Proofs of Restricted Shuffles

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A motivating example: Voting

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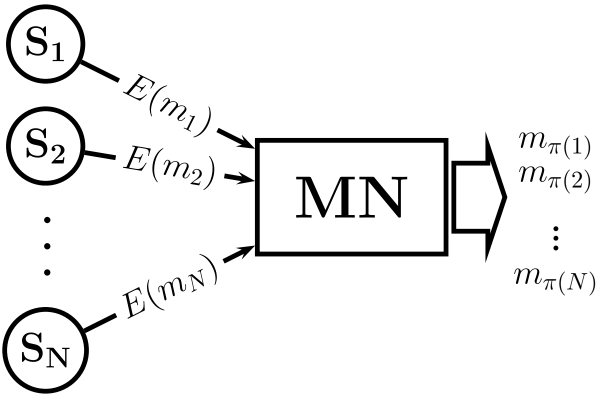
- How can we ensure that the voters remain anonymous when the votes are decrypted?
- There are two main ways to achieve this, homomorphic tallying [CGS97] and mixnets [Cha81].

Mixnets



Mixnets (2)

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- Chain of mixservers, each permutes and re-encrypts its list of inputs.

$$\xrightarrow{L_0} \boxed{\mathbf{T_1}} \xrightarrow{L_1} \boxed{\mathbf{T_2}} \xrightarrow{L_2} \cdots \xrightarrow{L_{k-1}} \boxed{\mathbf{T_k}} \xrightarrow{L_k}$$

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- Let each server produce an interactive zero-knowledge proof, a *proof of a shuffle* [SK95, Nef01, FS01].
- Like [FS01], we will construct a proof that a commitment contains a permutation matrix.
- One can then prove that the encrypted votes are permuted accordingly.

Test for permutation matrices

M permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M not permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\begin{aligned} \prod_{i=1}^{N} \langle \overline{m}_i, \overline{x} \rangle &= x_2(2x_1 - x_3)x_3 \\ &\neq x_1 x_2 x_3 \end{aligned}$$

Test for permutation matrices

Theorem (Permutation Matrix)

Let $M = (m_{i,j})$ be an $N \times N$-matrix over $\mathbb{Z}_q$ and $\overline{x} = (x_1, \ldots, x_N)$ be a list of variables. Then M is a permutation matrix if and only if

$$\prod_{i=1}^{N} \langle \overline{m}_i, \overline{x} \rangle = \prod_{i=1}^{N} x_i \quad \textit{and} \quad M\overline{1} = \overline{1} \ .$$

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Lemma (Schwartz-Zippel)

Let $f \in \mathbb{Z}_q[x_1, \ldots, x_N]$ be a non-zero polynomial of total degree d and let $e_1, \ldots, e_N$ be chosen randomly from $\mathbb{Z}_q$. Then

$$\Pr[f(e_1, \ldots, e_N) = 0] \leq \frac{d}{q} \enspace .$$

Recall Pedersen commitments

Let g, g_1 be randomly chosen generators in a group of prime order q. The Pedersen commitment of $m \in \mathbb{Z}_q$ is

$$\mathcal{C}(m, s) = g^s g_1^m$$

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- perfectly hiding
- computationally binding
- homomorphic, $\mathcal{C}(m, s)\,\mathcal{C}(m', s') = \mathcal{C}(m + m', s + s')$
  $\mathcal{C}(m, s)^e = \mathcal{C}(em, es)$

Generalized Pedersen commitments [FS01]

Let $g, g_1, \ldots, g_N$ be randomly chosen generators in a group of prime order q. We commit to a vector $\overline{m} = (m_1, \ldots, m_N)^{\mathrm{T}}$ by

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- homomorphic, $\mathcal{C}\left(\overline{m}, s\right) \mathcal{C}\left(\overline{m}', s'\right) = \mathcal{C}\left(\overline{m} + \overline{m}', s + s'\right)$

$$\mathcal{C}\left(\overline{m}, s\right)^e = \mathcal{C}\left(e\overline{m}, es\right)$$

Generalized Pedersen commitments

We commit column-wise to an $N \times N$-matrix $M = (m_{i,j})$, so $a = \mathcal{C}(M, \overline{s})$ is a list of N commitments satisfying

$$\mathcal{C}(M, \overline{s})^{\overline{e}} = \mathcal{C}(M\overline{e}, \langle \overline{s}, \overline{e} \rangle)$$

where we use the convention

$$a^{\overline{e}} = \prod_{i=1}^{N} a_i^{e_i} \quad .$$

A review of sigma proofs

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1. the view of the verifier can be simulated for any given challenge
2. a witness can be computed from any pair of accepting transcripts with the same random tape and distinct challenges

Example: Proof of knowledge of discrete logarithm

$\mathcal{P}$ wants to prove knowledge of x such that $y = g^x$

1. $\mathcal{P}$ chooses r at random and sends $\alpha = g^r$
2. $\mathcal{V}$ sends a random challenge c
3. $\mathcal{P}$ responds with $d = cx + r$

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There are similar protocols for proving any polynomial relation!

Proof of knowledge of permutation matrix

Given a matrix commitment a, $\mathcal{P}$ wants to prove knowledge of a **permutation matrix** M and randomness $\overline{s}$ such that $a = \mathcal{C}\left(M, \overline{s}\right)$.

1. $\mathcal{V}$ chooses a vector $\overline{e}$ randomly and sends it to $\mathcal{P}$.
2. $\mathcal{P}$ uses a sigma proof to prove knowledge of t, k and a vector $\overline{e}'$ such that

$$\mathcal{C}\left(\overline{e}', k\right) = a^{\overline{e}}$$
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$$\overline{e}' = M\overline{e}$$
$$\overline{1} = M\overline{1}$$
$$\prod_{i=1}^{N} \langle \overline{m}_i, \overline{e} \rangle = \prod_{i=1}^{N} e_i$$

Properties of the protocol

Theorem

The protocol is a honest verifier zero knowledge proof of knowledge of a permutation matrix M *such that* $a = \mathcal{C}(M, \bar{s})$*, assuming the commitment scheme is binding.*

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- The zero-knowledge property is easy.
- We must construct an extractor which computes a permutation matrix from accepting transcripts.

Sketch of proof

1. Run the extractor of the sigma proof N times with $\overline{e}_1, \ldots, \overline{e}_N$, each time extracting $\overline{e}'_i$ and k_i such that $\mathcal{C}\left(\overline{e}'_i, k_i\right) = a^{\overline{e}_i}$.

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3. Linear independence implies existence of $\alpha_{\ell,j} \in \mathbb{Z}_q$ such that $\sum_{j=1}^{N} \alpha_{\ell,j}\overline{e}_j$ is the ℓth standard unit vector in $\mathbb{Z}_q^N$.

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4. Then $\sum_{j=1}^{N} \alpha_{\ell,j}\overline{e}'_j$ is the ℓth column in M since

$$a_l = \prod_{j=1}^{N} a^{\alpha_{\ell,j}\overline{e}_j} = \prod_{j=1}^{N} \mathcal{C}\left(\overline{e}'_j, k_j\right)^{\alpha_{\ell,j}} = \mathcal{C}\left(\sum_{j=1}^{N} \alpha_{\ell,j}\overline{e}'_j, \sum_{j=1}^{N} \alpha_{\ell,j}k_j\right)$$

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$$\mathcal{C}\left(\overline{e}', k\right) = a^{\overline{e}} = \mathcal{C}\left(M\overline{e}, \langle \overline{s}, \overline{e} \rangle\right)$$

but $\overline{e}' \neq M\overline{e}$.

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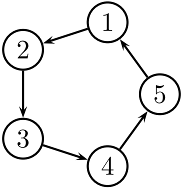
A rotation is precisely an automorphism of the directed cycle graph!



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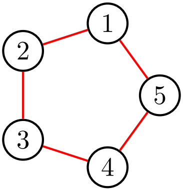

Let us look at the undirected cycle instead.

Restricting the permutation (graphs)

- Let $\mathscr{G}$ be a graph with vertices $V = \{1, 2, 3, \ldots, N\}$. Encode the edge set as

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- A permutation π is an automorphism of $\mathscr{G}$ if and only if

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- Apply Schwartz-Zippel ...

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$$F_{\mathscr{G}}(x_1, \ldots, x_N) = \sum_{(i,j) \in E} x_i x_j^2 = x_1 x_2^2 + x_2 x_3^2 + x_3 x_4^2 + x_4 x_5^2 + x_5 x_1^2$$

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Testing $F_{\mathscr{G}}(x_1, \ldots, x_N) = F_{\mathscr{G}}(x_{\pi(1)}, \ldots, x_{\pi(N)})$ determines whether π is a rotation.

Restricting the permutation (polynomials)

Theorem

Let F be any polynomial in $\mathbb{Z}_q[x_1, \ldots, x_N]$ and let S_F be the group of permutations π such that

$$F(x_1, \ldots, x_N) = F(x_{\pi(1)}, \ldots, x_{\pi(N)}) \ .$$

Then we can prove that the permutation is chosen from S_F.

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Problem Are there applications for other restrictions than rotations, e.g. automorphisms of a complete binary tree?

Questions?

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