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Proof of Grünbaum's Conjecture on Common Transversals for Translates

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Abstract. In 1958 B. Grünbaum made a conjecture concerning families of disjoint translates of a compact convex set in the plane: if such a family consists of at least five sets, and if any five of these sets are met by a common line, then some line meets all sets of the family. This paper gives a proof of the conjecture.

1. Introduction

Let K be a compact convex set in the plane, and let F be a family of pairwise disjoint translates of K (also in the plane). We say that F has property $T(k)$ if any k sets from F have a common transversal (i.e., a line meeting those k sets). Then Grünbaum's conjecture (G.C.) can be stated as follows [4]:

> If F has property $T(5)$, and consists of at least five sets, then there is a common transversal for all sets in F.

G.C. is known to hold for circles [1], squares [4], and then, of course, for ellipses and parallelograms. Danzer's paper was motivated by a question of Hadwiger [5], who gave an example (five circles, almost touching and with centers forming a regular pentagon) which shows that G.C. is best possible in the sense that "five" cannot be replaced by "four."

A weaker form of G.C. was recently proved by Katchalski [7]. He assumes $T(128)$ (and $|F| \geq 128$), and proves the existence of a transversal. This result is very nice, not only because it shows that there is a finite number which suffices, but also because it is then fairly easy to see that G.C. is decidable, as it suffices to check whether $T(5)$ implies $T(128)$ or not.

As usual we have to give a reference to [2], which is still a very valuable source for results and problems about transversals. We would also like to draw attention to Eckhoff's thesis [3], with its systematic study of basic transversal theory.

Our proof starts with some standard reductions, showing that if there is a counterexample there is one for which $|F|$ is finite, K is a centrally symmetric polygon, and the centers of the translates are convexly independent. In Section 3 it is shown that we can assume $|F|=6$, and in Section 4 the hexagon of centers is studied more closely. Then follows a discussion of which single partial transversals can exist (for five sets). In Section 6 we give rules which show that certain pairs of partial transversals cannot coexist. The results from Sections 5 and 6 are used to exclude myriads of combinations of six partial transversals, and in Section 7 the nonexcluded combinations are dealt with. The exclusion was made (independently) by hand and by computer. The computer work was organized by my colleague, Dr. Svein Mossige, who deserves a double thanks here, as it was the preparation for the computer work which provided the insight that made the checking by hand a feasible task.

2. Some Standard Reductions

Let a counterexample be given as $\{K+c_i;\, i\in I\}$, where K contains the origin. Then c_i can be thought of as a translation vector, or a point in $K+c_i$, at will. K is two-dimensional, as otherwise even $T(3)$ would suffice.

Let S be a point set in the plane. Then we define the K-height of S in the direction D as the quotient of the length of the orthogonal projection of S on a line in the direction D by the length of the projection of K on the same line. The condition $T(5)$ then clearly means: for any $i_1<\cdots<i_5$, the set $\{c_{i_1},\ldots,c_{i_5}\}$ has K-height ≤ 1 in some direction $D=D(i_1,\ldots,i_5)$. The fact that F is a counterexample means that $\{c_i\colon i\in I\}$ has K-height >1 in all directions, but for any given direction some finite set of c_i's will already have K-height >1 in that direction, and in an open set of neighboring directions. The compactness of S^1 thus shows that a finite subset of the c_i's has K-height >1 in all directions, i.e., we may assume $|F|=N<\infty$. (In view of Katchalski's result we could of course assume $N<129$ straight away.)

Now to symmetrization, which means to replace K by $\frac{1}{2}(K-K)$, while conserving the c_i. Projecting K and $\frac{1}{2}(K-K)$ on a line, we get intervals of equal lengths, so K-height and $\frac{1}{2}(K-K)$-height means the same for all directions. This means that any subfamily of $\{K+c_1,\ldots,K+c_N\}$ has a transversal in exactly those directions in which the corresponding subfamily of $\{\frac{1}{2}(K-K)+c_1,\ldots,\frac{1}{2}(K-K)+c_N\}$ has one. As two sets intersect if and only if they have a common transversal in every direction, the disjointness of the $K+c_i$ is also preserved on replacing K by $\frac{1}{2}(K-K)$. We may thus assume K to be centrally symmetric.

Remark. Symmetrization has another nice property which, though not essential to our proof of G.C., may be useful in studying geometric permutations of families of translates (see Section 6). Namely, a family and the symmetrized family admit exactly the same geometric permutations. The reader will easily see this by considering what happens as K changes into $\frac{1}{2}(K-K)$ through the sets $(1-t)K-tK, 0\leq t\leq\frac{1}{2}$, or by making an alternative calculation.

We may also assume K to be a polygon. For if we replace K by a slightly larger centrally symmetric polygon K', the K-heights of $\{c_1,\ldots,c_N\}$ will decrease so little that we will still have a counterexample.

It will be useful to have some independence conditions on the centers c_i. Firstly, we may assume that the $\binom{N}{2}$ directions c_ic_j, $i\neq j$, are all different. For we can replace K by $(1+\varepsilon)K$, for some small $\varepsilon>0$, and still have a counterexample. Then we are sure of enough freedom to move the centers c_i a little so that coinciding directions can be eliminated. Secondly, the c_i may be assumed convexly independent. For $\{c_1,\ldots,c_N\}$ and $\operatorname{conv}\{c_1,\ldots,c_N\}$ have the same K-height in any direction D, so the extremal points of $\operatorname{conv}\{c_1,\ldots,c_N\}$ will already furnish a counterexample to G.C.

3. Reduction to the Case $N=6$

The idea of this reduction is to manufacture, from the counterexample under study, a counterexample where some five of the sets have a unique common transversal T. Then, since we have a counterexample, some sixth set does not meet T. But the uniqueness of T shows that T would be the only common transversal for these six sets. It follows that these six sets already form a counterexample.

The manufacturing mentioned above is effected by Hadwiger's shrinking process [6], i.e., we replace, in our example, K by λK, where $\lambda\in[0,1]$, and get what we want by choosing λ as the infimum of those λ's for which $\lambda K+c_1,\ldots,\lambda K+c_N$ form a counterexample. This procedure will work when we have modified the given counterexample as follows.

First we arrange it so that K has a pair of parallel sides in each of the directions c_ic_j. This is done, for one direction at a time, by cutting two small triangles off K (or rather that mutilated version of the original K which we have at that stage). The triangles have to be small, so that no 5-tuple of sets loses a transversal in the process. It may also be necessary to blow up K a little before starting the cutting.

We now add, on each of the $2\binom{N}{2}$ sides introduced, an isosceles triangle with base on that side and height h_{ij} where opposite triangles get the same height so that symmetry is preserved. It is clear that there is a $\delta>0$ so that if the $\binom{N}{2}$ values h_{ij} are in $[0,\delta]$, we will still have a counterexample given by $K=K(h_{12},h_{13},\ldots)$ and $c_1,\ldots,c_N$. The projection of $K(h_{12},h_{13},\ldots)$ on a line orthogonal to c_ic_j will equal ($2h_{ij}$ + the corresponding length for K) (still provided δ is small). So this length, l_{ij}, is a free variable in a certain interval, and the $\binom{N}{2}$ variables are independent.

In particular, when given finitely many nonzero polynomials in these variables, we can find values of them for which none of the polynomials vanish. We choose the polynomials $d(c_t, c_r, c_s)l_{ij} - d(c_k, c_i, c_j)l_{rs}$, where $d(c_x, c_y, c_z)$ is the distance from c_x to the line c_yc_z, and we choose only the combinations where $|\{r, s, t\}| = |\{i, j, k\}| = 3 \leq |\{i, j, r, s\}|$.

We assume that the l_{ij} have been chosen so that none of our polynomials vanish. Then Hadwiger's procedure will work. The critical value of λ will be the largest one for which at least one 5-subset of $\{c_1, \ldots, c_N\}$ has (λK)-height≥ 1 in all directions, with equality at least once. Let $\{c_1, \ldots, c_5\}$, say, be such a set. As its (λK)-height is 1 in some direction, $K + c_1, \ldots, K + c_5$ have a transversal T in the orthogonal direction. We claim that T is the unique transversal for $\{c_1, \ldots, c_5\}$. Firstly, T must be tangent to at least three of the five sets $\lambda K + c_1, \ldots, \lambda K + c_5$. For otherwise T could be moved a little, so as to meet the five sets in interior points, and then $\{c_1, \ldots, c_5\}$ would have λK-height< 1 in some direction. Secondly, T cannot be tangent to more than three of the sets. For there would then be either three collinear points among $c_1, \ldots, c_5$, or two parallel segments defined by four of these points.

Let T be tangent to, say, $\lambda K + c_1$, $\lambda K + c_2$, and $\lambda K + c_3$, meeting the sets in this order. Then T separates $\lambda K + c_2$ from $\lambda K + c_1$ and $\lambda K + c_3$, as otherwise a contradiction could again be obtained by moving T a little. Then we clearly have (as $d(c_2, c_1, c_3)$ denotes the distance from c_2 to aff$(\{c_1, c_3\})$)

$$d(c_2, c_1, c_3)/\lambda l_{13} = 1$$

so that $\lambda = d(c_2, c_1, c_3)/l_{13}$. This means that T is the unique transversal for $\lambda K + c_1, \ldots, \lambda K + c_5$. For there is no other one parallel to T, and one in a different direction would give $\lambda = d(c_x, c_y, c_z)/l_{yz}$ for some (x, y, z) with $\{y, z\} \neq (1, 3)$, which would make one of the nonvanishing polynomials, described above, vanish.

4. The Four Possible "Shapes" of the Hexagon of Centers

Consider a convex n-gon P, having no two sides parallel. Assign to each side of P the unique vertex of P which has maximal distance from the line through that side, the *opposite* vertex of the side. The n-gon P' will be said to have the same *shape* as P if there is a bijection between the vertices of P and those of P', and also one between the edges, so that incidence and "opposition" between edges and vertices are preserved. Shape is an affine invariant and it will be important to us further on.

For a given n, all possible shapes are obtained as follows: start with a regular k-gon Q, for some *odd* $k \leq n$, and a distribution of $n - k$ points, $q_1, \ldots, q_{n-k}$, on the sides of Q. Then choose $n - k$ points $p_1, \ldots, p_{n-k}$, one near each q_i, so that

$\operatorname{conv}(Q \cup \{p_1, \ldots, p_{n-k}\})$ becomes a convex n-gon P. If the p_i are sufficiently near the q_i, only the vertices of Q will be opposite to the sides of P and the shape of P will only depend on the numbers of q_i's on the various sides of Q.

We prove by induction on n that all shapes are obtained as asserted. The case $n=3$ is trivial, so assume $n>3$. Choose, if possible, a vertex, say $(0,0)$ (with adjacent vertices $(0,1)$ and $(1,0)$), which is not opposite any side of P. Let u be the vertex opposite the side $(0,0)$-$(1,0)$, i.e., the highest vertex of P. Then u is also the rightmost vertex of P (i.e., opposite $(0,0)$-$(0,1)$). For otherwise the side uv (where v succeeds u in the clockwise order around P) would have a negative slope, so that its parallel through $(0,0)$ would exhibit $(0,0)$ as being the vertex opposite uv.

If we now omit the vertex $(0,0)$ from P, we get a convex $(n-1)$-gon P' where each side, except $(0,1)$-$(1,0)$, has the same opposite vertex as it had in P. $(0,1)$-$(1,0)$ clearly has the opposite vertex u. Having represented the shape of P' in the desired way, we can obviously add a vertex and get a representation for P.

If each vertex of P is opposite some side of P, we have to check that n is odd and that P has the shape of a regular n-gon. Consider an arbitrary vertex, say $(0,0)$, with its opposite side being $(1,-1)$-$(1,1)$. Let P have a sides in the first quadrant and $b\,(=n-a-1)$ sides in the fourth quadrant. A side in the first quadrant will have its opposite vertex in the fourth quadrant, as a tangent to P through the vertex and parallel to the side will have to run strictly below $(0,0)$ and $(1,0)$. Thus $a \le b$, which, together with the converse inequality, proves our assertion.

In the case $n=6$ it follows that there are three possible shapes with $k=3$ and one with $k=5$.

We list below, for each shape, a corresponding *shape sequence*. This is obtained by naming the vertices around the polygon as $1, \ldots, 6$, and forming the sequence $j(1), \ldots, j(6)$, where $j(i)$ is the vertex opposite the side $ii+1$.

S_1: 5, 5, 1, 1, 3, 3. $\quad$ S_2: 5, 5, 6, 1, 3, 4.

S_3: 3, 5, 2, 2, 3, 3. $\quad$ S_4: 3, 4, 2, 3, 3, 3.

In the next section we see how the shape sequence restricts the possibilities for the partial transversals of a given family F.

5. The Possibilities for Individual Partial Transversals for F

Consider disjoint convex sets $K_1, \ldots, K_n$ and a transversal T, meeting them in the order $i_1, \ldots, i_n$. Then, according to Katchalski *et al.* [9], we say that the sets admit the geometric permutation $(i_1, \ldots, i_n)$, and also the geometric permutation $(i_n, \ldots, i_1)$, which we identify with the first one. If we have more information about the sets, some of the $\frac{1}{2}n!$ initial possibilities can often be excluded. If we, for instance, for $n=4$, let $K_1, \ldots, K_4$ be the sets $K+c_1, \ldots, K+c_4$ from our counterexample, then (1324) is not possible. For if it occurs, we can shrink K (as in Section 3), so as to get in the situation where a transversal T (inducing 1324) separates one set from two of the others. Up to symmetry, only two cases are possible: T separates $K+c_2$ from $K+c_1$ and $K+c_4$, or from $K+c_3$ and $K+c_4$. Drawing a picture, the reader will realize that in each case the cyclic order of the c_i is violated.

It follows, by symmetry, that (2431), (3142), and (4213) are also excluded. If we represent the permutation (1324) by a 3-edge path with edges c_1c_3, c_3c_2, and c_2c_4, and the 11 others in a similar way, we find that the forbidden ones are those for which the path crosses itself.

Consider now five of the sets $K+c_1, \ldots, K+c_6$, and a permutation of them, represented by a 4-edge self-crossing path. An edge-crossing here involves only four of the sets, and so we find that this permutation cannot be a geometric one, as it induces a nongeometric one on four of the subsets. It turns out that 40 of the 60 candidates for a geometric permutation are excluded in this way.

But there are further limitations on the geometric permutations possible in the present situation. Consider, for example, the case when the shape sequence is 5, 5, 1, 1, 3, 3. Then we know that no geometric permutation for five of the sets can be of the form (... 1 ... 5 ... 2 ...). For if that were the case, then that common, nonseparating tangent T for $K+c_1$ and $K+c_2$ which is not a support line for $\operatorname{conv}(K+c_1 \cup \cdots \cup K+c_6)$, would meet $K+c_5$. But since the first element of the shape sequence is 5, we know that $K+c_5$ is the set which is furthest away from the support line just mentioned. It follows that $K+c_3$, $K+c_4$, and $K+c_6$ also meet T, and our counterexample fails.

Thus each term of the shape sequence limits the possibilities for three of the six 5-set transversals, and the possible number of combinations goes down from 20^6 to $4^3 \times 12^3$. For S_2, S_3, and S_4 we get the numbers $4\times 8^4 \times 12$, $4^3 \times 8 \times 12 \times 16$, and $4 \times 8^3 \times 20^2$, respectively.

6. Incompatible Pairs of Geometric Permutations

If A, B, X, and Y are disjoint convex sets in the plane, then $ABXY$ and $BAYX$ cannot both be geometric permutations, as was observed in [9]. The reader will readily prove this, as one is immediately reduced to the case when $A, \ldots, Y$ are (≤ 1)-dimensional. Below we list this incompatible pair, together with the other types of incompatible pairs which we have used in our proof:

I_1: $ABXY$, $BAYX$. $\quad$ I_2: $AXBY$, $AYCX$.

I_3: $AXPYB$, $YABX$. $\quad$ I_4: $AXYZ$, $AYPZX$.

I_5: $AXYPZ$, $AYZX$.

I_1 is valid for arbitrary convex sets. In I_2 we must have some restriction on the sets, and we have assumed them to be translates of one set. For I_3, I_4, and I_5 we use the fact that the pair in question comes from our counterexample.

Before starting the proofs of I_2–I_5, we remark that I_2 (for $B=C$) was also discovered by Katchalski *et al.* [8]. In each case we assume that a line L induces the first permutation listed, and a line M the second one, and deduce a contradiction. We start with some observations which are valid in all four cases.

Firstly, the sets may be assumed to be squares. For assume (temporarily) L to be horizontal, and consider the two horizontal tangents to one of the sets meeting L. By central symmetry, the set will have at least one central chord meeting both tangents. This chord meets L, and in each of the other translates meeting L, the parallel central chord will also meet L. The permutation induced on the chords corresponds, of course, to the one on the sets.

Now the horizontality of L was irrelevant (but linguistically convenient), and so we conclude that there will be another set of parallel central chords on which M induces the "same" permutations as the one given on the sets. Thus, cutting each set down to a parallelogram spanned by a pair of central chords, we find that our sets may be assumed to be parallelograms. These are nondegenerate, as a set of parallel segments admits only one geometric permutation. After an affine transformation we can thus assume that the sets are squares, with horizontal and vertical sides.

If we consider three squares, A, X, and Y, say, there are in a certain sense (to be discussed below) 468 possibilities for their relative positions. Assuming that L, say, induces the permutation AXY and M the permutation AYX, we see that there are just 16 possibilities. These are the two depicted in Fig. 1 and the ones obtained from them by a rotation and/or a reflection.

Relative positions are defined as follows. "X is to the right of A" means that X is strictly separated from A by a vertical line to the left of X. Similar expressions are also used with "left," "below," and "above." "X is further right than Y" (as in Fig. 1(a)) means that the center of X is further right than that of Y. Whether X is to the right of Y or not, is left open. Similar expressions are also used with "further left," "lower than," and "higher than." These expressions suffice to define relative positions for our purpose. (Note that by increasing and moving the squares a little, we can avoid ambiguous cases with collinear pairs of square sides.)

The figures are meant to give full information on relative positions. Thus Fig. 1 tells us that X is lower than A, and *also* not below A.

For the proof of the assertion about the 16 possibilities for X, Y, and A, we can clearly normalize by assuming that X is to the right of A and lower than A,

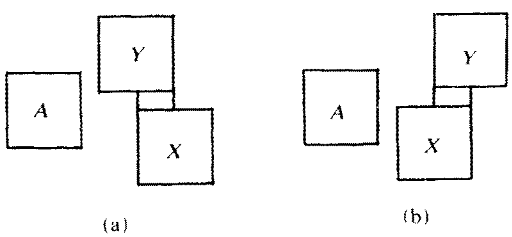

Fig. 1

and then prove that one of the situations of Fig. 1 must occur. First note that the permutation AXY shows that X meets $\mathrm{conv}(A \cup Y)$ and, *a fortiori*, the minimal rectangle, with horizontal and vertical sides, containing A and Y. Similarly Y meets the rectangle spanned by A and X. Thus, as X is to the right of A, Y can be neither to the left nor to the right of X. Since X is lower than A, it follows that Y is neither below X (nor above A). Thus Y is above X, which shows that A is not also above X, and so we have one of the situations of Fig. 1. Note that in the situation depicted in Fig. 1 L must be ascending and M must be descending.

Each of the cases I_2–I_5 is normalized by assuming that X is to the right of A and lower than A. Considering A, X, and Y in cases I_2, I_4, and I_5, and A, X, and B in case I_3, we see that L is ascending and M is descending.

The proof of I_2 is now immediate from Fig. 1. If we have the situation in Fig. 1(a), then L cannot exist. For the part of L which is assumed to run from X to Y would have to be contained in the rectangle shown as connecting X to Y. But this rectangle cannot meet B, as its height is smaller than the side-length of B, and our squares are disjoint. Similarly, in the situation in Fig. 1(b), M, which is descending, must meet C somewhere in the rectangle connecting Y and X, which cannot meet C, however.

In case I_3, with $AXPYB$ induced by L, and $YABX$ by M, we already know the two possibilities for A, X, and B (see Fig. 2). But, in the situation in Fig. 2(b), XYB (induced by L) and XBY (induced by M) show that Y must be to the right of B, so L and M are both impossible.

In the situation in Fig. 2(a) (for A, X, and B), it follows from YXA and YAX that Y is above or below both A and X (since A and X are separated vertically). But Y cannot be below X because of L which is ascending from A to X to Y. Thus Y is above A and X, and the situation will be as in Fig. 2(a) except that Y might be above B (and then, possibly, not to the left of B). But with Y above B, L cannot exist. Thus Fig. 2(a) is correct. The rectangle meeting just Y and X indicates where L must pass from X to Y, and hence meets P. The other four rectangles, connecting the pairs AY, YB, BX, and XA, cannot meet P and P is then clearly contained in $\mathrm{conv}(A \cup Y \cup B \cup X)$. But this cannot happen, as the centers of the squares are assumed to be convexly independent.

Figure 1 shows the possibilities for A, X, and Y also in case I_4, when $AXYZ$ is induced by L and $AYPZX$ by M. If we replace Y by Z, it becomes valid for

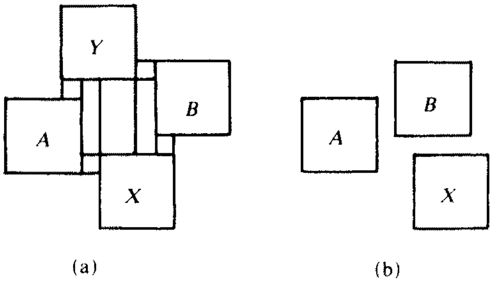

Fig. 2

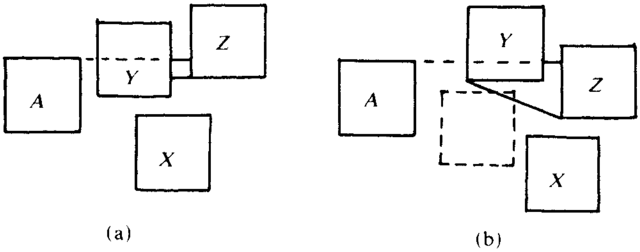

Fig. 3

A, X, and Z. Since Y and Z are both above X, but not above A, they must be separated vertically and hence, because of L, Y is to the left of Z. The two alternatives are shown in Fig. 3.

The situation in Fig. 3(a) is impossible as P cannot meet the rectangle between Y and Z, as required by M. In the situation in Fig. 3(b), P must meet the nonconvex pentagon connecting Y to Z. But only that part of it which is to the left of X is accessible to P and so P must be higher than and to the left of X. P must also be below Y (and thus lower than A), but it cannot be below A, and so it is to the right of A. P is thus situated as the dashed square in Fig. 3(b). We now use the fact that L and M are assumed to be two of the six partial transversals from our counterexample. Thus they do not meet the same five sets, and L does not meet P. But if L passes above P, it misses X, and if it passes below P, it misses A.

Figure 3 gives the possibilities for A, X, Y, and Z also in the case of I_5, when $AXYPZ$ is induced by L, and $AYZX$ by M. But now we cannot have the situation in Fig. 3(b), as there the part of L between Y and Z cannot meet P.

Figure 4 shows the case of Fig. 3(a), with P' and P'' showing the two alternatives for P (the domain connecting X, Y, and Z being inaccessible). Figure 4 is not completely exact, as it shows P' as being further right than Y, whereas P' might in fact be further left than Y.

If $P = P'$ we use the fact that the centers of the squares are convexly independent, i.e., Y is not in $\operatorname{conv}(A \cup X \cup Z \cup P')$. But Y is obviously "inside" seven of the lines defining this octagon. The eighth one is the common tangent (as

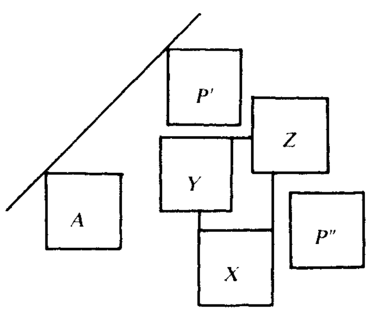

Fig. 4

drawn) to A and P', which also passes above Y, as A is to the left of and not below Y, while P' is above Y and not to the right of Y.

If $P = P''$ we use the fact that M does not meet P (just as L did not meet P in the case of I_4). As M is descending and meets X, it must pass below P''. Thus M meets the left vertical tangent to Z above P'' and the right one below P'', so that slope $(M) < -1$. On the other hand, M must leave and enter Y through its vertical sides, so that slope $(M) > -1$. This contradiction ends the discussion of I_5.

7. End of the Proof

Consider a counterexample F to G.C., having a given shape sequence for the hexagon of centers. The six partial transversals of F induce six geometric permutations, one for each five of the sets in F. From Section 5 we know which possibilities there are for each of these permutations, and in Section 6 we proved that certain types of pairs of permutations cannot coexist. Using this knowledge we checked by hand and, independently, by computer, which sixtuples of permutations are possible. The computer work was organized as follows. For a given shape sequence, six lists $L_1, \ldots, L_6$ of possible permutations were produced, and also 15 tables T_{ij}, showing which combinations of a permutation from L_i and one from L_j are incompatible. We now go through all possible combinations of six permutations, one from each L_i. If, for a given combination $\pi_1, \ldots, \pi_6$, say, i is the first index for which π_i is incompatible with some $\pi_j, j < i$, then this combination is discarded, simultaneously with all combinations of the form $\pi_1, \ldots \pi_i, \vartheta_{i+1}, \ldots, \vartheta_6$. The computer work took less than 5 minutes CPU-time on the Univac 1100/82 of the University of Bergen. Only two sixtuples come out:

$$3\,2\,4\,5\,6,\quad 4\,3\,5\,6\,1,\quad 5\,4\,6\,1\,2,\quad 6\,5\,1\,2\,3,\quad 1\,6\,2\,3\,4,\quad 2\,1\,3\,4\,5$$

and

$$2\,3\,4\,6\,5,\quad 3\,4\,5\,1\,6,\quad 4\,5\,6\,2\,1,\quad 5\,6\,1\,3\,2,\quad 6\,1\,2\,4\,3,\quad 1\,2\,3\,5\,4,$$

and they both occur when the shape sequence is S_1 or S_2.

We have to prove that these sixtuples cannot arise after all. Consider one of them, and consider the set $K + c_2$, say, and the set

$$K_2 = \operatorname{conv}(K + c_1 \cup K + c_3 \cup K + c_4 \cup K + c_5 \cup K + c_6).$$

The fifth permutation shows that $K + c_2$ meets $\operatorname{conv}(K + c_1 \cup K + c_3)$. The convex independence of the c_i, and their cyclic ordering, then shows that $K + c_2$ meets that boundary segment of K_2 which connects $K + c_1$ to $K + c_3$. $K + c_2$ thus penetrates partly into K_2 between $K + c_1$ and $K + c_3$.

Consider now the standard triangular lattice in the plane. We may assume that c_1, c_3, and c_5 are lattice points at pairwise distances of two lattice units, and that c_1c_3 is horizontal, with c_5 above c_1c_3 (see Fig. 5). Let $H + c_1$ be the regular hexagon formed by the lattice triangles around c_1, let L be the lower horizontal

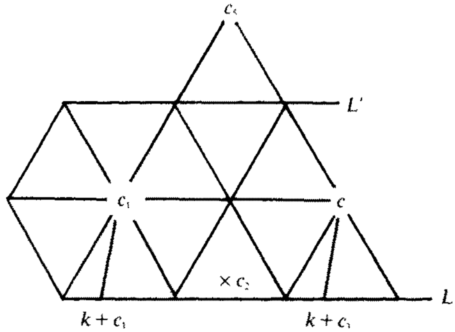

Fig. 5

tangent to $H+c_1$ and let $k+c_1$ be a point in $(K+c_1)\cap L$ (its existence being assumed for the moment). Then the area of the parallelogram with vertices $k+c_1$, $k+c_3$, c_1, and c_3 equals four (triangular units). But we shall see that, of that area, $K+c_1$ and $K+c_3$ together cover at least two units. Also (provided we have made the right choice) $K+c_2$ covers at least two units, which contradicts the disjointness of our sets.

In order to see that $K+c_1$ meets L, it suffices, by the central symmetry of K, to see that $K+c_1$ meets the reflection of L in c_1, i.e., the line L' through $\frac{1}{2}(c_1+c_5)$ and $\frac{1}{2}(c_3+c_5)$. But if $K+c_1$ were below L' then $K+c_5$ would lie above it, by symmetry, and so L' would separate $K+c_5$ from $K+c_1$ and $K+c_3$. This contradicts the existence of the second permutation, however, and so the point $k+c_1$ really exists.

The point $k+c_1$ even belongs to the lower horizontal side of $H+c_1$, not just to L. For if it were outside that side, then it is obvious that $-k+c_5$ would not be between $-k+c_1$ and $-k+c_3$, so that L' would induce a permutation different from 1, 5, 3. But the second permutation shows that the upper common horizontal tangent to $K+c_1$ and $K+c_3$ induces 1, 5, 3. This is a contradiction, as parallels cannot induce different permutations.

We have just seen that $K+c_1$ meets the horizontal sides of $H+c_1$. Symmetry then shows that the positive-sloped sides are also met by $K+c_1$. A similar argument to the one above shows that $K+c_5$ meets the lower horizontal side of $H+c_5$, and then symmetry shows that $K+c_1$ also meets the negative-sloped sides of $H+c_1$.

The fact that K meets the sides of H in points $\pm k$, $\pm k'$, and $\pm k''$ implies that $K\cap H$ has an area of at least four triangular units. For the area of the hexagon with vertices $\pm k$, $\pm k'$, and $\pm k''$ is a linear function of the parameter which describes the position of one pair of vertices, while the other pairs are constant. Thus it achieves its minimum (at least) when each of k, k', and k'' is a vertex of H.

Consider now the parallelogram with vertices c_1, c_3, $k+c_1$, and $k+c_3$. That part of it which is covered by $K+c_1$ and $K+c_3$ equals exactly half of the area of $K\cap H$, i.e., at least two triangular units. The part which is covered by $K+c_2$ will depend on the position of c_2. We shall see below that c_2 can be assumed to lie above L. The area we are interested in is that part of $K+c_2$ which lies in the strip between L and the line c_1c_3. The symmetry of K shows this area to be a (weakly) decreasing function of the distance from c_2 to the center of the strip. Thus its value is at least two triangular units, as we get at least that when c_2 lies on L.

It only remains to show that c_2, our chosen center, can be assumed to lie above L, i.e., that in a barycentric coordinate system, with basis c_1, c_3, and c_5, the c_5-coordinate of c_2 is greater than $-\frac{1}{2}$. We prove a little bit more, namely that for some i, the c_{i+3}-coordinate of c_i, with reference to the basis $c_{i-1}, c_{i+1}, c_{i+3}$ (everything is read mod 6, of course) equals at least $-\frac{1}{3}$. And if the largest of the c_{i+3}-coordinates equals $-\frac{1}{3}$, then they are all equal and $\operatorname{conv}\{c_1, \ldots, c_6\}$ is the affine image of a regular hexagon.

Affine invariance shows that we may assume c_1, c_3, and c_5 to be fixed points, and then minimize the maximal one of the six coordinates we are interested in, letting c_2, c_4, and c_6 vary under the given conditions. These say that $c_1, \ldots, c_6$ are the successive vertices of a nondegenerate convex hexagon.

The minimum in question must be assumed. For if c_2, c_4, and c_6 stay bounded, and c_2, say, approaches the line c_1c_3, then its c_5-coordinate tends to 0. And if c_2 approaches the line c_3c_5, say, while staying away from c_1c_3, convexity forces c_4 toward c_3c_5. Possible unboundedness of, say, c_2 is dealt with during the computation to follow.

Let A be a 3×3 matrix where a_{ij} is the c_{2j+3}-coordinate of the point c_{2i}, and let B be its inverse. The six numbers we are interested in are the diagonal elements of A and B, and we want to minimize their maximum. The conditions imply that A and B have negative diagonal elements and positive off-diagonal elements. Thus the equations

$$a_{i1}b_{1i} + a_{i2}b_{2i} + a_{i3}b_{3i} = 1, \qquad i = 1, 2, 3,$$

show that if $a_{ii} < -3$, then $b_{ii} > -\frac{1}{3}$, which deals with unboundedness, as promised. Hence our minimum is assumed, and we go on to consider a corresponding matrix pair A and $B\,(=A^{-1})$.

Firstly, $a_{11} = a_{22} = a_{33} = b_{11} = b_{22} = b_{33}$. For if a_{11}, say, is the smallest of these numbers, and they are not all equal, then we can increase a_{11} by moving the point c_2 a little toward c_5, along the segment c_2c_5. Then a_{22} and a_{33} do not change, while b_{11}, b_{22}, and b_{33} decrease. A slight move of any of the points c_i, $i = 1, 3, 5$, toward c_{i+3}, makes a_{22} and a_{33} decrease, too, and minimality is contradicted.

Put $a_{11} = \cdots = b_{33} = -x$. Then since $B = A^{-1}$, we have $x^2 - a_{23}a_{32} = x^2 - a_{31}a_{13} = x^2 - a_{12}a_{21}$, so that, since the row sums of A equal 1,

$$a_{23}(1 + x - a_{31}) = a_{31}(1 + x - a_{12}) = a_{12}(1 + x - a_{23}).$$

Expressing a_{12} in terms of a_{23} and a_{31}, and substituting this in the middle term above, we get from the first equation

$$(a_{23} - a_{31})((1 + x - a_{23})(1 + x) + a_{31}a_{23}) = 0.$$

But here the second factor is positive, as $1+x-a_{23}=a_{21}>0$, and so $a_{23}=a_{31}=a$, say. By symmetry $a_{12}=a$, and, also, $b_{23}=b_{31}=b_{12}=b$, say. Thus A and B are circulants with first rows $-x, a, 1+x-a$ and $-x, b, 1+x-b$, respectively. $AB=I$ now gives

$$x^2+a(1+x-b)+(1+x-a)b=1,$$

$$-bx-ax+(1+x-a)(1+x-b)=0,$$

i.e., $a+b=1+x$, $ab=x(1+x)$. Hence $a=\frac{1}{2}(1+x\pm\sqrt{(1+x)(1-3x)})$ so that $x\leq\frac{1}{3}$. But this was our main assertion. The additional assertion about the extremal case also follows from our argument.

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