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Propagation of a Shock Wave in General Relativity

——*Stationary Approximation*——

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Normal propagation of a shock wave in general relativity through the inhomogeneous gases is considered. Approximation in which the equillibrium medium is devided into infinitesimal layer of a uniform density (the Chisnell-Ôno method) is applied. In general relativity, energy density induces a gravitational force and matter deforms space-time. Both these effects are included simply in determining propagation of a shock wave. It is found that the growth of strength of the shock wave due to pressure gradient is suppressed compared with the Newtonian case. Both effects have influence on propagation of the shock wave near the surface of a star where shock strength becomes large. However, the effect owing to gravitational field induced by pressure is rather larger than that by deformation of the space-time. As an example propagation of the shock wave in the neutron star is calculated numerically and it is shown that shock strength is estimated to be 10～20% small compared with that in flat space-time.

§1. Introduction

General relativistic effects, which come mainly from the facts that energy density induces gravity and the space-time deviates from flat space-time, have many influences on phenomena of a cosmic scale. Fluid motion in general relativity has been investigated by several authors. Cahill and Taub[1] found that the flow becomes similarity one as far as Killing vector satisfies some condition. They obtained the solution for flow containing strong shock wave (similarity type) and also studied the type of solutions for various equation of state. Recently, general relativistic flows were investigated by many authors,[2] particularly for similarity flows. Bogoyavlensky[3] considered propagation of the strong blast wave, and he classified many types of solution in a particular form of metric and found the existence of oscillatory state for gases behind a shock wave. However, he did not consider propagation itself. On the other hand, except for the similarity solution, propagation of a shock wave was considered by direct numerical solution of Einstein equation.[4),5] They carried out direct numerical computation of collapse for a star and also confirmed numerically the generation of a shock wave in some stage of collapse. However, owing to numerical smoothing, clear structure of the shock wave remained ambiguous. Furthermore numerical method may be cumbersome. Thus, it is desirable to develop semi-analytical formulation, if possible.

Here, we consider propagation of shock wave including general relativistic effects by new formulation, i.e., an extension of the Chisnell-Ôno method,[6] which is reformulated in Newtonian case and its validity is discussed in § 2. In special relativity, it may become important that Lorentz factor is not neglected when a fluid velocity approaches the light velocity and energy itself is attracted by gravity. In general relativity, besides the factors mentioned above, it is much important that thermodynamic energy induces gravity and space-time is strongly deformed near compact objects (such as neutron star or black hole). In § 3 the Chisnell-Ôno method is generalized including effects of general relativity. Though this formulation is only applicable for the stationary propagation of the shock wave in static equilibrium of the stellar objects, its physical image may be clear and therefore it may be applicable to propagation of a shock wave in non-spherical explosion.[7]

Structure of the stellar objects through which the shock wave propagates is also influenced in general relativity. In § 4, we compute numerically the growth of strength of shock wave for a simple stellar structure in general relativity and the results are presented, compared with the Newtonian cases.

§ 2. Chisnell-Ôno method

Let us imagine a shock wave impinging a material layer of some thickness (in this case Δx).[6] Transmission and reflection of the shock wave occur at this layer. The thin layer considered (Fig. 1), after the transmission of the shock wave, gains momentum. However, this momentum may be negligibly small because of an infinitesimal jump of densities between this layer and the preceding one. Thus, immediately after the transmission of the shock wave, the pressure difference between regions 1 and 5, which is balanced with the gravitational force, equals that of regions 4 and 3. Thus we have

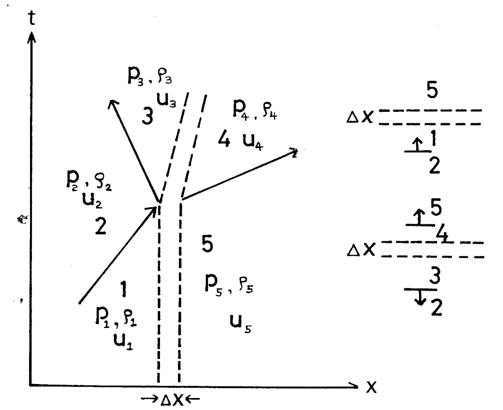

Fig. 1. Time (t) versus distance (x) diagram showing refraction and reflection of a shock wave.

$$(p_4 - p_3)/(p_5 - p_1) = 1\,. \qquad (1)$$

Next, the condition of contact discontinuity at the layer before and after the transmission of shock wave gives

$$u_5 = u_1 \quad \text{and} \quad u_4 = u_3\,. \qquad (2)$$

Now, let us introduce quantities ϕ and ψ, which mean relative velocity of flow 2 to flow 1 or 2 to the shock front,

$$\phi(12:1) = [(p_2 - p_1)(\varepsilon_2 - \varepsilon_1)/(p_2 + \varepsilon_1)(p_1 + \varepsilon_2)]^{1/2}, \tag{3·a}$$

$$\psi(1) = [(p_2 - p_1)(\varepsilon_2 + p_1)/(\varepsilon_2 - \varepsilon_1)(p_2 + \varepsilon_1)]^{1/2} \tag{3·b}$$

and

$$\psi(2) = [(p_2 - p_1)(\varepsilon_1 + p_2)/(\varepsilon_2 - \varepsilon_1)(p_1 + \varepsilon_2)]^{1/2}, \tag{3·c}$$

where ε is energy density. These relations can be derived from shock conditions at the front of the shock wave and are shown in the Appendicies for the case of general relativistic shock wave. By use of the same notations as defined in a previous paper, Eqs. (1) and (2) give, in linear approximation,

$$z_{23} = 1 + dz/z + (1 - 1/z)\, dp/p \tag{4}$$

and

$$\phi(12:1) - \phi(23:2) = \phi(54:5). \tag{5}$$

Substitution of the Newtonian version of Eq. (3·a), i.e., $\phi(12:1) = (z_{12} - 1)[p_1(1-A)/\rho_1(A + z_{12})]^{1/2}$, into Eq. (4) gives the equation determining strength of the shock wave versus pressure or radius of the objects. It is also obtained in the next section as a limit of Newtonian cases in general relativistic formulation. As is easily seen from Fig. 1, this method is fundamentally based on the existence of the static stellar structure. Therefore, we cannot apply this formulation to non-static case, for example, to propagation of shock wave through gases dynamically collapsing. The propagation of the shock wave in freely falling gases occurs frequently in astrophysical phenomena and this is studied by similarity method[8] for example. However, similarity solution assumes particular distribution of matter. On the other hand, contrary to the similarity formulation, the Chisnell-Ôno method is applicable to any distribution of matter. Thus, it is hoped to develope dynamical version of this method.

§ 3. The case of strongly gravitating gases

The general relativistic equation of hydrostatics in the stellar structure is given by

$$dp/dr = -G(\varepsilon + p/c^2)(M_r + 4\pi r^3 p/c^2)/(r^2 - 2GM_r/c^2) \tag{6}$$

for a spherically symmetric mass distribution. Here G is gravitational constant and c light velocity. We have used static Schwarzschild metric. Contribution to energy density comes both from rest mass density and from internal energy density. As is known, the right-hand side of Eq. (6) has a definite physical meaning. The first term in the numerator shows that internal energy and pressure of the gases as well as its rest mass are attracted by gravitational force. The second term in the numerator gives sources of gravity, in which not only the rest mass but also the thermodynamic energy are included. Lastly, the denominator expresses the deformation of space. Contrary to Newtonian cases, pressure gradient becomes large in general relativity. These factors may have influences on the propagation of the shock wave.

In the case considered, the gravitational mass M_r is given by

$$dM_r/dr = 4\pi r^2 \varepsilon / c^2. \tag{7}$$

Now, we separate the energy density into mass density and that of internal energy. The latter quantity, different from the Newtonian case, have had to be included in the calculation of propagation of shock wave. Thus we put $\varepsilon = \rho c^2 + e$. Then, by use of thermodynamic relation $e = p/(\gamma - 1)$, we must revise Eq. (1) to

$$\frac{p_4 - p_3}{p_5 - p_1} = \frac{1 + \alpha(p_3/p_1 - 1)}{1 - \beta(p_3/p_1 - 1)}. \tag{8}$$

Here, we approximate $\int 4\pi p r^2/(\gamma-1)\, dr \simeq 4\pi r^3 \bar{p}/3(\gamma-1)$. In numerical calculation, $\bar{p}$ may be replaced by p because of compactness of stellar body considered. The quantity α, defined by $\alpha = 1/[1 + 3M_r c^2(\gamma-1)/4\pi r^3(3\gamma-2)p]$, shows the ratio of contribution of pressure to source of gravity. On the other hand, the quantity β includes the effect of space deformation on the propagation and is given by $\beta = 8\pi G r^2 p/[3c^4(\gamma-1)(1 - r_g/r)]$, where r_g is the Schwarzschild radius of a stellar object. We here estimate order of magnitude of the parameters α and β. Although α has at most the value of about 0.6 in extreme relativistic gases, where $\gamma = 4/3$, β becomes infinitely large when radius of the stellar object approaches critical one, i.e., Schwarzschild radius. As is seen from both expressions of α and β, pressure plays an essential role. At the stellar interior where pressure is negligibly small, the factors α and β are nearly zero. On the other hand, in the case where pressure of gas is low but its volume is large, these factors must not be negligible. Such a case may occur on a cosmic scale in the early universe but is not considered here.

By making use of the notations previously defined and after some manipulation taking account of the first order term of dz and dp, Eq. (8) is reduced to

$$z_{23} = 1 + \frac{1 - \beta(z-1)}{1 + \beta z}\frac{dz}{z} + \frac{dp/p}{1 + \beta z}[1 - 1/z - (\alpha/z + \beta)(z-1)]. \tag{9}$$

In a flat space, where $\alpha = 0$ and $\beta = 0$, this relation is reduced to Eq. (4). Physically, z_{23} expresses the strength of the reflected shock wave at the infinitesimal layer (Fig. 1). This must have the value of about 1. Even for an extremely strong shock wave, z_{23} must be slightly different from 1. As z becomes larger, we see that Eq. (9) is reduced to $z_{23} = 1 - dz/z - dp/p$. Therefore, also in general relativity, the strength of the reflected shock wave is always nearly equal to 1.

Next, the contact condition (2) must be revised by taking account of the relativistic addition law of velocity. As is shown in Appendix A, various relations between velocities of flow and shock front are the same as special relativity when they are expressed in terms of proper velocity instead of coordinate velocity. Using

$$\phi(12:1) = (\beta_2 - \beta_1)/(1 - \beta_1\beta_2),$$

$$\phi(23:2) = (\beta_3 - \beta_2)/(1 - \beta_2\beta_3)$$

and

$$\phi(54:5) = (\beta_4 - \beta_5)/(1 - \beta_4\beta_5),$$

we can express the contact condition given previously in the following form:

$$\phi(12:1) - \phi(23:2) - \phi(54:5) + \phi(12:1)\cdot\phi(23:2)\cdot\phi(54:5) = 0. \qquad (10)$$

If we use for z_{23} Eq. (4) which is the non-relativistic version of Eq. (9) and use the relation for ϕ in the special relativistic case, i.e., Eq. (3·a), we can obtain the equation determining propagation of the special relativistic shock wave as was already found in Ref. 9). This is a very complex form as is given in Appendix B. Our interest, however, is in the effects induced by general relativity. Thus, to see clearly the difference between general relativity and Newtonian case, we take a much simpler case. We assume the same gases ahead of and behind the shock front ($\gamma_1 = \gamma_2$). Furthermore, we use non-relativistic version for the Rankine-Hugoniot relation as given in Appendix B. Then, substituting z_{23} and z_{54} into Eq. (10) and taking only the first order terms of dz, dp, $d\rho$, and velocity of sound, we find after all the equation

$$\frac{dz}{d\ln p} = \frac{1 - \dfrac{d\ln\rho}{d\ln p} + \dfrac{2}{z-1}\dfrac{1}{1+\beta z}\left[1 - \dfrac{1}{z} - \left(\dfrac{\alpha}{z} + \beta\right)(z-1)\right]\sqrt{\dfrac{(1+Az)z}{1+A}}}{\dfrac{1}{z+A} - \dfrac{2}{z-1} - \dfrac{2}{z-1}\left[\dfrac{1-\beta(z-1)}{1+\beta z}\right]\sqrt{\dfrac{1+Az}{(1+A)z}}} \qquad (11)$$

which determines the variation of strength of the shock wave propagating through the inhomogeneous gases.

Before computing numerically, we consider how the general relativistic effects appear in the above equation. We may see that α and β in the numerator of the right-hand side act as factors diminishing the growth rate of strength of the shock wave. Also β in the denominator may have a similar effect because of plus sign. These effects are confirmed numerically in the next section.

§ 4. Results and discussion

Strong gravitational field is realized in and near the neutron stars or the

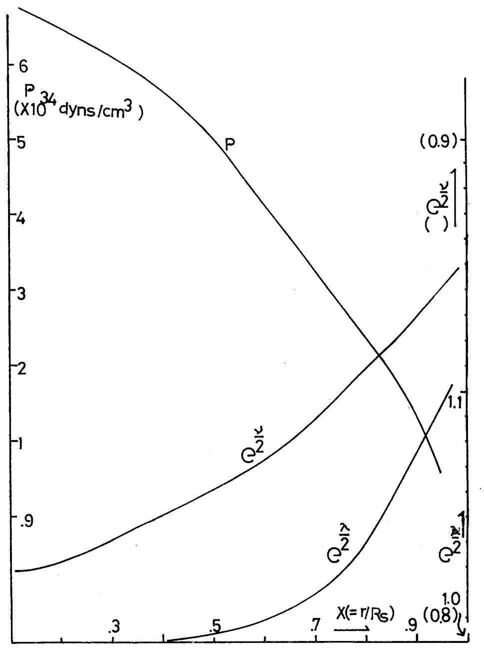

Fig. 2. Physical variables in the stellar interior used. Central density is 1.0×10^{15}(gr/cm^3). Metric coefficients and pressure versus non-dimensional radius $x(=r/R_s)$ are plotted.

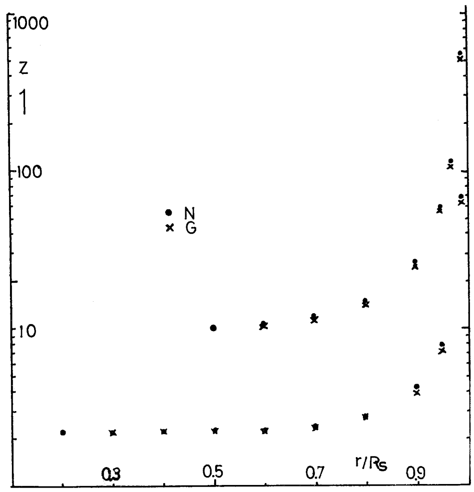

Fig. 3. Variation of strength of a shock wave versus radius. For comparison, the case in Newtonian mechanics (dot) are also plotted.

surface of the black hole. It may be thought that at the early stage of a super-nova explosion, the collapsing body becomes neutron star or black hole. During this process of dense body formation, propagation of a shock wave may be influenced by deformation of space-time. Therefore, as a numerical example we compute propagation of a shock wave in the neutron stars. To avoid complexity, we use the model of constant density. Although such a star is not realized astrophysically, we can estimate some physical characteristics for general relativistic structure of the star and here use it only in order to determine metric and pressure distribution. Then, we assume that gas is ordinary one of adiabatic index $\gamma=5/3$. Mass M_r, pressure p and metric coefficients $e^{\lambda/2}$ and $e^{\nu/2}$ are given by $M_r=4\pi/3\rho r^3$,

$$p=\frac{(\rho c^2/3)\,[[(1-R_g r^2/R_s^3)/(1-R_g/R_s)]^{1/2}-1]}{1-(1/3)\,[(1-R_g r^2/R_s^3)/(1-R_g/R_s)]^{1/2}},$$

$$e^{\lambda/2}=(1-R_g r^2/R_s^3)^{-1/2}$$

and

$$e^{\nu/2}=1.5\,(1-R_g/R_s)^{1/2}-(1-R_g r^2/R_s^3)^{1/2}/2\,.$$

Here R_s and R_g are radius of the neutron star and corresponding Schwarzschild radius, respectively. In Fig. 2, static physical quantities are plotted. As is well known, stars of mass larger than M_{critical} have no stable configuration. In this calculation we put $M_{\text{total}}=0.5$ (solar unit). Then R_s and R_g are 6.4×10^5(cm) and 1.4×10^5(cm), respectively. Density assumed is 1.0×10^{15}(gr/cm^3). Also we have assumed initial strength of the shock wave to be 2.2 starting from the center or to be 10 at the intermediate point of the stellar radius, respectively. In Fig. 3 the growth of strength of the shock wave both in the present case and in the Newtonian one is plotted. It is easily seen that the growth of shock strength in general relativity is suppressed compared with that in Newtonian case, but the suppression is comparatively small. This effect, however, is due to both factors of α and β. To see the effects due to α and β respectively, we represent the curves of strength z determined by the case when either of the two appears in Eq. (11) in Fig. 4. We see that the factor α makes the growth of the shock strength much suppressed. On the other hand, β has minor influence on the propagation of the shock wave in this model. In Fig. 4, we show the values of α and β although these are very small. As seen from the figure, deformation of space occurs near the surface of the star. On the other hand, although α has rather large values in the central part of the star the suppression due to this appears in the outer part because of large values of z. In supernovae explosion collapsing core may shrink rapidly and outside it there remains gaseous envelope through which a shock wave propagates. Thus, much larger effect may be expected in realistic models of star. Rate of suppression in z, however, can be estimated to be 10~20% less in the model used. Thus, we can conclude that stellar explosion in general relativity may be slightly more difficult to occur than in Newtonian mechanics.

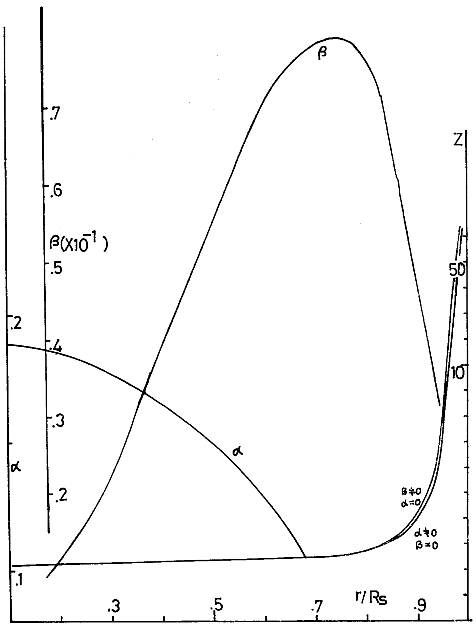

Fig. 4. Effects of α and β on the growth of the shock wave. α and β themselves are also represented.

Although the formulation considered in this paper is valid only in stationary generation of explosion energy, the conclusion obtained will be valid in real explosion. To justify the results, we consider this conclusion from another viewpoint. In general relativity proper volume becomes much larger than that in a flat space and then pressure due to thermal energy generated in the star during explosion is smaller.[11] Therefore, the shock strength may become weak in curved space

On the other hand, the time for a shock wave to travel the star of radius R_s is given by $T=(1/c)\int_0^{R_s} e^{\lambda/2}/\beta_s dr$, where β_s is velocity of the shock front. If the gas ahead of the shock wave is at rest, β_s is given by $\psi(1)$. From the above relation, we can see that the time required becomes long. Qualitatively, these facts agree with those mentioned previously.

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Appendix A

——ϕ and ψ versus Velocity Relation——

Let us define N^μ, a four-unit vector normal to the front of a shock wave in Schwarzschild coordinate system. Its line element is given by

$$ds^2=c^2e^\nu dT^2-e^\lambda dr^2-r^2 d\Omega^2. \tag{A·1}$$

Then, by introduction of energy-momentum tensor $T^{\mu\sigma}$, we have for conservation of energy and momentum at the front of the shock wave,

$$[T^{\mu\sigma}N_\sigma]=0. \tag{A·2}$$

The symbol [ ] defines the difference of variables ahead of and behind the shock front. Mass conservation is expressed in the form:

$$[\rho u^\mu N_\mu]=0. \tag{A·3}$$

ρ and u^μ are matter density and four-velocity of a fluid particle, respectively. Defining $U=u^\mu N_\mu$ and after some lengthy calculation, we obtain the following equations:

$$U_2^2=(p_2-p_1)(p_2+\varepsilon_1)/[(\varepsilon_2+p_2)(\varepsilon_2-\varepsilon_1+p_1-p_2)], \tag{A·4}$$

$$U_1^2=(p_2-p_1)(\varepsilon_2+p_1)/[(\varepsilon_1+p_1)(\varepsilon_2-\varepsilon_1+p_1-p_2)] \tag{A·5}$$

and

$$\rho_2/\rho_1=U_1/U_2. \tag{A·6}$$

Suffix 1 or 2 denotes gas ahead of or behind the shock front. Also, ε is given by $\varepsilon=\rho c^2+e$.

Next, these relations can be expressed by use of usual proper velocity of a fluid particle and a shock wave, β and β_s, respectively. Let us denote the relative velocity of a shock front to the fluid 1 or the fluid 2 to 1, as $\psi(1)$ or $\phi(12{:}\,1)$. These are given in the velocity form by

$$\psi(1)=V_2 \tag{A·7}$$

and

$$\phi(12{:}\,1)=(V_2-V_1)/(1-V_1V_2) \tag{A·8}$$

with $V_i^2=U_i^2/(1+U_i^2)$.

Now, expressing explicitly V or U by the proper velocities of the fluid and the shock front we obtain

$$V=(\beta_s-\beta)/(1-\beta_s\beta) \tag{A·9}$$

and

$$U=(\beta_s-\beta)/\{\sqrt{1-\beta^2}\sqrt{1-\beta_s^2}\}. \tag{A·10}$$

From Eq. (A·8) ϕ becomes $\phi=(\beta_2-\beta_1)/(1-\beta_1\beta_2)$. Finally, by substitution of Eqs. (A·4) and (A·5) into Eqs. (A·7) and (A·8), ψ and ϕ are expressed in the following forms:

$$\psi(1)=[(p_2-p_1)(\varepsilon_2+p_1)/(\varepsilon_2-\varepsilon_1)(p_2+\varepsilon_1)] \tag{A·11}$$

and

$$\phi(12{:}\,1)=[(p_2-p_1)(\varepsilon_2-\varepsilon_1)/(p_2+\varepsilon_1)(p_1+\varepsilon_2)]. \tag{A·12}$$

Appendix B

——*General Relativistic Rankine-Hugoniot Relation*——

This relation is the same as that for special relativistic shock wave, as far as we use proper velocity for fluid particle and shock front. We can find it from two conservation laws (A·2) and (A·3) and it is given by

$$(p_2+\varepsilon_2)^2/\rho_2^2-(p_1+\varepsilon_1)^2/\rho_1^2$$
$$=(p_2-p_1)[(p_1+\varepsilon_1)/\rho_1^2+(\varepsilon_2+p_2)/\rho_2^2]. \tag{B·1}$$

This relation can be reexpressed by means of the notations previousely defined[9] in the form:

$$b_1(\xi yz-1)[2+b_1\zeta(\xi yz+1)]=(z-1)[1+y+b_1\zeta(1+\xi y^2z)], \tag{B·2}$$

where $z=p_2/p_1$, $y=\rho_1/\rho_2$, $\zeta=p_1/(\rho_1 c^2)$, $b_1=\gamma_2/(\gamma_2-1)$ and $\xi=b_2/b_1$. The y-z relation in general case is so cumbersome that we must use the simplified relation corresponding to each situation occurred. Here, the most simple case, i.e., $\zeta \ll 1$, is considered. In this case y is given by

$$y=(1+Az)/(z+A) \tag{B·3}$$

with $A=(\gamma-1)/(\gamma+1)$.

Now, substituting this relation into ψ and ϕ, and expanding z_{54}, z_{23} in terms of z, p and ρ and the first order terms of dz and dp, we can obtain Eq. (11).

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