divisible by 4.

This generalised statement cannot by proven by the method Proof by Exhaustion since it would involve infinitely many integers. Looking at the Proof by Contradiction method, we commence with the negation of the premise (n is an integer and n^2+2 is divisible by 4) and demonstrate that this negation is false.

If n is an integer and n^2+2 is divisible by 4 we need to consider n as either even or odd.

After considering both cases we can make some conclusions.

1. Assume first that n is even. Then $n=2m$, for some integer m.
2. Thus, $n^2+2=\left(2m\right)^2+2=4m^2+2$
3. Since n^2+2 is divisible by 4, we have that $4m^2+2=4k$, for some integer k.
4. By dividing both sides by 2 we get $2m^2+1=2k$, where k and m^2 are integers.
5. So, there is an odd number that is equal to an even number (the conclusion is false).

1. Assume now that n is odd. Then $n=2m+1$ for some integer m.
2. Thus, $n^2+2=\left(2m+1\right)^2+2=4m^2+4m+3$.
3. Since n^2+2 is divisible by 4, we have that $4m^2+4m+3=4k$, for some integer k.
4. By dividing both sides by 2 we get $2m^2+2m+1.5=2\left(m^2+m\right)+1.5=2k$, k an integer.
5. So again, there is a decimal number that is equal to an even number (false conclusion).

Since both cases lead to a false conclusion we have proven that the original statement is true.

Difficulties with teaching and learning proof by contradiction

The literature base suggests that teaching and learning proof in general is a difficult endeavour for a variety of reasons. According to Lawson (2016, p. 13), proofs are difficult because:

> …it is usually far from obvious how to reach the conclusions from the assumptions. In particular, we are allowed to assume anything that has been previously been proved, which is daunting given the scale of the subject.

More specifically, proof by contradiction is a complex activity for students at various scholastic levels (Antonini, 2008). For instance, research efforts have uncovered how many students experience difficulties in understanding certain aspects of logic. To illustrate, Romano and Strachota (2016) found that secondary school graduates enrolled in first-year university courses struggle to understand the concepts of logical implication and its contraposition. In a study of 202 Chinese students (aged 17–20 years), Lin, Lee and Yu (2003) determined that 80% of their sample were unable to negate the quantifier 'only one' and that more than 70% lacked conceptual understanding of proof by contradiction. Studies conducted by Suppes (1962) and Suppes et al. (1962) revealed that students experience difficulty in recognising invalid proofs, and out of all sections tested that proof by contradiction was the worst scoring section. Antonini and Mariotti (2006) examined the complex relationship between the original statement to be proved (the principal statement) and a new statement (the secondary statement) that is actually proved. In their research, these authors noted that, through questionnaires and interviews conducted with Italian senior secondary students, the generation of false hypotheses in the initial stages of proving can lead to an impasse in the process itself. Proceeding to solve the proof with reasoning based on false assumptions induces cognitive strain, because the student does not know what is or what is not true. This finding supports the work of Durand-Guerrier (2003) who expressed that students who assume false hypotheses ‘block’ the deductive process as they are required to apply mathematical theory to absurd situations.

Reid and Dobbin (1998) suggested that the reasoning underlying proof by contradiction examples is less difficult that it is often thought to be. Underpinning their research is the notion that children use contradictions in playing games and in checking conjectures. On this basis, these authors posited that the difficulties students have with standard proofs by contradiction in mathematics may arise from issues of emotioning (the capacity to care about the decisions they make), especially the need from which their reasoning arises. In earlier work, Damasio (1994) conjectured that any decision people make is a choice between a vast number of possibilities, most of which are not even consciously considered because these have already been rejected by a pre-conscious emotioning process. Reid and Dobbin, as well as other authors (e.g. de Villiers, 1991; Hanna, 1989) have written about the needs which proving can satisfy, especially the need to explain. Reid and Dobbin argued that to prove the irrationality of $\sqrt{2}$, it is rare that the principal need of students is to verify. These authors claim that for students to feel a need to verify, they must first be uncertain of the result; in this example it is likely that there is any uncertainty at all. Rather, there is perhaps some other need than the need to verify which is driving students to prove the irrationality of $\sqrt{2}$. While acknowledging that conceptual difficulties associated with proof by contradiction examples are real, Reid and Dobbin contended it is likely that the need to function within a social context (e.g. proving within a classroom because a teacher has asked it, or to attain good marks) supersedes a need to verify or explain mathematical statements. As such, these authors emphasised that when teachers examine students’ reasoning it should not be done in isolation from their emotioning, and that perhaps those students who do know that $\sqrt{2}$ is irrational are in a better position to prove that it is so.

Conclusion

The purpose of this paper was to offer insight to educators about proof by contradiction as it pertains to the *Australian Curriculum: Mathematics*. In particular, this method of proof has been outlined in a step-by-step fashion, and some worked examples have been offered to amplify these steps and the theoretical approach overall. The review of literature focussed on the importance of proof as well as the affordances and caveats of this proof method within a context of secondary school teaching and learning. As advanced mathematics courses rely heavily on concepts of logic, the place of proof within secondary and tertiary education must remain centrally positioned. With increasing emphasis placed on Science Technology Engineering and Mathematics (STEM)–related careers, students must be explicitly introduced to concepts in proof and logic in order to succeed in STEM academic programs (Romano & Strachota, 2016). It is the author’s hope that this paper will be useful to mathematics educators within Australia and perhaps more broadly, as they model and explain how to apply the method of proof by contradiction to their students. More importantly, it is hoped that as students engage with examples they will not only master the method of *reductio ad absurdum* but concomitantly enhance their procedural understanding of logical implication.

Examples to try with secondary students

1. Prove by contradiction that if a^2 is a multiple of *p*, then *a* is a multiple of *p* (suppose that $a \in \mathbb{Z}$ and *p* is prime).
2. Prove by contradiction that $\sqrt{2}$ is irrational.
3. Using Proof by Contradiction, investigate the statement: “If the square of an integer is odd, then that integer is odd”.
4. For all integers n, prove that if n^3+5 is odd then n is even.
5. Prove by contradiction that if a^2-2a+7 is even, then *a* is odd (suppose that $a \in \mathbb{Z}$).
6. Using Proof by Contradiction, prove for ΔABC that if $\angle A$ is a right angle, then $\angle B$ cannot be an obtuse angle.
7. Prove that for every real number $x \in \left[0, \frac{\pi}{2}\right]$, we have $\sin(x)+\cos(x) \geq 1$.
8. Prove that no integers *a* and *b* exist for which $18a+6b=1$.
9. Prove that there are infinitely many prime numbers.
10. Using Proof by Contradiction, investigate the statement: “There is no greatest even integer”.

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