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Propagation and structural interpretation of non-plane waves

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SUMMARY

As a model for the 2-D horizontal propagation of seismic surface waves, we study the propagation of non-plane acoustic waves in homogeneous and inhomogeneous media. We find that their phase velocity depends not only on the medium but also on the local geometry of the wavefield, especially on the distribution of amplitudes around the point of observation. The phase velocity of a wave is therefore conceptually and in most cases numerically different from the phase velocity parameter in the wave equation, which is determined by the elastic properties of the medium. The same distinction must be made for seismic surface waves. Although it is a common observation that waves of the same period can propagate with different phase velocities over the same path, the fundamental character of this observation has apparently not been recognized, and the two phase velocities are frequently confused in the seismological literature. We derive a local relationship between the two phase velocities that permits a correct structural interpretation of acoustic waves in inhomogeneous media, and also of non-plane seismic surface waves in laterally homogeneous parts of the medium.

Key words: Eikonal equation, phase velocity, polar phase shift, structural interpretation, surface waves, wave equation.

TWO DEFINITIONS OF PHASE VELOCITY AND WAVENUMBER

Seismologists do not in general bother to define the terms 'phase velocity' or 'wavenumber' for anything other than plane waves. The general understanding is, however, as follows.

Consider a train of waves travelling along the surface of a flat earth. It may originate from an extended source and contain multiple interfering signals; we record it with an array of stations but imagine that we could have observed it at any point (x, y) of the surface. We decompose the recorded signals into time-harmonic components of frequency ω. Each component is characterized by a complex-valued amplitude distribution $F(x, y, \omega)$; we drop the argument ω and refer to $F(x, y)$ as a wavefield. A natural wavefield is unlikely to have zeroes or nodal lines in the strict sense; if they should occur, we exclude them from consideration—we do not define a phase velocity in these points. Elsewhere, we can form the complex logarithm of F: $\ln F(x, y) = a(x, y) + ib(x, y)$, with a and b real; a is the logarithmic amplitude and b is the phase of F. The local wavenumber vector $\mathbf{w}$ of a wavefield is defined as the negative gradient of the phase, thus

$$\mathbf{w}(x, y) = -\text{grad}\, b(x, y) = -\text{grad}\, \mathcal{Im} \ln F(x, y). \quad (1)$$

For reasons that will soon become clear, we avoid here the conventional symbol $\mathbf{k}$ for the wavenumber, From $w(x, y) = |\mathbf{w}(x, y)|$ we obtain the local phase velocity as

$$v(x, y) = \omega / w(x, y) = \omega / |\text{grad}\, \mathcal{Im} \ln F(x, y)|\,. \quad (2)$$

This is a general definition of the phase velocity of time-harmonic signals along the surface of the earth, for wavefields of arbitrary spatial geometry.

In seismological practice, the gradient of the phase is approximated by finite differences taken between the points of observation, and the resulting wavenumber or phase velocity attributed to a profile or an area 'between' these points. We must, however, keep in mind that w and v are properties of the wavefield, not of the medium. For example, w is zero for a standing plane wave but not for a propagating plane wave in the same medium. We will refer to w and v as dynamic parameters.

Another wavenumber parameter k whose relationship with w we are going to investigate appears in the Helmholtz equation

$$(\Delta + k^2)F(x, y) = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (3)$$

This equation governs the propagation of most classical wavefields in homogeneous, isotropic media: acoustic, electromagnetic, elastic compression and shear separately. If **f** is a vector field derived from a potential F that satisfies (3), then eq. (3) also applies to each cartesian component of **f** (but not to curvi-linear components such as longitudinal or transverse). Remarkably, the Helmholtz equation remains valid in acoustic media with non-uniform compressibility, and can be generalized to acoustic media with non-uniform density (Appendix 1). Single-mode seismic surface waves in a laterally homogeneous half-space obey the Helmholtz equation in the horizontal coordinates (Appendix 2). Our subsequent derivations therefore apply strictly to acoustic waves in 2-D and 3-D inhomogeneous media with constant density, and to single-mode seismic surface waves in laterally homogeneous media.

The wavenumber k in the Helmholtz equation is simply an abbreviation for ω/c where in the acoustic case $c=(\kappa/\rho)^{1/2}$ is the sound velocity calculated from the incompressibility κ and the density ρ of the medium; for surface waves c is the characteristic phase velocity of a mode. The important point is that k and c in the Helmholtz equation are structural parameters determined by the properties of the medium, independent of the specific geometry of the wavefield. Thus, k cannot be the same wavenumber as w. This becomes obvious if we solve (3) for k, respectively $k(x, y)$ in the acoustic case:

$$k(x, y)=[-F(x, y)^{-1}\,\Delta F(x, y)]^{1/2}. \quad (4)$$

Clearly, eqs (1) and (4) are not mathematical equivalents of each other, and the two wavenumbers are in general different. In homogeneous media, they agree only for ordinary propagating plane waves and linearly inhomogeneous plane waves, such as $(y-y_0)\exp(ikx)$. Other solutions of the wave equation with $w=k$ require exotic media and have no obvious physical significance (Appendix 3). Unfortunately, plane waves are the only paradigm treated in most seismological textbooks, so the conceptual and numerical difference between the two definitions of phase velocity is usually missed. In the real world of surface-wave seismology, plane waves do not occur, and we must carefully distinguish between dynamic and structural wavenumbers and phase velocities.

Generally, we may define the dynamic phase velocity as the local phase velocity of an individual wavefield. The structural phase velocity in a given point of the medium is that of a plane wave in a fictitious laterally homogeneous area around this point. In the case of guided or surface waves, such waves are known as local modes (e.g. Maupin 1988); each mode has its own structural velocity.

Measured, i.e. dynamic, phase velocities cannot directly be attributed to the structure and cannot be 'regionalized' or used as an input for a tomographic inversion. They must first be converted into structural phase velocities. Eq. (4) indicates that the second spatial derivatives of the wavefield must be known for this purpose while the dynamic phase velocity is determined from the first derivatives.

Equations (1) and (4) are both non-linear with respect to F. However, (4) is compatible with the superposition principle while (1) is not. Any non-vanishing superposition of waves with the same structural wavenumber will reproduce that wavenumber when inserted into (4), but will, in general, exhibit a spatially variable dynamic wavenumber

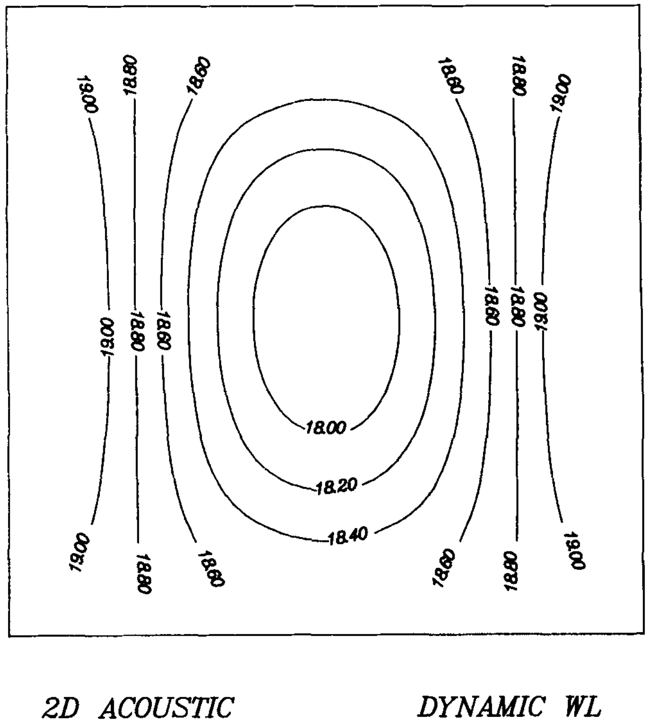


Figure 1. Dynamic wavelength waves of a non-plane (but nearly plane) wave in a homogeneous structure with a structural wavelength of 18.81 arbitrary length units (ALU). The wavefield is composed of three plane waves. A 'direct' wave incident from the top and carrying 99.5 per cent of the total energy is superimposed with two weak 'scattered' waves incident at ±60° and carrying 0.25 per cent of the energy each. The waves interfere destructively at the centre of the figure. The conventional method of interpretation would attribute the variations of the phase velocity to the structure.

that does not permit a structural interpretation. This is illustrated in Fig. 1, which shows the dynamic wavelength of a superposition of three interfering plane waves of the same wavelength; the dynamic wavelength is not uniform, and might be misinterpreted as indicating an inhomogeneous structure. It is now clear why 'interferences' have always been a problem in surface-wave dispersion analysis. The conventional definition (1) of the wavenumber vector is useless when waves propagating in different directions interfere. Eq. (4) is unaffected by this complication and remains valid even for standing waves. For these $b=$ const. $w=0$, and all information on the medium is contained in the logarithmic amplitude a. The structural parameters cannot, in general, be determined from phase measurements alone.

THE EIKONAL EQUATION AND THE DYNAMIC PHASE VELOCITY

Equation (2) is closely related to the eikonal equation

$$(\text{grad}\, T)^2=1/c^2 \quad (5)$$

as formulated for example by Aki & Richards (1980, p. 90). This becomes apparent when we define the traveltime function T so that it describes the traveltime of a phase front of our time-harmonic wave $F(x, y)\exp(i\omega t)$. Assuming zero phase at the source, the phase at any location must be zero at time $t = T(x, y)$. This is achieved with

$$T(x, y) = -b(x, y)/\omega. \tag{6}$$

Solving (5) for c, we obtain the same expression as for v in eq. (2). Thus, the eikonal equation can be given a precise meaning: it defines the dynamic phase velocity. The normal understanding is of course that c in (5) is the structural velocity and the eikonal equation is a short-wavelength approximation. In that sense, the present paper is an investigation into the accuracy of the eikonal equation for finite wavelengths. The classical interpretation of surface-wave phase velocities is based on the eikonal equation in place of the wave equation, and therefore inaccurate.

SOME HISTORICAL REMARKS

It is hard to believe that the conceptual difference between structural and dynamic phase velocities should not have been noticed. Yet I have not found any mention of it in the seismological literature, neither in the classical papers on surface-wave dispersion measurements nor in modern textbooks such as Aki & Richards (1980) or Ben-Menahem & Singh (1981).

It is of course a common observation that measured phase velocities are incompatible with a purely structural interpretation, and pertinent remarks are found in many places (e.g. Ewing & Press 1959; Knopoff, Mueller & Pilant 1966; Dziewonski & Hales 1972; Seidl & Mueller 1977). However, the problem is usually covered with a smoke screen by distinguishing between a well behaved (i.e. plane wave) component of the wavefield that can be evaluated with eq. (2), and ill-behaved components that must be ignored. It will suffice to quote from Aki & Richards (1980, p. 582): 'Even the fundamental-mode Rayleigh waves, which approach with different propagation directions because of lateral heterogeneity, must be considered as noise'. This concept is not very helpful. The real signal is an integral over a continuum of plane waves; how can we pick out one of these as 'the signal' and declare the others as noise?

Some insight into the problem could have been gained from studies of wave propagation on a sphere, such as the paper by Brune, Nafe & Alsop (1961) on the polar phase shift. The existence of the polar phase shift should have made it obvious that surface waves do not necessarily propagate with the structural velocity. But this was apparently not recognized as being a general property of wave propagation. I conclude this from Knopoff's (1969) attempt to prove that in a heterogeneous medium 'the apparent phase slowness (over a profile) is the average of the phase slownesses of the subparts of the medium', and from Schwab & Kausel's (1976a) paper on the quadripartite surface-wave method where structural and dynamic phase velocities are consistently confused. The same authors were well aware of the fact that surface waves on a spherically symmetric earth do not propagate with a constant velocity (Biswas & Knopoff 1970; Schwab & Kausel 1976b). They must have attributed the effect to the spherical geometry of the medium, rather than to the geometry of the wavefield.

THE RELATIONSHIP BETWEEN DYNAMIC AND STRUCTURAL WAVENUMBERS

The relationship between dynamic and structural wavenumbers follows directly from a comparison of eqs (1) and (4). For all wavefields $F = \exp(a + ib)$ that obey a Helmholtz equation we have from (4)

$$\begin{aligned} k^2 &= -\exp(-a - ib)\,\Delta \exp(a + ib) \\ &= -(\operatorname{grad} a + i \operatorname{grad} b)^2 - \Delta(a + ib). \end{aligned} \tag{7}$$

Splitting this expression into real and imaginary parts, and substituting $\mathbf{w}$ for $-\operatorname{grad} b$ according to (1), we obtain

$$k^2 = w^2 - (\operatorname{grad} a)^2 - \Delta a \tag{8}$$

and

$$2\mathbf{w} \cdot \operatorname{grad} a - \Delta b = 0. \tag{9}$$

Equation (8) is the desired relationship. The deviation of the dynamic wavenumber w from the structural wavenumber k depends only on spatial derivatives of the logarithmic amplitude, i.e. on the relative distribution of amplitudes around the point of observation. Observed dynamic wavenumbers can be corrected for the effects of non-uniform amplitude distribution with eq. (8). The relationship is local, i.e. we can determine the structural parameters locally without knowledge of the global properties of the medium or of the wavefield, especially without knowledge of the source. Unfortunately, the latter remark does not seem to apply to seismic surface waves, which do not generally fulfill the Helmholtz equation.

Each of the two correction terms in the right-hand side of eq. (8) can be associated with a well known feature of wave propagation. Let us first assume that a and b are linear functions, i.e. $\Delta a = 0$ and $\Delta b = 0$ but $\operatorname{grad} a \neq 0$. Then eq. (9) says that $\operatorname{grad} a$ must be orthogonal to $\mathbf{w}$, and F is an inhomogeneous plane wave. (8) now indicates that w is larger than k, i.e. the dynamic phase velocity is smaller than the structural one. This is in fact characteristic of inhomogeneous waves, and is usually included in the plane-wave formalism by giving the wavenumber vector one real and one imaginary component.

The other correction term, Δa, is responsible for the polar phase shift and related effects (Brune *et al.* 1961; Schwab & Kausel 1976b; Wielandt 1980). Consider a point where $\operatorname{grad} a = 0$, and the amplitude has a local extremum. If it is a maximum, then $\Delta a < 0$, w must be smaller than k, and the dynamic phase velocity larger than the structural one. This is in fact observed at caustics and foci of a wavefield (not only at the epicentre and its antipode on a spherical earth). In an amplitude minimum we have $\Delta a > 0$, and the dynamic wavenumber—the gradient of the phase—is greater than normal. Because a is the logarithmic amplitude, Δa will become large when the amplitude comes close to zero. The phase changes rapidly in the vicinity of an amplitude minimum, a phenomenon known as a 'phase jump'.

Equation (8) has important consequences for the experimental set up of phase-velocity measurements. We need data from which we can estimate not only the local gradient of the phase (as in the three-station method) but also the first and second spatial derivatives of the logarithmic amplitude. Under the assumption that the logarithmic amplitude can locally be approximated with a

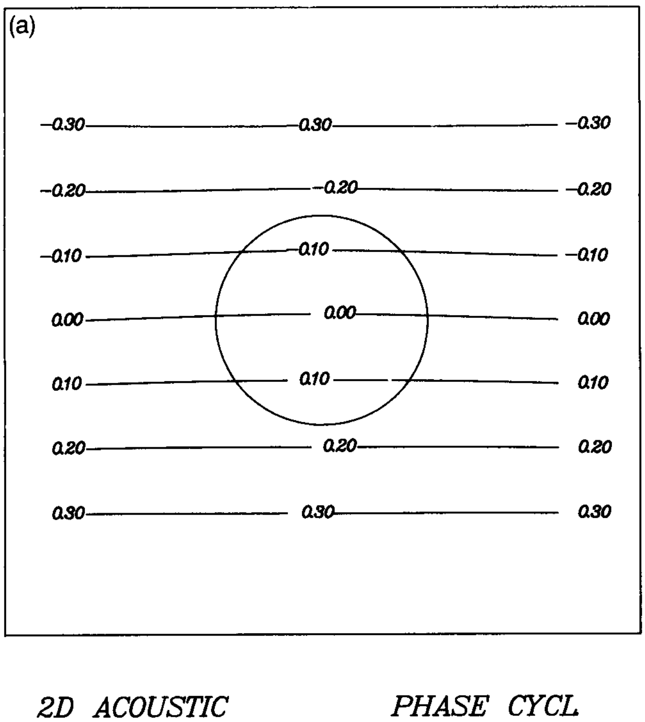


Figure 2(a). Progression of the phase front in time of steps of 0.1 cycles.

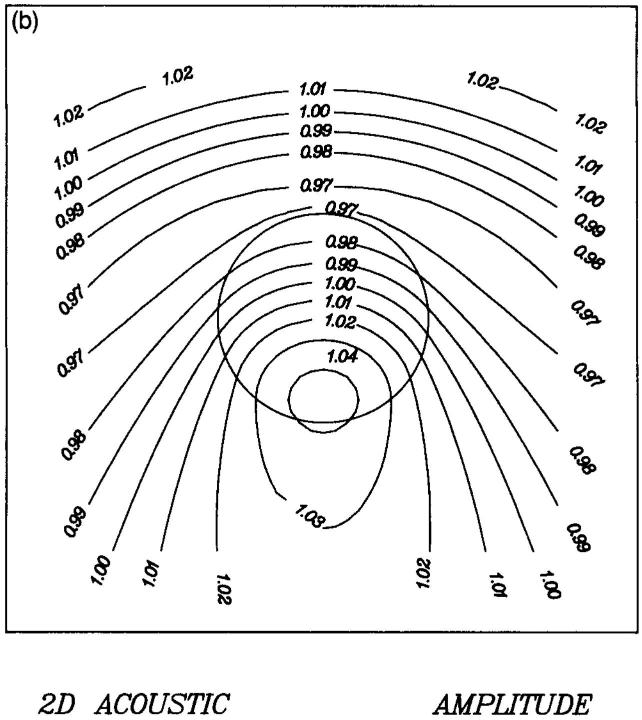


Figure 2(b). Relative amplitudes.

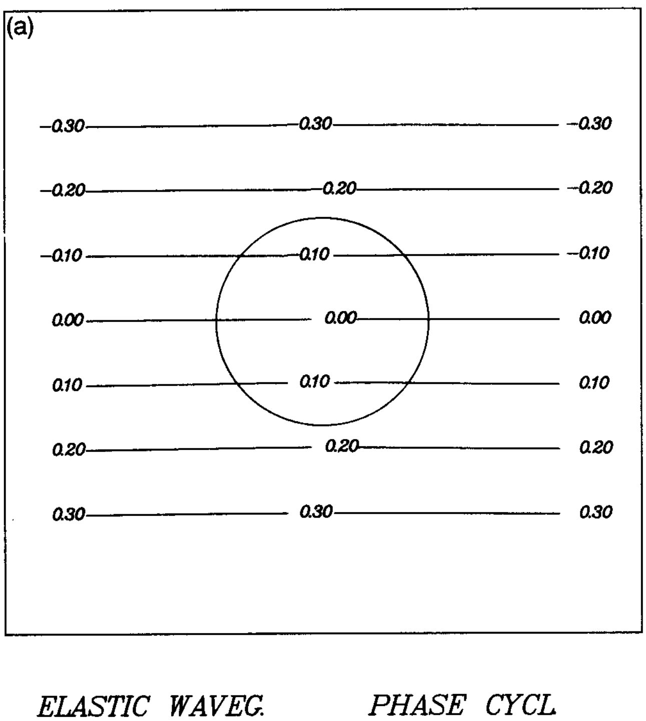


Figure 3(a). Progression of the phase front in time steps of 0.1 cycles.

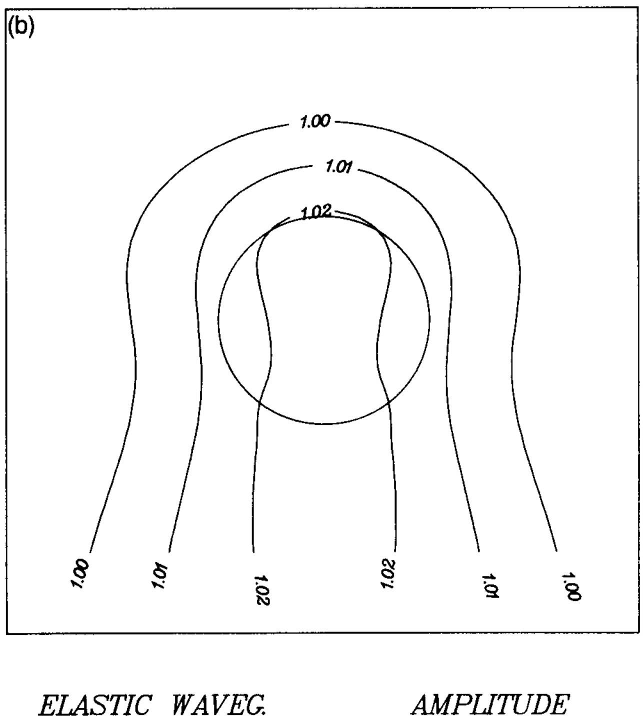


Figure 3(b). Relative amplitudes.

Figure 2 (top). Acoustic test case. A 2-D plane wave with a wavelength of 18.81 ALU is incident from the top onto a circular area in which the structural wavelength is 17.94 ALU, thus about 5 per cent lower. The anomaly has a diameter of 6 ALU.

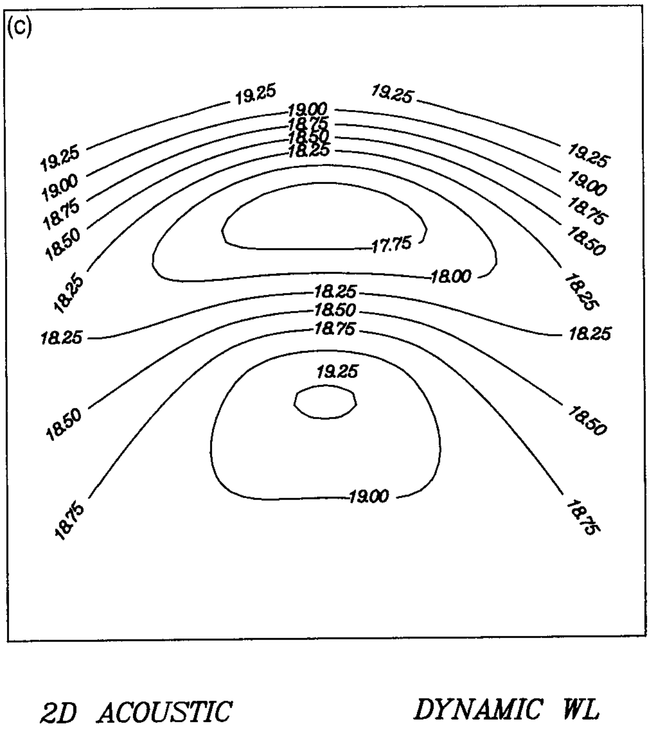


Figure 2(c). Dynamic wavelength.

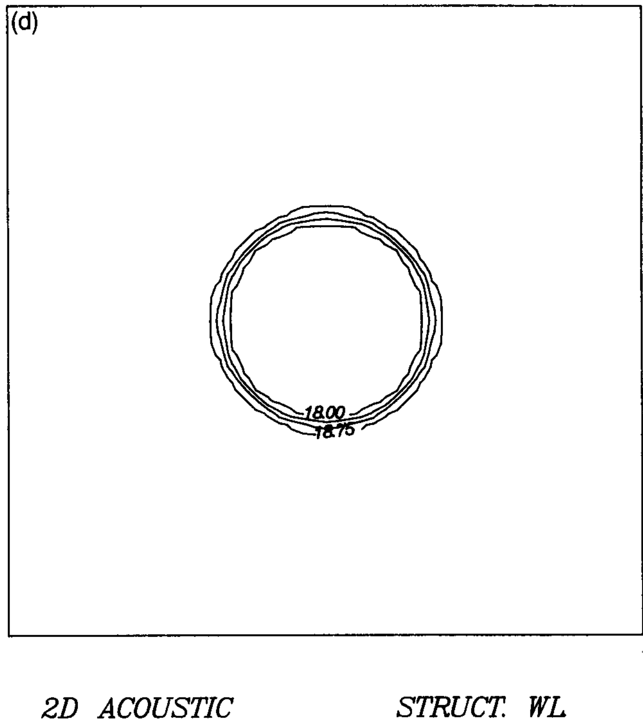
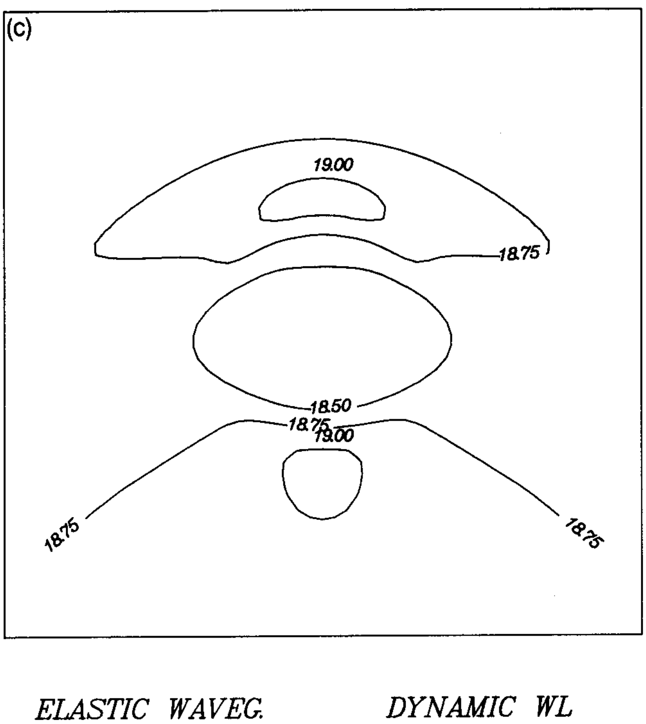


Figure 2(d). Structural wavelength, extracted from the wavefield with eq. (4).



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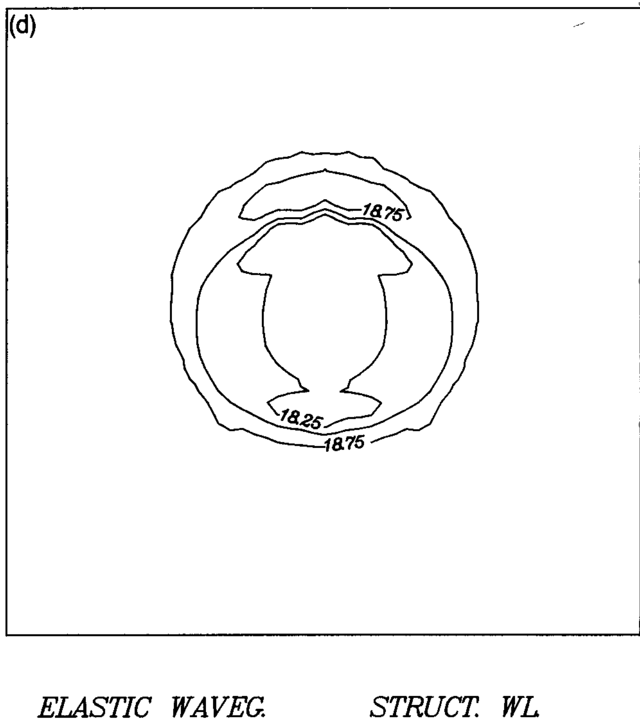

Figure 3(d). Structural wavelength, extracted from the wavefield with eq. (4).

Figure 3 (bottom). Elastic test case. The model structure is a layered waveguide representing the earth's continental crust, and has a vertical cylindrical inclusion in which the phase velocity is about 5 per cent lower. The horizontal geometry is identical to that of the acoustic test case, with a length unit of 1 km. The incident wave is the fundamental Rayleigh mode with a period of 2π seconds. The scattered wavefield contains all significant Rayleigh and Love modes. Details in Stange & Friederich (1992).

second-order polynomial in x and y, having six unknown coefficients, we would normally need six closely spaced stations to determine the structural velocity. Four or five stations would suffice in a suitable regular arrangement (e.g. four stations at the centre and corners of a regular triangle). Conventional two- or three-station measurements are, however, inadequate, and not even the four-station method proposed by Schwab & Kausel (1976a) is a solution. Of course, the required number and density of stations is a serious practical limitation; interpolation is another problem, to which we will come back later. If the number of stations is insufficient or the amplitudes are inaccurate (which is normal), then we have to guess on the amplitude distribution and can obtain only an approximate correction to the observed dynamic wavenumber. If no guess is possible, we have to assume that the correction is zero, in which case we are back to the conventional interpretation.

On the other hand, if we are somehow in a position to obtain the required data, then our derivations permit us—at least in the acoustic case—to remove the effect of interferences and to precisely image structures whose size is only a small fraction of a wavelength. This is illustrated with the following example.

STRUCTURAL INTERPRETATION OF AN ACOUSTIC WAVEFIELD

We apply eqs (1) and (4) to a synthetic wavefield that is an exact solution of the Helmholtz equation for a known structure (Fig. 2). The wavefield was computed with a 2-D version of the method described in Wielandt (1987). It represents a plane acoustic wave incident from the top of the figure onto a circular anomaly in an otherwise homogeneous background medium. The diameter of the anomaly is one third of the wavelength, and the sound velocity inside is about 5 per cent lower than outside. The wavefield is sampled at a grid of 41×41 points (unfortunately, somewhat unrealistic for seismic observations). Grid spacing is about 1/60 wavelength. We observe a slight distortion of the phase front (Fig. 2a) as the wave passes through the anomaly, and also amplitude disturbances (Fig. 2b) due to the interference between incident and scattered waves. A conventional phase-velocity measurement according to eq. (2) would simply evaluate the phase gradient in the direction of undisturbed wave propagation. The result, expressed as the local dynamic wavelength $\lambda = 2\pi/w$, is shown in Fig. 2(c). Obviously the conventional method is unable to resolve the structure; an extended diffraction pattern of apparent negative and positive anomalies appears in place of the uniform circular anomaly. In contrast, application of eq. (4) to the wavefield restores the distribution of the structural wavelength perfectly (Fig. 2d). Remaining small inaccuracies are due to spatial sampling.

It is even possible to reconstruct an acoustic medium in which both the sound velocity and the density are variable. The equivalent to the Helmholtz equation for such a medium is, according to Appendix 1,

$$(\Delta + k^2)F = \operatorname{grad} F \cdot \operatorname{grad} \ln \rho. \tag{10}$$

Proceeding in the same way as above, we find that the eqs (8) and (9) must be replaced by

$$k^2 = w^2 - (\operatorname{grad} a)^2 - \Delta a + \operatorname{grad} a \cdot \operatorname{grad} \ln \rho. \tag{11}$$

$$\operatorname{grad} b \cdot \operatorname{grad} \ln \rho = 2 \operatorname{grad} a \cdot \operatorname{grad} b + \Delta b. \tag{12}$$

This pair of equations permits an independent reconstruction of the density and wavenumber distributions in the whole medium, provided only that the density is known along a transverse profile, e.g. along the 'top' edge of the area represented in Fig. 2. The density is obtained by integrating (12) in the direction of wave propagation; the wavenumber then follows directly from (11). The results (not shown) are of the same quality as in Fig. 2(d).

APPLICATION TO SEISMIC SURFACE WAVES

As stated above, cartesian components of the particle motion of isolated modes of seismic surface waves (Love or Rayleigh) in a laterally homogeneous medium satisfy the Helmholtz equation. We can therefore use eq. (8) to correct observed phase velocities for the disturbances caused by interfering arrivals of the same mode. Moreover, we expect the acoustic Helmholtz equation to remain approximately valid for seismic surface waves when the structure is only slightly or smoothly inhomogeneous compared to the heterogeneity of the wavefield. This situation may arise near caustics, foci and seismic sources. In such a case we are certainly justified in considering the medium as locally homogeneous (i.e. neglect the logarithmic spatial derivatives of the elastic moduli against those of the displacement in the elastic equation of motion). A problematic situation for our concept is when the incident wavefield is reasonably smooth but the structure is inhomogeneous at a small scalelength; then the Helmholtz equation is a poor approximation for the elastic wave equation, and we cannot expect to obtain from eq. (4) a good image of the structure.

Nevertheless, since the method has worked so well in the acoustic case, we have made some numerical experiments with elastic surface waves. There is only a limited choice of relevant structures for which the elastic wavefield can be computed with some precision. We have investigated the case of a vertical cylindrical inclusion in a laterally homogeneous, layered waveguide whose parameters were chosen such as to represent the continental crust. The method of computation, a mode-matching technique, is described in Stange & Friederich (1992). The wavelength is about 19 km and the diameter of the cylinder is 6 km; the phase velocity (or wavelength) inside the cylinder is about 5 per cent lower than outside as in the acoustic model of Fig. 2, which is actually a 2-D equivalent of the seismic model. A fundamental-mode Rayleigh wave incident from the top of the figure propagates through the inclusion (Fig. 3a) and gives rise to a scattered wave of the same mode and a series of converted Rayleigh and Love modes, all of which contribute to the wavefield shown. To remove a very slight numerical mismatch between the inner and outer wavefields, the computed wavefield was smoothed by convolution with a Gaussian with an $1/e$ radius of 0.5 km, causing a slight loss of resolution in Figs 3 compared to Figs 2.

Figures 3(a), (b) and (c) represent phase, amplitude and local wavelength of the vertical displacement. By comparison with Figs 2(a), (b) and (c) it is apparent that for a scatterer of this size, the behaviour of acoustic waves and Rayleigh waves is quite different; acoustic scattering by a point scatterer is isotropic while Rayleigh wave scattering is predominantly forward (Snieder 1986; Stange & Friederich 1992).

Nevertheless, the interpretation of the wavefield with eq. (4) results in a reasonable structural image (Fig. 3d), which is certainly better than the image obtained with the conventional method (Fig. 3c). The improvement would be still larger if we had used multiple incident waves, for which the conventional method fails even in the absence of structural anomalies (Fig. 1).

THE WAVE EQUATION AS AN INVERSE PROBLEM

The structural interpretation of observed wavefields is normally treated as an inverse problem, the direct problem being the solution of the wave equation for a given structure. We can now also consider solving the acoustic wave equation as the inverse problem to direct structural interpretation. Given any approximate or trial solution, eqs (11) and (12) define a structure for which the given solution is exact. We can then check if this reconstruction is sufficiently close to the original structure. If so, then the approximate solution is good enough for a structural interpretation; otherwise it will introduce systematic errors. We have thus a criterion for the usefulness of an approximate solution for structural interpretation. It should also be possible to improve a trial solution iteratively according to this criterion; details have not yet been worked out.

INTERPOLATION AND INTERPRETATION

For an application of eqs (4) or (8) to real data, we must estimate second spatial derivatives of the wavefield. This is an unstable process that will in general require smoothing and interpolation. The result of the structural interpretation will critically depend on constraints imposed on the interpolating field.

It is a disadvantage of the direct approach that we can apply *a priori* constraints only to the wavefield, not to the structure. We do not know at present which constraints we would have to apply to the field in order to obtain a smooth and reasonable structure; smoothness of the interpolating field is an inadequate condition because strongly inhomogeneous wavefields can exist in smooth structures and vice versa. This, and the limitation to laterally homogeneous or slowly varying media, seem to make the direct approach impractical for the interpretation of real surface-wave data. Its strength lies in theoretical considerations; the direct approach exhibits the relationship between the observed field and the structural model more clearly than any other method of structural interpretation.

All methods of structural interpretation depend, in one way or other, on questionable assumptions with respect to the wavefield between the points of observation. The conventional two-station and three-station methods assume that the wavefield is a plane wave of known or unknown incidence. Existing amplitude information is ignored; it would in most cases vitiate the plane-wave hypothesis. In a network of four or more stations, not even the phase can, in general, be represented by that of a plane wave. We must then admit more complicated wavefields, consisting of interfering plane waves with different wavenumber vectors.

At this point we run into a fundamental problem of ambiguity. Any given set of observations can be interpolated

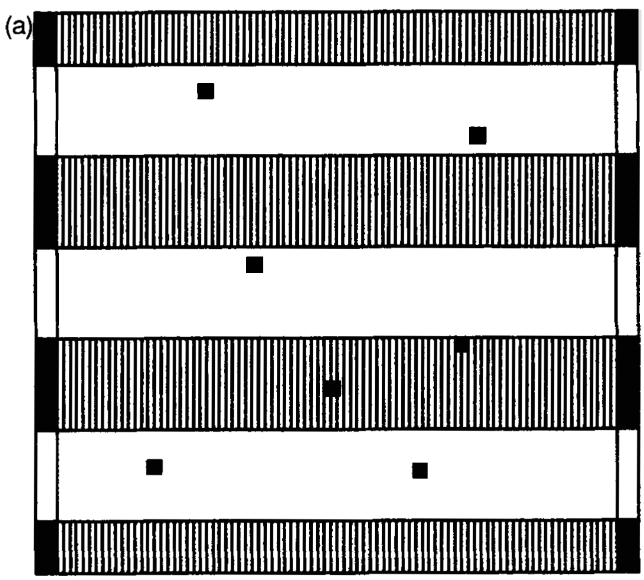

Figure 4(a). A plane wave of wavelength λ propagating over a network of seven randomly arranged stations. Negative instantaneous amplitudes are shaded. The area shown is 3 λ by 3 λ.

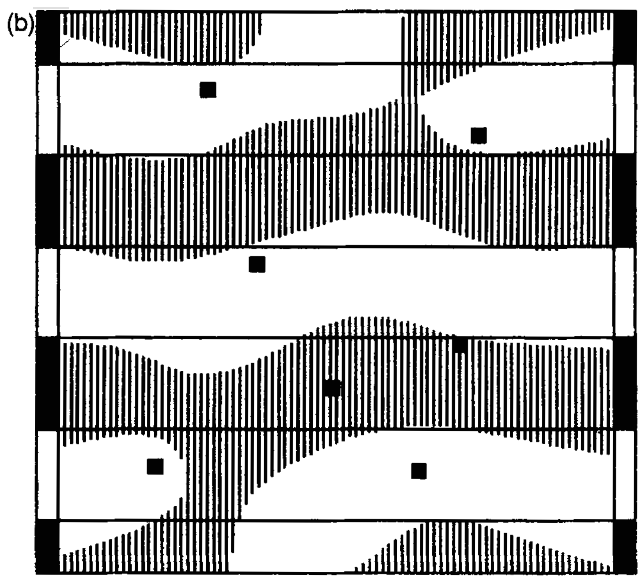

Figure 4(b). A non-plane wave of a different wavelength can produce exactly the same amplitudes and phases at all stations of the network. This wave is composed of nine plane waves of wavelength 1.2 λ, all propagating within ±45° of the original direction of propagation.

with interfering plane waves of any given wavenumber. Figs 4(a) and (b) show a numerical experiment with a homogeneous acoustic medium. Samples of a plane wave of wavenumber k_1 (Fig. 4a) are taken at seven randomly distributed stations and then interpolated with a superposition of plane waves of a different but uniform wavenumber $k_2 = 1.2\,k_1$ (Fig. 4b). A structural interpretation of the interpolating wavefield would suggest the presence of a homogeneous medium with the wavenumber k_2.

Interpolation with an incorrect wavenumber may even result in a field smoother than the original. This is evident if we consider the wavefield of Fig. 4(b) as the original and that of Fig. 4(a) as an incorrect, oversimplified interpolation.

We conclude that discrete samples of a 2-D wavefield do not contain stringent information on the underlying medium, and even the application of smoothness criteria to the interpolating field and the structure cannot restore this information uniquely. This non-uniqueness is common to all methods of structural interpretation; our numerical experiment is not related to any specific method. The mathematical situation in two dimensions is thus quite different from the 1-D case where the structural wavenumber of a homogeneous medium is uniquely determined (in fact, overdetermined) by samples of a wavefield at three points.

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APPENDIX I: THE HELMHOLTZ EQUATION FOR ACOUSTIC WAVES IN INHOMOGENEOUS MEDIA

Let $\rho(x, y)$ be the density and $\kappa(x, y)$ the incompressibility of an acoustic medium, $p(x, y, t)$ the pressure, and $\mathbf{u}(x, y, t)$ the displacement (particle motion) vector. For brevity we omit all arguments; what follows is also valid in three dimensions. Time derivatives are written with dots as usual. $\Delta = \operatorname{div}\operatorname{grad}$ is the Laplace operator. Then $p = -\kappa \operatorname{div} \mathbf{u}$, and the equation of motion is

$$\rho\ddot{\mathbf{u}} = -\operatorname{grad} p.$$

Forming the divergence of both sides, we obtain

$$\ddot{\mathbf{u}} \cdot \operatorname{grad} \rho + \rho \operatorname{div} \ddot{\mathbf{u}} = -\Delta p.$$

We now eliminate $\mathbf{u}$ from the equation using $\ddot{\mathbf{u}} = -\rho^{-1} \operatorname{grad} p$ and $\operatorname{div} \ddot{\mathbf{u}} = -\kappa^{-1} \ddot{p}$. After rearranging the terms, we have

$$\Delta p - \frac{\rho}{\kappa}\ddot{p} = \operatorname{grad} p \cdot \operatorname{grad} \ln \rho.$$

For time-harmonic waves with angular frequency ω this becomes

$$\left(\Delta + \frac{\omega^2 \rho}{\kappa}\right) p = \operatorname{grad} p \cdot \operatorname{grad} \ln \rho$$

which is our eq. (10), and reduces to the ordinary Helmholtz eq. (3) for constant ρ.

APPENDIX II: THE HELMHOLTZ EQUATION FOR ELASTIC SURFACE WAVES IN A LATERALLY HOMOGENEOUS HALFSPACE

The concept of 'modes' of wave propagation is connected with a specific method of solving the equation of motion, namely the separation of variables. We use here a slightly more general approach than found in most textbooks: we do not separate the horizontal coordinates. Our trial solution for the particle displacement vector is

$$\mathbf{u}(x, y, z, t) = [f(z) \cdot u(x, y), f(z) \cdot v(x, y), g(z) \cdot w(x, y)] \cdot h(t).$$

Each component of motion is assumed to be the product of three factors: a function of time, a function of the horizontal coordinates (this is what we have called a wavefield), and a function of depth. Separation of the variables x and y would leave only plane waves as solutions, and thus obscure the propagation effects we are studying. The trial solution must have vanishing tractions at the surface and vanishing amplitudes at infinite depth and must satisfy the equation of motion

$$\rho\ddot{\mathbf{u}} = \mathrm{div}\, \boldsymbol{T}$$

where $\boldsymbol{T}$ is the stress tensor (e.g. Ben-Menahem & Singh 1981, eq. 1.100). It is easily seen that the time dependence of a separated solution must be harmonic; we can then restrict ourselves to one frequency ω and omit the time factor $h(t) = \exp(i\omega t)$. For a given frequency, all solutions that have the same depth dependence belong to one mode.

Solutions where the vertical component disappears everywhere are Love modes. The boundary conditions at the free surface then require $u_x + v_y = 0$ and $f'(0) = 0$. (Subscripts indicate partial derivatives.) Inserting this into the equation of motion with elastic parameters depending only on depth, we find that $u^{-1}\Delta u$ and $v^{-1}\Delta v$ must equal the same constant value, say $-k^2$. Thus, u and v fulfill the Helmholtz equation with the same wavenumber. Then f must satisfy, with μ as the shear modulus,

$$(\mu f')' + (\rho\omega^2 - \mu k^2)f = 0$$

which is equivalent to the system of eqs (7.24) of Aki & Richards (1980). Together with the boundary conditions, this equation forms an eigenvalue problem that determines k.

Solutions with a non-vanishing vertical component are Rayleigh modes. The boundary conditions at the free surface then require $f'(0) \neq 0$. We normalize $f(z)$ and $g(z)$ by putting $f'(0) = 1$ and $g(0) = 1$. The absence of horizontal tractions at the surface requires $u = -w_x$ and $v = -w_y$. The wavefield of the vertical displacement is thus a potential for the wavefields of the horizontal displacement. Inserting these relationships into the equation of motion, we find that u, v and w must all satisfy the Helmholtz equation with the same wavenumber k. The remaining coupled second-order differential equations for $f(z)$ and $g(z)$ are equivalent to the system of eqs (7.28) of Aki & Richards (1980), and determine k as an eigenvalue when the boundary conditions are applied.

APPENDIX III: WAVEFIELDS WHICH PROPAGATE WITH THE STRUCTURAL PHASE VELOCITY

The wavefields in question, $F(x, y) = \exp[a(x, y) + ib(x, y)]$, must satisfy eq. (9) and

$$(\mathrm{grad}\, b)^2 = k^2 \text{ (coincidence of wavenumbers)} \qquad (13)$$

$$(\mathrm{grad}\, a)^2 + \Delta a = 0 \text{ (from eq. 8)}. \qquad (14)$$

With three differential equations for two real functions, the problem is over determined. One would first solve the independent eq. (14) for a, then (9) for b; then (13) determines the wavenumber of the medium. So it appears that in general eq. (13) cannot hold in an arbitrary medium, except in isolated points or lines.

The two known types of solutions for homogeneous media result from the trivial solution of (14), $a = \text{const}$, and from the 1-D solution, $a(y) = \ln y$, both of which can be combined with a linear phase $b(x) = ikx$. (All solutions can of course be rotated and translated, and constants added.) Separable 2-D solutions of (14) such as

$$a(x, y) = \ln \alpha x + \ln \beta y$$

$$a(x, y) = \ln \cosh \alpha x + \ln \cos \alpha y$$

$$a(x, y) = \ln \sinh \alpha x + \ln \cos \alpha y$$

(with constant α and β) require that $b(x, y)$ is a non-linear function; hence the medium cannot be homogeneous and must reflect the unphysical character of the amplitude distribution. Non-separable solutions are unlikely to have more physical relevance. To pursue this topic further seems not to be worthwhile.