

**PROOF OF THE 1-FACTORIZATION AND HAMILTON
DECOMPOSITION CONJECTURES IV: EXCEPTIONAL SYSTEMS
FOR THE TWO CLIQUES CASE**

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ABSTRACT. In a sequence of four papers, we prove the following results (via a unified approach) for all sufficiently large n :

- (i) [*1-factorization conjecture*] Suppose that n is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D -regular graph G on n vertices has a decomposition into perfect matchings. Equivalently, $\chi'(G) = D$.
- (ii) [*Hamilton decomposition conjecture*] Suppose that $D \geq \lfloor n/2 \rfloor$. Then every D -regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.
- (iii) We prove an optimal result on the number of edge-disjoint Hamilton cycles in a graph of given minimum degree.

According to Dirac, (i) was first raised in the 1950s. (ii) and (iii) answer questions of Nash-Williams from 1970. The above bounds are best possible. In the current paper, we prove results on the decomposition of sparse graphs into path systems. These are used in the proof of (i) and (ii) in the case when G is close to the union of two disjoint cliques.

1. INTRODUCTION

1.1. Background and results. In a sequence of four papers, we develop a unified approach to prove the following results on Hamilton decompositions and 1-factorizations. The first of these results confirms the so-called 1-factorization conjecture for all sufficiently large graphs. (A *1-factorization* of a graph G consists of a set of edge-disjoint perfect matchings covering all edges of G .) This conjecture was first stated explicitly by Chetwynd and Hilton [1, 2]. However, they wrote that according to Dirac, it was already discussed in the 1950s.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D -regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

The bound on the degree in Theorem 1.1 is best possible. Nash-Williams [9, 10] raised the related problem of finding a Hamilton decomposition in an even-regular graph. Here a decomposition of an (even-regular) graph G into Hamilton cycles

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consists of a set of edge-disjoint Hamilton cycles covering all edges of G . If G is a regular graph of odd degree, it is natural to ask for a perfect matching in G together with a decomposition of the remaining edges into Hamilton cycles.

Theorem 1.2. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor n/2 \rfloor$. Then every D -regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

Again, the bound on the degree in Theorem 1.2 is best possible and so the theorem confirms the conjecture of Nash-Williams for all sufficiently large graphs.

Finally (in combination with [6]), we also prove an optimal result on the number of edge-disjoint Hamilton cycles one can guarantee in a graph of given minimum degree, which (as a special case) answers another question of Nash-Williams. For a detailed discussion of the results and their background we refer to [7].

1.2. Overall structure of the argument. For all of our main results, we split the argument according to the structure of the graph G under consideration:

- (i) G is close to the complete balanced bipartite graph $K_{n/2, n/2}$;
- (ii) G is close to the union of two disjoint copies of a clique $K_{n/2}$;
- (iii) G is a ‘robust expander’.

Roughly speaking, G is a robust expander if for every set S of vertices, the neighbourhood of S is at least a little larger than $|S|$, even if we delete a small proportion of the edges of G . The main result of [8] states that every dense regular robust expander has a Hamilton decomposition. This immediately implies Theorems 1.1 and 1.2 in Case (iii).

Case (i) is proved in [3]. Most of the argument for Case (ii) is contained in [7], which also includes a more detailed discussion of the overall structure of the proof. Some of the results needed for Case (ii) (on decompositions into ‘exceptional path systems’) are proved in the current paper. Case (ii) is by far the hardest case for Theorems 1.1 and 1.2, as the extremal examples are all close to the disjoint union of two cliques. The arguments in [3, 7] make use of an ‘approximate decomposition’ result, which is proved in [4].

1.3. Contribution of the current paper. As mentioned above, the current paper is concerned with Case (ii), i.e. when G is close to the union of two cliques. More precisely, we say that a graph G on n vertices is ε -close to the union of two disjoint copies of $K_{n/2}$ if there exists $A \subseteq V(G)$ with $|A| = \lfloor n/2 \rfloor$ and such that $e(A, V(G) \setminus A) \leq \varepsilon n^2$.

We will prove results which are used in [7] to prove the following theorem, which is a common generalization of Theorems 1.1 and Theorems 1.2 in Case (ii). Essentially, this theorem guarantees a decomposition into Hamilton cycles and perfect matchings which contains as many Hamilton cycles as possible.

Theorem 1.3. *For every $\varepsilon_{\text{ex}} > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that $D \geq n - 2\lfloor n/4 \rfloor - 1$ and that G is a D -regular graph on n vertices which is ε_{ex} -close to the union of two disjoint copies of $K_{n/2}$. Let F*

be the size of a minimum cut in G . Then G can be decomposed into $\lfloor \min\{D, F\}/2 \rfloor$ Hamilton cycles and $D - 2\lfloor \min\{D, F\}/2 \rfloor$ perfect matchings.

When constructing the Hamilton cycles (and perfect matchings) guaranteed by Theorem 1.3, a crucial step is to obtain a decomposition of the ‘exceptional edges’. To define exceptional edges, we consider a suitable partition of $V(G)$ into sets A, A_0, B, B_0 so that A and B induce almost complete graphs on close to $n/2$ vertices and A_0, B_0 contain the (small number of) ‘exceptional vertices’ which have many neighbours in both $A' := A \cup A_0$ and $B' := B \cup B_0$. The exceptional edges are all those edges incident to A_0 and B_0 as well as all those edges joining A' to B' . These exceptional edges will be decomposed into ‘exceptional (path) systems’, and each such exceptional system will be extended into a Hamilton cycle. (Actually, the exceptional systems may contain some non-exceptional edges as well.)

The exceptional systems are constructed in the current paper. If we want to extend an exceptional system into a Hamilton cycle, one obvious necessary property is that the exceptional system needs to contain two independent edges between A' and B' . Another requirement will be that these exceptional systems are ‘localized’, i.e. given a partition of A and B into clusters, each exceptional system uses only vertices from $A_0 \cup B_0$ as well as from one of the clusters in both A and B . Some further constraints are due to the overall structure of the argument, which we outline below.

In [4], we show how one can extend a suitable set of exceptional systems to obtain an approximate decomposition of G , i.e. a set of edge-disjoint Hamilton cycles covering almost all edges of G . However, one does not have any control over the ‘leftover’ graph H , i.e. it is not clear how to extend this into a decomposition. In [8] this problem was solved by introducing the concept of a ‘robustly decomposable graph’ G^{rob} . Roughly speaking, this is a sparse regular graph with the following property: given *any* very sparse regular graph H with $V(H) = V(G^{\text{rob}})$ which is edge-disjoint from G^{rob} , one can guarantee that $G^{\text{rob}} \cup H$ has a Hamilton decomposition. This leads to a natural (and very general) strategy to obtain a decomposition of G :

- (1) find a (sparse) robustly decomposable graph G^{rob} in G and let G' denote the leftover;
- (2) find an approximate Hamilton decomposition of G' and let H denote the (very sparse) leftover;
- (3) find a Hamilton decomposition of $G^{\text{rob}} \cup H$.

G^{rob} is constructed in [7] using the ‘robust decomposition lemma’ of [8]. As an ‘input’ this lemma needs a suitable set of exceptional systems, which will be part of the decomposition found in this paper.

The nature of the decomposition of the exceptional edges into exceptional systems depends on the structure of the bipartite subgraph $G[A', B']$ of G : we say that G is ‘critical’ if many edges of $G[A', B']$ are incident to very few (exceptional) vertices. In our decomposition into exceptional systems, we will need to distinguish between (a) the non-critical case when $G[A', B']$ contains many edges, (b) the critical case when $G[A', B']$ contains many edges, and (c) the case when $G[A', B']$ contains only a few edges. The three lemmas guaranteeing this decomposition are the main results of this paper. In these lemmas, we will be able to assume that A_0 and

B_0 are independent sets of vertices, as suitable Hamilton cycles covering all edges of $G[A_0]$ and $G[B_0]$ are already found in [7].

2. NOTATION AND TOOLS

2.1. Notation. Given a graph G , we write $V(G)$ for its vertex set, $E(G)$ for its edge set, $e(G) := |E(G)|$ for the number of its edges and $|G| := |V(G)|$ for the number of its vertices. We write $\delta(G)$ for the minimum degree of G , $\Delta(G)$ for its maximum degree and $\chi'(G)$ for the edge-chromatic number of G . Given a vertex v of G and a set $A \subseteq V(G)$, we write $d_G(v, A)$ for the number of all those neighbours of v in G which lie in A . Given $A, B \subseteq V(G)$, we write $e_G(A)$ for the number of all those edges of G which have both endvertices in A and $e_G(A, B)$ for the number of AB -edges of G , i.e. for the number of all those edges of G which have one endvertex in A and its other endvertex in B . If $A \cap B = \emptyset$, we denote by $G[A, B]$ the bipartite subgraph of G whose vertex classes are A and B and whose edges are all AB -edges of G . We often omit the index G if the graph G is clear from the context.

Given a vertex set V and two edge-disjoint graphs G and H with $V(G), V(H) \subseteq V$, we write $G + H$ for the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. We write $G - H$ for the subgraph of G which is obtained from G by deleting all the edges in $E(G) \cap E(H)$. Given $A \subseteq V(G)$, we write $G - A$ for the graph obtained from G by deleting all vertices in A .

We say that a graph G has a *decomposition* into H_1, \dots, H_r if $G = H_1 + \dots + H_r$ and the H_i are pairwise edge-disjoint.

A *path system* is a graph Q which is the union of vertex-disjoint paths (some of them might be trivial). We say that P is a *path in Q* if P is a component of Q and, abusing the notation, sometimes write $P \in Q$ for this. We often view a matching M as a graph (in which every vertex has degree precisely one).

In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then this means that there are non-decreasing functions $f : (0, 1] \rightarrow (0, 1]$, $g : (0, 1] \rightarrow (0, 1]$ and $h : (0, 1] \rightarrow (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq g(b)$ and $1/n \leq h(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way. We will write $a = b \pm c$ as shorthand for $b - c \leq a \leq b + c$.

2.2. Tools. We will need the following Chernoff bound for binomial distribution (see e.g. [5, Corollary 2.3]). Recall that the binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1 - p$.

Proposition 2.1. *Suppose X has binomial distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-a^2\mathbb{E}X/3}$.*

We will also use the following special cases of Propositions 6.1 and 6.3 in [7] which, given a suitable graph G and a partition A', B' of $V(G)$, provide bounds on the number $e_G(A', B')$ of edges between A' and B' .

Proposition 2.2. *Let G be a graph on n vertices with $\delta(G) \geq D$ and let A', B' be a partition of $V(G)$. If $D \geq n - 2\lfloor n/4 \rfloor - 1$, then $e_G(A', B') \geq D$ unless $n = 0 \pmod{4}$, $D = n/2 - 1$ and $|A'| = |B'| = n/2$.*

Proposition 2.3. *Let G be a D -regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$. Let A', B' be a partition of $V(G)$ with $|A'|, |B'| \geq D/2$ and $\Delta(G[A', B']) \leq D/2$. Then*

$$e_{G-U}(A', B') \geq \begin{cases} D - 28 & \text{if } D \geq n/2, \\ D/2 - 28 & \text{if } D = (n-1)/2 \end{cases}$$

for every $U \subseteq V(G)$ with $|U| \leq 3$.

Finally, we will also need the following result, which is a simple consequence of Vizing's theorem and was first observed by McDiarmid and independently by de Werra (see e.g. [11]).

Proposition 2.4. *Let G be a graph with $\chi'(G) \leq m$. Then G has a decomposition into m matchings M_1, \dots, M_m with $|e(M_i) - e(M_j)| \leq 1$ for all $i, j \leq m$.*

3. EXCEPTIONAL SYSTEMS, (K, m, ε_0) -PARTITIONS AND EXCEPTIONAL SCHEMES

In this section, we formally introduce ‘exceptional (path) systems’. Their first property is that the (interiors of) their paths cover all exceptional vertices.

Suppose that A, A_0, B, B_0 forms a partition of a vertex set V of size n such that $|A| = |B|$. Let $V_0 := A_0 \cup B_0$. An *exceptional cover* J is a graph which satisfies the following properties:

- (EC1) J is a path system with $V_0 \subseteq V(J) \subseteq V$.
- (EC2) $d_J(v) = 2$ for every $v \in V_0$ and $d_J(v) \leq 1$ for every $v \in V(J) \setminus V_0$.
- (EC3) $e_J(A), e_J(B) = 0$.

We say that J is an *exceptional system with parameter ε_0* , or an *ES* for short, if J satisfies the following properties:

- (ES1) J is an exceptional cover.
- (ES2) One of the following is satisfied:
 - (HES) The number of AB -paths in J is even and positive. In this case we say J is a *Hamilton exceptional system*, or *HES* for short.
 - (MES) $e_J(A', B') = 0$. In this case we say J is a *matching exceptional system*, or *MES* for short.
- (ES3) J contains at most $\sqrt{\varepsilon_0 n}$ AB -paths.

Note that by (EC2) every AB -path in J must be a maximal path in J . In [7] we will extend each Hamilton exceptional system J into a Hamilton cycle using only edges induced by A and edges induced by B . This is the reason for condition (HES) since the number of AB -paths in J corresponds to the number of genuine ‘connections’ between A and B . In [7], matching exceptional systems will always be extended into two edge-disjoint perfect matchings.

In general, we construct an exceptional system by first choosing an exceptional system candidate (defined below) and then extending it to an exceptional system. More precisely, suppose that A, A_0, B, B_0 forms a partition of a vertex set V . Let $V_0 := A_0 \cup B_0$. A graph F is called an *exceptional system candidate with parameter ε_0* , or an *ESC* for short, if F satisfies the following properties:

- (ESC1) F is a path system with $V_0 \subseteq V(F) \subseteq V$ and such that $e_F(A), e_F(B) = 0$.
- (ESC2) $d_F(v) \leq 2$ for all $v \in V_0$ and $d_F(v) = 1$ for all $v \in V(F) \setminus V_0$.
- (ESC3) $e_F(A', B') \leq \sqrt{\varepsilon_0}n/2$. In particular, $|V(F) \cap A|, |V(F) \cap B| \leq 2|V_0| + \sqrt{\varepsilon_0}n/2$.
- (ESC4) One of the following holds:
 - (HESC) Let $b(F)$ be the number of maximal paths in F with one endpoint in A' and the other in B' . Then $b(F)$ is even and $b(F) > 0$. In this case we say that F is a *Hamilton exceptional system candidate*, or *HESC* for short.
 - (MESC) $e_F(A', B') = 0$. In this case, F is called a *matching exceptional system candidate* or *MESC* for short.

Note that if $d_F(v) = 2$ for all $v \in V_0$, then F is an exceptional system. Also, if F is a Hamilton exceptional system candidate with $e(F) = 2$, then F consists of two independent $A'B'$ -edges. Moreover, note that (EC2) allows an exceptional cover J (and so also an exceptional system J) to contain vertices in $A \cup B$ which are isolated in J . However, (ESC2) does not allow for this in an exceptional system candidate F .

Similarly to condition (HES), in (HESC) the parameter $b(F)$ counts the number of ‘connections’ between A' and B' . In order to extend a Hamilton exceptional system candidate into a Hamilton cycle without using any additional $A'B'$ -edges, it is clearly necessary that $b(F)$ is positive and even.

The following result shows that we can extend an exceptional system candidate into an exceptional system by adding suitable A_0A - and B_0B -edges. Its easy proof is included in [7, Lemma 7.2].

Lemma 3.1. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1$ and that $n \in \mathbb{N}$. Let G be a graph on n vertices so that*

- (i) A, A_0, B, B_0 forms a partition of $V(G)$ with $|A_0 \cup B_0| \leq \varepsilon_0 n$;
- (ii) $d(v, A) \geq \sqrt{\varepsilon_0}n$ for all $v \in A_0$ and $d(v, B) \geq \sqrt{\varepsilon_0}n$ for all $v \in B_0$.

Let F be an exceptional system candidate with parameter ε_0 . Then there exists an exceptional system J with parameter ε_0 such that $F \subseteq J \subseteq G + F$ and such that every edge of $J - F$ lies in $G[A_0, A] + G[B_0, B]$. Moreover, if F is a Hamilton exceptional system candidate, then J is a Hamilton exceptional system. Otherwise J is a matching exceptional system.

As mentioned earlier, the exceptional systems we seek will need to be ‘localized’. For a formal definition, let $K, m \in \mathbb{N}$ and $\varepsilon_0 > 0$. A (K, m, ε_0) -*partition* \mathcal{P} of a set V of vertices is a partition of V into sets A_0, A_1, \dots, A_K and B_0, B_1, \dots, B_K such that $|A_i| = |B_i| = m$ for all $i \geq 1$ and $|A_0 \cup B_0| \leq \varepsilon_0 |V|$. The sets A_1, \dots, A_K and B_1, \dots, B_K are called *clusters* of \mathcal{P} and A_0, B_0 are called *exceptional sets*. We often write V_0 for $A_0 \cup B_0$ and think of the vertices in V_0 as ‘exceptional vertices’. Unless stated otherwise, whenever \mathcal{P} is a (K, m, ε_0) -partition, we will denote the clusters by A_1, \dots, A_K and B_1, \dots, B_K and the exceptional sets by A_0 and B_0 . We will also

write $A := A_1 \cup \dots \cup A_K$, $B := B_1 \cup \dots \cup B_K$, $A' := A_0 \cup A_1 \cup \dots \cup A_K$ and $B' := B_0 \cup B_1 \cup \dots \cup B_K$.

Given a (K, m, ε_0) -partition \mathcal{P} and $1 \leq i, i' \leq K$, we say that J is an (i, i') -localized Hamilton exceptional system (abbreviated as (i, i') -HES) if J is a Hamilton exceptional system and $V(J) \subseteq V_0 \cup A_i \cup B_{i'}$. In a similar way, we define

- (i, i') -localized matching exceptional systems $((i, i')$ -MES),
- (i, i') -localized exceptional systems $((i, i')$ -ES),
- (i, i') -localized Hamilton exceptional system candidates $((i, i')$ -HESC),
- (i, i') -localized matching exceptional system candidates $((i, i')$ -MESc),
- (i, i') -localized exceptional system candidates $((i, i')$ -ESC).

To make clear with which partition we are working, we sometimes also say that J is an (i, i') -localized Hamilton exceptional system with respect to \mathcal{P} etc.

Finally, we define an ‘exceptional scheme’, which will be the structure within which we find our localized exceptional systems. Given a graph G on n vertices and a partition \mathcal{P} of $V(G)$, we call (G, \mathcal{P}) a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme if the following properties are satisfied:

- (ESch1) \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$.
- (ESch2) $e(A), e(B) = 0$.
- (ESch3) If $v \in A$ then $d(v, B') < \varepsilon_0 n$ and if $v \in B$ then $d(v, A') < \varepsilon_0 n$.
- (ESch4) For all $v \in V(G)$ and all $1 \leq i \leq K$ we have $d(v, A_i) = (d(v, A) \pm \varepsilon n)/K$ and $d(v, B_i) = (d(v, B) \pm \varepsilon n)/K$.
- (ESch5) For all $1 \leq i, i' \leq K$ we have

$$\begin{aligned} e(A_0, A_i) &= (e(A_0, A) \pm \varepsilon \max\{e(A_0, A), n\})/K, \\ e(B_0, A_i) &= (e(B_0, A) \pm \varepsilon \max\{e(B_0, A), n\})/K, \\ e(A_0, B_i) &= (e(A_0, B) \pm \varepsilon \max\{e(A_0, B), n\})/K, \\ e(B_0, B_i) &= (e(B_0, B) \pm \varepsilon \max\{e(B_0, B), n\})/K, \\ e(A_i, B_{i'}) &= (e(A, B) \pm \varepsilon \max\{e(A, B), n\})/K^2. \end{aligned}$$

4. CONSTRUCTING LOCALIZED EXCEPTIONAL SYSTEMS

Given a D -regular graph G and a (K, m, ε_0) -partition \mathcal{P} of $V(G)$, let $G' := G - G[A] - G[B]$ and suppose that (G', \mathcal{P}) is an exceptional scheme. Roughly speaking, the aim of this section is to decompose G' into edge-disjoint exceptional systems. In [7], each of these exceptional systems J will then be extended into a Hamilton cycle (in the case when J is a Hamilton exceptional system) or into two perfect matchings (in the case when J is a matching exceptional system). We will ensure that all but a small number of these exceptional systems are localized (with respect to \mathcal{P}).

Rather than decomposing G' in a single step, we actually need to proceed in two steps: initially, we find a small number of exceptional systems J which have some additional useful properties (e.g. the number of $A'B'$ -edges of J is either zero or two). In [7] these exceptional systems will be used to construct the robustly decomposable graph G^{rob} . (Recall that the role of G^{rob} in [7] was also discussed in Section 1.3.)

Some of the additional properties of the exceptional systems contained in G^{rob} then allow us to find the desired decomposition of $G^\diamond := G' - G^{\text{rob}}$ into exceptional systems.

In order to construct the required (localized) exceptional systems, we will distinguish three cases:

- (a) the case when G is ‘non-critical’ and contains at least D $A'B'$ -edges (see Lemma 4.2 in Section 4.2);
- (b) the case when G is ‘critical’ and contains at least D $A'B'$ -edges (see Lemma 4.10 in Section 4.3);
- (c) the case when G contains less than D $A'B'$ -edges (see Lemma 4.14 in Section 4.4).

Each of the three lemmas above is formulated in such a way that we can apply it twice in [7]: firstly to obtain the small number of exceptional systems needed for the robustly decomposable graph G^{rob} and secondly for the decomposition of the graph $G^\diamond := G - G^{\text{rob}} - G[A] - G[B]$ into exceptional systems.

4.1. Critical graphs. Let G be a D -regular graph and let A', B' be a partition of $V(G)$. Roughly speaking, G is critical if most of its $A'B'$ -edges are incident to only a few vertices. More precisely, we say that G is *critical* (with respect to A', B' and D) if both of the following hold:

- $\Delta(G[A', B']) \geq 11D/40$;
- $e(H) \leq 41D/40$ for all subgraphs H of $G[A', B']$ with $\Delta(H) \leq 11D/40$.

One example of a critical graph is the following: G_{crit} consists of two disjoint cliques on $(n-1)/2$ vertices with vertex set A and B respectively, where $n = 4k + 1$ for some $k \in \mathbb{N}$. In addition, there is a vertex a which is adjacent to exactly half of the vertices in each of A and B . Also, add a perfect matching M between those vertices of A and those vertices in B not adjacent to a . Let $A' := A \cup \{a\}$, $B' := B$ and $D := (n-1)/2$. Then G_{crit} is critical, and D -regular with $e(A', B') = D$. Note that $e(M) = D/2$.

To obtain a Hamilton decomposition of G_{crit} , we will need to decompose $G_{\text{crit}}[A', B']$ into $D/2$ Hamilton exceptional system candidates F_s (which need to be matchings of size exactly two in this case). In this example, this decomposition is essentially unique: every F_s has to consist of exactly one edge in M and one edge incident to a . Note that in this way, every edge between a and B yields a ‘connection’ (i.e. a maximal path) between A' and B' required in (ESC4).

The following lemma collects some properties of critical graphs. In particular, there is a set W consisting of between one and three vertices with many neighbours in both A and B (such as the vertex a in G_{crit} above). As in the example of G_{crit} , we will need to use $A'B'$ -edges incident to one or two vertices in W to provide connections between A' and B' when constructing the Hamilton exceptional system candidates in the critical case (b).

Lemma 4.1. *Suppose that $0 < 1/n \ll 1$ and that $D, n \in \mathbb{N}$ are such that*

$$(4.1) \quad D \geq n - 2\lfloor n/4 \rfloor - 1 = \begin{cases} n/2 - 1 & \text{if } n \equiv 0 \pmod{4}, \\ (n-1)/2 & \text{if } n \equiv 1 \pmod{4}, \\ n/2 & \text{if } n \equiv 2 \pmod{4}, \\ (n+1)/2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Let G be a D -regular graph on n vertices and let A', B' be a partition of $V(G)$ with $|A'|, |B'| \geq D/2$ and $\Delta(G[A', B']) \leq D/2$. Suppose that G is critical. Let W be the set of vertices $w \in V(G)$ such that $d_{G[A', B']}(w) \geq 11D/40$. Then the following properties are satisfied:

- (i) $1 \leq |W| \leq 3$.
- (ii) Either $D = (n-1)/2$ and $n \equiv 1 \pmod{4}$, or $D = n/2 - 1$ and $n \equiv 0 \pmod{4}$. Furthermore, if $n \equiv 1 \pmod{4}$, then $|W| = 1$.
- (iii) $e_G(A', B') \leq 17D/10 + 5 < n$.
- (iv)

$$e_{G-W}(A', B') \leq \begin{cases} 3D/4 + 5 & \text{if } |W| = 1, \\ 19D/40 + 5 & \text{if } |W| = 2, \\ D/5 + 5 & \text{if } |W| = 3. \end{cases}$$

- (v) There exists a set W' of vertices such that $W \subseteq W'$, $|W'| \leq 3$ and for all $w' \in W'$ and $v \in V(G) \setminus W'$ we have

$$d_{G[A', B']}(w') \geq \frac{21D}{80}, \quad d_{G[A', B']}(v) \leq \frac{11D}{40} \quad \text{and} \quad d_{G[A', B']}(w') - d_{G[A', B']}(v) \geq \frac{D}{240}.$$

Proof. Let w_1, \dots, w_4 be vertices of G such that

$$d_{G[A', B']}(w_1) \geq \dots \geq d_{G[A', B']}(w_4) \geq d_{G[A', B']}(v)$$

for all $v \in V(G) \setminus \{w_1, \dots, w_4\}$. Let $W_4 := \{w_1, \dots, w_4\}$. Suppose that $d_{G[A', B']}(w_4) \geq 21D/80$. Let H be a spanning subgraph of $G[A', B']$ such that $d_H(w_i) = \lceil 21D/80 \rceil$ for all $i \leq 4$ and such that every vertex $v \in V(G) \setminus W_4$ satisfies $N_H(v) \subseteq W_4$. Thus $\Delta(H) = \lceil 21D/80 \rceil$ and so $e(H) \leq 41D/40$ since G is critical. On the other hand, $e(H) \geq 4 \cdot \lceil 21D/80 \rceil - 4$, a contradiction. (Here we subtract four to account for the edges of H' between vertices in W .) Hence, $d_{G[A', B']}(w_4) < 21D/80$ and so $|W| \leq 3$. But $|W| \geq 1$ since G is critical. So (i) holds.

Let j be minimal such that $d_{G[A', B']}(w_j) \leq 21D/80$. So $1 < j \leq 4$. Choose an index i with $1 \leq i < j$ such that $W \subseteq \{w_1, \dots, w_i\}$ and $d_{G[A', B']}(w_i) - d_{G[A', B']}(w_{i+1}) \geq D/240$. Then the set $W' := \{w_1, \dots, w_i\}$ satisfies (v).

Let H' be a spanning subgraph of $G[A', B']$ such that $G[A' \setminus W, B' \setminus W] \subseteq H'$ and $d_{H'}(w) = \lfloor 11D/40 \rfloor$ for all $w \in W$. Similarly as before, $e(H') \leq 41D/40$ since G is critical. Thus

$$\begin{aligned} 41D/40 &\geq e(H') \geq e(H' - W) + \lfloor 11D/40 \rfloor |W| - 2 \\ &= e_{G-W}(A', B') + \lfloor 11D/40 \rfloor |W| - 2. \end{aligned}$$

This in turn implies that

$$(4.2) \quad e_{G-W}(A', B') \leq (41 - 11|W|)D/40 + 5.$$

Together with (i) this implies (iv). If $D \geq n/2$, then by Proposition 2.3 we have $e_{G-W}(A', B') \geq D - 28$. This contradicts (iv). Thus (4.1) implies that $D = (n-1)/2$ and $n \equiv 1 \pmod{4}$, or $D = n/2 - 1$ and $n \equiv 0 \pmod{4}$. If $n \equiv 1 \pmod{4}$ and $D = (n-1)/2$, then Proposition 2.3 implies that $e_{G-W}(A', B') \geq D/2 - 28$. Hence, by (iv) we deduce that $|W| = 1$ and so (ii) holds. Since $|W| \leq 3$ and $\Delta(G[A', B']) \leq D/2$, we have

$$e_G(A', B') \leq e_{G-W}(A', B') + \frac{|W|D}{2} \stackrel{(4.2)}{\leq} \frac{(41 + 9|W|)D}{40} + 5 \leq \frac{17D}{10} + 5 < n.$$

(The last inequality follows from (ii).) This implies (iii). \square

4.2. Non-critical case with $e(A', B') \geq D$. Recall from the beginning of Section 4 that our aim is to find a decomposition of $G - G[A] - G[B]$ into suitable exceptional systems (in particular, most of these exceptional systems have to be localized). The following lemma implies that this can be done if G is not critical and $e(A', B') \geq D$. We will prove this lemma in this subsection and will then consider the remaining two cases in the next two subsections.

Lemma 4.2. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \lambda, 1/K \ll 1$, that $D \geq n/3$, that $0 \leq \phi \ll 1$ and that $D, n, K, m, \lambda n/K^2, (D - \phi n)/(2K^2) \in \mathbb{N}$. Suppose that the following conditions hold:*

- (i) G is a D -regular graph on n vertices.
- (ii) \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$ such that $D \leq e_G(A', B') \leq \varepsilon_0 n^2$ and $\Delta(G[A', B']) \leq D/2$. Furthermore, G is not critical.
- (iii) G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$, $e_{G_0}(A', B') \leq \phi n$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.
- (iv) Let $G^\diamond := G - G[A] - G[B] - G_0$. $e_{G^\diamond}(A', B')$ is even and $(G^\diamond, \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme.

Then there exists a set \mathcal{J} consisting of $(D - \phi n)/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G^\diamond which satisfies the following properties:

- (a) Together all the Hamilton exceptional systems in \mathcal{J} cover all edges of G^\diamond .
- (b) For all $1 \leq i, i' \leq K$, the set \mathcal{J} contains $(D - (\phi + 2\lambda)n)/(2K^2)$ (i, i') -HES. Moreover, $\lambda n/K^2$ of these (i, i') -HES J are such that $e_J(A', B') = 2$.

Note that (b) implies that \mathcal{J} contains λn Hamilton exceptional systems which might not be localized. On the other hand, the lemma is ‘robust’ in the sense that we can remove a sparse subgraph G_0 before we find the decomposition \mathcal{J} into Hamilton exceptional systems. (In particular, as discussed at the beginning of the section, we can remove the graph G^{rob} before applying the lemma.)

We will split the proof of Lemma 4.2 into the following four steps:

Step 1 We first decompose G^\diamond into edge-disjoint ‘localized’ subgraphs $H(i, i')$ and $H'(i, i')$ (where $1 \leq i, i' \leq K$). More precisely, each $H(i, i')$ only contains $A_0 A_i$ -edges and $B_0 B_{i'}$ -edges of G^\diamond while all edges of $H'(i, i')$ lie in $G^\diamond[A_0 \cup$

$A_i, B_0 \cup B_{i'}$], and all the edges of G° are distributed evenly amongst the $H(i, i')$ and $H'(i, i')$ (see Lemma 4.3). We will then move a small number of $A'B'$ -edges between the $H'(i, i')$ in order to obtain graphs $H''(i, i')$ such that $e(H''(i, i'))$ is even (see Lemma 4.4).

Step 2 We decompose each $H''(i, i')$ into $(D - \phi n)/(2K^2)$ Hamilton exceptional system candidates (see Lemma 4.6).

Step 3 Most of the Hamilton exceptional system candidates constructed in Step 2 will be extended into an (i, i') -HES (see Lemma 4.7).

Step 4 The remaining Hamilton exceptional system candidates will be extended into Hamilton exceptional systems, which need not be localized (see Lemma 4.8). (Altogether, these will be the λn Hamilton exceptional systems in \mathcal{J} which are not mentioned in Lemma 4.2(b).)

4.2.1. *Step 1: Constructing the graphs $H''(i, i')$.* The next lemma from [7, Lemma 9.2] will be used to find a decomposition of G° into suitable ‘localized subgraphs’ $H(i, i')$ and $H'(i, i')$ as described in Step 1 above.

Lemma 4.3. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll 1/K \ll 1$ and that $n, K, m \in \mathbb{N}$. Let (G, \mathcal{P}) be a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme with $|G| = n$ and $e_G(A_0), e_G(B_0) = 0$. Then G can be decomposed into edge-disjoint spanning subgraphs $H(i, i')$ and $H'(i, i')$ of G (for all $1 \leq i, i' \leq K$) such that the following properties hold, where $G(i, i') := H(i, i') + H'(i, i')$:*

- (a₁) *Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.*
- (a₂) *All edges of $H'(i, i')$ lie in $G[A_0 \cup A_i, B_0 \cup B_{i'}]$.*
- (a₃) *$e(H'(i, i')) = (e_G(A', B') \pm 4\varepsilon \max\{n, e_G(A', B')\})/K^2$.*
- (a₄) *$d_{H'(i, i')}(v) = (d_{G[A', B']}(v) \pm 2\varepsilon n)/K^2$ for all $v \in V_0$.*
- (a₅) *$d_{G(i, i')}(v) = (d_G(v) \pm 4\varepsilon n)/K^2$ for all $v \in V_0$.*

Let $H(i, i')$ and $H'(i, i')$ be the graphs obtained by applying Lemma 4.3 to G° . As mentioned before, we would like to decompose each $H'(i, i')$ into Hamilton exceptional system candidates. In order to do this, $e(H'(i, i'))$ must be even. The next lemma shows that we can ensure this property without destroying the other properties of the $H'(i, i')$ too much by moving a small number of edges between the $H'(i, i')$.

Lemma 4.4. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \varepsilon' \ll \lambda, 1/K \ll 1$, that $D \geq n/3$, that $0 \leq \phi \ll 1$ and that $D, n, K, m, (D - \phi n)/(2K^2) \in \mathbb{N}$. Define α by*

$$(4.3) \quad 2\alpha n := \frac{D - \phi n}{K^2} \quad \text{and let} \quad \gamma := \alpha - \frac{2\lambda}{K^2}.$$

Suppose that the following conditions hold:

- (i) *G is a D -regular graph on n vertices.*
- (ii) *\mathcal{P} is a (K, m, ε_0) -partition of $V(G)$ such that $D \leq e_G(A', B') \leq \varepsilon_0 n^2$ and $\Delta(G[A', B']) \leq D/2$. Furthermore, G is not critical.*
- (iii) *G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$, $e_{G_0}(A', B') \leq \phi n$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.*
- (iv) *Let $G^\circ := G - G[A] - G[B] - G_0$. $e_{G^\circ}(A', B')$ is even and (G°, \mathcal{P}) is a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme.*

Then G° can be decomposed into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ of G° (for all $1 \leq i, i' \leq K$) such that the following properties hold, where $G'(i, i') := H(i, i') + H''(i, i')$:

- (b₁) Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.
- (b₂) $H''(i, i') \subseteq G^\circ[A', B']$. Moreover, all but at most $\varepsilon'n$ edges of $H''(i, i')$ lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (b₃) $e(H''(i, i'))$ is even and $2\alpha n \leq e(H''(i, i')) \leq 11\varepsilon_0 n^2 / (10K^2)$.
- (b₄) $\Delta(H''(i, i')) \leq 31\alpha n / 30$.
- (b₅) $d_{G'(i, i')}(v) = (2\alpha \pm \varepsilon')n$ for all $v \in V_0$.
- (b₆) Let \tilde{H} be any spanning subgraph of $H''(i, i')$ which maximises $e(\tilde{H})$ under the constraints that $\Delta(\tilde{H}) \leq 3\gamma n / 5$, $H''(i, i')[A_0, B_0] \subseteq \tilde{H}$ and $e(\tilde{H})$ is even. Then $e(\tilde{H}) \geq 2\alpha n$.

Proof. Since $\phi \ll 1/3 \leq D/n$, we deduce that

$$(4.4) \quad \alpha \geq 1/(7K^2), \quad (1 - 14\lambda)\alpha \leq \gamma < \alpha \quad \text{and} \quad \varepsilon \ll \varepsilon' \ll \lambda, 1/K, \alpha, \gamma \ll 1.$$

Note that (ii) and (iii) together imply that

$$(4.5) \quad e_{G^\circ}(A', B') \geq D - \phi n \stackrel{(4.3)}{=} 2K^2\alpha n \stackrel{(4.4)}{\geq} n/4.$$

By (i) and (iii), each $v \in V_0$ satisfies

$$(4.6) \quad d_{G^\circ}(v) = D - \phi n \stackrel{(4.3)}{=} 2K^2\alpha n.$$

Apply Lemma 4.3 to decompose G° into subgraphs $H(i, i')$, $H'(i, i')$ (for all $1 \leq i, i' \leq K$) satisfying the following properties, where $G(i, i') := H(i, i') + H'(i, i')$:

- (a'₁) Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.
- (a'₂) All edges of $H'(i, i')$ lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (a'₃) $e(H'(i, i')) = (1 \pm 16\varepsilon)e_{G^\circ}(A', B')/K^2$. In particular,

$$2(1 - 16\varepsilon)\alpha n \leq e(H'(i, i')) \leq (1 + 16\varepsilon)\varepsilon_0 n^2 / K^2.$$

$$(a'_4) \quad d_{H'(i, i')}(v) = (d_{G^\circ[A', B']}(v) \pm 2\varepsilon n) / K^2 \text{ for all } v \in V_0.$$

$$(a'_5) \quad d_{G(i, i')}(v) = (2\alpha \pm 4\varepsilon/K^2)n \text{ for all } v \in V_0.$$

Indeed, (a'₃) follows from (4.5), Lemma 4.3(a₃) and (ii), while (a'₅) follows from (4.6) and Lemma 4.3(a₅). We now move some $A'B'$ -edges of G° between the $H'(i, i')$ such that the graphs $H''(i, i')$ obtained in this way satisfy the following conditions:

- Each $H''(i, i')$ is obtained from $H'(i, i')$ by adding or removing at most $32K^2\varepsilon\alpha n \leq \sqrt{\varepsilon}n$ edges.
- $e(H''(i, i')) \geq 2\alpha n$ and $e(H''(i, i'))$ is even.

Note that this is possible by (a'₃) and since $\alpha n \in \mathbb{N}$ and $e_{G^\circ}(A', B') \geq 2K^2\alpha n$ is even by (iv).

We will show that the graphs $H(i, i')$ and $H''(i, i')$ satisfy conditions (b₁)–(b₆). Clearly both (b₁) and (b₂) hold. (a'₃) implies that

$$(4.7) \quad e(H''(i, i')) = (1 \pm 16\varepsilon)e_{G^\circ}(A', B')/K^2 \pm \sqrt{\varepsilon}n \stackrel{(4.4), (4.5)}{=} (1 \pm \varepsilon')e_{G^\circ}(A', B')/K^2.$$

Together with (ii) and our choice of the $H''(i, i')$ this implies (b₃). (b₅) follows from (a'₅) and the fact that $d_{G'(i, i')}(v) = d_{G(i, i')}(v) \pm \sqrt{\varepsilon}n$. Similarly, (a'₄) implies that for all $v \in V_0$ we have

$$(4.8) \quad d_{H''(i, i')}(v) = (d_{G^\circ[A', B']}(v) \pm \varepsilon'n)/K^2.$$

Recall that $\Delta(G[A', B']) \leq D/2$ by (ii). Thus

$$\Delta(H''(i, i')) \stackrel{(4.8)}{\leq} \frac{D/2 + \varepsilon'n}{K^2} \stackrel{(4.3)}{=} \left(\alpha + \frac{\phi + 2\varepsilon'}{2K^2} \right) n \stackrel{(4.4)}{\leq} \frac{31\alpha n}{30},$$

so (b₄) holds.

So it remains to verify (b₆). To do this, fix $1 \leq i, i' \leq K$ and set $H'' := H''(i, i')$. Let \tilde{H} be a subgraph of H'' as defined in (b₆). We need to show that $e(\tilde{H}) \geq 2\alpha n$. Suppose the contrary that $e(\tilde{H}) < 2\alpha n$. We will show that this contradicts the assumption that G is not critical. Roughly speaking, the argument will be that if \tilde{H} is sparse, then so is H'' . This in turn implies that G° is also sparse, and thus any subgraph of $G[A', B']$ of comparatively small maximum degree is also sparse, which leads to a contradiction.

Let X be the set of all those vertices x for which $d_{\tilde{H}}(x) \geq 3\gamma n/5 - 2$. So $X \subseteq V_0$ by (iv) and (ESch3). Note that if $X = \emptyset$, then $\tilde{H} = H''$ and so $e(\tilde{H}) \geq 2\alpha n$ by (b₃). If $|X| \geq 4$, then $e(\tilde{H}) \geq 4(3\gamma n/5 - 2) - 4 \geq 2\alpha n$ by (4.4). Hence $1 \leq |X| \leq 3$. Note that $\tilde{H} - X$ contains all but at most one edge from $H'' - X$. Together with the fact that $\tilde{H}[X]$ contains at most two edges (since $|X| \leq 3$ and \tilde{H} is bipartite) this implies that

$$\begin{aligned} (4.9) \quad 2\alpha n > e(\tilde{H}) &\geq e(\tilde{H} - X) + \left(\sum_{x \in X} d_{\tilde{H}}(x) \right) - 2 \geq e(H'' - X) - 1 + |X|(3\gamma n/5 - 2) - 2 \\ &\geq e(H'') - \sum_{x \in X} d_{H''}(x) + |X|(3\gamma n/5 - 2) - 3 \\ &= e(H'') - \sum_{x \in X} (d_{H''}(x) - 3\gamma n/5 + 2) - 3 \end{aligned}$$

and so

$$(4.10) \quad e(H'') \stackrel{(4.8)}{<} 2\alpha n + \sum_{x \in X} \left(\frac{d_{G^\circ[A', B']}(x) + \varepsilon'n}{K^2} - 3\gamma n/5 + 2 \right) + 3.$$

Note that (b₄) and (4.9) together imply that if $e(H'') \geq 4\alpha n$ then $e(\tilde{H}) \geq e(H'') - |X|(31\alpha n/30 - 3\gamma n/5 + 2) - 3 \geq 2\alpha n$. Thus $e(H'') < 4\alpha n$ and by (4.7) we have $e_{G^\circ}(A', B') \leq 4K^2\alpha n/(1 - \varepsilon') \leq 5K^2\alpha n \leq 3n$. Hence

$$\begin{aligned} (4.11) \quad e_{G^\circ}(A', B') &\stackrel{(4.7)}{\leq} K^2 e(H'') + \varepsilon' e_{G^\circ}(A', B') \leq K^2 e(H'') + 3\varepsilon'n \\ &\stackrel{(4.10)}{\leq} D - \phi n + 7\varepsilon'n + \sum_{x \in X} (d_{G^\circ[A', B']}(x) - K^2(3\gamma n/5)). \end{aligned}$$

Let G' be any subgraph of $G^\diamond[A', B']$ which maximises $e(G')$ under the constraint that $\Delta(G') \leq K^2(3\gamma/5 + 2\varepsilon')n$. Note that if $d_{G^\diamond[A', B']}(v) \geq K^2(3\gamma/5 + 2\varepsilon')n$, then $v \in V_0$ (by (iv) and (ESch3)) and so $d_{H''}(v) > 3\gamma n/5$ by (4.8). This in turn implies that $v \in X$. Hence

$$\begin{aligned} e(G') &\leq e_{G^\diamond}(A', B') - \sum_{x \in X} (d_{G^\diamond[A', B']}(x) - K^2(3\gamma/5 + 2\varepsilon')n) + 2 \\ (4.12) \quad &\stackrel{(4.11)}{\leq} D - \phi n + 7K^2\varepsilon'n. \end{aligned}$$

Note that (4.8) together with the fact that $X \neq \emptyset$ implies that

$$\Delta(G[A', B']) \geq \Delta(G^\diamond[A', B']) \geq K^2(3\gamma n/5 - 2) - \varepsilon'n \stackrel{(4.3), (4.4)}{\geq} 11D/40.$$

Since G is not critical this means that there exists a subgraph G'' of $G[A', B']$ such that $\Delta(G'') \leq 11D/40 \leq K^2(3\gamma/5 + 2\varepsilon')n$ and $e(G'') \geq 41D/40$. Thus

$$D - \phi n + 7K^2\varepsilon'n \stackrel{(4.12)}{\geq} e(G') \geq e(G'') - e_{G_0}(A', B') \geq 41D/40 - \phi n,$$

which is a contradiction. Therefore, we must have $e(\tilde{H}) \geq 2\alpha n$. Hence (b₆) is satisfied. \square

4.2.2. Step 2: Decomposing $H''(i, i')$ into Hamilton exceptional system candidates. Our next aim is to decompose each $H''(i, i')$ into αn Hamilton exceptional system candidates (this will follow from Lemma 4.6). Before we can do this, we need the following result on decompositions of bipartite graphs into ‘even matchings’. We say that a matching is *even* if it contains an even number of edges, otherwise it is *odd*.

Proposition 4.5. *Suppose that $0 < 1/n \ll \gamma \leq 1$ and that $n, \gamma n \in \mathbb{N}$. Let H be a bipartite graph on n vertices with $\Delta(H) \leq 2\gamma n/3$ and where $e(H) \geq 2\gamma n$ is even. Then H can be decomposed into γn edge-disjoint non-empty even matchings, each of size at most $3e(H)/(\gamma n)$.*

Proof. First note that since $e(H) \geq 2\gamma n$, it suffices to show that H can be decomposed into at most γn edge-disjoint non-empty even matchings, each of size at most $3e(H)/(\gamma n)$. Indeed, by splitting these matchings further if necessary, one can obtain precisely γn non-empty even matchings.

Set $n' := \lfloor 2\gamma n/3 \rfloor$. König’s theorem implies that $\chi'(H) \leq n'$. So Proposition 2.4 implies that there is a decomposition of H into n' edge-disjoint matchings $M_1, \dots, M_{n'}$ such that $|e(M_s) - e(M_{s'})| \leq 1$ for all $s, s' \leq n'$. Hence we have

$$2 \leq \frac{e(H)}{n'} - 1 \leq e(M_s) \leq \frac{e(H)}{n'} + 1 \leq \frac{3e(H)}{\gamma n}$$

for all $s \leq n'$. Since $e(H)$ is even, there are an even number of odd matchings. Let M_s and $M_{s'}$ be two odd matchings. So $e(M_s), e(M_{s'}) \geq 3$ and thus there exist two disjoint edges $e \in M_s$ and $e' \in M_{s'}$. Hence, $M_s - e$, $M_{s'} - e'$ and $\{e, e'\}$ are three even matchings. Thus, by pairing off the odd matchings and repeating this process, the proposition follows. \square

Lemma 4.6. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \gamma < 1$, that $\gamma + \gamma' < 1$ and that $n, \gamma n, \gamma' n \in \mathbb{N}$. Let H be a bipartite graph on n vertices with vertex classes $A \dot{\cup} A_0$ and $B \dot{\cup} B_0$, where $|A_0| + |B_0| \leq \varepsilon_0 n$. Suppose that*

- (i) $e(H)$ is even, $\Delta(H) \leq 16\gamma n/15$ and $\Delta(H[A, B]) < (3\gamma/5 - \varepsilon_0)n$.

Let H' be a spanning subgraph of H which maximises $e(H')$ under the constraints that $\Delta(H') \leq 3\gamma n/5$, $H[A_0, B_0] \subseteq H'$ and $e(H')$ is even. Suppose that

- (ii) $2(\gamma + \gamma')n \leq e(H') \leq 10\varepsilon_0\gamma n^2$.

Then there exists a decomposition of H into edge-disjoint Hamilton exceptional system candidates $F_1, \dots, F_{\gamma n}, F'_1, \dots, F'_{\gamma' n}$ with parameter ε_0 such that $e(F'_s) = 2$ for all $s \leq \gamma' n$.

Since we are in the non-critical case with many edges between A' and B' , we will be able to assume that the subgraph H' satisfies (ii).

Roughly speaking, the idea of the proof of Lemma 4.6 is to apply the previous proposition to decompose H' into a suitable number of even matchings M_i (using the fact that it has small maximum degree). We then extend these matchings into Hamilton exceptional system candidates to cover all edges of H . The additional edges added to each M_i will be vertex-disjoint from M_i and form vertex-disjoint 2-paths uvw with $v \in V_0$. So the number of connections from A' to B' remains the same (as H is bipartite). Each matching M_i will already be a Hamilton exceptional system candidate, which means that M_i and its extension will have the correct number of connections from A' to B' (which makes this part of the argument simpler than in the critical case).

Proof of Lemma 4.6. Set $A' := A_0 \cup A$ and $B' := B_0 \cup B$. We first construct the F'_s . If $\gamma' = 0$, there is nothing to do. So suppose that $\gamma' > 0$. Note that each F'_s has to be a matching of size 2 (this follows from the definition of a Hamilton exceptional system candidate and the fact that $e(F'_s) = 2$). Since H' is bipartite and so

$$\frac{e(H')}{\chi'(H')} = \frac{e(H')}{\Delta(H')} \geq \frac{2(\gamma + \gamma')n}{3\gamma n/5} > \frac{10}{3},$$

we can find a 2-matching F'_1 in H' . Delete the edges in F'_1 from H' and choose another 2-matching F'_2 . We repeat this process until we have chosen $\gamma' n$ edge-disjoint 2-matchings $F'_1, \dots, F'_{\gamma' n}$.

We now construct $F_1, \dots, F_{\gamma n}$ in two steps: first we construct matchings $M_1, \dots, M_{\gamma n}$ in H' and then extend each M_i into the desired F_i . Let H_1 and H'_1 be obtained from H and H' by removing all the edges in $F'_1, \dots, F'_{\gamma' n}$. So now $2\gamma n \leq e(H'_1) \leq 10\varepsilon_0\gamma n^2$ and both $e(H_1)$ and $e(H'_1)$ are even. Thus Proposition 4.5 implies that there is a decomposition of H'_1 into edge-disjoint non-empty even matchings $M_1, \dots, M_{\gamma n}$, each of size at most $30\varepsilon_0 n$.

Note that each M_i is a Hamilton exceptional system candidate with parameter ε_0 . So if $H'_1 = H_1$, then we are done by setting $F_s := M_s$ for each $s \leq \gamma n$. Hence, we may assume that $H'' := H_1 - H'_1 = H - H'$ contains edges. Let X be the set of all those vertices $x \in A_0 \cup B_0$ for which $d_{H''}(x) > 0$. Note that each $x \in X$ satisfies $N_{H''}(x) \subseteq A \cup B$ (since $H[A_0, B_0] \subseteq H'$). This implies that each $x \in X$ satisfies

$d_{H'}(x) \geq \lfloor 3\gamma n/5 \rfloor - 1$ or $d_{H''}(x) = 1$. (Indeed, suppose that $d_{H'}(x) \leq \lfloor 3\gamma n/5 \rfloor - 2$ and $d_{H''}(x) \geq 2$. Then we can move two edges incident to x from H'' to H' . The final assumption in (i) and the assumption on $d_{H'}(x)$ together imply that we would still have $\Delta(H') \leq 3\gamma n/5$, a contradiction.) Since $\Delta(H) \leq 16\gamma n/15$ by (i) this in turn implies that $d_{H''}(x) \leq 7\gamma n/15 + 2$ for all $x \in X$.

Let \mathcal{M} be a random subset of $\{M_1, \dots, M_{\gamma n}\}$ where each M_i is chosen independently with probability $2/3$. By Proposition 2.1, with high probability, the following assertions hold:

$$(4.13) \quad r := |\mathcal{M}| = (2/3 \pm \varepsilon_0)\gamma n \\ |\{M_s \in \mathcal{M} : d_{M_s}(v) = 1\}| = 2d_{H'_1}(v)/3 \pm \varepsilon_0\gamma n \quad \text{for all } v \in V(H).$$

By relabeling if necessary, we may assume that $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$. For each $s \leq r$, we will now extend M_s to a Hamilton exceptional system candidate F_s with parameter ε_0 by adding edges from H'' . Suppose that for some $1 \leq s \leq r$ we have already constructed F_1, \dots, F_{s-1} . Set $H''_s := H'' - \sum_{j < s} F_j$. Let W_s be the set of all those vertices $w \in X$ for which $d_{M_s}(w) = 0$ and $d_{H''_s}(w) \geq 32\varepsilon_0 n \geq 2|A_0 \cup B_0| + e(M_s)$. Recall that $X \subseteq A_0 \cup B_0$ and $N_{H''_s}(w) \subseteq N_{H''}(w) \subseteq A \cup B$ for each $w \in X$ and thus also for each $w \in W_s$. Thus there are $|W_s|$ vertex-disjoint 2-paths uwu' with $w \in W_s$ and $u, u' \in N_{H''_s}(w) \setminus V(M_s)$. Assign these 2-paths to M_s and call the resulting graph F_s . Observe that F_s is a Hamilton exceptional system candidate with parameter ε_0 . Therefore, we have constructed F_1, \dots, F_r by extending M_1, \dots, M_r .

We now construct $F_{r+1}, \dots, F_{\gamma n}$. For this, we first prove that the above construction implies that the current ‘leftover’ H''_{r+1} has small maximum degree. Indeed, note that if $w \in W_s$, then $d_{H''_{s+1}}(w) = d_{H''_s}(w) - 2$. By (4.13), for each $x \in X$, the number of $M_s \in \mathcal{M}$ with $d_{M_s}(x) = 0$ is

$$\begin{aligned} r - |\{M_s \in \mathcal{M} : d_{M_s}(x) = 1\}| &\geq (2/3 - \varepsilon_0)\gamma n - (2d_{H'_1}(x)/3 + \varepsilon_0\gamma n) \\ &\geq 2\gamma n/3 - 2d_{H'}(x)/3 - 2\varepsilon_0\gamma n \\ &\geq 2\gamma n/3 - 2/3 \cdot \lfloor 3\gamma n/5 \rfloor - 2\varepsilon_0\gamma n \\ &\geq (4/15 - 2\varepsilon_0)\gamma n > d_{H''}(x)/2. \end{aligned}$$

Hence, we have $d_{H''_{r+1}}(x) < 32\varepsilon_0 n$ for all $x \in X$ (as we remove 2 edges at x each time we have $d_{M_s}(x) = 0$ and $d_{H''_s}(x) \geq 32\varepsilon_0 n$). Note that by definition of H' , all but at most one edge in H'' must have an endpoint in X . So for $x \notin X$, $d_{H''}(x) \leq |X| + 1 \leq |A_0 \cup B_0| + 1 \leq \varepsilon_0 n + 1$. Therefore, $\Delta(H''_{r+1}) < 32\varepsilon_0 n$.

Let $H''' := H_1 - (F_1 + \dots + F_r)$. So H''' is the union of H''_{r+1} and all the M_s with $r < s \leq \gamma n$. Since each of H_1 and F_1, \dots, F_r contains an even number of edges, $e(H''')$ is even. In addition, $M_s \subseteq H'''$ for each $r < s \leq \gamma n$, so $e(H''') \geq 2(\gamma n - r)$. By (4.13), since $\Delta(H''_{r+1}) \leq 32\varepsilon_0 n$, we deduce that for every vertex $v \in V(H''')$, we have

$$d_{H'''}(v) \leq \left(\frac{d_{H'_1}(v)}{3} + \varepsilon_0\gamma n \right) + \Delta(H''_{r+1}) \leq \frac{3\gamma n/5}{3} + \varepsilon_0\gamma n + 32\varepsilon_0 n \leq \frac{2(\gamma n - r)}{3}$$

In the second inequality, we used that $d_{H_1}(v) \leq d_{H'}(v)$. Moreover, we have

$$e(H''') = e(H''_{r+1}) + e(M_{r+1} + \cdots + M_{\gamma n}) \leq 32\varepsilon_0 n^2 + 30\varepsilon_0 n(\gamma n - r) \leq 62\varepsilon_0 n^2.$$

Thus, by Proposition 4.5 applied with H''' and $\gamma - r/n$ playing the roles of H and γ , there exists a decomposition of H''' into $\gamma n - r$ edge-disjoint non-empty even matchings $F_{r+1}, \dots, F_{\gamma n}$, each of size at most $3e(H''')/(\gamma n - r) \leq \sqrt{\varepsilon_0}n/2$. Thus each such F_s is a Hamilton exceptional system candidate with parameter ε_0 . This completes the proof. \square

4.2.3. Step 3: Constructing the localized exceptional systems. The next lemma will be used to extend most of the exceptional system candidates guaranteed by Lemma 4.6 into localized exceptional systems. These extensions are required to be ‘faithful’ in the following sense. Suppose that F is an exceptional system candidate. Then J is a *faithful extension* of F if the following holds:

- J contains F and $F[A', B'] = J[A', B']$.
- If F is a Hamilton exceptional system candidate, then J is a Hamilton exceptional system and the analogue holds if F is a matching exceptional system candidate.

Lemma 4.7. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1$, that $0 \leq \gamma \leq 1$ and that $n, K, m, \gamma n \in \mathbb{N}$. Let \mathcal{P} be a (K, m, ε_0) -partition of a set V of n vertices. Let $1 \leq i, i' \leq K$. Suppose that H and $F_1, \dots, F_{\gamma n}$ are pairwise edge-disjoint graphs which satisfy the following conditions:*

- (i) $V(H) = V$ and H contains only A_0A_i -edges and $B_0B_{i'}$ -edges.
- (ii) Each F_s is an (i, i') -ESC with parameter ε_0 .
- (iii) Each $v \in V_0$ satisfies $d_{H+\sum F_s}(v) \geq (2\gamma + \sqrt{\varepsilon_0})n$.

Then there exist edge-disjoint (i, i') -ES $J_1, \dots, J_{\gamma n}$ with parameter ε_0 in $H + \sum F_s$ such that J_s is a faithful extension of F_s for all $s \leq \gamma n$.

Proof. For each $s \leq \gamma n$ in turn, we extend F_s into an (i, i') -ES J_s with parameter ε_0 in $H + \sum F_s$ such that J_s and $J_{s'}$ are edge-disjoint for all $s' < s$. Since H does not contain any $A'B'$ -edges, the J_s will automatically satisfy $J_s[A', B'] = F_s[A', B']$. Suppose that for some $1 \leq s \leq \gamma n$ we have already constructed J_1, \dots, J_{s-1} . Set $H_s := H - \sum_{s' < s} J_{s'}$. Consider any $v \in V_0$. Since v has degree at most 2 in an exceptional system and in an exceptional system candidate, (iii) implies that

$$d_{H_s}(v) \geq d_{H+\sum F_s}(v) - 2\gamma n \geq \sqrt{\varepsilon_0}n.$$

Together with (i) this shows that condition (ii) in Lemma 3.1 holds (with H_s playing the role of G). Since \mathcal{P} is a (K, m, ε_0) -partition of V , Lemma 3.1(i) holds too. Hence we can apply Lemma 3.1 to obtain an exceptional system J_s with parameter ε_0 in $H_s + F_s$ such that J_s is a faithful extension of F_s . (i) and (ii) ensure that J_s is an (i, i') -ES, as required. \square

4.2.4. *Step 4: Constructing the remaining exceptional systems.* Due to condition (iii), Lemma 4.7 cannot be used to extend *all* the exceptional system candidates returned by Lemma 4.6 into localized exceptional systems. The next lemma will be used to deal with the remaining exceptional system candidates (the resulting exceptional systems will not be localized).

Lemma 4.8. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon' \ll \lambda \ll 1$ and that $n, \lambda n \in \mathbb{N}$. Let A, A_0, B, B_0 be a partition of a set V of n vertices such that $|A_0| + |B_0| \leq \varepsilon_0 n$ and $|A| = |B|$. Suppose that $H, F_1, \dots, F_{\lambda n}$ are pairwise edge-disjoint graphs which satisfy the following conditions:*

- (i) $V(H) = V$ and H contains only A_0A -edges and B_0B -edges.
- (ii) Each F_s is an exceptional system candidate with parameter ε_0 .
- (iii) For all but at most $\varepsilon'n$ indices $s \leq \lambda n$ the graph F_s is either a matching exceptional system candidate with $e(F_s) = 0$ or a Hamilton exceptional system candidate with $e(F_s) = 2$. In particular, all but at most $\varepsilon'n$ of the F_s satisfy $d_{F_s}(v) \leq 1$ for all $v \in V_0$.
- (iv) All $v \in V_0$ satisfy $d_{H+\sum F_s}(v) = 2\lambda n$.
- (v) All $v \in A \cup B$ satisfy $d_{H+\sum F_s}(v) \leq 2\varepsilon_0 n$.

Then there exists a decomposition of $H + \sum F_s$ into edge-disjoint exceptional systems $J_1, \dots, J_{\lambda n}$ with parameter ε_0 such that J_s is a faithful extension of F_s for all $s \leq \lambda n$.

Proof. Let $V_0 := A_0 \cup B_0$ and let $v_1, \dots, v_{|V_0|}$ denote the vertices of V_0 . We will decompose H into graphs J'_s in such a way that the graphs $J_s := J'_s + F_s$ satisfy $d_{J_s}(v_i) = 2$ for all $i \leq |V_0|$ and $d_{J_s}(v) \leq 1$ for all $v \in A \cup B$. Hence each J_s will be an exceptional system with parameter ε_0 . Condition (i) guarantees that J_s will be a faithful extension of F_s . Moreover, the J_s will form a decomposition of $H + \sum F_s$. We construct the decomposition of H by considering each vertex v_i of $A_0 \cup B_0$ in turn.

Initially, we set $V(J'_s) = E(J'_s) = \emptyset$ for all $s \leq \lambda n$. Suppose that for some $1 \leq i \leq |V_0|$ we have already assigned (and added) all the edges of H incident with each of v_1, \dots, v_{i-1} to the J'_s . Consider v_i . Without loss of generality assume that $v_i \in A_0$. Note that $N_H(v_i) \subseteq A$ by (i). Define an auxiliary bipartite graph Q_i with vertex classes V_1 and V_2 as follows: $V_1 := N_H(v_i)$ and V_2 consists of $2 - d_{F_s}(v_i)$ copies of F_s for each $s \leq \lambda n$. Moreover, Q_i contains an edge between $v \in V_1$ and $F_s \in V_2$ if and only if $v \notin V(F_s + J'_s)$.

We now show that Q_i contains a perfect matching. For this, note that $|V_1| = 2\lambda n - d_{\sum F_s}(v_i) = |V_2|$ by (iv). (v) implies that for each $v \in V_1 \subseteq A$ we have $d_{\sum(F_s + J'_s)}(v) \leq d_{H+\sum F_s}(v) \leq 2\varepsilon_0 n$. So v lies in at most $2\varepsilon_0 n$ of the graphs $F_s + J'_s$. Therefore, $d_{Q_i}(v) \geq |V_2| - 4\varepsilon_0 n \geq |V_2|/2$ for all $v \in V_1$. (The final inequality follows since (iii) and (iv) together imply that $d_H(v_i) = 2\lambda n - d_{\sum F_s}(v_i) \geq 2\lambda n - (\lambda n - \varepsilon'n) - 2\varepsilon'n \geq \lambda n/2$ and so $|V_2| = |V_1| \geq \lambda n/2$.) On the other hand, since each $F_s + J'_s$ is an exceptional system candidate with parameter ε_0 , (ESC3) implies that $|V(F_s + J'_s) \cap A| \leq (\sqrt{\varepsilon_0}/2 + 2\varepsilon_0)n \leq \sqrt{\varepsilon_0}n$ for each $F_s \in V_2$. Therefore $d_{Q_i}(F_s) \geq |V_1| - |V(F_s + J'_s) \cap A| \geq |V_1|/2$ for each $F_s \in V_2$. Thus we can apply Hall's theorem to find a perfect matching M in Q_i . Whenever M contains an edge between v and

F_s , we add the edge $v_i v$ to J'_s . This completes the desired assignment of the edges of H at v_i to the J'_s . \square

4.2.5. *Proof of Lemma 4.2.* In our proof of Lemma 4.2 we will use the following result, which is a consequence of Lemmas 4.7 and 4.8. Given a suitable set of exceptional system candidates in an exceptional scheme, the lemma extends these into exceptional systems which form a decomposition of the exceptional scheme. We prove the lemma in a slightly more general form than needed for the current case, as we will also use it in the other two cases.

Lemma 4.9. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \varepsilon' \ll \lambda, 1/K \ll 1$, that $1/(7K^2) \leq \alpha < 1/K^2$ and that $n, K, m, \alpha n, \lambda n/K^2 \in \mathbb{N}$. Let*

$$\gamma := \alpha - \frac{\lambda}{K^2} \quad \text{and} \quad \gamma' := \frac{\lambda}{K^2}.$$

Suppose that the following conditions hold:

- (i) (G^*, \mathcal{P}) is a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme with $|G^*| = n$.
- (ii) G^* is the edge-disjoint union of $H(i, i')$, $F_1(i, i')$, \dots , $F_{\gamma n}(i, i')$ and $F'_1(i, i')$, \dots , $F'_{\gamma' n}(i, i')$ over all $1 \leq i, i' \leq K$.
- (iii) Each $H(i, i')$ contains only $A_0 A_i$ -edges and $B_0 B_{i'}$ -edges.
- (iv) Each $F_s(i, i')$ is an (i, i') -ESC with parameter ε_0 .
- (v) Each $F'_s(i, i')$ is an exceptional system candidate with parameter ε_0 . Moreover, for all but at most $\varepsilon' n$ indices $s \leq \gamma' n$ the graph $F'_s(i, i')$ is either a matching exceptional system candidate with $e(F'_s(i, i')) = 0$ or a Hamilton exceptional system candidate with $e(F'_s(i, i')) = 2$.
- (vi) $d_{G^*}(v) = 2K^2 \alpha n$ for all $v \in V_0$.
- (vii) For all $1 \leq i, i' \leq K$ let $G^*(i, i') := H(i, i') + \sum_{s \leq \gamma n} F_s(i, i') + \sum_{s \leq \gamma' n} F'_s(i, i')$. Then $d_{G^*(i, i')}(v) = (2\alpha \pm \varepsilon')n$ for all $v \in V_0$.

Then G^ has a decomposition into $K^2 \alpha n$ edge-disjoint exceptional systems*

$$J_1(i, i'), \dots, J_{\gamma n}(i, i') \quad \text{and} \quad J'_1(i, i'), \dots, J'_{\gamma' n}(i, i')$$

with parameter ε_0 , where $1 \leq i, i' \leq K$, such that $J_s(i, i')$ is an (i, i') -ES which is a faithful extension of $F_s(i, i')$ for all $s \leq \gamma n$ and $J'_s(i, i')$ is a faithful extension of $F'_s(i, i')$ for all $s \leq \gamma' n$.

Proof. Fix any $i, i' \leq K$ and set $H := H(i, i')$ and $F_s := F_s(i, i')$ for all $s \leq \gamma n$. Our first aim is to apply Lemma 4.7 in order to extend each of $F_1, \dots, F_{\gamma n}$ into a (i, i') -HES. (iii) and (iv) ensure that conditions (i) and (ii) of Lemma 4.7 hold. To verify Lemma 4.7(iii), note that by (v) and (vii) each $v \in V_0$ satisfies

$$\begin{aligned} d_{H + \sum F_s}(v) &= d_{G^*(i, i')}(v) - d_{\sum_s F'_s(i, i')}(v) \geq (2\alpha - \varepsilon')n - (\gamma' - \varepsilon')n - 2\varepsilon' n \\ &= (2\alpha - \gamma' - 2\varepsilon')n \geq (2\gamma + \sqrt{\varepsilon_0})n. \end{aligned}$$

(Here the first inequality follows since (v) implies that $d_{F'_s(i, i')}(v) \leq 1$ for all but at most $\varepsilon' n$ indices $s \leq \gamma' n$.) Thus we can indeed apply Lemma 4.7 to find edge-disjoint (i, i') -ES $J_1(i, i'), \dots, J_{\gamma n}(i, i')$ with parameter ε_0 in $H + \sum F_s$ such that $J_s(i, i')$ is a

faithful extension of F_s for all $s \leq \gamma n$. We repeat this procedure for all $1 \leq i, i' \leq K$ to obtain $K^2 \gamma n$ edge-disjoint (localized) exceptional systems.

Our next aim is to apply Lemma 4.8 in order to construct the $J'_s(i, i')$. Let H_0 be the union of $H(i, i') - (J_1(i, i') + \dots + J_{\gamma n}(i, i'))$ over all $i, i' \leq K$. Relabel the $F'_s(i, i')$ (for all $s \leq \gamma' n$ and all $i, i' \leq K$) to obtain exceptional system candidates $F'_1, \dots, F'_{\lambda n}$. Note that by (vi) each $v \in V_0$ satisfies

$$(4.14) \quad d_{H_0 + \sum F'_s}(v) = d_{G^*}(v) - 2K^2 \gamma n = 2K^2 \alpha n - 2K^2 \gamma n = 2\lambda n.$$

Thus condition (iv) of Lemma 4.8 holds with H_0, F'_s playing the roles of H, F_s . (iii) and (v) imply that conditions (i)–(iii) of Lemma 4.8 hold with $K^2 \varepsilon'$ playing the role of ε' . To verify Lemma 4.8(v), note that each $v \in A$ satisfies $d_{H_0 + \sum F'_s}(v) \leq d_{G^*}(v, A_0) + d_{G^*}(v, B') \leq 2\varepsilon_0 n$ by (iii), (i) and (ESch3). Similarly each $v \in B$ satisfies $d_{H_0 + \sum F'_s}(v) \leq 2\varepsilon_0 n$. Thus we can apply Lemma 4.8 with $H_0, F'_s, K^2 \varepsilon'$ playing the roles of H, F_s, ε' to obtain a decomposition of $H_0 + \sum_s F'_s$ into λn edge-disjoint exceptional systems $J'_1, \dots, J'_{\lambda n}$ with parameter ε_0 such that J'_s is a faithful extension of F'_s for all $s \leq \lambda n$. Recall that each F'_s is a $F'_{s'}(i, i')$ for some $i, i' \leq K$ and some $s' \leq \gamma' n$. Let $J'_{s'}(i, i') := J'_s$. Then all the $J_s(i, i')$ and all the $J'_s(i, i')$ are as required in the lemma. \square

We will now combine Lemmas 4.4, 4.6 and 4.9 in order to prove Lemma 4.2.

Proof of Lemma 4.2. Let G^\diamond be as defined in Lemma 4.2(iv). Choose a new constant ε' such that $\varepsilon \ll \varepsilon' \ll \lambda, 1/K$. Set

$$(4.15) \quad 2\alpha n := \frac{D - \phi n}{K^2}, \quad \gamma_1 := \alpha - \frac{2\lambda}{K^2} \quad \text{and} \quad \gamma'_1 := \frac{2\lambda}{K^2}.$$

Similarly as in the proof of Lemma 4.4, since $\phi \ll 1/3 \leq D/n$, we have

$$(4.16) \quad \alpha \geq 1/(7K^2), \quad (1 - 14\lambda)\alpha \leq \gamma_1 < \alpha \quad \text{and} \quad \varepsilon \ll \varepsilon' \ll \lambda, 1/K, \alpha, \gamma_1 \ll 1.$$

Apply Lemma 4.4 with γ_1 playing the role of γ in order to obtain a decomposition of G^\diamond into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ (for all $1 \leq i, i' \leq K$) which satisfy the following properties, where $G'(i, i') := H(i, i') + H''(i, i')$:

- (b1) Each $H(i, i')$ contains only $A_0 A_i$ -edges and $B_0 B_{i'}$ -edges.
- (b2) $H''(i, i') \subseteq G^\diamond[A', B']$. Moreover, all but at most $\varepsilon' n$ edges of $H''(i, i')$ lie in $G^\diamond[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (b3) $e(H''(i, i'))$ is even and $2\alpha n \leq e(H''(i, i')) \leq 11\varepsilon_0 n^2 / (10K^2)$.
- (b4) $\Delta(H''(i, i')) \leq 31\alpha n / 30$.
- (b5) $d_{G'(i, i')}(v) = (2\alpha \pm \varepsilon') n$ for all $v \in V_0$.
- (b6) Let \tilde{H} any spanning subgraph of $H''(i, i')$ which maximises $e(\tilde{H})$ under the constraints that $\Delta(\tilde{H}) \leq 3\gamma_1 n / 5$, $H''(i, i')[A_0, B_0] \subseteq \tilde{H}$ and $e(\tilde{H})$ is even. Then $e(\tilde{H}) \geq 2\alpha n$.

Fix any $1 \leq i, i' \leq K$. Set $H := H(i, i')$ and $H'' := H''(i, i')$. Our next aim is to decompose H'' into suitable ‘localized’ Hamilton exceptional system candidates. For this, we will apply Lemma 4.6 with H'', γ_1, γ'_1 playing the roles of H, γ, γ' . Note that $\Delta(H'') \leq 31\alpha n / 30 \leq 16\gamma_1 n / 15$ by (b4) and (4.16). Moreover, $\Delta(H''[A, B]) \leq$

$\Delta(G^\circ[A, B]) \leq \varepsilon_0 n$ by (iv) and (ESch3). Since $e(H'')$ is even by (b₃), it follows that condition (i) of Lemma 4.6 holds. Condition (ii) of Lemma 4.6 follows from (b₆) and the fact that any \tilde{H} as in (b₆) satisfies $e(\tilde{H}) \leq e(H'') \leq 11\varepsilon_0 n^2 / (10K^2) \leq 10\varepsilon_0 \gamma_1 n^2$ (the last inequality follows from (4.16)). Thus we can indeed apply Lemma 4.6 in order to decompose H'' into αn edge-disjoint Hamilton exceptional system candidates $F_1, \dots, F_{\gamma_1 n}, F'_1, \dots, F'_{\gamma'_1 n}$ with parameter ε_0 such that $e(F'_s) = 2$ for all $s \leq \gamma'_1 n$. Next we set

$$\gamma_2 := \alpha - \frac{\lambda}{K^2} \quad \text{and} \quad \gamma'_2 := \frac{\lambda}{K^2}.$$

Condition (b₂) ensures that by relabeling the F_s 's and F'_s 's we obtain αn edge-disjoint Hamilton exceptional system candidates $F_1(i, i'), \dots, F_{\gamma_2 n}(i, i'), F'_1(i, i'), \dots, F'_{\gamma'_2 n}(i, i')$ with parameter ε_0 such that properties (a') and (b') hold:

- (a') $F_s(i, i')$ is an (i, i') -HESC for every $s \leq \gamma_2 n$. Moreover, at least $\gamma'_2 n$ of the $F_s(i, i')$ satisfy $e(F_s(i, i')) = 2$.
- (b') $e(F'_s(i, i')) = 2$ for all but at most $\varepsilon' n$ of the $F'_s(i, i')$.

Indeed, we can achieve this by relabeling each F_s which is a subgraph of $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$ as one of the $F_{s'}(i, i')$ and each F_s for which is not the case as one of the $F'_{s'}(i, i')$.

Our next aim is to apply Lemma 4.9 with $G^\circ, \gamma_2, \gamma'_2$ playing the roles of G^*, γ, γ' . Clearly conditions (i) and (ii) of Lemma 4.9 hold. (iii) follows from (b₁). (iv) and (v) follow from (a') and (b'). (vi) follows from Lemma 4.2(i),(iii). Finally, (vii) follows from (b₅) since $G'(i, i')$ plays the role of $G^*(i, i')$. Thus we can indeed apply Lemma 4.9 to obtain a decomposition of G° into $K^2 \alpha n$ edge-disjoint Hamilton exceptional systems $J_1(i, i'), \dots, J_{\gamma_2 n}(i, i')$ and $J'_1(i, i'), \dots, J'_{\gamma'_2 n}(i, i')$ with parameter ε_0 , where $1 \leq i, i' \leq K$, such that $J_s(i, i')$ is an (i, i') -HES which is a faithful extension of $F_s(i, i')$ for all $s \leq \gamma_2 n$ and $J'_s(i, i')$ is a faithful extension of $F'_s(i, i')$ for all $s \leq \gamma'_2 n$. Then the set \mathcal{J} of all these Hamilton exceptional systems is as required in Lemma 4.2. \square

4.3. Critical case with $e(A', B') \geq D$. The aim of this section is to prove the following analogue of Lemma 4.2 for the case when G is critical and $e_G(A', B') \geq D$. For this, recall that G is critical if $\Delta(G[A', B']) \geq 11D/40$ and $e(H) \leq 41D/40$ for all subgraphs H of $G[A', B']$ such that $\Delta(H) \leq 11D/40$. By Lemma 4.1(ii) we know that in this case $D = (n-1)/2$ or $D = n/2 - 1$.

Lemma 4.10. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \lambda, 1/K \ll 1$, that $D \geq n - 2\lfloor n/4 \rfloor - 1$, that $0 \leq \phi \ll 1$ and that $n, K, m, \lambda n/K^2, (D - \phi n)/(400K^2) \in \mathbb{N}$. Suppose that the following conditions hold:*

- (i) G is a D -regular graph on n vertices.
- (ii) \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$ such that $e_G(A', B') \geq D$ and $\Delta(G[A', B']) \leq D/2$. Furthermore, G is critical. In particular, $e_G(A', B') < n$ and $D = (n-1)/2$ or $D = n/2 - 1$ by Lemma 4.1(ii) and (iii).
- (iii) G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$, $e_{G_0}(A', B') \leq \phi n$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.

- (iv) Let $G^\diamond := G - G[A] - G[B] - G_0$. $e_{G^\diamond}(A', B')$ is even and $(G^\diamond, \mathcal{P})$ is a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme.
- (v) Let w_1 and w_2 be (fixed) vertices such that $d_{G[A', B']}(w_1) \geq d_{G[A', B']}(w_2) \geq d_{G[A', B']}(v)$ for all $v \in V(G) \setminus \{w_1, w_2\}$. Suppose that

$$(4.17) \quad d_{G^\diamond[A', B']}(w_1), d_{G^\diamond[A', B']}(w_2) \leq (D - \phi n)/2.$$

Then there exists a set \mathcal{J} consisting of $(D - \phi n)/2$ edge-disjoint Hamilton exceptional systems with parameter ε_0 in G^\diamond which satisfies the following properties:

- (a) Together the Hamilton exceptional systems in \mathcal{J} cover all edges of G^\diamond .
- (b) For each $1 \leq i, i' \leq K$, the set \mathcal{J} contains $(D - (\phi + 2\lambda)n)/(2K^2)$ (i, i') -HES. Moreover, $\lambda n/K^2$ of these (i, i') -HES are such that
 - (b₁) $e_J(A', B') = 2$ and
 - (b₂) $d_{J[A', B']}(w) = 1$ for all $w \in \{w_1, w_2\}$ with $d_{G[A', B']}(w) \geq 11D/40$.

Similarly as for Lemma 4.2, (b) implies that \mathcal{J} contains λn Hamilton exceptional systems which might not be localized. Another similarity is that when constructing the robustly decomposable graph G^{rob} in [7], we only use those J_s which have some additional useful properties, namely (b₁) and (b₂) in this case. This gives us a way of satisfying (4.17) in the second application of Lemma 4.10 in [7] (i.e. after the removal of G^{rob}), by ‘tracking’ the degrees of the high degree vertices w_1 and w_2 . Indeed, if $d_{G[A', B']}(w_2) \geq 11D/40$, then (b₂) will imply that $d_{G^{\text{rob}}[A', B']}(w_i)$ is large for $i = 1, 2$. This in turn means that after removing G^{rob} , in the leftover graph G^\diamond , $d_{G^\diamond[A', B']}(w_i)$ is comparatively small, i.e. condition (4.17) will hold in the second application of Lemma 4.10.

Condition (4.17) itself is natural for the following reason: suppose for example that it is violated for w_1 and that $w_1 \in A_0$. Then for some Hamilton exceptional system J returned by the lemma, both edges of J incident to w_1 will have their other endpoint in B' . So (the edges at) w_1 cannot be used as a ‘connection’ between A' and B' in the Hamilton cycle which will extend J , and it may be impossible to find such a connection elsewhere.

The overall strategy for the proof of Lemma 4.10 is similar to that of Lemma 4.2. As before, it consists of four steps. In Step 1, we use Lemma 4.11 instead of Lemma 4.4. In Step 2, we use Lemma 4.13 instead of Lemma 4.6. We still use Lemma 4.9 which combines Steps 3 and 4.

4.3.1. Step 1: Constructing the graphs $H''(i, i')$. The next lemma is an analogue of Lemma 4.4. We will apply it with the graph G^\diamond from Lemma 4.10(iv) playing the role of G . Note that instead of assuming that our graph G given in Lemma 4.10 is critical, the lemma assumes that $e_{G^\diamond}(A', B') \leq 2n$. This is a weaker assumption, since if G is critical, then $e_{G^\diamond}(A', B') \leq e_G(A', B') < n$ by Lemma 4.1(iii). Using only this weaker assumption has the advantage that we can also apply the lemma in the proof of Lemma 4.14, i.e. the case when $e_G(A', B') < D$. (b₇) is only used in the latter application.

Lemma 4.11. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll 1/K \ll 1$ and that $n, K, m \in \mathbb{N}$. Let (G, \mathcal{P}) be a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme with $|G| = n$ and $e_G(A_0), e_G(B_0) = 0$.*

Let W_0 be a subset of V_0 of size at most 2 such that for each $w \in W_0$, we have

$$(4.18) \quad K^2 \leq d_{G[A', B']}(w) \leq e_G(A', B')/2.$$

Suppose that $e_G(A', B') \leq 2n$ is even. Then G can be decomposed into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ of G (for all $1 \leq i, i' \leq K$) such that the following properties hold, where $G'(i, i') := H(i, i') + H''(i, i')$:

- (b₁) Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.
- (b₂) $H''(i, i') \subseteq G[A', B']$. Moreover, all but at most $20\epsilon n/K^2$ edges of $H''(i, i')$ lie in $G[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (b₃) $e(H''(i, i')) = 2 \lceil e_G(A', B')/(2K^2) \rceil$ or $e(H''(i, i')) = 2 \lfloor e_G(A', B')/(2K^2) \rfloor$.
- (b₄) $d_{H''(i, i')}(v) = (d_{G[A', B']}(v) \pm 25\epsilon n)/K^2$ for all $v \in V_0$.
- (b₅) $d_{G'(i, i')}(v) = (d_G(v) \pm 25\epsilon n)/K^2$ for all $v \in V_0$.
- (b₆) Each $w \in W_0$ satisfies $d_{H''(i, i')}(w) = \lceil d_{G[A', B']}(w)/K^2 \rceil$ or $d_{H''(i, i')}(w) = \lfloor d_{G[A', B']}(w)/K^2 \rfloor$.
- (b₇) Each $w \in W_0$ satisfies $2d_{H''(i, i')}(w) \leq e(H''(i, i'))$.

Proof. Since $e_G(A', B')$ is even, there exist unique non-negative integers b and q such that $e_G(A', B') = 2K^2b + 2q$ and $q < K^2$. Hence, for all $1 \leq i, i' \leq K$, there are integers $b_{i, i'} \in \{2b, 2b + 2\}$ such that $\sum_{i, i' \leq K} b_{i, i'} = e_G(A', B')$. In particular, the number of pairs i, i' for which $b_{i, i'} = b + 2$ is precisely q . We will choose the graphs $H''(i, i')$ such that $e(H''(i, i')) = b_{i, i'}$. (In particular, this will ensure that (b₃) holds.) The following claim will help to ensure (b₆) and (b₇).

Claim. For each $w \in W_0$ and all $i, i' \leq K$ there is an integer $a_{i, i'} = a_{i, i'}(w)$ which satisfies the following properties:

- $a_{i, i'} = \lceil d_{G[A', B']}(w)/K^2 \rceil$ or $a_{i, i'} = \lfloor d_{G[A', B']}(w)/K^2 \rfloor$.
- $2a_{i, i'} \leq b_{i, i'}$.
- $\sum_{i, i' \leq K} a_{i, i'} = d_{G[A', B']}(w)$.

To prove the claim, note that there are unique non-negative integers a and p such that $d_{G[A', B']}(w) = K^2a + p$ and $p < K^2$. Note that $a \geq 1$ by (4.18). Moreover,

$$(4.19) \quad 2(K^2a + p) = 2d_{G[A', B']}(w) \stackrel{(4.18)}{\leq} e_G(A', B') = 2K^2b + 2q.$$

This implies that $a \leq b$. Recall that $b_{i, i'} \in \{2b, 2b + 2\}$. So if $b > a$, then the claim holds by choosing any $a_{i, i'} \in \{a, a + 1\}$ such that $\sum_{i, i' \leq K} a_{i, i'} = d_{G[A', B']}(w)$. Hence we may assume that $a = b$. Then (4.19) implies that $p \leq q$. Therefore, the claim holds by setting $a_{i, i'} := a + 1$ for exactly p pairs i, i' for which $b_{i, i'} = 2b + 2$ and setting $a_{i, i'} := a$ otherwise. This completes the proof of the claim.

Apply Lemma 4.3 to decompose G into subgraphs $H(i, i')$, $H'(i, i')$ (for all $i, i' \leq K$) satisfying the following properties, where $G(i, i') = H(i, i') + H'(i, i')$:

- (a₁') Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.
- (a₂') All edges of $H'(i, i')$ lie in $G[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (a₃') $e(H'(i, i')) = (e_G(A', B') \pm 8\epsilon n)/K^2$.
- (a₄') $d_{H'(i, i')}(v) = (d_{G[A', B']}(v) \pm 2\epsilon n)/K^2$ for all $v \in V_0$.
- (a₅') $d_{G(i, i')}(v) = (d_G(v) \pm 4\epsilon n)/K^2$ for all $v \in V_0$.

Indeed, (a₃') follows from Lemma 4.3(a₃) and our assumption that $e_G(A', B') \leq 2n$.

Clearly, (a₁') implies that the graphs $H(i, i')$ satisfy (b₁). We will now move some $A'B'$ -edges of G between the $H(i, i')$ such that the graphs $H''(i, i')$ obtained in this way satisfy the following conditions:

- Each $H''(i, i')$ is obtained from $H(i, i')$ by adding or removing at most $20\epsilon n/K^2$ edges of G .
- $e(H''(i, i')) = b_{i, i'}$.
- $d_{H''(i, i')}(w) = a_{i, i'}(w)$ for each $w \in W_0$, where $a_{i, i'}(w)$ are integers satisfying the claim.

Write $W_0 =: \{w_1\}$ if $|W_0| = 1$ and $W_0 =: \{w_1, w_2\}$ if $|W_0| = 2$. If $W_0 \neq \emptyset$, then (a₄') implies that $d_{H(i, i')}(w_1) = a_{i, i'}(w_1) \pm (2\epsilon n/K^2 + 1)$. For each $i, i' \leq K$, we add or remove at most $2\epsilon n/K^2 + 1$ edges incident to w_1 such that the graphs $H''(i, i')$ obtained in this way satisfy $d_{H''(i, i')}(w_1) = a_{i, i'}(w_1)$. Note that since $a_{i, i'}(w_1) \geq \lfloor d_{G[A', B']}(w_1)/K^2 \rfloor \geq 1$ by (4.18), we can do this in such a way that we do not move the edge $w_1 w_2$ (if it exists). Similarly, if $|W_0| = 2$, then for each $i, i' \leq K$ we add or remove at most $2\epsilon n/K^2 + 1$ edges incident to w_2 such that the graphs $H''(i, i')$ obtained in this way satisfy $d_{H''(i, i')}(w_2) = a_{i, i'}(w_2)$. As before, we do this in such a way that we do not move the edge $w_1 w_2$ (if it exists).

Thus $d_{H''(i, i')}(w_1) = a_{i, i'}(w_1)$ and $d_{H''(i, i')}(w_2) = a_{i, i'}(w_2)$ for all $1 \leq i, i' \leq K$ (if w_1, w_2 exist). In particular, together with the claim this implies that $d_{H''(i, i')}(w_1), d_{H''(i, i')}(w_2) \leq b_{i, i'}/2$. Thus the number of edges of $H''(i, i')$ incident to W_0 is at most

$$(4.20) \quad \sum_{w \in W_0} d_{H''(i, i')}(w) \leq b_{i, i'}.$$

(This holds regardless of the size of W_0 .) On the other hand, (a₃') implies that for all $i, i' \leq K$ we have

$$e(H''(i, i')) = (e_G(A', B') \pm 8\epsilon n)/K^2 \pm 2(2\epsilon n/K^2 + 1) = b_{i, i'} \pm 13\epsilon n/K^2.$$

Together with (4.20) this ensures that we can add or delete at most $13\epsilon n/K^2$ edges which do not intersect W_0 to or from each $H''(i, i')$ in order to ensure that $e(H''(i, i')) = b_{i, i'}$ for all $i, i' \leq K$. Hence, (b₃), (b₆) and (b₇) hold. Moreover,

$$(4.21) \quad e(H''(i, i') - H(i, i')) \leq |W_0|(2\epsilon n/K^2 + 1) + 13\epsilon n/K^2 \leq 20\epsilon n/K^2.$$

So (b₂) follows from (a₂'). Finally, (b₄) and (b₅) follow from (4.21), (a₄') and (a₅'). \square

4.3.2. Step 2: Decomposing $H''(i, i')$ into Hamilton exceptional system candidates.

Before we can prove an analogue of Lemma 4.6, we need the following result. It will allow us to distribute the edges incident to the (up to three) vertices w_i of high degree in $G[A', B']$ in a suitable way among the localized Hamilton exceptional system candidates F_j . The degrees of these high degree vertices w_i will play the role of the a_i . The c_j will account for edges (not incident to w_i) which have already been assigned to the F_j . (b) and (c) will be used to ensure (ESC4), i.e. that the total number of 'connections' between A' and B' is even and positive.

Lemma 4.12. *Let $1 \leq q \leq 3$ and $0 \leq \eta < 1$ and $r, \eta r \in \mathbb{N}$. Suppose that $a_1, \dots, a_q \in \mathbb{N}$ and $c_1, \dots, c_r \in \{0, 1, 2\}$ satisfy the following conditions:*

- (i) $c_1 \geq \dots \geq c_r \geq c_1 - 1$.
- (ii) $\sum_{i \leq q} a_i + \sum_{j \leq r} c_j = 2(1 + \eta)r$.
- (iii) $31r/60 \leq a_1, a_2 \leq r$ and $31r/60 \leq a_3 \leq 31r/30$.

Then for all $i \leq q$ and all $j \leq r$ there are $a_{i,j} \in \{0, 1, 2\}$ such that the following properties hold:

- (a) $\sum_{j \leq r} a_{i,j} = a_i$ for all $i \leq q$.
- (b) $c_j + \sum_{i \leq q} a_{i,j} = 4$ for all $j \leq \eta r$ and $c_j + \sum_{i \leq q} a_{i,j} = 2$ for all $\eta r < j \leq r$.
- (c) For all $j \leq r$ there are at least $2 - c_j$ indices $i \leq q$ with $a_{i,j} = 1$.

Proof. We will choose $a_{i,1}, \dots, a_{i,r}$ for each $i \leq q$ in turn such that the following properties (α_i) – (ρ_i) hold, where we write $c_j^{(i)} := c_j + \sum_{i' \leq i} a_{i',j}$ for each $0 \leq i \leq q$ (so $c_j^{(0)} = c_j$):

- (α_i) If $i \geq 1$ then $\sum_{j \leq r} a_{i,j} = a_i$.
- (β_i) $4 \geq c_1^{(i)} \geq \dots \geq c_r^{(i)}$.
- (γ_i) If $\sum_{j \leq r} c_j^{(i)} < 2r$, then $|c_j^{(i)} - c_{j'}^{(i)}| \leq 1$ for all $j, j' \leq r$.
- (δ_i) If $\sum_{j \leq r} c_j^{(i)} \geq 2r$, then $c_j^{(i)} \geq 2$ for all $j \leq \eta r$ and $c_j^{(i)} = 2$ for all $\eta r < j \leq r$.
- (ρ_i) If $1 \leq i \leq q$ and $c_j^{(i-1)} < 2$ for some $j \leq r$, then $a_{i,j} \in \{0, 1\}$.

We will then show that the $a_{i,j}$ defined in this way are as required in the lemma.

Note that (i) and the fact that $c_1, \dots, c_r \in \{0, 1, 2\}$ together imply (β_0) – (δ_0) . Moreover, (α_0) and (ρ_0) are vacuously true. Suppose that for some $1 \leq i \leq q$ we have already defined $a_{i',j}$ for all $i' < i$ and all $j \leq r$ such that $(\alpha_{i'})$ – $(\rho_{i'})$ hold. In order to define $a_{i,j}$ for all $j \leq r$, we distinguish the following cases.

Case 1: $\sum_{j \leq r} c_j^{(i-1)} \geq 2r$.

Recall that in this case $c_j^{(i-1)} \geq 2$ for all $j \leq r$ by (δ_{i-1}) . For each $j \leq r$ in turn we choose $a_{i,j} \in \{0, 1, 2\}$ as large as possible subject to the constraints that

- $a_{i,j} + c_j^{(i-1)} \leq 4$ and
- $\sum_{j' \leq j} a_{i,j'} \leq a_i$.

Since $c_j^{(i)} = a_{i,j} + c_j^{(i-1)}$, (β_i) follows from (β_{i-1}) and our choice of the $a_{i,j}$. (γ_i) is vacuously true. To verify (δ_i) , note that $c_j^{(i)} \geq c_j^{(i-1)} \geq 2$ by (δ_{i-1}) . Suppose that the second part of (δ_i) does not hold, i.e. that $c_{\eta r+1}^{(i)} > 2$. This means that $a_{i,\eta r+1} > 0$. Together with our choice of the $a_{i,j}$ this implies that $c_j^{(i)} = 4$ for all $j \leq \eta r$. Thus

$$2(1 + \eta)r = 4\eta r + 2(r - \eta r) < \sum_{j \leq r} c_j^{(i)} = \sum_{j \leq r} a_{i,j} + \sum_{i' < i} a_{i',j} + \sum_{j \leq r} c_j \leq \sum_{i' < i} a_{i'} + \sum_{j \leq r} c_j$$

contradicting (ii). Thus the second part of (δ_i) holds too. Moreover, $c_{\eta m+1}^{(i)} = c_{\eta m+1}^{(i-1)} = 2$ also means that $a_{i,\eta m+1} = 0$. So $\sum_{j' \leq \eta m} a_{i,j'} = a_i$, i.e. (α_i) holds. (ρ_i) is vacuously true since $c_j^{(i-1)} \geq 2$ by (δ_{i-1}) .

Case 2: $2r - a_i \leq \sum_{j \leq r} c_j^{(i-1)} < 2r$.

If $i \in \{1, 2\}$ then together with (iii) this implies that

$$(4.22) \quad \sum_{j \leq r} c_j^{(i-1)} \geq r \geq a_i.$$

If $i = 3$ then

$$(4.23) \quad \sum_{j \leq r} c_j^{(i-1)} \geq \sum_{j \leq r} \sum_{i' \leq 2} a_{i',j} = a_1 + a_2 \geq \frac{31r}{30} \geq a_3$$

by (iii). In particular, in both cases we have $\sum_{j \leq r} c_j^{(i-1)} \geq r$. Together with (γ_{i-1}) this implies that $c_j^{(i-1)} \in \{1, 2\}$ for all $j \leq r$. Let $0 \leq r' \leq r$ be the largest integer such that $c_{r'}^{(i-1)} = 2$. So $r' < r$ and $\sum_{j \leq r} c_j^{(i-1)} = r + r'$. Together with (4.22) and (4.23) this in turn implies that $a_i \leq r + r'$ (regardless of the value of i).

Set $a_{i,j} := 1$ for all $r' < j \leq r$. Note that

$$\sum_{r' < j \leq r} a_{i,j} = r - r' = 2r - \sum_{j \leq r} c_j^{(i-1)} \leq a_i,$$

where the final inequality comes from the assumption of Case 2. Take $a_{i,1}, \dots, a_{i,r'}$ to be a sequence of the form $2, \dots, 2, 0, \dots, 0$ (in the case when $a_i - \sum_{r' < j \leq r} a_{i,j}$ is even) or $2, \dots, 2, 1, 0, \dots, 0$ (in the case when $a_i - \sum_{r' < j \leq r} a_{i,j}$ is odd) which is chosen in such a way that $\sum_{j \leq r'} a_{i,j} = a_i - \sum_{r' < j \leq r} a_{i,j} = a_i - r + r'$. This can be done since $a_i \leq r + r'$ implies that the right hand side is at most $2r'$.

Clearly, (α_i) , (β_i) and (ρ_i) hold. Since $\sum_{j \leq r} c_j^{(i)} = a_i + \sum_{j \leq r} c_j^{(i-1)} \geq 2r$ as we are in Case 2, (γ_i) is vacuously true. Clearly, our choice of the $a_{i,j}$ guarantees that $c_j^{(i)} \geq 2$ for all $j \leq r$. As in Case 1 one can show that $c_j^{(i)} = 2$ for all $\eta r < j \leq r$. Thus (δ_i) holds.

Case 3: $\sum_{j \leq r} c_j^{(i-1)} < 2r - a_i$.

Note that in this case

$$2r > \sum_{j \leq r} c_j^{(i-1)} + a_i = \sum_{i' \leq i} a_{i',j} + \sum_{j \leq r} c_j,$$

and so $i < q$ by (ii). Together with (iii) this implies that $a_i \leq r$. Thus for all $j \leq r$ we can choose $a_{i,j} \in \{0, 1\}$ such that (α_i) – (γ_i) and (ρ_i) are satisfied. (δ_i) is vacuously true.

This completes the proof of the existence of numbers $a_{i,j}$ (for all $i \leq q$ and all $j \leq r$) satisfying (α_i) – (ρ_i) . It remains to show that these $a_{i,j}$ are as required in the lemma. Clearly, (α_1) – (α_q) imply that (a) holds. Since $c_j^{(q)} = c_j + \sum_{i \leq q} a_{i,j}$ the

second part of (b) follows from (δ_q) . Since $c_j^{(q)} \leq 4$ for each $j \leq \eta r$ by (β_q) , together with (ii) this in turn implies that the first part of (b) must hold too. If $c_j < 2$, then (ρ_1) – (ρ_q) and (b) together imply that for at least $2 - c_j$ indices i we have $a_{i,j} = 1$. Therefore, (c) holds. \square

We can now use the previous lemma to decompose the bipartite graph induced by A' and B' into Hamilton exceptional system candidates.

Lemma 4.13. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \alpha < 1$, that $0 \leq \eta < 199/200$ and that $n, \alpha n/200, \eta \alpha n \in \mathbb{N}$. Let H be a bipartite graph on n vertices with vertex classes $A \dot{\cup} A_0$ and $B \dot{\cup} B_0$ where $|A_0| + |B_0| \leq \varepsilon_0 n$. Furthermore, suppose that the following conditions hold:*

- (c₁) $e(H) = 2(1 + \eta)\alpha n$.
- (c₂) *There is a set $W' \subseteq V(H)$ with $1 \leq |W'| \leq 3$ and such that*

$$e(H - W') \leq 199\alpha n/100 \text{ and } d_H(w) \geq 13\alpha n/25 \text{ for all } w \in W'.$$
- (c₃) *There exists a set $W_0 \subseteq W'$ with $|W_0| = \min\{2, |W'|\}$ and such that $d_H(w) \leq \alpha n$ for all $w \in W_0$ and $d_H(w') \leq 41\alpha n/40$ for all $w' \in W' \setminus W_0$.*
- (c₄) *For all $w \in W'$ and all $v \in V(H) \setminus W'$ we have $d_H(w) - d_H(v) \geq \alpha n/150$.*
- (c₅) *For all $v \in A \cup B$ we have $d_H(v) \leq \varepsilon_0 n$.*

Then there exists a decomposition of H into edge-disjoint Hamilton exceptional system candidates $F_1, \dots, F_{\alpha n}$ such that $e(F_s) = 4$ for all $s \leq \eta \alpha n$ and $e(F_s) = 2$ for all $\eta \alpha n < s \leq \alpha n$. Furthermore, at least $\alpha n/200$ of the F_s satisfy the following two properties:

- $d_{F_s}(w) = 1$ for all $w \in W_0$,
- $e(F_s) = 2$.

Roughly speaking, the idea of the proof is first to find the F_s which satisfy the final two properties. Let H_1 be the graph obtained from H by removing the edges in all these F_s . We will decompose $H_1 - W'$ into matchings M_j of size at most two. Next, we extend these matchings into Hamilton exceptional system candidates F_j using Lemma 4.12. In particular, if $e(M_j) < 2$, then we will use one or more edges incident to W' to ensure that the number of $A'B'$ -connections is positive and even, as required by (ESC4). (Note that it does not suffice to ensure that the number of $A'B'$ -edges is positive and even for this.)

Proof. Set $H' := H - W'$, $W_0 := \{w_1, w_{|W_0|}\}$ and $W' := \{w_1, \dots, w_{|W'|}\}$. Hence, if $|W'| = 3$, then $W' \setminus W_0 = \{w_3\}$. Otherwise $W' = W_0$.

We will first construct $e_H(W')$ Hamilton exceptional system candidates F_s , such that each of them is a matching of size two and together they cover all edges in $H[W']$. So suppose that $e_H(W') > 0$. Thus $|W'| = 2$ or $|W'| = 3$. If $|W'| = 2$, let f denote the unique edge in $H[W']$. Note that

$$e(H') \geq e(H) - (d_H(w_1) + d_H(w_2) - 1) \geq 2(1 + \eta)\alpha n - (2\alpha n - 1) \geq 1$$

by (c₁) and (c₃). So there exists an edge f' in H' . Therefore, $M'_1 := \{f, f'\}$ is a matching. If $|W'| = 3$, then $e_H(W') \leq 2$ as H is bipartite. Since by (c₂) each

$w \in W'$ satisfies $d_H(w) \geq 13\alpha n/25$, it is easy to construct $e_H(W')$ 2-matchings $M'_1, M'_{e_H(W')}$ such that $d_{M'_s}(w) = 1$ for all $w \in W'$ and all $s \leq e_H(W')$ and such that $H[W'] \subseteq M'_1 \cup M'_{e_H(W')}$. Set $F_{\alpha n - s + 1} := M'_s$ for all $s \leq e_H(W')$ (regardless of the size of W').

We now greedily choose $\alpha n/200 - e_H(W')$ additional 2-matchings $F_{199\alpha n/200+1}, \dots, F_{\alpha n - e_H(W')}$ in H which are edge-disjoint from each other and from $F_{\alpha n}, F_{\alpha n - e_H(W') + 1}$ and such that $d_{F_s}(w) = 1$ for all $w \in W_0$ and all $199\alpha n/200 < s \leq \alpha n - e_H(W')$. To see that this can be done, recall that by (c₂) we have $d_H(w) \geq 13\alpha n/25$ for all $w \in W'$ (and thus for all $w \in W_0$) and that (c₁) and (c₃) together imply that $e(H - W_0) \geq 2(1 + \eta)\alpha n - \alpha n > \alpha n$ if $|W_0| = 1$.

Thus $F_{199\alpha n/200+1}, \dots, F_{\alpha n}$ are Hamilton exceptional system candidates satisfying the two properties in the ‘furthermore part’ of the lemma. Let H_1 and H'_1 be the graphs obtained from H and H' by deleting all the $\alpha n/100$ edges in these Hamilton exceptional system candidates. Set

$$(4.24) \quad r := 199\alpha n/200 \quad \text{and} \quad \eta' := \eta\alpha n/r = 200\eta/199.$$

Thus $0 \leq \eta' < 1$ and we now have

$$(4.25) \quad H_1[W'] = \emptyset, \quad e(H_1) = e(H) - \alpha n/100 = 2(1 + \eta')r \quad \text{and} \quad e(H'_1) \leq 2r.$$

(To verify the last inequality note that $e(H'_1) \leq e(H - W') \leq 2r$ by (c₂).) Also, (c₂) and (c₄) together imply that for all $w \in W'$ and all $v \in V(H) \setminus W'$ we have

$$(4.26) \quad d_{H_1}(w) \geq \alpha n/2 \geq 4\varepsilon_0 n \quad \text{and} \quad d_{H_1}(w) - d_{H_1}(v) \geq 2\varepsilon_0 n.$$

Moreover, by (c₂) and (c₃), each $w \in W_0$ satisfies

$$(4.27) \quad \begin{aligned} 31r/60 &\leq 13\alpha n/25 - \alpha n/200 \leq d_H(w) - d_{H-H_1}(w) = d_{H_1}(w) \\ &\leq \alpha n - \alpha n/200 = r. \end{aligned}$$

Similarly, if $|W'| = 3$ and so w_3 exists, then

$$(4.28) \quad \begin{aligned} 31r/60 &\leq 13\alpha n/25 - \alpha n/200 \leq d_H(w_3) - d_{H-H_1}(w_3) = d_{H_1}(w_3) \\ &\leq 41\alpha n/40 \leq 31r/30. \end{aligned}$$

(4.26) and (4.27) together imply that $d_{H'_1}(v) \leq d_{H_1}(v) < d_{H_1}(w_1) \leq r$ for all $v \in V(H) \setminus W'$. Thus $\chi'(H'_1) \leq \Delta(H'_1) \leq r$. Together with Proposition 2.4 this implies that H'_1 can be decomposed into r edge-disjoint matchings M_1, \dots, M_r such that $|m_j - m_{j'}| \leq 1$ for all $1 \leq j, j' \leq r$, where we set $m_j := e(M_j)$.

Our next aim is to apply Lemma 4.12 with $|W'|$, $d_{H_1}(w_i)$, m_j , η' playing the roles of q , a_i , c_j , η (for all $i \leq |W'|$ and all $j \leq r$). Since $\sum_{j \leq r} m_j = e(H'_1) \leq 2r$ by (4.25) and since $|m_j - m_{j'}| \leq 1$, it follows that $m_j \in \{0, 1, 2\}$ for all $j \leq r$. Moreover, by relabeling the matchings M_j if necessary, we may assume that $m_1 \geq m_2 \geq \dots \geq m_r$. Thus condition (i) of Lemma 4.12 holds. (ii) holds too since $\sum_{i \leq |W'|} d_{H_1}(w_i) + \sum_{j \leq r} m_j = e(H_1) = 2(1 + \eta')r$ by (4.25). Finally, (iii) follows from (4.27) and (4.28). Thus we can indeed apply Lemma 4.12 in order to obtain numbers $a_{i,j} \in \{0, 1, 2\}$ (for all $i \leq |W'|$ and $j \leq r$) which satisfy the following properties:

$$(a') \quad \sum_{j \leq r} a_{i,j} = d_{H_1}(w_i) \text{ for all } i \leq |W'|.$$

- (b') $m_j + \sum_{i \leq |W'|} a_{i,j} = 4$ for all $j \leq \eta'r$ and $m_j + \sum_{i \leq |W'|} a_{i,j} = 2$ for all $\eta'r < j \leq r$.
- (c') If $m_j < 2$ then there exist at least $2 - m_j$ indices i such that $a_{i,j} = 1$.

For all $j \leq r$, our Hamilton exceptional system candidate F_j will consist of the edges in M_j as well as of $a_{i,j}$ edges of H_1 incident to w_i (for each $i \leq |W'|$). So let $F_j^0 := M_j$ for all $j \leq r$. For each $i = 1, \dots, |W'|$ in turn, we will now assign the edges of H_1 incident with w_i to $F_1^{i-1}, \dots, F_r^{i-1}$ such that the resulting graphs F_1^i, \dots, F_r^i satisfy the following properties:

- (α_i) If $i \geq 1$, then $e(F_j^i) - e(F_j^{i-1}) = a_{i,j}$.
- (β_i) F_j^i is a path system. Every vertex $v \in A \cup B$ is incident to at most one edge of F_j^i . For every $v \in V_0 \setminus W'$ we have $d_{F_j^i}(v) \leq 2$. If $e(F_j^i) \leq 2$, we even have $d_{F_j^i}(v) \leq 1$.
- (γ_i) Let b_j^i be the number of vertex-disjoint maximal paths in F_j^i with one endpoint in A' and the other in B' . If $a_{i,j} = 1$ and $i \geq 1$, then $b_j^i = b_j^{i-1} + 1$. Otherwise $b_j^i = b_j^{i-1}$.

We assign the edges of H_1 incident with w_i to $F_1^{i-1}, \dots, F_r^{i-1}$ in two steps. In the first step, for each index $j \leq r$ with $a_{i,j} = 2$ in turn, we assign an edge of H_1 between w_i and V_0 to F_j^{i-1} whenever there is such an edge left. More formally, to do this, we set $N_0 := N_{H_1}(w_i)$. For each $j \leq r$ in turn, if $a_{i,j} = 2$ and $N_{j-1} \cap V_0 \neq \emptyset$, then we choose a vertex $v \in N_{j-1} \cap V_0$ and set $F_j^i := F_j^{i-1} + w_iv$, $N_j := N_{j-1} \setminus \{v\}$ and $a'_{i,j} := 1$. Otherwise, we set $F_j^i := F_j^{i-1}$, $N_j := N_{j-1}$ and $a'_{i,j} := a_{i,j}$.

Therefore, after having dealt with all indices $j \leq r$ in this way, we have that

$$(4.29) \quad \text{either } a'_{i,j} \leq 1 \text{ for all } j \leq r \text{ or } N_r \cap V_0 = \emptyset \text{ (or both).}$$

Note that by (b') we have $e(F_j^i) \leq m_j + \sum_{i' \leq i} a_{i',j} \leq 4$ for all $j \leq r$. Moreover, (a') implies that $|N_r| = \sum_{j \leq r} a'_{i,j}$. Also, $N_r \setminus V_0 = N_{H_1}(w_i) \setminus V_0$, and so $N_{H_1}(w_i) \setminus N_r \subseteq V_0$. Hence

$$(4.30) \quad |N_r| = |N_{H_1}(w_i)| - |N_{H_1}(w_i) \setminus N_r| \geq d_{H_1}(w_i) - |V_0| \geq d_{H_1}(w_i) - \varepsilon_0 n.$$

In the second step, we assign the remaining edges of H_1 incident with w_i to F_1^i, \dots, F_r^i . We achieve this by finding a perfect matching M in a suitable auxiliary graph.

Claim. *Define a graph Q with vertex classes N_r and V' as follows: V' consists of $a'_{i,j}$ copies of F_j^i for each $j \leq r$. Q contains an edge between $v \in N_r$ and $F_j^i \in V'$ if and only if v is not an endpoint of an edge in F_j^i . Then Q has a perfect matching M .*

To prove the claim, note that

$$(4.31) \quad |V'| = \sum_{j \leq r} a'_{i,j} = |N_r| \stackrel{(4.30)}{\geq} d_{H_1}(w_i) - \varepsilon_0 n.$$

Moreover, since $F'_j \subseteq H$ is bipartite and so every edge of F'_j has at most one endpoint in N_r , it follows that

$$(4.32) \quad d_Q(F'_j) \geq |N_r| - e(F'_j) \geq |N_r| - 4$$

for each $F'_j \in V'$. Consider any $v \in N_r$. Clearly, there are at most $d_{H_1}(v)$ indices $j \leq r$ such that v is an endpoint of an edge of F'_j . If $v \in N_r \setminus V_0 \subseteq A \cup B$, then by (c₅), v lies in at most $2d_{H_1}(v) \leq 2d_H(v) \leq 2\varepsilon_0 n$ elements of V' . (The factor 2 accounts for the fact that each F'_j occurs in V' precisely $a'_{i,j} \leq 2$ times.) So

$$d_Q(v) \geq |V'| - 2\varepsilon_0 n \stackrel{(4.31)}{\geq} d_{H_1}(w_i) - 3\varepsilon_0 n \stackrel{(4.26)}{\geq} \varepsilon_0 n.$$

If $v \in N_r \cap V_0$, then (4.29) implies that $a'_{i,j} \leq 1$ for all $j \leq r$. Thus

$$d_Q(v) \geq |V'| - d_{H_1}(v) \stackrel{(4.31)}{\geq} (d_{H_1}(w_i) - d_{H_1}(v)) - \varepsilon_0 n \stackrel{(4.26)}{\geq} 2\varepsilon_0 n - \varepsilon_0 n = \varepsilon_0 n.$$

To summarize, for all $v \in N_r$ we have $d_Q(v) \geq \varepsilon_0 n$. Together with (4.32) and the fact that $|N_r| = |V'|$ by (4.31) this implies that Q contains a perfect matching M by Hall's theorem. This proves the claim.

For each $j \leq r$, let F_j^i be the graph obtained from F'_j by adding the edge $w_i v$ whenever the perfect matching M (as guaranteed by the claim) contains an edge between v and F'_j .

Let us now verify $(\alpha_i) - (\gamma_i)$ for all $i \leq |W'|$. Clearly, $(\alpha_0) - (\gamma_0)$ hold and $b_j^0 = m_j$. Now suppose that $i \geq 1$ and that $(\alpha_{i-1}) - (\gamma_{i-1})$ hold. Clearly, (α_i) holds by our construction of F_1^i, \dots, F_r^i . Now consider any $j \leq r$. If $a_{i,j} = 0$, then (β_i) and (γ_i) follow from (β_{i-1}) and (γ_{i-1}) . If $a_{i,j} = 1$, then the unique edge in $F_j^i - F_j^{i-1}$ is vertex-disjoint from any edge of F_j^{i-1} (by the definition of Q) and so (β_i) holds. Moreover, $b_j^i = b_j^{i-1} + 1$ and so (γ_i) holds. So suppose that $a_{i,j} = 2$. Then the unique two edges in $F_j^i - F_j^{i-1}$ form a path $P = v' w_i v''$ of length two with internal vertex w_i . Moreover, at least one of the edges of P , $w_i v''$ say, was added to F_j^{i-1} in the second step of our construction of F_j^i . Thus $d_{F_j^i}(v'') = 1$. The other edge $w_i v'$ of P was either added in the first or in the second step. If $w_i v'$ was added in the second step, then $d_{F_j^i}(v') = 1$. Altogether this shows that in this case (γ_i) holds and (β_i) follows from (β_{i-1}) . So suppose that $w_i v'$ was added to F_j^{i-1} in the first step of our construction of F_j^i . Thus $v' \in V_0 \setminus W'$. But since $a_{i,j} = 2$, (b') implies that $e(F_j^{i-1}) = m_j + \sum_{i' < i} a_{i',j} \leq 2$. Together with (β_{i-1}) this shows that $d_{F_j^{i-1}}(v) \leq 1$ for all $v \in V_0 \setminus W'$. Hence $d_{F_j^{i-1}}(v') \leq 1$ and so $d_{F_j^i}(v') \leq 2$. Together with (β_{i-1}) this implies (β_i) . (Note that if $e(F_j^{i-1}) = 0$, then the above argument actually shows that $d_{F_j^i}(v') \leq 1$, as required.) Moreover, the above observations also guarantee that (γ_i) holds. Thus F_1^i, \dots, F_r^i satisfy $(\alpha_i) - (\gamma_i)$.

After having assigned the edges of H_1 incident with w_i for all $i \leq |W'|$, we have obtained graphs $F_1^{|W'|}, \dots, F_r^{|W'|}$. Let $F_j := F_j^{|W'|}$ for all $j \leq r$. Note that by $(\gamma_{|W'|})$

for all $j \leq r$ the number of vertex-disjoint maximal $A'B'$ -paths in F_j is precisely $b_j^{|W'|}$.

We now claim that $b_j^{|W'|}$ is positive and even. To verify this, recall that $b_j^0 = m_j$. Let odd_j be the number of $a_{i,j}$ with $a_{i,j} = 1$ and $i \leq |W'|$. So $b_j^{|W'|} = m_j + \text{odd}_j$. Together with (c') this immediately implies that $b_j^{|W'|} \geq 2$. Moreover, since $a_{i,j} \in \{0, 1, 2\}$ we have

$$b_j^{|W'|} = m_j + \text{odd}_j = m_j + \sum_{i \leq |W'|, a_{i,j} \text{ is odd}} a_{i,j}.$$

Together with (b') this now implies that $b_j^{|W'|}$ is even. This proves the claim.

Together with (a'), (b') and $(\alpha_i), (\beta_i)$ for all $i \leq |W'|$ this in turn shows that F_1, \dots, F_r form a decomposition of H_1 into edge-disjoint Hamilton exceptional system candidates with $e(F_j) = 4$ for all $j \leq \eta'r$ and $e(F_j) = 2$ for all $\eta'r < j \leq r$. Recall that $\eta'r = \eta\alpha n$ by (4.24) and that we have already constructed Hamilton exceptional system candidates $F_{199\alpha n/200+1}, \dots, F_{\alpha n}$ which satisfy the ‘furthermore statement’ of the lemma, and thus in particular consist of precisely two edges. This completes the proof of the lemma. \square

4.3.3. Proof of Lemma 4.10. We will now combine Lemmas 4.11, 4.13 and 4.9 in order to prove Lemma 4.10. This will complete the construction of the required exceptional sequences in the case when G is both critical and $e(G[A', B']) \geq D$.

Proof of Lemma 4.10. Let G° be as defined in Lemma 4.10(iv). Our first aim is to decompose G° into suitable ‘localized’ subgraphs via Lemma 4.11. Choose a new constant ε' such that $\varepsilon \ll \varepsilon' \ll \lambda, 1/K$ and define α by

$$(4.33) \quad 2\alpha n := \frac{D - \phi n}{K^2}.$$

Recall from Lemma 4.10(ii) that $D = (n - 1)/2$ or $D = n/2 - 1$. Together with our assumption that $\phi \ll 1$ this implies that

$$(4.34) \quad \frac{1 - 2/n - 2\phi}{4K^2} \leq \alpha \leq \frac{1 - 2\phi}{4K^2} \quad \text{and} \quad \varepsilon \ll \varepsilon' \ll \lambda, 1/K, \alpha \ll 1.$$

Note that by Lemma 4.10(ii) and (iii) we have $e_{G^\circ}(A', B') \geq D - \phi n = 2K^2\alpha n$. Together with Lemma 4.1(iii) this implies that

$$(4.35) \quad 2K^2\alpha n \leq e_{G^\circ}(A', B') \leq e_G(A', B') \leq 17D/10 + 5 \stackrel{(4.33)}{\leq} 18K^2\alpha n/5 \stackrel{(4.34)}{<} n.$$

Moreover, recall that by Lemma 4.10(i) and (iii) we have

$$(4.36) \quad d_{G^\circ}(v) = 2K^2\alpha n \quad \text{for all } v \in V_0.$$

Let W be the set of all those vertices $w \in V(G)$ with $d_{G[A', B']}(w) \geq 11D/40$. So W is as defined in Lemma 4.1 and $1 \leq |W| \leq 3$ by Lemma 4.1(i). Let $W' \subseteq V(G)$ be as

guaranteed by Lemma 4.1(v). Thus $W \subseteq W'$, $|W'| \leq 3$,

(4.37)

$$d_{G[A',B']}(w') \geq \frac{21D}{80}, \quad d_{G[A',B']}(v) \leq \frac{11D}{40} \quad \text{and} \quad d_{G[A',B']}(w') - d_{G[A',B']}(v) \geq \frac{D}{240}.$$

for all $w' \in W'$ and all $v \in V(G) \setminus W'$. In particular, $W' \subseteq V_0$. (This follows since Lemma 4.10(iii),(iv) and (ESch3) together imply that $d_{G[A',B']}(v) = d_{G^\circ[A',B']}(v) + d_{G_0[A',B']}(v) \leq \varepsilon_0 n + e_{G_0}(A', B') \leq \varepsilon_0 n + \phi n$ for all $v \in A \cup B$.) Let w_1, w_2, w_3 be vertices of G such that

$$d_{G[A',B']}(w_1) \geq d_{G[A',B']}(w_2) \geq d_{G[A',B']}(w_3) \geq d_{G[A',B']}(v)$$

for all $v \in V(G) \setminus \{w_1, w_2, w_3\}$, where w_1 and w_2 are as in Lemma 4.10(v). Hence W consists of $w_1, \dots, w_{|W|}$ and W' consists of $w_1, \dots, w_{|W'|}$. Set $W_0 := \{w_1, w_2\} \cap W'$. Since $d_{G_0}(v) = \phi n$ for each $v \in V_0$ (and thus for each $v \in W_0$), each $w \in W_0$ satisfies

$$(4.38) \quad K^2 \leq 21D/80 - \phi n \stackrel{(4.37)}{\leq} d_{G^\circ[A',B']}(w) \leq K^2 \alpha n \stackrel{(4.35)}{\leq} e_{G^\circ}(A', B')/2.$$

(Here the third inequality follows from Lemma 4.10(v).) Apply Lemma 4.11 to G° in order to obtain a decomposition of G° into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ (for all $1 \leq i, i' \leq K$) which satisfy the following properties, where $G'(i, i') := H(i, i') + H''(i, i')$:

- (b'₁) Each $H(i, i')$ contains only $A_0 A_i$ -edges and $B_0 B_{i'}$ -edges.
- (b'₂) $H''(i, i') \subseteq G^\circ[A', B']$. Moreover, all but at most $20\varepsilon n/K^2$ edges of $H''(i, i')$ lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (b'₃) $e(H''(i, i')) = 2 \lceil e_{G^\circ}(A', B')/(2K^2) \rceil$ or $e(H''(i, i')) = 2 \lfloor e_{G^\circ}(A', B')/(2K^2) \rfloor$. In particular, $2\alpha n \leq e(H''(i, i')) \leq 19\alpha n/5$ by (4.35).
- (b'₄) $d_{H''(i, i')}(v) = (d_{G^\circ[A',B']}(v) \pm 25\varepsilon n)/K^2$ for all $v \in V_0$.
- (b'₅) $d_{G'(i, i')}(v) = (d_{G^\circ}(v) \pm 25\varepsilon n)/K^2 = (2\alpha \pm 25\varepsilon/K^2)n$ for all $v \in V_0$ by (4.36).
- (b'₆) Each $w \in W_0$ satisfies $d_{H''(i, i')}(w) \leq \lceil d_{G^\circ[A',B']}(w)/K^2 \rceil \leq \alpha n$ by (4.38).

Our next aim is to apply Lemma 4.13 to each $H''(i, i')$ to obtain suitable Hamilton exceptional system candidates (in particular almost all of them will be ‘localized’). So consider any $1 \leq i, i' \leq K$ and let $H'' := H''(i, i')$. We claim that there exists $0 \leq \eta \leq 9/10$ such that H'' satisfies the following conditions (which in turn imply conditions (c₁)–(c₅) of Lemma 4.13):

- (c'₁) $e(H'') = 2(1 + \eta)\alpha n$ and $\eta\alpha n \in \mathbb{N}$.
- (c'₂) $e(H'' - W') \leq 199\alpha n/100$ and $d_{H''}(w) \geq 13\alpha n/25$ for all $w \in W'$.
- (c'₃) $d_{H''}(w) \leq \alpha n$ for all $w \in W_0$ and $d_{H''}(w') \leq 41\alpha n/40$ for all $w' \in W' \setminus W_0$.
- (c'₄) For all $w \in W'$ and all $v \in V(G) \setminus W'$ we have $d_{H''}(w) - d_{H''}(v) \geq \alpha n/150$.
- (c'₅) For all $v \in A \cup B$ we have $d_{H''}(v) \leq \varepsilon_0 n$.

Clearly, (b'₃) implies the first part of (c'₁). Since $e(H'')$ is even by (b'₃) and $\alpha n \in \mathbb{N}$, it follows that $\eta\alpha n \in \mathbb{N}$. To verify the first part of (c'₂), note that (b'₃) and (b'₄) together

imply that

$$\begin{aligned}
 e(H'' - W') &= e(H'') - \sum_{w \in W'} d_{H''}(w) + e(H''[W']) \\
 &\leq 2 \lceil e_{G^\circ}(A', B') / (2K^2) \rceil - \sum_{w \in W'} (d_{G^\circ[A', B']}(w) - 25\epsilon n) / K^2 + 3 \\
 &\leq (e_{G^\circ - W'}(A', B') + 80\epsilon n) / K^2.
 \end{aligned}$$

Together with Lemma 4.1(iv) this implies that

$$e(H'' - W') \leq (e_{G - W'}(A', B') + 80\epsilon n) / K^2 \leq ((3D/4 + 5) + 80\epsilon n) / K^2 \leq 199\alpha n / 100.$$

To verify the second part of (c'₂), note that by (4.37) and Lemma 4.10(iii) each $w \in W'$ satisfies $d_{G^\circ[A', B']}(w) \geq d_{G[A', B']}(w) - \phi n \geq 21D/80 - \phi n$. Together with (b'₄) this implies $d_{H''}(w) \geq 26\alpha n/50$. Thus (c'₂) holds. By (b'₆) we have $d_{H''}(w) \leq \alpha n$ for all $w \in W_0$. If $w' \in W' \setminus W_0$, then Lemma 4.10(ii) implies $d_{G[A', B']}(w') \leq D/2 \leq 51K^2\alpha n/50$. Thus, $d_{H''}(w') \leq 41\alpha n/40$ by (b'₄). Altogether this shows that (c'₃) holds. (c'₄) follows from (4.37), (b'₄) and the fact that $d_{G^\circ[A', B']}(v) \geq d_{G[A', B']}(v) - \phi n$ for all $v \in V(G)$ by Lemma 4.10(iii). (c'₅) holds since $d_{H''}(v) \leq d_{G^\circ[A', B']}(v) \leq \epsilon_0 n$ for all $v \in A \cup B$ by (ESch3).

Now we apply Lemma 4.13 in order to decompose H'' into αn edge-disjoint Hamilton exceptional system candidates $F_1, \dots, F_{\alpha n}$ such that $e(F_s) \in \{2, 4\}$ for all $s \leq \alpha n$ and such that at least $\alpha n/200$ of F_s satisfy $e(F_s) = 2$ and $d_{F_s}(w) = 1$ for all $w \in W_0$. Let

$$\gamma := \alpha - \frac{\lambda}{K^2} \quad \text{and} \quad \gamma' := \frac{\lambda}{K^2}.$$

Recall that by (b'₂) all but at most $20\epsilon n/K^2 \leq \epsilon' n$ edges of H'' lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_i]$. Together with (4.34) this ensures that we can relabel the F_s if necessary to obtain αn edge-disjoint Hamilton exceptional system candidates $F_1(i, i'), \dots, F_{\gamma n}(i, i')$ and $F'_1(i, i'), \dots, F'_{\gamma' n}(i, i')$ such that the following properties hold:

- (a') $F_s(i, i')$ is an (i, i') -HESC for every $s \leq \gamma n$. Moreover, $\gamma' n$ of the $F_s(i, i')$ satisfy $e(F_s(i, i')) = 2$ and $d_{F_s(i, i')}(w) = 1$ for all $w \in W_0$.
- (b') $e(F'_s(i, i')) = 2$ for all but at most $\epsilon' n$ of the $F'_s(i, i')$.
- (c') $e(F_s(i, i')), e(F'_s(i, i')) \in \{2, 4\}$.

For (b') and the 'moreover' part of (a'), we use that $\alpha n/200 - \epsilon' n \geq 2\lambda n/K^2 = 2\gamma' n$. Our next aim is to apply Lemma 4.9 with G° playing the role of G^* to extend the above exceptional system candidates into exceptional systems. Clearly conditions (i) and (ii) of Lemma 4.9 hold. (iii) follows from (b'₁). (iv) and (v) follow from (a')–(c'). (vi) follows from Lemma 4.10(i), (iii). Finally, (vii) follows from (b'₅) since $G'(i, i')$ plays the role of $G^*(i, i')$. Thus we can indeed apply Lemma 4.9 to obtain a decomposition of G° into $K^2\alpha n$ edge-disjoint Hamilton exceptional systems $J_1(i, i'), \dots, J_{\gamma n}(i, i')$ and $J'_1(i, i'), \dots, J'_{\gamma' n}(i, i')$ with parameter ϵ_0 , where $1 \leq i, i' \leq K$, such that $J_s(i, i')$ is an (i, i') -HES which is a faithful extension of $F_s(i, i')$ for all $s \leq \gamma n$ and $J'_s(i, i')$ is a faithful extension of $F'_s(i, i')$ for all $s \leq \gamma' n$. Then the set \mathcal{J} of all these exceptional systems is as required in Lemma 4.10. (Since W_0

contains $\{w_1, w_2\} \cap W$, the ‘moreover part’ of (a’) implies the ‘moreover part’ of Lemma 4.10(b). \square

4.4. The case when $e_G(A', B') < D$. The aim of this section is to prove the following analogue of Lemma 4.2 for the case when $e_G(A', B') < D$. In this case, we do not need to prove any auxiliary lemmas first, as we can apply those proved in the other two cases (Lemmas 4.9 and 4.11).

Recall that Proposition 2.2 implies that in the current case we have $n \equiv 0 \pmod{4}$, $D = n/2 - 1$ and $|A'| = |B'| = n/2$.

Lemma 4.14. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll \varepsilon \ll \lambda, 1/K \ll 1$, that $0 \leq \phi \ll 1$ and that $n/4, K, m, \lambda n/K^2, (n/2 - 1 - \phi n)/(2K^2) \in \mathbb{N}$. Suppose that the following conditions hold:*

- (i) G is an $(n/2 - 1)$ -regular graph on n vertices.
- (ii) \mathcal{P} is a (K, m, ε_0) -partition of $V(G)$ such that $\Delta(G[A', B']) \leq n/4$ and $|A'| = |B'| = n/2$.
- (iii) G_0 is a subgraph of G such that $G[A_0] + G[B_0] \subseteq G_0$ and $d_{G_0}(v) = \phi n$ for all $v \in V_0$.
- (iv) Let $G^\circ := G - G[A] - G[B] - G_0$. $e_{G^\circ}(A', B')$ is even and (G°, \mathcal{P}) is a $(K, m, \varepsilon_0, \varepsilon)$ -exceptional scheme.
- (v) $\Delta(G^\circ[A', B']) \leq e_{G^\circ}(A', B')/2 \leq (n/2 - 1 - \phi n)/2$.

Then there exists a set \mathcal{J} consisting of $(n/2 - 1 - \phi n)/2$ edge-disjoint exceptional systems in G° which satisfies the following properties:

- (a) Together the exceptional systems in \mathcal{J} cover all edges of G° . Each J_s in \mathcal{J} is either a Hamilton exceptional system with $e_{J_s}(A', B') = 2$ or a matching exceptional system.
- (b) For all $1 \leq i, i' \leq K$, the set \mathcal{J} contains $(n/2 - 1 - (\phi n + 2\lambda))/(2K^2)$ (i, i') -ES.

As in the other two cases, in [7] we will use some of the exceptional systems in (b) to construct the robustly decomposable graph G^{rob} . Unlike the critical case with $e_G(A', B') \geq D$, there is no need to ‘track’ the degrees of the vertices w_i of high degree in $G[A', B']$ this time (this is due to the very special structure of the exceptional systems produced in this case).

Proof. Let ε' be a new constant such that $\varepsilon \ll \varepsilon' \ll \lambda, 1/K$ and set

$$(4.39) \quad 2\alpha n := \frac{n/2 - 1 - \phi n}{K^2}.$$

Similarly as in the proof of Lemma 4.10 we have

$$(4.40) \quad \varepsilon \ll \varepsilon' \ll \lambda, 1/K, \alpha \ll 1.$$

We claim that G° can be decomposed into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ (for all $1 \leq i, i' \leq K$) which satisfy the following properties, where $G'(i, i') := H(i, i') + H''(i, i')$:

- (b'₁) Each $H(i, i')$ contains only A_0A_i -edges and $B_0B_{i'}$ -edges.

- (b'₂) $H''(i, i') \subseteq G^\circ[A', B']$. Moreover, all but at most $\varepsilon'n$ edges of $H''(i, i')$ lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$.
- (b'₃) $e(H''(i, i'))$ is even and $e(H''(i, i')) \leq 2\alpha n$.
- (b'₄) $\Delta(H''(i, i')) \leq e(H''(i, i'))/2$.
- (b'₅) $d_{G'(i, i')}(v) = (2\alpha \pm \varepsilon')n$ for all $v \in V_0$.

To see this, let us first consider the case when $e_{G^\circ}(A', B') \leq 300\varepsilon n$. Apply Lemma 4.3 to G° in order to obtain a decomposition of G° into edge-disjoint spanning subgraphs $H(i, i')$ and $H'(i, i')$ (for all $1 \leq i, i' \leq K$) which satisfy Lemma 4.3(a₁)–(a₅). Set $H''(1, 1) := \bigcup_{i, i' \leq K} H'(i, i') = G^\circ[A', B']$ and $H''(i, i') := \emptyset$ for all other pairs $1 \leq i, i' \leq K$. Then (b'₁) follows from (a₁). (b'₂) follows from our definition of the $H''(i, i')$ and our assumption that $e_{G^\circ}(A', B') \leq 300\varepsilon n < \varepsilon'n < \alpha n$. Together with Lemma 4.14(iv) this also implies (b'₃). (b'₄) follows from Lemma 4.14(v). Note that by Lemma 4.14(i) and (iii), every $v \in V_0$ satisfies $d_{G^\circ}(v) = n/2 - 1 - \phi n = 2K^2\alpha n$. So, writing $G(i, i') := H(i, i') + H'(i, i')$, (a₅) implies that

$$d_{G'(i, i')}(v) = d_{G(i, i')}(v) \pm 300\varepsilon n = (2\alpha \pm 4\varepsilon/K^2)n \pm 300\varepsilon n = (2\alpha \pm \varepsilon')n.$$

Thus (b'₅) holds too.

So let us next consider the case when $e_{G^\circ}(A', B') > 300\varepsilon n$. Let W_0 be the set of all those vertices $v \in V(G)$ for which $d_{G^\circ[A', B']}(v) \geq 3e_{G^\circ}(A', B')/8$. Then clearly $|W_0| \leq 2$. Moreover, each $v \in V(G) \setminus W_0$ satisfies

$$(4.41) \quad d_{G^\circ[A', B']}(v) + 26\varepsilon n < 3e_{G^\circ}(A', B')/8 + e_{G^\circ}(A', B')/8 = e_{G^\circ}(A', B')/2.$$

Recall from Lemma 4.14(v) that each $w \in W_0$ satisfies $d_{G^\circ[A', B']}(w) \leq e_{G^\circ}(A', B')/2$. So we can apply Lemma 4.11 to G° in order to obtain a decomposition of G° into edge-disjoint spanning subgraphs $H(i, i')$ and $H''(i, i')$ (for all $1 \leq i, i' \leq K$) which satisfy Lemma 4.11(b₁)–(b₇). Then (b₁) and (b₂) imply (b'₁) and (b'₂). (b'₃) follows from (b₃), (4.39) and Lemma 4.14(v). Note that (b₃), (b₄) and (4.41) together imply that

$$(4.42) \quad d_{H''(i, i')}(v) \leq \frac{e_{G^\circ}(A', B')/2 - \varepsilon n}{K^2} \leq \frac{e(H''(i, i'))}{2}$$

for all $v \in V_0 \setminus W_0$. Note that each $v \in A \cup B$ satisfies $d_{H''(i, i')}(v) \leq d_{G^\circ[A', B']}(v) \leq \varepsilon_0 n$ by Lemma 4.14(iv) and (ESch3). Together with the fact that $e(H''(i, i')) \geq 2\lfloor 300\varepsilon n/(2K^2) \rfloor \geq 2\varepsilon_0 n$ by (b₃), this implies that (4.42) also holds for all $v \in A \cup B$. Together with (b₇) this implies (b'₄). (b'₅) follows from (b₅) and the fact that by Lemma 4.14(i) and (iii) every $v \in V_0$ satisfies $d_{G^\circ}(v) = n/2 - 1 - \phi n = 2K^2\alpha n$. So (b'₁)–(b'₅) hold in all cases.

We now decompose the localized subgraphs $H''(i, i')$ into exceptional system candidates. For this, fix $i, i' \leq K$ and write H'' for $H''(i, i')$. By (b'₄) we have $\Delta(H'') \leq e(H'')/2$ and so $\chi'(H'') \leq e(H'')/2$. Apply Proposition 2.4 with $e(H'')/2$ playing the role of m to decompose H'' into $e(H'')/2$ edge-disjoint matchings, each of size 2. Note that $\alpha n - e(H'')/2 \geq 0$ by (b'₃). So we can add some empty matchings to obtain a decomposition of H'' into αn edge-disjoint $M_1, \dots, M_{\alpha n}$ such that each M_s is either

empty or has size 2. Let

$$\gamma := \alpha - \frac{\lambda}{K^2} \quad \text{and} \quad \gamma' := \frac{\lambda}{K^2}.$$

Recall from (b₂') that all but at most $\varepsilon'n \leq \gamma'n$ edges of H'' lie in $G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$. Hence by relabeling if necessary, we may assume that $M_s \subseteq G^\circ[A_0 \cup A_i, B_0 \cup B_{i'}]$ for every $s \leq \gamma n$. So by setting $F_s(i, i') := M_s$ for all $s \leq \gamma n$ and $F'_s(i, i') := M_{\gamma n + s}$ for all $s \leq \gamma'n$ we obtain a decomposition of H'' into edge-disjoint exceptional system candidates $F_1(i, i'), \dots, F_{\gamma n}(i, i')$ and $F'_1(i, i'), \dots, F'_{\gamma'n}(i, i')$ such that the following properties hold:

- (a') $F_s(i, i')$ is an (i, i') -ESC for every $s \leq \gamma n$.
- (b') Each $F_s(i, i')$ is either a matching exceptional system candidate with $e(F_s(i, i')) = 0$ or a Hamilton exceptional system candidate with $e(F_s(i, i')) = 2$. The analogue holds for each $F'_{s'}(i, i')$.

Our next aim is to apply Lemma 4.9 with G° playing the role of G^* , to extend the above exceptional system candidates into exceptional systems. Clearly conditions (i) and (ii) of Lemma 4.9 hold. (iii) follows from (b₁'). (iv) and (v) follow from (a') and (b'). (vi) follows from Lemma 4.14(i),(iii). Finally, (vii) follows from (b₅') since $G'(i, i')$ plays the role of $G^*(i, i')$ in Lemma 4.9. Thus we can indeed apply Lemma 4.9 to obtain a decomposition of G° into $K^2\alpha n$ edge-disjoint exceptional systems $J_1(i, i'), \dots, J_{\gamma n}(i, i')$ and $J'_1(i, i'), \dots, J'_{\gamma'n}(i, i')$, where $1 \leq i, i' \leq K$, such that $J_s(i, i')$ is an (i, i') -ES which is a faithful extension of $F_s(i, i')$ for all $s \leq \gamma n$ and $J'_{s'}(i, i')$ is a faithful extension of $F'_{s'}(i, i')$ for all $s \leq \gamma'n$. Then the set \mathcal{J} of all these exceptional systems is as required in Lemma 4.14. \square

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