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Propagation of chaos for the Burgers equation

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§ 0. Introduction.

In [5] H. P. McKean considered systems of many particles obeying stochastic differential equations:

$$dX_i^n=\frac{1}{n-1}\sum_{j\neq i}^{n}e(X_i^n, X_j^n)dB_i+\frac{1}{n-1}\sum_{j\neq i}^{n}f(X_i^n, X_j^n)dt \qquad (i=1, 2, \cdots, n). \tag{0.1}$$

Under the conditions of smoothness and boundedness of the coefficients, he proved that if the initial values $X_i^n(0)$ are i. i. d. random variables then any fixed finite particles converge to independent copies of a one-dimensional diffusion process determined by an equation

$$dX(t)=e[X(t), p_t]dB(t)+f[X(t), p_t]dt, \tag{0.2}$$

where $e[x, \mu]=\int_R e(x, y)\mu(dy)$ and p_t is a distribution of $X(t)$. A point of the proof is in studying the processes $\{X_i^n(t)\}$ in the infinite product probability space $\prod_{i=1}^{\infty}\{\mu_i, P_i\}$ of identically and independently distributed initial distribution and Brownian motions, and applying Hewitt-Savage's 0-1 law to these diffusion processes on this probability space. There is another approach to this problem employed by H. Tanaka and A. S. Sznitman [9], [8]. They discussed probability measure-valued processes $(1/n)\sum_{i=1}^{n}\delta_{X_i^n(t)}$ from the point of view of a martingale problem. In these arguments the smoothness of the coefficients e, f is crucial. However an interesting case of $e(x)=1$ and $f(x)=(\lambda/2)\delta(x)$ is excluded. In this case the expected limit process satisfies

$$dZ(t)=dB(t)+\frac{\lambda}{2}p_t(Z(t))dt, \tag{0.3}$$

and $p_t(x)$ is a solution of the Burgers equation

$$\frac{\partial}{\partial t}p=\frac{1}{2}\nabla^2 p-\frac{\lambda}{2}\nabla p^2 \qquad \left(\nabla=\frac{\partial}{\partial x}\right). \tag{0.4}$$

(0.4) is uniquely solvable for any initial distribution in the following way:

$$p_t(x)=-\frac{1}{\lambda}\cdot\frac{\partial}{\partial x}\left\{\log\int_R g_t(x-y)e^{-\lambda\int_{-\infty}^{y}\nu(dz)}dy\right\}, \qquad g_t(x)=\frac{1}{(2\pi t)^{1/2}}e^{-x^2/2}.$$

If the measure ν is smooth enough, then the above formula turns to

$$p_t(x)=\frac{\int_R g_t(x-y)\exp\left(-\lambda\int_{-\infty}^{y}\nu(dz)\right)\nu(dy)}{\int_R g_t(x-y)\exp\left(-\lambda\int_{-\infty}^{y}\nu(dz)\right)dy}.$$

Therefore it is easily seen that $p_t(x)\leqq e^{|\lambda|}/(2\pi t)^{1/2}$ holds, which assures the uniqueness and existence for the equation (0.3). This diffusion process will be called Burgers diffusion process $\{Z(t)\}$. The generator of the diffusion process with $e(x)=1$ and $f(x)=(\lambda/2)\delta(x)$ is at least formally given by

$$L_n=\frac{1}{2}\Delta+\frac{\lambda}{2(n-1)}\sum_{i\neq j}^{n}\delta(x_i-x_j)\nabla_j\,, \tag{0.5}$$

where $\Delta=\sum_{j=1}^{n}\nabla_j^2$ and $\nabla_j=\partial/\partial x_j$. However if we set

$$H_{ij}(x)=H(x_i-x_j)\,,\qquad H(\xi)=\begin{cases}\left(\frac{1}{2}\right)\lambda, & \xi>0\\ 0, & \xi=0\\ \left(-\frac{1}{2}\right)\lambda, & \xi<0,\end{cases}$$

then (0.5) can be written as

$$\begin{aligned}L_n&=\frac{1}{2}\Delta+\frac{1}{2(n-1)}\sum_{i,j}(\nabla_i H_{ij})\nabla_j\\ &=\frac{1}{2}\Delta+\frac{1}{2(n-1)}\sum_{i,j}\nabla_i(H_{ij}\nabla_j)\,.\end{aligned}$$

Since the last representation of L_n is of the divergence form, one can define the diffusion process $\{X_n(t)\}$ with generator L_n employing a result from theory of partial differential equations, which will be seen in § 1.

Our purpose of this paper is to establish "propagation of chaos" for the above diffusion process. This terminology is used as the propagation of being chaotic in the following sense: Let S be a separable metric space. Let $\{\mu_n\}$ be a family of symmetric probability measures on S^n and μ be a probability measure on S. Then a sequence $\{\mu_n\}$ is μ-chaotic if for any bounded continuous functions $f_1, \cdots, f_m$ on S, $\langle\mu_n, \otimes_{i=1}^{m} f_i\rangle\rightarrow\prod_{i=1}^{m}\langle\mu, f_i\rangle$ as $n\rightarrow\infty$.

Set $\pi(d\xi)=(1/(2\pi)^{1/2})e^{-\xi^2/2}d\xi$ and $\pi(dx)=\pi(dx_1)\cdots\pi(dx_n)$ $(x=(x_1, \cdots, x_n))$.

THEOREM. *Suppose* $\{X_n(t)=(X_n^1(t), \cdots, X_n^n(t))\}$ *to be the diffusion process on* $\boldsymbol{R}^n$ *with generator* L_n, *whose initial distribution is of the form* $\varphi_n(x)\pi(dx)$. *Assume the sequence* $\{\varphi_n(x)\pi(dx)\}$ *is* $\phi(\xi)\pi(d\xi)$*-chaotic and* φ_n *is a symmetric function in* $L^2(\boldsymbol{R}^n, \pi)$ *satisfying*

$$\int_{R^m}\left\{\int_{R^{n-m}}\varphi_n(x)\pi(dx_{m+1})\cdots\pi(dx_n)\right\}^2\pi(dx_1)\cdots\pi(dx_m)\leqq C\mu^{2m}$$

with positive constants C and μ for any $m\leqq n$. Let $T=(\lambda\mu)^{-4}-1$. Then a sequence of the distributions of $\{X_n(\cdot)\}$ in $C([0, T)\to \boldsymbol{R}^n)\cong\{C([0, T)\to\boldsymbol{R})\}^n$ is $Z(\cdot)$-chaotic, where $\{Z(\cdot)\}$ is the Burgers diffusion process with initial distribution $\phi(d\xi)\pi(d\xi)$ defined in the above.

We sketch the story of the proof of Theorem. Since it seems difficult to show Theorem in some probabilistic method, we approach Theorem by estimating marginal distributions of $\{X_n(\cdot)\}$ analytically. Because of the homogeneity of H our process $\{X_n(\cdot)\}$ has the same space-time invariance as Brownian motion. Therefore a new process $Y_n(t)=e^{-t}X_n(e^{2t}-1)$ turns to be a diffusion process with generator

$$\hat{L}_n=2L_n-\sum_{i=1}^{n}x_i\nabla_i$$

$$=\Delta-\sum_{i=1}^{n}x_i\nabla_i+\frac{1}{n-1}\sum_{i,j=1}^{n}\nabla_i H_{i,j}\nabla_j\,.$$

Since the first two terms is the generator of Ornstein-Uhlenbeck process, we can write the dual semi-group $\hat{T}_n^*(t)$ of $\{Y_n(t)\}$ in $L^2(\boldsymbol{R}^n, \pi)$ as

$$\text{(0.6)}\qquad \begin{aligned}\hat{T}_n^*(t)\varphi&=\hat{T}_n^0(t)\varphi-\frac{1}{n-1}\sum_{i,j=1}^{n}\int_0^t ds\hat{T}_n^0(t-s)\nabla_i^*H_{ij}\nabla_j^*\hat{T}_n^*(s)\varphi\\&=\hat{T}_n^0(t)\varphi+G_n\hat{T}_n^*(t)\varphi\,,\qquad(\nabla_i^*=-x_i+\nabla_i)\,,\end{aligned}$$

where $\hat{T}_n^0(t)$ is the semi-group of the generator $\Delta-\sum_{i=1}^n x_i\nabla_i$ and $*$ denotes the dual operation in $L^2(\boldsymbol{R}^n, \pi)$. Therefore $\hat{T}_n^*(t)$ can be expanded in a Neumann series

$$\hat{T}_n^*(t)\varphi=\sum_{p=0}^{\infty}G_n^p\hat{T}_n^0(t)\varphi\qquad\text{in}\quad L^2(\boldsymbol{R}^n, \pi)\,.$$

If we denote the projection from $L^2(\boldsymbol{R}^n, \pi)$ to $L^2(\boldsymbol{R}^m, \pi)$ by P_m $(1\leqq m\leqq n)$, then

$$\text{(0.7)}\qquad P_m\hat{T}_n^*(t)=\sum_{p=0}^{\infty}P_mG_n^p\hat{T}_n^0(t)\,.$$

If we choose φ as any function satisfying those assumptions of Theorem, then it is easy to see that for each fixed $p\geqq 0$, $P_mG_n^p\hat{T}_n^0(t)\varphi_n$ converges weakly in $L^2(\boldsymbol{R}^m, \pi)$ as $n\to\infty$. Therefore if we dominate $\|P_mG_n^p\hat{T}_n^*(t)\varphi_n\|$ by some convergent series uniformly with respect to n, (which shall be done in §2), then one can compute the limit of $P_m\hat{T}_n^*(t)\varphi_n$ as $n\to\infty$. If we denote the limit by $f_m(t, x)$, then from (0.6) they satisfy

$$\begin{aligned}f_m(t, x)&=\hat{T}_m^0(t)(\phi(x_1)\cdots\phi(x_m))\\&\quad-\sum_{i=1}^{m}\int_0^t P_m\hat{T}_{m+1}^0(t-s)\nabla_i^*H_{i,m+1}\nabla_{m+1}^*f_{m+1}(s, x)ds\,.\end{aligned}$$

In §3 it will be shown that this infinite system equations can be solved uniquely up to $\hat{T}=-2\log(|\lambda|\mu)$ and they coincide with products of the solutions for a transformed Burgers equation. In this way we can prove the theorem.

Our thanks are due to Professor H. Tanaka who led our interest to this problem.

§1. Definition of the diffusion process with generator L_n.

Since our diffusion process has singular drifts, the construction is not easy. The key point is to notice its divergence form, that is, the generator L_n can be written as

$$L_n = \frac{1}{2}\Delta + \frac{1}{2(n-1)} \sum_{i,j=1}^{n} (\nabla_i H_{ij})\nabla_j$$

$$= \frac{1}{2}\Delta + \frac{1}{2(n-1)} \sum_{i,j=1}^{n} \nabla_i (H_{ij}\nabla_j)$$

using the skew symmetry of H_{ij}. Let $\{a_{ij}(x), 1 \leqq i, j \leqq n\}$ be real valued bounded functions satisfying

$$\nu \sum_{i=1}^{n} \xi_i^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j$$

for some $\nu > 0$ and any $\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n$. We do not assume that $\{a_{ij}\}$ is symmetric.

DEFINITION 1.1. A continuous function $p(t, x, y)$ on $(0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ is said to be a fundamental solution of $\partial/\partial t - \sum_{i,j=1}^{n} \nabla_i a_{ij} \nabla_j$ if it satisfies the following conditions;

(i) $$p(t, x, y) \geqq 0 \quad \text{and} \quad \int_{\boldsymbol{R}^n} p(t, x, y)dy = 1$$

for any $(t, x, y) \in (0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^n$.

(ii) Let φ be a continuous function on $\boldsymbol{R}^n$ with compact support, and set $u(t, x) = \int_{\boldsymbol{R}^n} p(t, x, y)\varphi(y)dy$. Then $u(t, x) \to \varphi(x)$ uniformly on $\boldsymbol{R}^n$ as $t \to 0$ and

$$\sum_{i=1}^{n} \int_a^b \int_{\boldsymbol{R}^n} |\nabla_i u(t, x)|^2 dx\, dt < \infty,$$

$$\int_0^\infty \int_{\boldsymbol{R}^n} \left\{ u \frac{\partial}{\partial t}\phi - \sum_{i,j=1}^{n} a_{ij}\nabla_i u \nabla_j \phi \right\} dx\, dt = 0$$

for all $0 < a < b < \infty$ and continuously differentiable function $\phi(t, x)$ on $(0, \infty) \times \boldsymbol{R}^n$ with compact support.

Following D.G. Aronson ([1], [2]), J. Nash ([6]), $\cdots$, we have

LEMMA 1.1. *The fundamental solution $p(t, x, y)$ exists uniquely and satisfies*

(i) $$\int_{\boldsymbol{R}^n} p(t, x, y)p(s, y, z)dz = p(s+t, x, z)$$

(ii) $$c_1(2\pi t)^{(-1/2)n}e^{-c_2|x-y|^2/2t}\leqq p(t,\ x,\ y)\leqq c_3(2\pi t)^{(-1/2)n}e^{-c_4|x-y|^2/2t}$$

for all $(t, x, y)\in(0, \infty)\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ *with some positive constants* $c_1, \cdots, c_4$. *Therefore it is possible to define a Markov process* $\{X_n(t);\ t\geqq 0\}$ *with generator* L_n *in the sense of Definition* 1.1. *It should be noted that this Markov process has continuous sample paths a.s. by virtue of* (ii) *of Lemma* 1.1.

Next we show the tightness of the marginal processes

$$\{(X_n^1(t),\ X_n^2(t),\ \cdots,\ X_n^m(t))\},\qquad m\leqq n<\infty\,.$$

LEMMA 1.2. *Let* $p(t, x, y)$ *be a fundamental solution of* $\partial/\partial t-\sum_{i,j=1}^n\nabla_i a_{ij}\nabla_j$. *Suppose* $a_{ij}(x)=(1/2)\delta_{ij}+b_{ij}(x)$, *where* $\{\delta_{ij}\}$ *is the Kronecker symbol and* $\{b_{ij}(x)\}$ *are skew symmetric matrices satisfying* $|b_{ij}(x)|\leqq M/n$ *for some positive constant* M. *Then*

$$\int_{R^n}\Big\{\sum_{i=1}^n|x_i-y_i|^p\Big\}p(t,\ x,\ y)dy\leqq Ct^{p/2}n$$

for any $x\in\boldsymbol{R}^n$, $t>0$ *and* n *with some positive constant* C *depending only on* M *and non-negative integers* p.

If we apply this result to our diffusion processes $X_n(t)=(X_n^1(t), X_n^2(t), \cdots, X_n^n(t))$ starting with any symmetric initial distribution, we can show

$$E\Big\{\sum_{i=1}^m|X_n^i(t)-X_n^i(s)|^3\Big\}\leqq mC|t-s|^{3/2},$$

which implies the tightness of the marginal processes $\{(X_n^1(t), X_n^2(t), \cdots, X_n^m(t))\}$, $m\leqq n$. The above estimates will be published elsewhere [7].

§ 2. Uniform estimates of the solutions.

Let $S(c)f(x)=f(cx)$ for $c>0$. Then it is easy to see $L_nS(c)=c^2S(c)L_n$, therefore the corresponding semi-group $T_n(t)$ also satisfies $T_n(t)S(c)=S(c)T_n(c^2t)$. Taking Ornstein-Uhlenbeck process into consideration, set $\hat{T}_n(t)=T_n(e^{2t}-1)S(e^{-t})$. Then the above invariance implies the semi-group property of $\hat{T}_n(t)$ and its generator becomes

$$\begin{aligned}\hat{L}_n&=2L_n-\sum_i x_i\nabla_i\\ &=\Delta-\sum_i x_i\nabla_i+\frac{1}{n-1}\sum_{i\neq j}\nabla_i H_{ij}\nabla_j\end{aligned}$$

at least formally. Let $\{X_n(t)\}$ be a sample path of our diffusion process with generator L_n. Then the diffusion process $\{Y_n(t)\}$ with generator $\hat{L}_n$ is $Y_n(t)=e^{-t}X_n(e^{2t}-1)$ or conversely $X_n(t)=(1+t)^{1/2}Y_n(\log(t+1)/2)$. Hence to prove the convergence of the marginal processes of $\{X_n(t)\}$ as $n\to\infty$ it is sufficient to show that of $\{Y_n(t)\}$.

One-dimensional Hermite polynomials are defined by

$$H_n(x)=\frac{(-1)^n}{(n!)^{1/2}}e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2} \qquad (n\geqq 0).$$

Then the following relations are known:

(i) $$H_n'(x)=n^{1/2}H_{n-1}(x)$$

(ii) $$-H_n'(x)+xH_n(x)=(n+1)^{1/2}H_{n+1}(x)$$

(iii) $$\int_R H_n(x)H_m(x)\pi(dx)=\delta_{n,m}.$$

Multi-dimensional Hermite polynomials are defined by

$$H_\alpha(x)=H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)\cdots H_{\alpha_n}(x_n) \qquad \text{for} \quad \alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)\in \boldsymbol{Z}_+^n,$$

where $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$. The above relations immediately imply

(iv) $$\nabla_i H_\alpha(x)=\alpha_i^{1/2}H_{\alpha-e_i}(x), \qquad e_i=(0, \cdots, \overset{i}{1}, \cdots, 0)$$

(v) $$\nabla_i^* H_\alpha(x)=(\alpha_i+1)^{1/2}H_{\alpha+e_i}(x), \qquad \nabla_i^*=-\nabla_i+x_i\cdot$$

(vi) $$\int_{R^n}H_\alpha(x)H_\beta(x)\pi(dx)=\delta_{\alpha,\beta}, \quad \pi(dx)=\pi(dx_1)\pi(dx_2)\cdots\pi(dx_n).$$

$\{H_\alpha(x); \alpha\in \boldsymbol{Z}_+^n\}$ forms a complete orthonormal system in $L^2(\boldsymbol{R}^n, \pi)$. Moreover if we set $\hat{L}_n^0=-\sum_{i=1}^n \nabla_i^*\nabla_i$, then H_α satisfies $\hat{L}_n^0 H_\alpha=-|\alpha|H_\alpha$ ($|\alpha|=\sum_{i=1}^n\alpha_i$). Therefore the semi-group $\hat{T}_n^0(t)$ of $\hat{L}_n^0$ has a kernel $\hat{g}_t(x, y)$ with respect to $\pi(dx)$ represented by

$$\hat{g}_t(x, y)=\sum_\alpha e^{-|\alpha|t}H_\alpha(x)H_\alpha(y).$$

Let $u_t(x)\pi(dx)$ be a distribution of $Y_n(t)$. Then u_t satisfies $(\partial/\partial t)u=\hat{L}_n^* u$, where $\hat{L}_n^*$ is dual for $\hat{L}_n$ in $L^2(\boldsymbol{R}^n, \pi)$. If u takes an initial distribution $\varphi(x)\pi(dx)$, then it is easy to see that u_t can be determined by

(2.1) $$u_t=u_t^0-\frac{1}{n-1}\sum_{i\neq j}^n\int_0^t \hat{T}_n^0(t-s)\nabla_i^* H_{ij}\nabla_j^* u_s ds,$$
$$u_t^0=\hat{T}_n^0(t)\varphi.$$

Denote the norm in $L^2(\boldsymbol{R}^n, \pi)$ by $\|\cdot\|$. We consider this equation in a Hilbert space

$$H_T(\boldsymbol{R}^n, \pi)=\left\{u:[0, T]\times\boldsymbol{R}^n\to\boldsymbol{R};\ \|u\|_T^2=\int_0^T\sum_{k=1}^n\|\nabla_k^* u(s, \cdot)\|^2 ds<\infty\right\}.$$

Applying (iv), (v) and (vi) we easily see

(2.2) $$\|u\|_T^2=\sum_\alpha(|\alpha|+n)\int_0^T(u_s, H_\alpha)^2 ds$$
$$=n\int_0^T ds\|u(s, \cdot)\|^2+\sum_{k=1}^n\int_0^T ds\|\nabla_k u(s, \cdot)\|^2.$$

For $u\in H_T(\boldsymbol{R}^n, \pi)$ define

$$(2.3)\qquad G_n u_t=-\frac{1}{n-1}\sum_{i\neq j}^{n}\int_0^t \hat{T}_n^0(t-s)\nabla_i^* H_{ij}\nabla_j^* u_s ds$$

$$=\frac{1}{n-1}\sum_{i\neq j}^{n}F_n(i, j)u_t .$$

LEMMA 2.1. *For* $\varphi\in L^2(\boldsymbol{R}^n, \pi)$ $\|\hat{T}_n^0(\cdot)\varphi\|_T\leqq(n(T+1))^{1/2}\|\varphi\|$.

PROOF. Since $(\hat{T}_n^0(s)\varphi, H_\alpha)=e^{-s|\alpha|}(\varphi, H_\alpha)$, we see from (2.2)

$$\|\hat{T}_n^0(\cdot)\varphi\|_T^2=\sum_\alpha(|\alpha|+n)\int_0^T ds\, e^{-2s|\alpha|}(\varphi, H_\alpha)^2$$

$$=\sum_\alpha(|\alpha|+n)\frac{1}{2|\alpha|}(1-e^{-2T|\alpha|})(\varphi, H_\alpha)^2$$

$$\leqq n(T+1)\|\varphi\|^2.$$

LEMMA 2.2. (i) *$F_n(i, j)$ is a bounded operator on $H_T(\boldsymbol{R}^n, \pi)$.*

(ii) *G_n is a bounded operator on $H_T(\boldsymbol{R}^n, \pi)$ satisfying*

$$\|G_n^p\|_T\leqq e^{nT/2}\Big(\frac{|\lambda|n}{n-1}\Big)^p \qquad \textit{for}\quad p=1, 2, \cdots.$$

(iii) *For* $u\in H_T(\boldsymbol{R}^n, \pi)$ $\|G_n u_t\|\leqq\dfrac{|\lambda|n}{n-1}\|u\|_t$.

(iv) $$\|G_n^p u_t\|\leqq e^{nt/2}\Big(\frac{|\lambda|n}{n-1}\Big)^p\|u\|_t \qquad \textit{for}\quad p=1, 2, \cdots.$$

PROOF. Introduce an operator $\theta_n u=e^{nt/2}u$ on $H_T(\boldsymbol{R}^n, \pi)$ and set $G_{n,n}=\theta_n^{-1}G_n\theta_n$. Then

$$(2.4)\qquad \|G_{n,n}\|_T\leqq\frac{|\lambda|n}{n-1}$$

holds. In fact, applying (2.2), we have for $u\in H_T(\boldsymbol{R}^n, \pi)$

$$\sum_{k=1}^n\|\nabla_k^* G_{n,n}u_t\|^2=\sum_\alpha(G_{n,n}u_t, H_\alpha)^2(|\alpha|+n)$$

$$=\sum_\alpha(|\alpha|+n)\Big\{\frac{-1}{n-1}\int_0^t e^{-(n/2+|\alpha|)(t-s)}\Big(\sum_{i\neq j}^n\nabla_i^* H_{ij}\nabla_j^* u_s, H_\alpha\Big)ds\Big\}^2.$$

By Schwarz inequality we see

$$\leqq\frac{1}{(n-1)^2}\sum_\alpha(|\alpha|+n)\int_0^t ds\, e^{-(n+|\alpha|)(t-s)}\int_0^t ds\, e^{-|\alpha|(t-s)}\Big(\sum_{i\neq j}^n\nabla_i^* H_{ij}\nabla_j^* u_s, H_\alpha\Big)^2$$

$$\leqq\frac{1}{(n-1)^2}\int_0^t ds\, e^{-|\alpha|(t-s)}\sum_{i=1}^n\Big(\sum_{i\neq j}^n H_{ij}\nabla_j^* u_s, H_{\alpha-e_i}\Big)^2\alpha_i^{1/2}$$

$$\leqq\frac{1}{(n-1)^2}\sum_\alpha|\alpha|\int_0^t ds\, e^{-|\alpha|(t-s)}\sum_{i=1}^n\Big(\sum_{i\neq j}^n H_{ij}\nabla_j^* u_s, H_{\alpha-e_i}\Big)^2.$$

Therefore

$$\|G_{n,n}u\|_T^2=\int_0^T dt \sum_{k=1}^n \|\nabla_k^* G_{n,n}u_t\|^2$$

$$\leqq \frac{1}{(n-1)^2}\sum_\alpha \int_0^T ds \sum_{i=1}^n \Big(\sum_{i\neq j} H_{ij}\nabla_j^* u_s,\ H_{\alpha-e_i}\Big)^2$$

$$\leqq \frac{1}{(n-1)^2}\sum_{i=1}^n \int_0^T \Big\|\sum_{i\neq j} H_{ij}\nabla_j^* u_s\Big\|^2 ds\,.$$

Since $\sum_{i=1}^n \Big(\sum_{j=1}^n H_{ij}f_j\Big)\leqq n^2\lambda^2 \sum_{j=1}^n f_j^2$, the above is dominated by

$$\leqq \frac{\lambda^2 n^2}{(n-1)^2}\sum_{i=1}^n \int_0^T \|\nabla_j^* u_s\|^2 ds$$

$$\leqq \frac{\lambda^2 n^2}{(n-1)^2}\|u\|_T^2\,.$$

This proves (2.4). Since $G_n^p=\theta_n G_{n,n}^p \theta_n^{-1}$ we have (ii). (iii) can be shown similarly as (ii) and (iv) is a direct consequence of (ii) and (iii). The boundedness of $F_n(i, j)$ is already shown implicitly in the above process of the proof.

This lemma shows us that the equation (2.1) can be solved uniquely by a Neumann series $u=\sum_{p=0}^\infty G_n^p u^0$ in $H_T(\boldsymbol{R}^n, \pi)$ and the solution is in $L^2(\boldsymbol{R}^n, \pi)$ for each fixed $t\in[0, T]$ if the initial value $u^0\in H_T(\boldsymbol{R}^n, \pi)$. Obviously the estimate (ii) is not sufficient to prove the theorem because the bound goes to ∞ exponentially fast as $n\to\infty$. However if we take projections onto fixed subspaces $L^2(\boldsymbol{R}^m, \pi)$, then it is possible to obtain a bound independent of n. This will be shown through a number of lemmas.

Now we introduce two new spaces. For $1\leqq r\leqq n$ let

$${}_rL^2(\boldsymbol{R}^n, \pi)=\{\varphi\in L^2(\boldsymbol{R}^n, \pi)\,;\ \varphi \text{ is symmetric with respect to the last } n-r \text{ variables}\},$$

$${}_rH_T(\boldsymbol{R}^n, \pi)=\{u\in H_T(\boldsymbol{R}^n, \pi)\,;\ u(t, \cdot)\in {}_rL^2(\boldsymbol{R}^n, \pi) \text{ for a.e. } t<T\}.$$

Then it is easy to see that $\hat{T}_n^0(t)$ and $\hat{T}_n^*(t)$ map ${}_rL^2(\boldsymbol{R}^n, \pi)$ into itself and G_n maps ${}_rH_T(\boldsymbol{R}^n, \pi)$ into itself. Let P_m be the projection operator from $L^2(\boldsymbol{R}^n, \pi)$ to $L^2(\boldsymbol{R}^m, \pi)$ $(0\leqq m\leqq n)$, namely

$$P_m f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)$$

$$=\int_{\boldsymbol{R}^{n-m}} \pi(dx_{m+1})\cdots \pi(dx_n) f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)\,.$$

We interpret P_m for $m\geqq n$ as the identity on $L^2(\boldsymbol{R}^n, \pi)$. Let $\theta_k f=e^{kt/2}f$ and $G_{n,k}=\theta_k^{-1}G_n\theta_k$. Then we have

LEMMA 2.3. *For* $u\in{}_rH_T(\boldsymbol{R}^n, \pi)$ *and* $r\leqq m\leqq n$,

(i)
$$P_mG_nu_t=\frac{m-1}{n-1}G_mP_mu_t-\frac{n-m}{n-1}\sum_{i=1}^{m}\int_0^t ds\,\hat{T}_m^o(t-s)\nabla_i^*P_mH_{i,m+1}\nabla_{m+1}^*P_{m+1}u_s$$
$$=\frac{m-1}{n-1}G_mP_mu_t+\frac{n-m}{n-1}\sum_{i=1}^{m}P_mF_{m+1}(i, m+1)P_{m+1}u_t,$$

(ii) *if* $m\leqq k$, *then*
$$P_mG_{n,k}u_t=\frac{m-1}{n-1}G_{n,k}P_mu_t-\frac{n-m}{n-1}\sum_{i=1}^{m}\int_0^t ds\,\hat{T}_m^o(t-s)$$
$$\cdot e^{-k(t-s)/2}\nabla_i^*P_mH_{i,m+1}\nabla_{m+1}^*P_{m+1}u_s.$$

PROOF. Since P_m and $\hat{T}_n^o(t)$ are commutative, we see
$$P_mG_nu_t=\frac{-1}{n-1}\sum_{i\neq j}^{n}\int_0^t dsP_m\hat{T}_n^o(t-s)\nabla_i^*H_{ij}\nabla_j^*u_s$$
$$=\frac{-1}{n-1}\sum_{i\neq j}^{n}\int_0^t ds\,\hat{T}_m^o(t-s)P_m\nabla_i^*H_{ij}\nabla_j^*u_s\,.$$

Observing
$$P_m\nabla_i^*=\begin{cases}0, & i\geqq m+1\\ \nabla_i^*P_m, & 1\leqq i\leqq m,\end{cases}$$
we have
$$P_mG_nu_t=\frac{-1}{n-1}\sum_{i\neq j}^{m}\int_0^t ds\,\hat{T}_m^o(t-s)\nabla_i^*H_{ij}\nabla_j^*P_mu_s$$
$$+\frac{-1}{n-1}\sum_{i=1}^{m}\sum_{j=m+1}^{n}\int_0^t ds\,\hat{T}_m^o(t-s)\nabla_i^*P_mH_{ij}\nabla_j^*u_s.$$

Since u_s is symmetric with respect to the last $n-r$ variables and $r\leqq m$, we have (i). (ii) is an immediate consequence of (i).

By virtue of the above lemma we obtain a recursive formula for P_mG_nu, that is:

LEMMA 2.4. *For* $u\in{}_rH_T(\boldsymbol{R}^n, \pi)$ *and* $r\leqq m\leqq n$, $m\leqq k$,
$$\|P_mG_{n,k}u\|_T\leqq\frac{|\lambda|m}{n-1}\|P_mu\|_T+\frac{|\lambda|(n-m)}{n-1}\left(\frac{m}{m+1-r}\right)^{1/2}\|P_{m+1}u\|_T\,.$$

PROOF. From (ii) of Lemma 2.3 we have
$$\|P_mG_{n,k}u\|_T\leqq\frac{m-1}{n-1}\|G_{m,k}P_mu\|_T$$
$$+\frac{n-m}{n-1}\left\|\sum_{i=1}^{m}\int_0^t ds\,\hat{T}_m^o(t-s)e^{-k(t-s)/2}\nabla_i^*P_mH_{i,m+1}\nabla_{m+1}^*P_{m+1}u_s\right\|_T.$$

The norm of $G_{m,k}P_mu$ can be computed by almost the same way as (ii) of Lemma 2.2 and we obtain
$$\frac{m-1}{n-1}\|G_{m,k}P_mu\|_T\leqq\frac{|\lambda|m}{n-1}\|P_mu\|_T\qquad\text{for}\quad k\geqq m\,.$$

The second term turns out to be

$$\frac{n-m}{n-1}\Big(\sum_\alpha\int_0^T dt(|\alpha|+m)\Big\{\int_0^t ds\, e^{-(k/2+|\alpha|)(t-s)}\sum_{i=1}^m (P_m H_{i,m+1}\nabla^*_{m+1}P_{m+1}u_s,\ H_{\alpha-e_i})\alpha_i^{1/2}\Big\}^2\Big)^{1/2}$$

$$\leqq\frac{n-m}{n-1}\Big(\sum_\alpha\int_0^T\frac{|\alpha|+m}{|\alpha|+k}\frac{|\alpha|}{|\alpha|}\sum_{i=1}^m (P_m H_{i,m+1}\nabla^*_{m+1}P_{m+1}u_s,\ H_{\alpha-e_i})^2\Big)^{1/2}$$

$$\leqq\frac{n-m}{n-1}\Big(\int_0^T ds\sum_{i=1}^m\|P_m H_{i,m+1}\nabla^*_{m+1}P_{m+1}u_s\|^2\Big)^{1/2}$$

$$\leqq\frac{n-m}{n-1}|\lambda|\Big(m\int_0^T ds\|\nabla^*_{m+1}p_{m+1}u_s\|^2\Big)^{1/2}.$$

Since $P_{m+1}u\in {}_rH_T(\boldsymbol{R}^{m+1},\ \pi)$, the last term is equal to

$$=\frac{n-m}{n-1}|\lambda|\Big(\frac{m}{m+1-r}\int_0^T ds\sum_{k=r+1}^{m+1}\|\nabla_k^*P_{m+1}u_s\|^2\Big)^{1/2}$$

$$\leqq\frac{n-m}{n-1}|\lambda|\Big(\frac{m}{m+1-r}\Big)^{1/2}\|P_{m+1}u\|_T,$$

which proves the lemma.

Let $\delta=\dfrac{n}{n-1}$ and $\rho(r,\ m,\ p)=\Big(\dfrac{m\cdots(m+p-1)}{(m-r+1)\cdots(m-r+p)}\Big)^{1/2}\vee 1$.

LEMMA 2.5. *For $r\leqq m\leqq n$, $1\leqq p\leqq n-m$ and $u\in {}_rH_T(\boldsymbol{R}^n,\ \pi)$,*

$$\|P_m G_n^p u\|_T\leqq e^{T(m+p-1)/2}(\delta|\lambda|)^p\rho(r,\ m,\ p)\sum_{k=0}^{p}{}^{\prime}\alpha_k\|P_{m+k}u\|_T$$

holds, where α_k is a positive number satisfying $\sum_{k=0}^p\alpha_k=1$.

PROOF. From Lemma 2.4 it follows that

$$\|P_m G_{n,k}u\|_T\leqq|\lambda|\delta\Big(\frac{m}{n}\|P_m u\|_T+\frac{n-m}{n}\Big(\frac{m}{m+1-r}\Big)^{1/2}\|P_{m+1}u\|_T\Big) \tag{2.5}$$

for any $k\geqq m$. However a trivial identity

$$G_n^p=\theta_m G_{n,m}\theta_1 G_{n,m+1}\theta_1\cdots\theta_1 G_{n,m+p-1}\theta_{m+p-1}^{-1}$$

and $\|\theta_k\|_T\leqq e^{kT/2}$, $\|\theta_k^{-1}\|_T\leqq 1$ imply the expected estimate if we use (2.5) repeatedly. α_k is the p-th transition probability from m to $m+k$ of a random walk with $n\times n$ transition matrix

$$\begin{pmatrix} \frac{1}{n}, & \frac{n-1}{n}, & 0, & \cdots\cdots\cdots, & 0 \\ 0, & \frac{2}{n}, & \frac{n-2}{n}, & 0,\ \cdots, & 0 \\ \vdots & & & & \\ 0, & \cdots\cdots, & 0, & \frac{n-1}{n}, & \frac{1}{n} \\ 0, & \cdots\cdots\cdots, & & 0, & 1 \end{pmatrix}.$$

Now we can estimate $P_m G_n^p \hat{T}_n^o(t)$ itself.

LEMMA 2.6. *Suppose* $\varphi_n \in {}_rL^2(\boldsymbol{R}^n, \pi)$ *and* $\|P_m\varphi_n\| \leqq C\mu^m$ *for* $m=0, 1, 2, \cdots, n$ *with some constants* C *and* $\mu \geqq 1$. *Then*

$$\|P_m G_n^p \hat{T}_n^o(\cdot)\varphi_n\|_T \leqq (m+p)^{1/2}(T+1)^{1/2} C\mu^m e^{T(m-1)/2} \rho(r, m, p)(e^{T/2}\delta|\lambda|\mu)^p$$

for any $1 \leqq p \leqq n-m$ *and* $r \leqq m \leqq n$.

PROOF. From Lemma 2.1 we see

$$\|P_k \hat{T}_n^o(\cdot)\varphi_n\|_T = \|\hat{T}_k^o(\cdot)P_k\varphi_n\|_T \leqq (k(T+1))^{1/2}\|P_k\varphi_n\|$$

for any $k \leqq n$. Then the above estimate follows immediately from Lemma 2.5.

LEMMA 2.7. *Let* φ_n *be that of Lemma* 2.6. *Then for* $r \leqq m \leqq n$ *and* $1 \leqq p \leqq n-m$,

$$\|P_m G_n^p \hat{T}_n^o(t)\varphi_n\| \leqq (m+p)^{1/2}(t+1)^{1/2} C\mu^m e^{t(m-1)/2} \rho(r, m, p)(e^{t/2}\delta|\lambda|\mu)^p.$$

PROOF. From (i) of Lemma 2.3 it follows that

$$\|P_m G_n u_t\| \leqq \frac{m-1}{n-1}\|G_m P_m u_t\| + \frac{n-m}{n-1}\left\| \sum_{i=1}^m \int_0^t ds\, \hat{T}_m^o(t-s)\nabla_i^* P_m H_{i,m+1} \nabla_{m+1}^* P_{m+1} u_s \right\|.$$

Applying (iii) of Lemma 2.2, we have

$$\leqq \frac{|\lambda| m}{n-1}\|P_m u\|_t + \frac{|\lambda|(n-m)}{n-1}\left(\frac{m}{m+1-r}\right)^{1/2}\|P_{m+1}u\|_t.$$

Therefore

$$\|P_m G_n u_t\| \leqq |\lambda|\delta\left(\frac{m}{n}\|P_m u\|_t + \frac{n-m}{n}\left(\frac{m}{m+1-r}\right)^{1/2}\|P_{m+1}u\|_t\right).$$

Then substituting $u_t = \hat{T}_n^o(t)\varphi_n$ we see

$$\begin{aligned}\|P_m G_n^p \hat{T}_n^o(t)\varphi_n\| &= \|P_m G_n G_n^{p-1}\hat{T}_n^o(t)\varphi_n\| \\ &\leqq |\lambda|\delta\left(\frac{m}{n}\|P_m G_n^{p-1}\hat{T}_n^o(t)\varphi_n\|_t + \frac{n-m}{n}\left(\frac{m}{m+1-r}\right)^{1/2}\|P_{m+1}G_n^{p-1}\hat{T}_n^o(t)\varphi_n\|_t\right),\end{aligned}$$

which combined with Lemma 2.6 shows the lemma.

The main lemma of this section is

LEMMA 2.8. *Let* φ_n *be that of Lemma* 2.6. *Then for any* $\tilde{\mu} > \mu$ *there exists a positive constant* C_1 *such that*

$$\|P_m G_n^p \hat{T}_n^o(t)\varphi_n\| \leqq C_1(e^{t/2}\tilde{\mu})^m (e^{t/2}|\lambda|\tilde{\mu})^p$$

for any $t \in [0, T]$, $r \leqq m \leqq n$ *and* $p \geqq 1$. C_1 *depends only on* T *and the constants* C, $\mu/\tilde{\mu}$, r.

PROOF. If $p > n-m$, then (iv) of Lemma 2.2 and Lemma 2.1 show

$$\|P_m G_n^p \hat{T}_n^o(t)\varphi_n\| \leqq \|G_n^p \hat{T}_n^o(t)\varphi_n\| \leqq e^{nt/2}(\delta|\lambda|)^p \|\hat{T}_n^o(\cdot)\varphi_n\|_t$$
$$\leqq e^{nt/2}(\delta|\lambda|)^p (n(t+1))^{1/2} C\mu^n$$
$$\leqq C(m+p)^{1/2}(t+1)^{1/2}(e^{t/2}\mu)^m (e^{t/2}\delta|\lambda|\mu)^p$$
$$\leqq C_1(e^{t/2}\tilde{\mu})^m (e^{t/2}\tilde{\mu}|\lambda|)^p,$$

with some constant C_1. On the other hand, if $1\leqq p\leqq n-m$, then from Lemma 2.7 we have

$$\|P_m G_n^p \hat{T}_n^o(t)\varphi_n\| \leqq C(m+p)^{1/2}(1+t)^{1/2}\delta^p \rho(r, m, p)(e^{t/2}\mu)^m (e^{t/2}\mu|\lambda|)^p.$$

Since for any $0<\theta<1$ there exists a constant C_1 such that $\rho(r, m, p)\theta^{m+p}\leqq C_1$ for any $1\leqq p$, $0\leqq r\leqq m\leqq n$ (C_1 may depend on r), we have the estimate of the lemma.

§ 3. Proof of the theorem.

Recalling the definition of $F_m(i, j)$ in § 2, we define for $p\geqq 1$

$$F_{m+p,m}=\sum_{\substack{1\leq i_1\leq m+1\\ 1\leq i_2\leq m+2\\ \vdots \\ 1\leq i_p\leq m+p}} P_m F_{m+1}(i_1, m+1) P_{m+1} F_{m+2}(i_2, m+2)\cdots P_{m+p-1} F_{m+p}(i_p, m+p).$$

If $p=0$, we set $F_{m+p,m}=\mathrm{id}$. Then $F_{m+p,m}$ is a bounded operator from $H_T(\boldsymbol{R}^{m+p}, \pi)$ to $H_T(\boldsymbol{R}^m, \pi)$. Let

$$W_{T,r}=\{\boldsymbol{f}=(f_1, f_2, \cdots, f_m, \cdots);\ f_m\in {}_rH_T(\boldsymbol{R}^m, \pi) \text{ and } P_l f_m=f_l \text{ for } r\leqq l\leqq m\}.$$

LEMMA 3.1. *For* $\boldsymbol{f}\in W_{T,r}$ *we have*

$$P_m G_n^p f_n \longrightarrow F_{m+p,m} f_{m+p} \quad \text{in} \quad H_T(\boldsymbol{R}^m, \pi) \tag{3.1}$$

as $n\to\infty$ *for any* $m\geqq r$.

PROOF. (3.1) is trivial for $p=0$. Suppose that (3.1) is valid up to p. From Lemma 2.3 it follows that

$$P_m G_n^{p+1} f_n=\frac{m-1}{n-1} G_m P_m G_n^p f_n+\frac{n-m}{n-1}\sum_{i=1}^m P_m F_{m+1}(i, m+1) P_{m+1} G_n^p f_n.$$

As we have seen in Lemma 2.2, G_m and $F_{m+1}(i, m+1)$ are bounded operators in $H_T(\boldsymbol{R}^m, \pi)$ and $H_T(\boldsymbol{R}^{m+1}, \pi)$. Therefore together with the assumption of the induction this shows

$$P_m G_n^{p+1} f_n \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^m P_m F_{m+1}(i, m+1) F_{m+1, m+1+p} f_{m+1+p}$$
$$=F_{m+p+1,m} f_{m+p+1},$$

which completes the proof.

For $\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(r)}$, and $\phi\in L^2(\boldsymbol{R}, \pi)$ set

$$
{}_rL^2(\boldsymbol{R}^m,\ \pi)\ni\phi_m^{(r)}(x)=\begin{cases}\phi(x_1)\phi(x_2)\cdots\phi(x_m) & \text{if}\ \ 1\leqq m\leqq r\,,\\ \phi^{(1)}(x_1)\phi^{(2)}(x_2)\cdots\phi^{(r)}(x_r)\phi(x_{r+1})\cdots\phi(x_m) & \\ & \text{if}\ \ m>r\,.\end{cases}
$$

Then $\{\hat{T}_m^{\circ}(t)\phi_m^{(r)}(x)\}_{m=1}^{\infty}\in W_{T,r}$. Suppose that we are given a sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that

$$
\varphi_n\in{}_rL^2(\boldsymbol{R}^n,\ \pi)\,,\quad \|P_m\varphi_n\|\leqq C\mu^m\qquad (\mu=\|\phi\|=\|\phi\|_{L^2(\boldsymbol{R},\pi)})
$$

$$
P_m\varphi_n\xrightarrow[n\to\infty]{}\phi_m^{(r)}\qquad \text{weakly in } L^2(\boldsymbol{R}^m,\ \pi)\,.
$$

Then we obtain

LEMMA 3.2. *Let* $\hat{T}=-2\log(|\lambda|\mu)$. *Then for any* $m=1,\ 2,\ \cdots$ *and any fixed* $T<\hat{T}$, *there exists a weak limit of* $P_m\hat{T}_n^*(\cdot)\varphi_n$ *as* $n\to\infty$ *in* $H_T(\boldsymbol{R}^m,\ \pi)$, *and hence the sequence* $P_m\hat{T}_n^*(t)\varphi_n$ *converges weakly in* $L^2(\boldsymbol{R}^m,\ \pi)$ *for each fixed* $t<\hat{T}$. *If we denote this limit by* $f_m(t,\ x)$, *then they satisfy*

$$
f_m(t,\ x)=\hat{T}_m^{\circ}(t)\phi_m^{(r)}(x)+\sum_{i=1}^{m}P_mF_{m+1}(i,\ m+1)f_{m+1}(t,\ x)\tag{3.2}
$$

for $m=r,\ r+1,\ \cdots$. *Moreover they have estimates*

$$
\|f_m\|_T\leqq C(e^{T/2}\tilde{\mu})^m,\qquad m=1,\ 2,\ \cdots,\tag{3.3}
$$

for any $T<\hat{T}$ *and* $\tilde{\mu}>\mu$, *where* C *is a constant independent of* m.

PROOF. As we have mentioned in § 0, employing the estimate in Lemma 2.2, we see $\hat{T}_n^*(t)=\sum_{p=0}^{\infty}G_n^p\hat{T}_n^{\circ}(t)$ both in $H_T(\boldsymbol{R}^n,\ \pi)$ and $L^2(\boldsymbol{R}^n,\ \pi)$. Hence $P_m\hat{T}_n^*(t)\varphi_n=\sum_{p=0}^{\infty}P_mG_n^p\hat{T}_n^{\circ}(t)\varphi_n$ is valid. However from Lemma 2.7 and Lemma 2.8 it follows that if $|\lambda|\mu<1$ and $T,\ t<\hat{T}$, then $\|P_mG_n^p\hat{T}_n^{\circ}(\cdot)\varphi_n\|_T$ and $\|P_mG_n^p\hat{T}_n^{\circ}(t)\varphi_n\|$ are dominated by convergent sequences uniformly with respect to n. Therefore we have only to show the existence of weak limits of each term $P_mG_m^p\hat{T}_n^{\circ}(t)\varphi_n$ as $n\to\infty$. For $p=0$, that is, $P_m\hat{T}_n^{\circ}(t)\varphi_n$ obviously converges weakly to $P_m\hat{T}_m^{\circ}(t)\phi_m^{(r)}$ both in $H_T(\boldsymbol{R}^m,\ \pi)$ and $L^2(\boldsymbol{R}^m,\ \pi)$. Suppose that $P_mG_n^p\hat{T}_n^{\circ}(t)\varphi_n$ converges to f_m both in the above spaces. Since from Lemma 2.3 we have for $m\geqq r$

$$
P_mG_n^{p+1}\hat{T}_n^{\circ}(t)\varphi_n=\frac{m-1}{n-1}G_mP_mG_n^p\hat{T}_n^{\circ}(t)\varphi_n
$$

$$
+\frac{n-m}{n-1}\sum_{i=1}^{m}P_mF_{m+1}(i,\ m+1)P_{m+1}G_n^p\hat{T}_n^{\circ}(t)\varphi_n\,,
$$

the boundedness of G_m and $F_{m+1}(i,\ m+1)$ (see Lemma 2.1 and Lemma 2.2) implies the weak convergence of $P_mG_n^{p+1}\hat{T}_n^{\circ}(t)\varphi_n$ as $n\to\infty$. Thus we can prove the convergence of $P_m\hat{T}_n^*(t)\varphi_n$ for each fixed m.

On the other hand, since $u_t=\hat{T}_n^*(t)\varphi_n$ satisfies (2.1), from Lemma 2.3 we have

$$P_m \hat{T}_n^*(t)\varphi_n = P_m \hat{T}_n^0(t)\varphi_n + P_m G_n \hat{T}_n^*(t)\varphi_n$$
$$= P_m \hat{T}_n^0(t)\varphi_n + \frac{m-1}{n-1} G_m P_m \hat{T}_n^*(t)\varphi_n$$
$$+ \frac{n-m}{n-1} \sum_{i=1}^{m} P_m F_{m+1}(i, m+1) P_{m+1} \hat{T}_n^*(t)\varphi_n ,$$

for $m \geqq r$. Therefore letting $n \to \infty$, we obtain the equation (3.2). The estimate (3.3) immediately follows from Lemma 2.6.

For $\psi \in L^2(\boldsymbol{R}, \pi)$ such that $\psi \geqq 0$ and $\int_{\boldsymbol{R}} \psi\pi = 1$, let $\hat{p}_\psi$ be the solution of

$$\frac{\partial \hat{p}}{\partial t} = -\nabla^* \nabla \hat{p} + \lambda \nabla^* (\hat{p}^2 \pi) \tag{3.4}$$
$$\hat{p}_{0+} = \psi .$$

Then it is not difficult to see that p_ψ defined by the identity $\hat{p}_\psi(t, x) = p_\psi(e^{2t}-1, e^t x)\pi(x)^{-1} e^t$ satisfies the Burgers equation:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p - \frac{\lambda}{2} \nabla p^2$$
$$p_{0+} = \psi\pi .$$

As we have proved in § 0, since the Burgers equation is uniquely solvable for any probability distribution $\psi\pi dx$, so is the equation (3.4) for any such a ψ.

LEMMA 3.3. *$\hat{p}_\psi$ satisfies*

$$\|\hat{p}_\psi\|_T^2 = \int_0^T \|\nabla^* \hat{p}_\psi(t, \cdot)\|_{L^2(\boldsymbol{R}, \pi)}^2 dt < \infty$$

for any $T > 0$.

PROOF. Suppose that the initial function ψ is a smooth function with compact support on $\boldsymbol{R}$. Then for any compact interval I of $(0, \infty)$, we can show below that there exists a constant C_1 depending on I, ψ and λ and satisfying

$$|\nabla^* \hat{p}_\psi(t, x)| \leqq C_1 , \tag{3.5}$$

for any $t \in I$ and $x \in \boldsymbol{R}$. Let

$$f(t, x) = \int_{\boldsymbol{R}} g_t(x-y) \exp\Big(-\lambda \int_{-\infty}^{y} \psi(z)\pi(dz)\Big) dy .$$

Since

$$\nabla f(t, x) = -\lambda \int_{\boldsymbol{R}} g_t(x-y)\psi(y)\pi(y) \exp\Big(-\lambda \int_{-\infty}^{y} \psi(z)\pi(z)dz\Big) dy ,$$

for any $\delta > 1$ there exist constants C_2, C_3 such that

$$|\nabla f(t, x)| \leqq C_2 \int_{\operatorname{supp}\psi} g_t(x-y) dy$$

$$\leqq C_3 \exp(-x^2/2\delta t)$$

holds for any $t \in I$ and $x \in \boldsymbol{R}$. Similarly we have

$$|\nabla^2 f(t, x)| \leqq C_4 \exp(-x^2/2\delta t).$$

On the other hand, we easily see $f(t, x) \geqq e^{-|\lambda|}$. However p_ϕ can be expressed by f as

$$p_\phi = -\frac{\nabla f}{\lambda f} \quad \text{and hence} \quad \nabla p_\phi = -\frac{f\nabla^2 f - (\nabla f)^2}{\lambda f^2}.$$

Therefore we have for $t \in I$ and $x \in \boldsymbol{R}$

$$|\nabla p_\phi(t, x)| \leqq C_5 \exp(-x^2/2\delta t)$$

with a constant C_5 depending on I, and ϕ. In view of the relation between p_ϕ and $\hat{p}_\phi$, we see

$$|\nabla^* \hat{p}_\phi(t, x)| \leqq C_6 \exp\left(-\frac{e^{2t}x^2}{2\delta(e^{2t}-1)} + \frac{x^2}{2}\right).$$

However, if we choose $\delta > 1$ sufficiently close to 1, then the coefficient of x^2 in the exponent turns to be negative for any $t \in I$. Consequently we have the estimate (3.5). In particular this implies

$$C_1^2 \leqq \int_{\boldsymbol{R}} |\nabla^* \hat{p}_\phi(t, x)|^2 \pi(dx) = \int_{\boldsymbol{R}} |\hat{p}_\phi(t, x)|^2 \pi(dx) + \int_{\boldsymbol{R}} |\nabla \hat{p}_\phi(t, x)|^2 \pi(dx),$$

therefore

(3.6) $$\int_{\boldsymbol{R}} |\nabla \hat{p}_\phi(t, x)|^2 \pi(dx) \leqq C_1^2.$$

To obtain a uniform bound for $\hat{p}_\phi(t, x)$ itself, note first

$$p_\phi(t, x) \leqq e^{|\lambda|} g_t^* \phi \pi(x).$$

This inequality has already been discussed in § 0. Hence

(3.7) $$\hat{p}_\phi(t, x) \leqq e^{|\lambda|} \hat{T}_1^0(t) \phi(x).$$

However

$$\hat{T}_1^0(t)\phi(x) = \int_{\boldsymbol{R}} \hat{g}_t(x, y) \phi(y) \pi(dy)$$
$$\leqq \left(\int_{\boldsymbol{R}} \hat{g}_t(x, y)^2 \pi(dy)\right)^{1/2} \|\phi\|$$
$$= \hat{g}_{2t}(x, x)^{1/2} \|\phi\|$$

holds. Moreover $\hat{g}_t(x, y) = e^t g_{e^{2t}-1}(x - e^t y) \pi(y)^{-1}$ implies $\hat{g}_t(x, x) \leqq C_1^* t^{-1/2} \pi(x)^{-1}$ with a constant C_1^* independent of $t > 0$ and $x \in \boldsymbol{R}$. Consequently we have

(3.8) $$\hat{p}_\phi(t,\ x)^2 \leqq C_2^* t^{-1/2}\pi(x)^{-1}\|\phi\|^2$$

with a constant C_2^* depending only on λ.

Now multiply $\hat{p}_\phi$ to the both sides of (3.4) and integrate them by $dt\pi(dx)$ on $(\varepsilon,\ T)\times \boldsymbol{R}$. Then we obtain

(3.9) $$2\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2 = \|\hat{p}_\phi(\varepsilon,\ \cdot)\|^2 - \|\hat{p}_\phi(T,\ \cdot)\|^2 + 2\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)(\nabla \hat{p}_\phi(t,\ x))\hat{p}_\phi(t,\ x)^2\pi(dx).$$

The integrants are integrable by virtue of (3.5), (3.6) and (3.8). The right-hand side can be dominated from above as follows:

$$e^{2|\lambda|}\|\phi\|^2 + 2|\lambda|\left\{\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2\right\}^{1/2}\left\{\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)\hat{p}_\phi(t,\ x)^4\pi(x)\right\}^{1/2}$$

$$\leqq e^{2|\lambda|}\|\phi\|^2 + 2|\lambda|e^{|\lambda|}\left\{\int_\varepsilon^T C_2^* t^{-1/2}dt\right\}^{1/2}\|\phi\|^2\left\{\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2\right\}^{1/2}$$

$$\leqq 2\left(C_3^* + C_4^*\left\{\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2\right\}^{1/2}\right)\|\phi\|^2$$

with some constants C_3^* and C_4^* depending only on λ and T. In the above computation we have used (3.7), (3.8) and the $L^2(\pi)$-contractivity of $\hat{T}_1^0(t)$. Combining the above estimate and (3.9), we have

(3.10) $$\int_\varepsilon^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2 \leqq \frac{1}{4}C_4^*\|\phi\|^4 + \left(\frac{C_4^*}{2} + C_3^*\right)\|\phi\|^2 \leqq C_5^*\|\phi\|^4,$$

with a constant $C_5^* = (3/4)C_4^* + C_3^*$. In the last inequality in (3.10) we have used the fact that $\phi\pi$ is a probability density and hence $\|\phi\| \geqq 1$. Since (3.10) holds with a constant C_5^* independent of ϕ and ε, letting $\varepsilon \downarrow 0$ and approximating an arbitrary ϕ by smooth functions with compact support on $\boldsymbol{R}$ suitably, we can conclude

$$\int_0^T dt\int_{\boldsymbol{R}}\pi(dx)|\nabla \hat{p}_\phi(t,\ x)|^2 \leqq C_5^*\|\phi\|^4.$$

From (3.7) we see that the L^2-norm of $\hat{p}_\phi$ itself with respect to $dt\pi(dx)$ on $(0,\ T)\times \boldsymbol{R}$ is dominated by $T^{1/2}e^{|\lambda|}\|\phi\|$, which completes the proof.

LEMMA 3.4. *Suppose* $\boldsymbol{f} = (f_1,\ f_2,\ \cdots,\ f_m,\ \cdots) \in \bigcap_{T<\hat{T}} W_{T,r}$ *satisfies the equation* (3.2). *If*

(3.11) $$\|f_m\|_T \leqq C(e^{T/2}\tilde{\mu})^m \qquad (m=1,\ 2,\ \cdots)$$

for any $T<\hat{T}$ *and* $\tilde{\mu}>\mu$ *with some constant* C, *then the equation* (3.2) *has exactly one solution* $\{f_m\}$ (f_m *is that of Lemma* 3.2) *and it coincides with*

$$(3.12)\qquad \begin{cases} \prod_{i=1}^{r}\left(\int_R \phi^{(i)}(y)\hat{q}_\phi(0,\ y,\ t,\ x_i)\pi(dy)\right) & \text{if}\quad m=r\,, \\ \prod_{i=1}^{r}\left(\int_R \phi^{(i)}(y)\hat{q}_\phi(0,\ y,\ t,\ x_i)\pi(dy)\right)\prod_{i=r+1}^{m}\hat{p}_\phi(t,\ x_i) & \text{if}\quad m>r\,, \end{cases}$$

where $\hat{q}_\phi(s,\ y,\ t,\ x)$ is the fundamental solution for a linear parabolic operator $(\partial/\partial t)+\nabla^*\nabla-\lambda\hat{p}_\phi\nabla$.

PROOF. Suppose that we have two solutions $\boldsymbol{f}$ and $\boldsymbol{g}$ in $\bigcap_{T<\hat{T}} W_{T,r}$ satisfying the equation (3.2) and the condition (3.11). Then $\boldsymbol{h}=\boldsymbol{f}-\boldsymbol{g}$ satisfies

$$h_m=\sum_{i=1}^{m} P_m F_{m+1}(i,\ m+1)h_{m+1} \qquad \text{for}\quad m\geqq r\,,$$

$$\|h_m\|_T\leqq C(e^{T/2}\tilde{\mu})^m\,.$$

Therefore we have

$$\begin{aligned} h_m &= \sum_{i=1}^{m} P_m F_{m+1}(i,\ m+1)\sum_{j=1}^{m+1} P_{m+1}F_{m+2}(j,\ m+2)h_{m+2} \\ &\ \ \vdots \\ &= F_{m+p,m}h_{m+p} \qquad \text{for}\quad p=0,\ 1,\ 2,\ \cdots. \end{aligned}$$

However from Lemma 2.5 and Lemma 3.1 we see

$$\begin{aligned} \|F_{m+p,m}h_{m+p}\|_T &\leqq e^{T(m-1)/2}\rho(r,\ m,\ p)(\delta|\lambda|)^p\sum_{k=0}^{p}\alpha_k\|P_{m+k}h_{m+p}\|_T \\ &\leqq e^{T(m-1)/2}\rho(r,\ m,\ p)(\delta|\lambda|)^p\|h_{m+p}\|_T \\ &\leqq C'(e^T\tilde{\mu})^m(e^{T/2}|\lambda|\tilde{\mu})^p. \end{aligned}$$

Here we have used the fact $\|P_m u\|_T\leqq\|u\|_T$ if $u\in H_T(\boldsymbol{R}^n,\ \pi)$ and $m\leqq n$. Therefore if we fix $T<\hat{T}=-2\log(|\lambda|\mu)$ and choose $\tilde{\mu}$ close enough to μ, then $h_m=F_{m+p,m}h_{m+p}\to 0$ in $H_T(\boldsymbol{R}^m,\ \pi)$ as $p\to\infty$, which implies $h_m=0$ for $m\geqq r$. Now denote the functions in (3.12) by $g_m(t,\ x)$. Then Lemma 3.3 says $g_m\in H_T(\boldsymbol{R}^m,\ \pi)$. Moreover g_m satisfies the equation (3.2) because for $m\geqq r$

$$\begin{aligned} \sum_{i=1}^{m} P_m F_{m+1}(i,\ m+1)g_{m+1}(t,\ x) &= -\sum_{i=1}^{m} P_m\int_0^t ds\hat{T}^0_{m+1}(t-s)\nabla_i^* H_{i,m+1}\nabla_{m+1}^* g_{m+1}(s,\ \cdot) \\ &= \lambda\sum_{i=1}^{m}\int_0^t ds\hat{T}^0_m(t-s)\nabla_i^*(g_m(s,\ \cdot)\hat{p}_\phi(s,\ x_i))\,. \end{aligned}$$

Here we have used the fact $\int_R \hat{g}_t(x,\ y)\pi(dy)=1$. Since $\int_R \phi^{(i)}(y)\hat{q}_\phi(0,\ y,\ t,\ x)\pi(dy)$ satisfies

$$\frac{\partial f}{\partial t}=-\nabla^*\nabla f+\lambda\nabla^*(\hat{p}_\phi f)\,,\qquad f(0+,\ \cdot)=\phi^{(i)},$$

we have

$$\lambda \sum_{i=1}^{m} \int_0^t ds \hat{T}_m^o(t-s) \nabla_i^* (g_m(s, \cdot) \hat{p}_\phi(s, \cdot))$$

$$= \sum_{i=1}^{m} \int_0^t ds \hat{T}_m^o(t-s) \nabla_i^* \nabla_i g_m(s, \cdot) + \int_0^t ds \hat{T}_m^o(t-s) \frac{\partial}{\partial s} g_m(s, \cdot)$$

$$= -\int_0^t ds \frac{\partial}{\partial t} \hat{T}_m^o(t-s) g_m(s, \cdot) + g_m(t, \cdot) + \int_0^t \frac{\partial}{\partial t} \hat{T}_m^o(t-s) g_m(s, \cdot) ds - \hat{T}_m^o(t) \phi_m^{(r)}$$

$$= -\hat{T}_m^o(t) \phi_m^{(r)} + g_m(t, \cdot),$$

which implies

$$g_m(t, \cdot) = \hat{T}_m^o(t) \phi_m^{(r)} + \sum_{i=1}^{m} P_m F_{m+1}(i, m+1) g_{m+1}.$$

On the other hand, it is easy to check that

$$\|g_m\|_T \leqq C(\sup_{0 \leqq t \leqq T} \|\hat{p}_\phi(t, \cdot)\|_{L^2(\boldsymbol{R}, \pi)})^m \qquad \text{for} \quad m=1, 2, \cdots,$$

with some constant C. However, since $\hat{p}_4(t, \cdot) \to \phi$ as $t \to 0$ in $L^2(\boldsymbol{R}, \pi)$, for any $\tilde{\mu} > \mu$, we can find $T>0$ such that $\|g_m\|_T \leqq C\tilde{\mu}^m$ is valid. Therefore the uniqueness of the equation (3.2) implies $g_m(t, \cdot) = f_m(t, \cdot)$ $(m \geqq r)$ at least for small enough $t<T$. Especially we have $\|g_m\|_t \leqq C(e^{t/2}\tilde{\mu})^m$ for all small t. Then we can start from $0<t_0<T$ instead of $t=0$ and make the same argument as before up to some time t_1 strictly bigger than t_0. Continuing this argument we can reach any time smaller than $\hat{T}$ because as long as the uniqueness holds, we have a bound $\|g_m\|_t \leqq C(e^{t/2}\tilde{\mu})^m$. This completes the proof of the lemma.

The final step of the proof of the theorem. All we have to do is to show the following fact:

"Let $h_i(x) = h_i^{(1)}(x_1) \cdots h_i^{(m)}(x_m) \in L^2(\boldsymbol{R}^m, \pi)$ $(i=1, 2, \cdots, k)$ and φ_n be that appearing in the statement of the theorem. Then for any time sequence $0<t_1<t_2< \cdots <t_k<\hat{T}$,

$$E_{\varphi_n \pi} h_1(Y_n(t_1)) h_2(Y_n(t_2)) \cdots h_k(Y_n(t_k)) \tag{3.13}$$

$$\longrightarrow \prod_{j=1}^{m} E_{\phi\pi} h_1^{(j)}(\hat{Z}(t_1)) h_2^{(j)}(\hat{Z}(t_2)) \cdots h_k^{(j)}(\hat{Z}(t_k))$$

$$= \prod_{j=1}^{m} \int_{\boldsymbol{R}^k} \pi(dy_1) \cdots \pi(dy_k) \hat{p}_\phi(t_1, y_1) \hat{q}_\phi(t_1, y_1, t_2, y_2) \cdots$$

$$\cdots \hat{q}_\phi(t_{k-1}, y_{k-1}, t_k, y_k) h_1^{(j)}(y_1) \cdots h_k^{(j)}(y_k)."$$

However the first term of (3.13) is equal to

$$(h_1, P_m \hat{T}_n^*(t_1) h_2 \hat{T}_n^*(t_2-t_1) h_3 \cdots \hat{T}_n^*(t_{k-1}-t_{k-2}) h_k \hat{T}_n^*(t_k-t_{k-1}) \varphi_n),$$

where $(\ ,\ )=(\ ,\ )_{L^2(\boldsymbol{R}^m,\pi)}$. Therefore applying Lemma 3.4 inductively, we can prove (3.13).

REMARK 1. Our method is applicable also when the generator L_n has a form of

$$\frac{1}{2}\Delta+\frac{1}{n-1}\sum_{i,j}\nabla_i H_{ij}\nabla_j$$

and $H_{ij}(x)=H(x_i,\ x_j)$ for a skew-symmetric bounded function H on $\boldsymbol{R}^2$ satisfying $H(r\xi,\ r\eta)=H(\xi,\ \eta)$ for any $r>0$. Moreover any finite dimensional distributions of the limit process $\{Z(t)\}$ can be described by using p_ϕ and q_ϕ defined by weak solutions for

$$\begin{cases} \dfrac{\partial p_\phi}{\partial t}=\dfrac{1}{2}\Delta p_\phi-\nabla(H[x,\ p_\phi]p_\phi), \qquad H[x,\ p]=\displaystyle\int_{\boldsymbol{R}}H(x,\ y)\nabla_y p(y)dy \\ p_\phi(0,\ x)=\phi(x)\pi(x) \\ \dfrac{\partial q_\phi}{\partial t}=\dfrac{1}{2}\Delta q-\nabla(H[x,\ p_\phi]q_\phi). \end{cases}$$

However we can not exclude the possibility of the explosion of the above p_ϕ or q_ϕ in $L^2(\boldsymbol{R},\ dx)$ after a certain time T. Therefore it is not easy to define in general case the limit process $\{Z(t)\}$ on the whole time interval $[0,\ \infty)$.

REMARK 2. If the generator has the form of

$$L_n=-\sum_{i=1}^{n}\nabla_i^*\nabla_i+\frac{1}{n-1}\sum_{i,j}^{n}\nabla_i^*F(x_i,\ x_j)\nabla_j,$$

we can prove the propagation of chaos for the corresponding diffusion processes without any restriction on time T. In fact L_n is of the form of divergence in $L^2(\boldsymbol{R}^n,\ \pi)$ and the corresponding estimate appearing in Lemma 2.2 does not contain the term $e^{nT/2}$.

REMARK 3. It should be remarked that a complete system of eigenfunctions for the operator L_n has been obtained by E. Gutkin and M. Kac [4].

REMARK 4. P. Calderoni and M. Pulvirenti [3] have shown that if the delta function appearing as the drift coefficient is approximated by smooth functions simultaneously as the increase of the number of the particles, then propagation of chaos is valid.

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