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Propagation of Chaos for Weakly Interacting Mild Solutions to Stochastic Partial Differential Equations

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Abstract
This article investigates the propagation of chaos property for weakly interacting mild solutions to semilinear stochastic partial differential equations whose coefficients might not satisfy Lipschitz conditions. Furthermore, we establish existence and uniqueness results for mild solutions to SPDEs with distribution dependent coefficients, so-called McKean–Vlasov SPDEs.

Keywords Interacting particle system · Propagation of chaos · McKean–Vlasov equation · SPDE · Weak solution · Martingale solution · Uniqueness in law · Pathwise uniqueness · compact semigroup · Factorization method · Compactness method

Mathematics Subject Classification 60H15 · 60G48 · 60B10

1 Introduction

A system of particles can be modeled as interacting stochastic processes. When the number N of particles gets large the process level usually contains too much information for a statistical description and it is interesting to change the point of view by passing to the macroscopic picture which means looking at the system on an average level. More specifically, the idea is to consider the empirical distribution of the particles and to study its limiting behavior when the number of particles tends to infinity. Under suitable assumptions on the system it is often possible to describe the limit via a so-called *McKean–Vlasov (MKV) equation*. The macroscopic behavior is also closely related to the so-called *propagation of chaos property*, which roughly speaking means that an asymptotic i.i.d. property of the initial distributions propagates to later times.

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1.1 Contributions of the Article

In this paper we study two questions related to MKV limits of interacting stochastic partial differential equations (SPDEs). First, we consider an N-particle system $X^{N,1}, \dots, X^{N,N}$ given by the weakly interacting SPDEs

$$dX_t^{N,i} = AX_t^{N,i}dt + \mu(t, X_t^{N,i}, \mathscr{X}_t^N)dt + \sigma(t, X_t^{N,i}, \mathscr{X}_t^N)dW_t^i, \quad i = 1, \dots, N,$$

where

$$\mathscr{X}_t^N \triangleq \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}, \quad t \in \mathbb{R}_+,$$

are the empirical distributions, A is the generator of a C_0-semigroup $S = (S_t)_{t\geq 0}$ and $W^1, \dots, W^N$ are independent standard cylindrical Brownian motions. The natural candidate for a MKV limit of this particle system is the law of the MKV SPDE

$$dX_t = AX_tdt + \mu(t, X_t, P_t^X)dt + \sigma(t, X_t, P_t^X)dW_t, \tag{1.1}$$

where P_t^X denotes the law of X_t and W is a standard cylindrical Brownian motion. Under certain assumptions on the initial distribution, compactness of S, and linear growth and continuity assumptions on the coefficients μ and σ, the first main contribution (Theorem 3.4) of this paper is the following:

If the MKV SPDE (1.1) satisfies uniqueness in law, then a unique law $\mathscr{X}^0$ exists and $\mathscr{X}^N \to \mathscr{X}^0$ in mean where $\mathscr{X}^N$ and $\mathscr{X}^0$ are considered as random variables with values in a Wasserstein space of probability measures on $C([0, T], E)$, where $T > 0$ is an arbitrary finite time horizon and E is the state space of the particles. Furthermore, we provide a propagation of chaos result and we derive similar results (Theorem 3.6) under Lipschitz conditions without a compactness assumption on the semigroup S.

Besides interacting SPDEs we also investigate weak existence, pathwise uniqueness and uniqueness in law for the MKV SPDE as given in (1.1). More precisely, in Theorem 2.5 we prove weak existence under a continuity (using the weak topology for the measure variable) and a linear growth assumption on μ and σ and a compactness condition on the semigroup S. For suitably integrable initial data, we replace the weak topology in the continuity assumption by a Wasserstein topology and we also relax the linear growth conditions, see Theorem 2.8. Furthermore, in Theorem 2.11 we establish pathwise uniqueness and uniqueness in law under a modified Lipschitz condition. Finally, in Theorem 2.12 we provide an existence and uniqueness result under a classical Lipschitz condition which requires no additional assumptions on the semigroup S.

1.2 Comments on Related Literature

Convergence to the MKV limit for finite dimensional equations was systematically studied in [14]. For weakly interacting one-dimensional stochastic heat equations with Lipschitz coefficients which are linear in the measure variable, propagation of chaos was proved in [27]. The heat equation is included in our framework, see Example 2.6. Using our notation from above, convergence to the MKV limit for coefficients of the type

$$\mu(t, x, \nu) \equiv \mu_1(t, x) + \int \mu_2(x, y)\nu(dy), \qquad \sigma(t, x, \nu) \equiv \sigma(t, x), \tag{1.2}$$

was proved in [2, Theorem 5.3] under assumptions on the initial distributions which are similar to ours (see Condition (I) below and Eq. (3.10) in [2]), certain assumptions on A, and linear growth and Lipschitz conditions on μ_1, μ_2 and σ, see Remark 2.4 for comments. Notice that the diffusion coefficient in (1.2) is independent of the measure variable and that the drift coefficient depends linearly on it. In this paper we present results for coefficients with a more general structure. In particular, in our main Theorem 3.4 we impose no Lipschitz conditions. For i.i.d. initial data the propagation of chaos result [2, Theorem 5.3] is covered by Theorem 3.6, whose proof appears to us more straightforward. More precisely, it adapts a coupling argument from the finite dimensional case [24]. Compared to [2], we establish stronger convergence results in the sense that we prove convergence in mean for random variables in a Wasserstein space, while convergence in probability and the weak topology are used in [2]. The basic structure of the proof for [2, Theorem 5.3] is similar to those of Theorem 3.4 in the sense that we also prove tightness and then use a martingale problem argument. The proof for tightness in [2] is an adaption of Kolmogorov's tightness criterion, while we use the compactness method from [15]. In addition, we prove tightness for random variables with values in a suitable Wasserstein space which is not done in [2]. On a technical level, also our martingale problem argument distinguishes from those in [2], as we work under different assumptions on the coefficients.

Various existence and uniqueness results for MKV SPDEs were proved in [1, 2, 17, 18, 31]. With the exception of [18], the conditions in these references are of Lipschitz type. Theorem 2.5 is closely related to [18, Theorem 2.1], which is an existence result for MKV SPDEs with uniformly bounded continuous coefficients (where the weak topology is used for the measure variable). Compared to this theorem, we require less assumptions on the parameters A, μ and σ (see Remark 2.4 for some comments). Furthermore, Theorem 2.8 extends [18, Theorem 2.1] in the direction that it only requires a continuity assumption for a Wasserstein instead of the weak topology. Thanks to this extension our result covers for instance linear (in the measure variable) coefficients of the type

$$\mu(t, x, \nu) \equiv \int \mu^\circ(t, x, y)\nu(dy)$$

for unbounded μ°. Similar to those of [18, Theorem 2.1], the proof of Theorem 2.5 relies on an approximation scheme and the compactness method from [15]. In contrast to [18], we use a martingale problem argument to identify the limit and we establish moment estimates to reduce assumptions on μ and σ. The martingale problem argument is robust w.r.t. the linearity A, while the argument in [18] uses some properties of A. When compared to existence results for classical SPDEs, Theorem 2.5 can be viewed as an extension of the main results from [15] to a McKean–Vlasov framework. For finite dimensional MKV equations, general existence and uniqueness results were proved in [13]. Our uniqueness result extends a theorem from [13] and we also adapt the basic proof strategy to our infinite dimensional setting.

1.3 Structure of the Article and Comments on Notation

The article is structured as follows: In Sect. 2 we introduce our setting and present existence and uniqueness results for MKV SPDEs. The convergence of the particle system to its MKV limit and the propagation of chaos property are discussed in Sect. 3. In the remaining sections we present the proofs for our results.

Before we turn to the main body of this paper, let us also comment on notation and terminology. In general, we follow the seminal monograph of Da Prato and Zabczyk [9]. We also refer to this monograph for background information on stochastic integration in infinite dimensions. Further standard references on infinite dimensional stochastic analysis are the monographs [16, 26].

Convention *If not indicated otherwise, $C > 0$ denotes a generic constant which is allowed to depend on all fixed parameters in the respective context. We also use the convention that C might change from line to line.*

2 Existence and Uniqueness Results for McKean–Vlasov SPDEs

Let $E = (E, \langle \cdot, \cdot \rangle_E, \| \cdot \|_E)$ and $H = (H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)$ be separable real Hilbert spaces, denote the space of linear bounded operators $H \to E$ by $L(H, E)$ and the space of Hilbert–Schmidt operators $H \to E$ by $L_2(H, E)$. Let $M_c(E)$ be the space of probability measures on $(E, \mathscr{B}(E))$ endowed with the weak topology, i.e. the topology of convergence in distribution. For $p \geq 1$ let $M_w^p(E)$ be the set of all $\nu \in M_c(E)$ such that

$$\|\nu\|_p \triangleq \Big(\int \|y\|_E^p \nu(dy) \Big)^{1/p} < \infty.$$

We endow $M_w^p(E)$ with the p-Wasserstein topology ([5, Section 5.1]), which turns $M_w^p(E)$ into a Polish space. Next, we introduce a quadruple (A, μ, σ, η) of *coefficients*.

(i) Let $A \colon D(A) \subseteq E \to E$ be the generator of a C_0-semigroup $S = (S_t)_{t \geq 0}$ on E and denote its adjoint by $A^* \colon D(A^*) \subseteq E \to E$.
(ii) Let $\mu \colon \mathbb{R}_+ \times E \times M_c(E) \to E$ and $\sigma \colon \mathbb{R}_+ \times E \times M_c(E) \to L(H, E)$ be Borel functions. To be precise, for σ we mean that for every $h \in H$ the E-valued function σh is Borel.
(iii) Let $\eta \in M_c(E)$.

In the following we use the notation $P_t^X \triangleq P \circ X_t^{-1}$ for $t \in \mathbb{R}_+$.

Definition 2.1 We call a triplet $(\mathbb{B}, W, X)$ a *martingale solution* to the MKV SPDE with coefficients (A, μ, σ, η) if $\mathbb{B}$ is a filtered probability space with right-continuous and complete filtration which supports a standard cylindrical Brownian motion W and a continuous E-valued adapted process X such that the following hold:

(i) $X_0 \sim \eta$, i.e. X_0 has law η.
(ii) Almost surely for all $t \in \mathbb{R}_+$

$$\int_0^t \|S_{t-s}\mu(s, X_s, P_s^X)\|_E ds + \int_0^t \|S_{t-s}\sigma(s, X_s, P_s^X)\|_{L_2(H,E)}^2 ds < \infty.$$

(iii) Almost surely for all $t \in \mathbb{R}_+$

$$X_t = S_t X_0 + \int_0^t S_{t-s}\mu(s, X_s, P_s^X) ds + \int_0^t S_{t-s}\sigma(s, X_s, P_s^X) dW_s.$$

We call X a *solution process* and its law, seen as a Borel probability measure on $C(\mathbb{R}_+, E)$ endowed with the local uniform topology, a *solution measure*. The pair $(\mathbb{B}, W)$ is called a *driving system*. Furthermore, for $p \geq 1$ we call the solution measure a *p-solution measure* if $P_t^X \in M_w^p(E)$ for all $t \in \mathbb{R}_+$ and $(t \mapsto P_t^X) \in C(\mathbb{R}_+, M_w^p(E))$. In the same manner, we say that X is a *p-solution process* if its law is a p-solution measure and in this case we call $(\mathbb{B}, W, X)$ a *p-martingale solution*.

Remark 2.2 In case one is only interested in p-martingale solutions it suffices that μ and σ are defined on $\mathbb{R}_+ \times E \times M^p_w(E)$. Of course, in this case also the initial law η has to be taken from $M^p_w(E)$.

From now on we fix

$$0 < \alpha < 1/2 \quad \text{and} \quad p' > 1/\alpha.$$

Let (A, μ, σ, η) be coefficients for a MKV SPDE. We formulate the following conditions:

(A1) A generates a compact C_0-semigroup $S = (S_t)_{t\geq 0}$, i.e. S_t is compact for all $t > 0$.

(A2) For all $y^* \in D(A^*)$ and $t > 0$ the maps $\langle \mu(t, \cdot, \cdot), y^* \rangle_E$ and $\|\sigma^*(t, \cdot, \cdot)y^*\|_H$ are continuous on $E \times M_c(E)$.

(A3) For every $T > 0$ there exists a Borel function $\mathfrak{f} = \mathfrak{f}_T : (0, T] \to [0, \infty]$ and a constant $C_T > 0$ such that

$$\int_0^T \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds < \infty,$$

and

$$\|S_t\sigma(s, x, \nu)\|_{L_2(H,E)} \leq \mathfrak{f}(t)\big(1 + \|x\|_E\big),$$

and

$$\|\mu(s, x, \nu)\|_E + \|\sigma(s, x, \nu)\|_{L(H,E)} \leq C_T\big(1 + \|x\|_E\big), \tag{2.1}$$

for all $0 < t, s \leq T$, $x \in E$ and $\nu \in M_c(E)$.

Remark 2.3 In (A3) the dependence of the function $\mathfrak{f}$ on a time horizon $T > 0$ localizes the time variable of the coefficient σ.

Condition (A3) is closely connected to the factorization method of Da Prato, Kwapien, and Zabczyk [10], which we also use in our proofs. In the following remark we relate (A3) to some conditions appearing in the literature on MKV SPDEs.

Remark 2.4 (i) In case the linear growth condition (2.1) holds, (A1) and (A3) are implied by

$$\int_0^t \frac{\|S_s\|^2_{L_2(E)} ds}{s^{2\alpha}} < \infty, \quad \forall t > 0, \tag{2.2}$$

which is a classical condition appearing for instance in [8, 15] for SPDEs without measure dependence.

Suppose that A is a negative definite self-adjoint operator[1] and that there exists a $\delta \in (0, 1)$ such that

$$\sum_{k=1}^\infty \lambda_k^{-1+\delta} < \infty, \tag{2.3}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are all eigenvalues of $-A$, counting multiplicities, with $-Ae_k = \lambda_k e_k$ for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of E. This assumption appears as (a1) in [18]. If it is in force, (2.2) holds with $\alpha = \delta/2$, since

$$\int_0^t \frac{\|S_s\|^2_{L_2(E)} ds}{s^\delta} = \sum_{k=1}^\infty \int_0^t \frac{e^{-2\lambda_k s} ds}{s^\delta} \leq \int_0^\infty \frac{e^{-2z} dz}{z^\delta} \sum_{k=1}^\infty \lambda_k^{-1+\delta} < \infty, \quad t > 0.$$

[1] A negative definite self-adjoint operator generates a contraction semigroup, see [32, Proposition 6.14].

In the paper [2] the linearity A is also assumed to be negative definite and self-adjoint[2] but only a weaker form of (2.3) is imposed. Let us compare the linear growth assumptions on the diffusion coefficient from [2] to our linear growth condition (A3). The article [2] works with a finite time horizon $T > 0$ and the diffusion coefficient $\sigma(t, x, \nu) \equiv \sigma(t, x)$ is presumed to be independent of the measure variable and such that there are non-negative numbers $b_1, b_2, \dots$ such that

$$\|\sigma^*(t, x)e_k\|_H^2 \le b_k^2\big(1 + \|x\|_E^2\big), \quad t \in [0, T], \ k = 1, 2, \dots,$$

and

$$\sum_{k=1}^{\infty} b_k^2 \lambda_k^{-\theta} < \infty$$

for some $\theta \in (0, 1)$, cf. Eq. (2.44) in [2]. Under this condition the first part of (A3) holds with

$$\mathfrak{f}(t) = \mathfrak{f}_T(t) \triangleq C\Big(\sum_{k=1}^{\infty} e^{-2\lambda_k t} b_k^2\Big)^{1/2}, \quad t \in (0, T],$$

and $\alpha \triangleq (1 - \theta)/2 \in (0, 1/2)$, since

$$\|S_t\sigma(s, x)\|_{L_2(H,E)}^2 = \sum_{k=1}^{\infty} \|\sigma^*(s, x)S_t e_k\|_H^2 \le \sum_{k=1}^{\infty} e^{-2\lambda_k t} b_k^2(1 + \|x\|_E^2),$$

for $0 < t, s \le T$, $x \in E$, and

$$\int_0^t \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds = C \sum_{k=1}^{\infty} \int_0^t \frac{e^{-2\lambda_k s} b_k^2 ds}{s^{1-\theta}} \le C \int_0^{\infty} \frac{e^{-2z} dz}{z^{1-\theta}} \sum_{k=1}^{\infty} b_k^2 \lambda_k^{-\theta} < \infty$$

for $t \in [0, T]$. The linear growth condition for the drift coefficient in [2] is of the same type as those for the diffusion coefficient (but the drift can depend on the measure variable). In particular, an interaction with the semigroup S is allowed. This feature is not included in (A3). Via Condition (L1) below, we also introduce a condition which allows an interaction of the drift coefficient μ and the semigroup S.

As pointed out in [8, Remark 5.10], (2.2) is very close to the necessary condition for the existence of solutions to Cauchy problems. More precisely, the stochastic Cauchy problem

$$dX_t = AX_t dt + dW_t$$

has a mild solution (with not necessarily continuous paths) if and only if

$$\int_0^t \|S_t\|_{L_2(E)}^2 dt < \infty, \quad t > 0, \tag{2.4}$$

see [34, Theorem 7.1]. Moreover, if A is self-adjoint, negative definite and has a discrete spectrum, then [19, Theorem 1] implies that (2.4) is sufficient for the existence of a mild solution with continuous paths, see also [8, Proposition 9.30] and [32, Proposition 5.12].

[2] More precisely, S is assumed to be a contraction semigroup such that $(-A)^{-1}$ exists as a bounded self-adjoint operator with discrete spectrum. By virtue of [23, Theorem 5.30, p. 169], the second part implies that A is self-adjoint and hence, so is S by [32, Proposition 6.13]. Then, as S is a contraction semigroup, A is negative definite by [32, Proposition 6.14].

(ii) Another classical type of linear growth condition is the following: For every $T > 0$ there exists a constant $C_T > 0$ such that

$$\|\mu(s, x, \nu)\|_E + \|\sigma(s, x, \nu)\|_{L_2(H,E)} \leq C_T\big(1 + \|x\|_E\big) \tag{2.5}$$

for all $s \in [0, T]$, $x \in E$ and $\nu \in M_c(E)$. Compared to (2.1), this condition uses the Hilbert–Schmidt norm instead of the operator norm. Under this condition, (A3) holds with $\mathfrak{f}(t) = \|S_t\|_{L(E)}$ for $t > 0$, as there are constants $M \geq 1$ and $\omega \in \mathbb{R}_+$ such that $\|S_t\|_{L(E)} \leq Me^{\omega t}$ for all $t > 0$.

Our first main result is the following:

Theorem 2.5 *Suppose that (A1), (A2) and (A3) hold. Then, for every $\eta \in M_c(E)$ there exists a martingale solution to the MKV SPDE with coefficients (A, μ, σ, η).*

The proof of Theorem 2.5 is given in Sect. 4. Let us mention two typical situations where the above theorem can be applied.

Example 2.6 [15] Let $\mathscr{O}$ be a bounded region in $\mathbb{R}^d$ with smooth boundary and set $E \triangleq L^2(\mathscr{O})$. If A is a strongly elliptic operator of order $2m > d$ (with Dirichlet boundary conditions), then there exists an α such that (2.2) holds, see [15, Example 3]. Thus, by virtue of part (i) of Remark 2.4, Theorem 2.5 shows that for $d = 1$ the McKean–Vlasov stochastic heat equation

$$dX_t = \Delta X_t dt + \mu(t, X_t, P^X_t)dt + \sigma(t, X_t, P^X_t)dW_t, \tag{2.6}$$

driven by a cylindrical standard Brownian motion W, has a martingale solution in case μ and σ satisfy the continuity condition (A2) and the linear growth condition (2.1), which only depends on the operator norm.

Example 2.7 Let $E = L^2(\mathscr{O})$ be as in Example 2.6. If A is a strongly elliptic second order operator (with Dirichlet boundary conditions), then (A1) holds, see [15, Remark 1] and [9, Appendix A.5.2]. Thus, by virtue of part (ii) of Remark 2.4, Theorem 2.5 shows that the McKean–Vlasov stochastic heat equation (2.6), driven by a standard cylindrical Brownian motion W, has a martingale solution in case μ and σ satisfy the continuity condition (A2) and the linear growth condition (2.5). This observation is independent of the dimension d (recall that $\mathscr{O} \subset \mathbb{R}^d$). Compared to the result mentioned in Example 2.6, the linear growth condition (2.5) entails the Hilbert–Schmidt norm while (2.1) only depends on the operator norm. In case the driving noise W from (2.6) is a Brownian motion with a trace class covariance operator Q, Theorem 2.5 implies that (2.6) has a martingale solution under (A2) and (2.1), again independently of the dimension d. To see this, notice that in this situation the equation (2.6) can be written as

$$dX_t = \Delta X_t dt + \mu(t, X_t, P^X_t)dt + \sigma(t, X_t, P^X_t)Q^{1/2}dB_t,$$

where B is a cylindrical standard Brownian motion, and that its diffusion coefficient $\sigma Q^{1/2}$ satisfies a linear growth condition for the Hilbert–Schmidt norm once σ satisfies a corresponding condition for the operator norm.

Provided the initial value satisfies a suitable integrability condition, we can relax the continuity assumptions on μ and σ in the measure variable by replacing the weak with a Wasserstein topology. Furthermore, we can strengthen the linear growth condition. Take $1 \leq p^\circ < p'$.

(A4) For all $y^* \in D(A^*)$ and $t > 0$ the maps $\langle \mu(t, \cdot, \cdot), y^* \rangle_E$ and $\|\sigma^*(t, \cdot, \cdot)y^*\|_H$ are continuous on $E \times M^{p^\circ}_w(E)$.

(A5) For every $T > 0$ there exists a Borel function $\mathfrak{f} = \mathfrak{f}_T : (0, T] \to [0, \infty]$ and a constant $C_T > 0$ such that

$$\int_0^T \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds < \infty,$$

and

$$\begin{aligned} \|S_t \sigma(s, x, \nu)\|_{L_2(H,E)} &\le \mathfrak{f}(t)\big(1 + \|x\|_E + \|\nu\|_{p'}\big), \\ \|\mu(t, x, \nu)\|_E + \|\sigma(t, x, \nu)\|_{L(H,E)} &\le C_T\big(1 + \|x\|_E + \|\nu\|_{p'}\big), \end{aligned} \tag{2.7}$$

for all $0 < t, s \le T$, $x \in E$ and $\nu \in M_w^{p'}(E)$.

Theorem 2.8 *Suppose that $\eta \in M_w^{p'}(E)$ and that μ and σ are only defined on $\mathbb{R}_+ \times E \times M_w^{p^\circ}(E)$. Furthermore, assume that (A1), (A4) and (A5) hold. Then, the MKV SPDE (A, μ, σ, η) has a p'-martingale solution.*

Theorem 2.8 can be proved similar to Theorem 2.5 and we outline the necessary changes in Sect. 5.

Remark 2.9 Replacing $M_c(E)$ by $M_w^{p^\circ}(E)$ for the continuity assumptions on the coefficients μ and σ is a useful generalization. For instance, consider the coefficient

$$\mu(t, x, \nu) \equiv \int \mu^*(s, x, y)\nu(dy)$$

for a measurable function μ^*. This coefficient is well-defined for all $\nu \in M_c(E)$ only if $y \mapsto \mu^*(t, x, y)$ is bounded. However, if we restrict our attention to $\nu \in M_w^p(E)$ for some $p \ge 1$, we can allow unbounded μ^* under a suitable growth assumption on $y \mapsto \mu^*(t, x, y)$.

Next, we also provide a uniqueness result for MKV SPDEs. We fix $p \ge 2$ and we assume that μ and σ are defined on $\mathbb{R}_+ \times E \times M_w^p(E)$.

Definition 2.10 Let $\eta \in M_w^p(E)$.

(i) We say that the MKV SPDE (A, μ, σ, η) satisfies *p-uniqueness in law* if there is at most one p-solution measure.
(ii) We say that the MKV SPDE (A, μ, σ, η) satisfies *p-pathwise uniqueness* if for any two p-martingale solutions $(\mathbb{B}, W, X)$ and $(\mathbb{B}, W, Y)$ we have a.s. $X = Y$.

Let w_p be the p-Wasserstein metric, i.e. for $\nu, \eta \in M_w^p(E)$ set

$$\mathsf{w}_p(\nu, \eta) \triangleq \inf_{F \in \Pi(\nu,\eta)} \Big(\int \|x - y\|_E^p F(dx, dy)\Big)^{1/p},$$

where $\Pi(\nu, \eta)$ is the set of Borel probability measures F on $E \times E$ such that $F(dx \times E) = \nu(dx)$ and $F(E \times dx) = \eta(dx)$.

(U1) For every $T, m > 0$ there are two Borel functions $\mathfrak{f} = \mathfrak{f}_{T,m} : (0, T] \to [0, \infty]$ and $\mathfrak{g} = \mathfrak{g}_{T,m} : (0, T] \to [0, \infty]$ such that

$$\int_0^T \Big(\Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 + \big[\mathfrak{g}(s)\big]^{p/(p-1)}\Big) ds < \infty,$$

and an increasing continuous function $\kappa = \kappa_{T,m} \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(0) = 0$, $k(x) > 0$ for $x > 0$, and

$$\forall \varepsilon > 0 \quad \int_0^\varepsilon \frac{dx}{|\kappa(x^{1/p})|^p} = \infty,$$

such that

$$\begin{aligned} \|S_t(\sigma(s,x,\nu) - \sigma(t,y,\eta))\|_{L_2(H,E)} &\le \mathfrak{f}(t)\big(\|x-y\|_E + \kappa(\mathsf{w}_p(\nu,\eta))\big), \\ \|S_t(\mu(s,x,\nu) - \mu(t,y,\eta))\|_E &\le \mathfrak{g}(t)\big(\|x-y\|_E + \kappa(\mathsf{w}_p(\nu,\eta))\big), \end{aligned}$$

and

$$\begin{aligned} \|S_t\sigma(s,x,\nu)\|_{L_2(H,E)} &\le \mathfrak{f}(t)\big(1 + \|x\|_E\big), \\ \|S_t\mu(s,x,\nu)\|_E &\le \mathfrak{g}(t)\big(1 + \|x\|_E\big), \end{aligned}$$

for all $0 < t, s \le T$, $x, y \in E$ and $\nu, \eta \in M^p_w(E)$ with $\|\nu\|_p, \|\eta\|_p \le m$.

Irrespective of p, the identity $\kappa(x) = x$ is a possible choice for κ and hence (U1) can be seen as a generalized Lipschitz condition.

Below we use (U1) together with the condition that $p > 1/\alpha$, which excludes the case $p = 2$. The following condition includes the case $p = 2$.

(U2) For every $T, m > 0$ there exists a constant $C = C_{T,m} > 0$ and an increasing continuous function $\kappa = \kappa_{T,m} \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(0) = 0$, $k(x) > 0$ for $x > 0$, and

$$\forall \varepsilon > 0 \quad \int_0^\varepsilon \frac{dx}{|\kappa(x^{1/p})|^p} = \infty,$$

such that

$$\begin{aligned} \|\mu(s,x,\nu) - \mu(t,y,\eta)\|_E &+ \|\sigma(s,x,\nu) - \sigma(t,y,\eta)\|_{L_2(H,E)} \\ &\le C\big(\|x-y\|_E + \kappa(\mathsf{w}_p(\nu,\eta))\big), \end{aligned}$$

and

$$\|\mu(s,x,\nu)\|_E + \|\sigma(s,x,\nu)\|_{L_2(H,E)} \le C\big(1 + \|x\|_E\big),$$

for all $0 < t, s \le T$, $x, y \in E$ and $\nu, \eta \in M^p_w(E)$ with $\|\nu\|_p, \|\eta\|_p \le m$.

Our next main result is the following:

Theorem 2.11 *Suppose that either $p > 1/\alpha$ and (U1) hold, or that $p \ge 2$ and (U2) hold. Then, for every $\eta \in M^p_w(E)$ the MKV SPDE (A, μ, σ, η) satisfies p-uniqueness in law and p-pathwise uniqueness.*

A different uniqueness result was established in [18]. We prove Theorem 2.11 in Sect. 6. The Theorems 2.8 and 2.11 can be combined to an existence and uniqueness statement. However, for the existence part we always require that A generates a compact semigroup. We now also provide a more classical existence and uniqueness result for equations with Lipschitz coefficients which needs no compactness assumption.

(L1) For every $T > 0$ there are two Borel functions $\mathfrak{f} = \mathfrak{f}_T \colon (0, T] \to [0, \infty]$ and $\mathfrak{g} = \mathfrak{g}_T \colon (0, T] \to [0, \infty]$ such that

$$\int_0^T \Big(\Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 + \big[\mathfrak{g}(s)\big]^{p/(p-1)}\Big) ds < \infty,$$

and

$$\|S_t(\sigma(s,x,\nu)-\sigma(s,y,\eta))\|_{L_2(H,E)} \le \mathfrak{f}(t)\big(\|x-y\|_E + \mathsf{w}_p(\nu,\eta)\big),$$
$$\|S_t(\mu(s,x,\nu)-\mu(s,y,\eta))\|_E \le \mathfrak{g}(t)\big(\|x-y\|_E + \mathsf{w}_p(\nu,\eta)\big),$$

and

$$\|S_t\sigma(s,x,\nu)\|_{L_2(H,E)} \le \mathfrak{f}(t)\big(1+\|x\|_E+\|\nu\|_p\big),$$
$$\|S_t\mu(s,x,\nu)\|_E \le \mathfrak{g}(t)\big(1+\|x\|_E+\|\nu\|_p\big),$$

for all $0 < t, s \le T$, $x, y \in E$ and $\nu, \eta \in M^p_w(E)$.

Below we use (L1) together with the condition that $p > 1/\alpha$, which excludes the case $p = 2$. The following Lipschitz condition includes the case $p = 2$.

(L2) For every $T > 0$ there exists a constant $C = C_T > 0$ such that

$$\|\mu(t,x,\nu)-\mu(t,y,\eta)\|_E + \|\sigma(t,x,\nu)-\sigma(t,y,\eta)\|_{L_2(H,E)} \le C\big(\|x-y\|_E + \mathsf{w}_p(\nu,\eta)\big),$$

for all $0 < t \le T$, $x, y \in E$ and $\nu, \eta \in M^p_w(E)$. Moreover, the functions $\|\mu(\cdot,0,\delta_0)\|_E$ and $\|\sigma(\cdot,0,\delta_0)\|_{L_2(H,E)}$ are bounded on compact subsets of $\mathbb{R}_+$.

Recall that S is called a *generalized contraction semigroup* if there exists an $\omega \in \mathbb{R}$ such that $\|S_t\|_{L(E)} \le e^{\omega t}$ for all $t \in \mathbb{R}_+$.

Theorem 2.12 *Suppose that $\eta \in M^p_w(E)$ and that either $p > 1/\alpha$ and (L1) hold, or that $p \ge 2$, that (L2) holds and that S is a generalized contraction. Then, the MKV SPDE (A, μ, σ, η) has a p-martingale solution and it satisfies p-uniqueness in law and p-pathwise uniqueness. Moreover, the MKV SPDE can be realized on any driving system* $(\mathbb{B}, W)$.

Theorem 2.12 can be seen as a version of [5, Theorem 4.21] for an infinite dimensional setting and its proof is similar. For completeness we provide it in Appendix B.

For negative definite self-adjoint A satisfying a (generalized) variant of (2.2), a related existence and uniqueness result is given by [31, Theorem 3.1]. Notice that any semigroup with negative definite self-adjoint generator is a contraction ([32, Proposition 6.14]).

3 Propagation of Chaos for Weakly Interacting SPDEs

In this section we discuss the chaotic property of weakly interacting particles which are modeled as mild solutions to SPDEs. The section is split into two parts. In the first we derive a result under a continuity condition on the coefficients and a uniqueness assumption on the law of the limiting MKV SPDE. In the second part we provide a result under Lipschitz conditions on the coefficients.

3.1 The Chaotic Property Under Continuity and Uniqueness Assumptions

To fix our setting, we assume that α, p', A, μ and σ are as in Sect. 2 with the important exception that μ and σ are only defined on $\mathbb{R}_+ \times E \times M^{p^\circ}_w(E)$ for some $1 \le p^\circ < p'$. For

$N \in \mathbb{N}$ we define a map $L^N \colon E^{\otimes N} \to M_c(E)$ by

$$L^N(x_1, \ldots, x_N) \triangleq \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \quad (x_1, \ldots, x_N) \in E^{\otimes N}.$$

Let us start with a condition for the initial laws:

(I) Let $\eta^N \in M_c(E^{\otimes N})$ be symmetric[3] such that there exists a measure $\eta \in M_c(E)$ with

$$\eta^N \circ (L^N)^{-1} \longrightarrow \delta_\eta \tag{3.1}$$

weakly[4] as $N \to \infty$. Moreover,

$$\sup_{N \in \mathbb{N}} \int \|\mathsf{X}_1^N(x)\|_E^{p'} \eta^N(dx) < \infty,$$

where $\mathsf{X}_1^N \colon E^{\otimes N} \to E$ denotes the projection to the first coordinate.

Remark 3.1 According to [33, Proposition I.2.2], (3.1) holds if and only if the sequence $\eta^1, \eta^2, \ldots$ is η-*chaotic*, i.e. for every $k \in \mathbb{N}$ and $\phi_1, \phi_2, \ldots, \phi_k \in C_b(E)$

$$\int_{E^{\otimes N}} \prod_{i=1}^k \phi_i(\mathsf{X}_i^N(x))\eta^N(dx) \to \prod_{i=1}^k \int_E \phi_i(x)\eta(dx), \quad N \to \infty.$$

Moreover, it is also equivalent to the above for $k = 2$.

The following condition deals with the existence of weakly interacting particles whose chaotic behavior we investigate in the remainder of this section.

(EUP) For $N \in \mathbb{N}$ there exists a filtered probability space $\mathbb{B}^N$ with right-continuous and complete filtration which supports independent standard cylindrical Brownian motions $W^1 \equiv W^{N,1}, \ldots, W^N \equiv W^{N,N}$ and mild solution processes $X^{N,1}, \ldots, X^{N,N}$ to the SPDE

$$dX_t^{N,i} = AX_t^{N,i}dt + \mu(t, X_t^{N,i}, \mathscr{X}_t^N)dt + \sigma(t, X_t^{N,i}, \mathscr{X}_t^N)dW_t^i,$$

with $(X_0^{N,1}, \ldots, X_0^{N,N}) \sim \eta^N$ and

$$\mathscr{X}_t^N \triangleq \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}} = L^N(X_t^{N,1}, \ldots, X_t^{N,N}), \quad t \in \mathbb{R}_+.$$

Moreover, the SPDE

$$dY_t = \Big[\bigoplus_{i=1}^N A\Big]Y_t dt + \Big[\bigoplus_{i=1}^N \mu(t, Y_t, L^N(Y_t))\Big]dt + \Big[\bigoplus_{i=1}^N \sigma(t, Y_t, L^N(Y_t))\Big]dW_t, \tag{3.2}$$

satisfies uniqueness in law. Here, $\bigoplus_{i=1}^N$ denotes the Hilbert space direct sum.

[3] Let Π_N be the set of all perturbations of $\{1, \ldots, N\}$. A probability measure η on $(E^{\otimes N}, \mathscr{B}(E^{\otimes N}))$ is called *symmetric* if $\eta(\pi(B)) = \eta(B)$ for every $B \in \mathscr{B}(E^{\otimes N})$ and $\pi \in \Pi_N$. Here, $\pi(B) \triangleq \{(e_{\pi(1)}, \ldots, e_{\pi(N)}) : e \in B\}$.

[4] Notice that the probability measures in (3.1) are elements of $M_c(M_c(E))$ and that the weak topology refers to the topology of convergences in distribution on the space $M_c(M_c(E))$.

Remark 3.2 Together with the symmetry assumption for η^N from (I), the uniqueness in law assumption for (3.2) from (EUP) implies that the law of $(X^{N,1}, \ldots, X^{N,N})$ is symmetric.

Conditions for (EUP) can be deduce from the Theorems 2.5 and A.1. More conditions for existence and (pathwise) uniqueness can be found in Part II of the monograph [9]. Pathwise uniqueness entails uniqueness in law by the Yamada–Watanabe theorem [28, Theorem 2]. Many results for uniqueness in law require more regular coefficients than its counterparts for existence. One approach to relax such regularity assumptions is based on Girsanov's theorem. For instance, it can be used to deduce existence and uniqueness in law for equations of the type

$$dX_t = (AX_t + \mu(X_t))dt + dW_t$$

from the corresponding properties of

$$dX_t = AX_t dt + dW_t.$$

We refer to Appendix I from [26] for a detailed application of this method.

Next, we formulate a uniqueness condition for the limiting MKV SPDE.

(UL) The MKV SPDE with coefficients (A, μ, σ, η) satisfies p'-uniqueness in law.

Finally, we also formulate a version of (A4) from Sect. 2.

(C) For all $y^* \in D(A^*)$ and $t > 0$ the maps $\langle \mu(t, \cdot, \cdot), y^* \rangle_E$ and $\|\sigma^*(t, \cdot, \cdot)y^*\|_H$ are continuous on $E \times M_w^{p^\circ}(E)$, and the maps $\langle \mu, y^* \rangle_E$ and $\|\sigma^* y^*\|_H$ are bounded on compact subsets of $\mathbb{R}_+ \times E \times M_w^{p^\circ}(E)$.

Remark 3.3 The final local boundedness condition from (C) is not implied by the linear growth condition (A5), because $M_w^{p'}(E) \subset M_w^{p^\circ}(E)$ since $p^\circ < p'$.

For $T > 0$ let $\mathsf{w}_T^{p^\circ}$ be the p°-Wasserstein metric on $M_w^{p^\circ}(C([0, T], E))$ where $C([0, T], E)$ is endowed with the uniform topology. The following theorem is the main result in this section. It formalizes the chaotic behavior of the weakly interacting SPDEs from (EUP).

Theorem 3.4 *Suppose that (A1), (A5), (I), (EUP), (UL) and (C) hold. Then, the MKV SPDE with coefficients (A, μ, σ, η) has a p'-martingale solution with unique law $\mathscr{X}^0$. Moreover, for all $T > 0$*

$$\lim_{N \to \infty} E\Big[\big|\mathsf{w}_T^{p^\circ}(\mathscr{X}^N, \mathscr{X}^0)\big|^{p^\circ}\Big] = 0, \tag{3.3}$$

and the particles $X^{N,i}$ are $\mathscr{X}^0$-chaotic, i.e. for every $k \in \mathbb{N}$ we have

$$(X^{N,1}, \ldots, X^{N,k}) \to (Y^1, \ldots, Y^k) \tag{3.4}$$

weakly as $N \to \infty$, where $Y^1, \ldots, Y^k$ are i.i.d. with $Y^1 \sim \mathscr{X}^0$.

The proof of Theorem 3.4 is given in Sect. 7. Equation (3.4) means that the particles $X^{N,i}$ become asymptotically i.i.d. as $N \to \infty$. In the proof of Theorem 3.4 we establish the existence part without invoking results from Sect. 2. Theorem 2.11 provides some conditions for (UL). More conditions for (UL) can be found in [18].

3.2 The Chaotic Property Under Lipschitz Conditions

In this section we discuss the chaotic behavior for weakly interacting SPDEs with Lipschitz coefficients.

Let A, μ and σ be as in Sect. 2 but μ and σ need only be defined on $\mathbb{R}_+ \times E \times M^p_w(E)$. Take a filtered probability space $\mathbb{B} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ which supports a sequence $W^1, W^2, \ldots$ of independent standard cylindrical Brownian motions and a sequence $\xi^1_0, \xi^2_0, \ldots$ of $\mathscr{F}_0$-measurable i.i.d. random variables with $\xi^1_0 \sim \eta \in M^p_w(E)$. The following proposition shows that (EUP) is implied by the global Lipschitz condition (L1).

Proposition 3.5 *Assume that (L1) holds. For $N \in \mathbb{N}$ and $i = 1, \ldots, N$, on $\mathbb{B}$ there exists a unique (up to indistinguishability) mild solution process $X^{N,i}$ to the SPDE*

$$dX^{N,i}_t = AX^{N,i}_t dt + \mu(t, X^{N,i}_t, \mathscr{X}^N_t)dt + \sigma(t, X^{N,i}_t, \mathscr{X}^N_t)dW^i_t, \quad X^{N,i} = \xi^i_0,$$

with

$$\mathscr{X}^N_t \triangleq \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}_t}, \quad t \in \mathbb{R}_+.$$

Proposition 3.5 follows from Theorem A.1. For completeness we give a proof in Appendix C. The following theorem is a version of Theorem 3.4 for the present setting. Compared to Theorem 3.4 its scope is slightly different as the semigroup S needs not to be compact but the coefficients have to be Lipschitz.

For $T > 0$ recall that w^p_T is the p-Wasserstein metric on $M^p_w(C([0, T], E))$ where $C([0, T], E)$ is endowed with the uniform topology. In case $p > 1/\alpha$ and (L1) holds, or $p \geq 2$, (L2) holds and that S is a generalized contraction, Theorem 2.12 implies the existence of a p-martingale solution to the MKV SPDE (A, μ, σ, η) with a unique law $\mathscr{X}^0$.

Theorem 3.6 *Assume that either $p > 1/\alpha$ and that (L1) hold, or that $p \geq 2$, (L2) holds and that S is a generalized contraction. For every $T > 0$ it holds that*

$$\lim_{N\to\infty} E\Big[\big|\mathsf{w}^p_T(\mathscr{X}^N, \mathscr{X}^0)\big|^p\Big] = 0. \tag{3.5}$$

Moreover, the particles $X^{N,i}$ are $\mathscr{X}^0$-chaotic, see Theorem 3.4 for this terminology.

Theorem 3.6 can be proved as its finite dimensional counterpart [24, Theorem 3.3]. For completeness we give a proof in Appendix D. Except of our assumption that we use i.i.d. initial data, Theorem 3.6 generalizes [2, Theorem 5.3] to more general particle systems. In particular, the convergence in Theorem 3.6 is stronger. In the following section we comment in some detail on the proofs of Theorems 3.4 and 3.6.

3.3 Comments on the Proofs of Theorems 3.4 and 3.6

Thanks to the i.i.d. initial conditions and the Lipschitz assumptions on the coefficients, Theorem 3.6 can be proved by a coupling argument with independent solutions to MKV SPDEs. To sketch the idea (cf. [24] for bibliographic notes), assume that the coefficients μ and σ satisfy Lipschitz conditions as in (L1) and consider the particle system $X^{N,i}$ from Proposition 3.5 with i.i.d. initial conditions. On the same probability space that supports the particle system, let $Y^1, Y^2, \ldots$ be independent solution processes to the MKV SPDE

(A, μ, σ, η), where Y^i is driven by the same noise W^i as the particle $X^{N,i}$. The existence of the sequence $Y^1, Y^2, \ldots$ is guaranteed by the Lipschitz conditions via Theorem 2.12. Using a Gronwall argument that hinges on the Lipschitz assumptions, we prove for any fixed number $k \in \mathbb{N}$ and any time $T > 0$ that

$$E\Big[\max_{i=1,\ldots,k} \sup_{s\in[0,T]} \|X_s^{N,i} - Y_s^i\|_E^p\Big] \to 0, \quad N \to \infty, \tag{3.6}$$

see Appendix D for details. This implies propagation of chaos.

The Lipschitz assumptions are used to realize $Y^1, Y^2, \ldots$ on the same probability space as the particles and to couple them via the same driving noise. Further, the Gronwall argument for (3.6) relies on them.

To prove Theorem 3.4, which provides propagation of chaos for less regular than Lipschitz coefficients, we use a technique based on tightness and martingale problem arguments. A similar strategy was used in [2]. We refer to Sect. 1.2 for comments on the relation of our results and proofs to those from [2].

In infinite dimensional settings tightness is often more difficult to establish than in finite dimensional cases, as moment estimates are less easy to use, because balls are not compact in infinite dimensional normed spaces. We adapt the compactness method from [15] to overcome this difficulty. Further, to use our continuity assumption for the Wasserstein topology in the martingale problem argument, we extend the tightness result to a suitable Wasserstein space with the help of moment estimates.

Notice that the empirical distributions $\{\mathscr{X}^N : N \in \mathbb{N}\}$ are probability measures on a Wasserstein space that consists itself of probability measures (on a path space). Using a general martingale problem for MKV SPDEs, we prove that almost all realizations of any accumulation point $\mathscr{X}^*$ are solution measures to the MKV SPDE (A, μ, σ, η). Here, it is important to reduce the martingale problem to a countable set of test functions. As we assume that the MKV SPDE (A, μ, σ, η) has a unique solution measure (that is our assumption (UL)), we can conclude that $\mathscr{X}^*$ is almost surely constant. From this we deduce $\mathscr{X}^N \to \mathscr{X}^*$ in probability and, with a suitable moment estimate, also convergence in mean.

4 Proof of Theorem 2.5

The proof is split into several steps. In Step 0 we recall the factorization formula from [10] and prepare some estimates. Then, in Step 1 we define an approximation sequence, in Step 2 we establish some moment estimates, in Step 3 we verify tightness of the approximation sequence and in Step 4 we investigate a martingale problem. In the fifth and final step we use a representation theorem for cylindrical continuous local martingales to complete the proof.

Step 0: A short recap of the factorization formula. Fix a finite time horizon $T > 0$. For $p' > 1$, $1/p' < \lambda \le 1$ and $h \in L^{p'}([0, T], E)$ we set

$$R_\lambda h(t) \triangleq \int_0^t (t-s)^{\lambda-1} S_{t-s} h(s) ds, \quad t \in [0, T].$$

Notice that R_λ is indeed well-defined, as

$$\begin{aligned} &\int_0^t (t-s)^{\lambda-1} \|S_{t-s}h(s)\|_E ds \\ &\le \Big(\int_0^T s^{p'(\lambda-1)/(p'-1)} \|S_s\|_{L(E)}^{p'/(p'-1)} ds\Big)^{(p'-1)/p'} \Big(\int_0^t \|h(s)\|_E^{p'} ds\Big)^{1/p'}, \end{aligned} \tag{4.1}$$

by Hölder's inequality. The first integral is finite as $p'(\lambda-1)/(p'-1) > -1 \iff \lambda > 1/p'$. The inequality (4.1) implies that

$$\|R_\lambda h(t)\|_E \le \Big(\int_0^T s^{p'(\lambda-1)/(p'-1)}\|S_s\|_{L(E)}^{p'/(p'-1)}ds\Big)^{(p'-1)/p'}\Big(\int_0^t \|h(s)\|_E^{p'}ds\Big)^{1/p'}, \quad (4.2)$$

which shows that R_λ is a bounded linear operator on $L^{p'}([0,T],E)$.

Lemma 4.1 [15, Proposition 1] *For any $1/p' < \lambda \le 1$, R_λ is a bounded linear operator from $L^{p'}([0,T],E)$ into $C([0,T],E)$. Moreover, if the semigroup S is compact, then R_λ is compact.*

Next, take some $0 < \alpha < 1/2$ and $p' > 2$ large enough such that $1/p' < \alpha$. Moreover, let $\mathfrak{f}\colon (0,T] \to [0,\infty]$ be a Borel function such that

$$\int_0^T \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds < \infty,$$

let ϕ be a predictable $L(H,E)$-valued process and ψ a predictable real-valued process such that

$$\|S_t\phi_s\|_{L_2(H,E)} \le \mathfrak{f}(t)|\psi_s|$$

for all $0 < t, s \le T$, and

$$E\Big[\int_0^T |\psi_s|^{p'}ds\Big] < \infty.$$

Set $\gamma \triangleq p'/(p'-1)$. Then, using Hölder's inequality in the second and last line, and Young's inequality in the third line, we obtain

$$\begin{aligned}
&\int_0^T (T-t)^{\alpha-1}\Big(E\Big[\int_0^t (t-s)^{-2\alpha}\|S_{t-s}\phi_s\|_{L_2(H,E)}^2 ds\Big]\Big)^{1/2}dt\\
&\quad\le \Big(\int_0^T s^{\gamma(\alpha-1)}ds\Big)^{1/\gamma}\Big(\int_0^T\Big(\int_0^T\Big[\frac{\mathfrak{f}(t-s)}{(t-s)^\alpha}\Big]^2 E\big[|\psi_s|^2\big]ds\Big)^{p'/2}dt\Big)^{1/p'}\\
&\quad\le \Big(\int_0^T s^{\gamma(\alpha-1)}ds\Big)^{1/\gamma}\Big(\int_0^T\Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds\Big)^{1/2}\Big(\int_0^T E\big[|\psi_s|^2\big]^{p'/2}ds\Big)^{1/p'}\\
&\quad\le \Big(\int_0^T s^{\gamma(\alpha-1)}ds\Big)^{1/\gamma}\Big(\int_0^T\Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds\Big)^{1/2}\Big(E\Big[\int_0^T |\psi_s|^{p'}ds\Big]\Big)^{1/p'}.
\end{aligned}$$

Since $\gamma(\alpha-1) > -1 \iff \alpha > \frac{1}{p'}$, the term in the last line is finite and the factorization formula [9, Theorem 5.10] yields that

$$\int_0^t S_{t-s}\phi_s dW_s = \frac{\sin(\pi\alpha)}{\pi}R_\alpha Y(t), \quad t \in [0,T], \quad (4.3)$$

with

$$Y_t \triangleq \int_0^t (t-s)^{-\alpha}S_{t-s}\phi_s dW_s,$$

where W is a standard cylindrical Brownian motion. In this formula the process Y has to be understood in the sense of the stochastic Fubini theorem ([9, Theorem 4.33] or [28, Proposition 6.1]). In particular, the latter yields that the stochastic convolution $\int_0^t S_{t-s}\phi_s dW_s$ is well-defined.

At the beginning of this step we defined R_α on $L^{p'}([0,T],E)$. We now show that a.a. paths of Y are in $L^{p'}([0,T],E)$. Then, we can also conclude from Lemma 4.1 that the stochastic convolution has a continuous version. For every $t \in [0,T]$, we estimate

$$\begin{aligned} E\Big[\int_0^t \|Y_s\|_E^{p'} ds\Big] &\leq c_{p'} \int_0^t E\Big[\Big(\int_0^s (s-r)^{-2\alpha} \|S_{s-r}\phi_r\|^2_{L_2(H,E)} dr\Big)^{p'/2}\Big] ds \\ &\leq c_{p'} E\Big[\int_0^t \Big(\int_0^s \Big[\frac{\mathfrak{f}(s-r)}{(s-r)^\alpha}\Big]^2 |\psi_r|^2 dr\Big)^{p'/2} ds\Big] \\ &\leq c_{p'} \Big(\int_0^t \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds\Big)^{p'/2} E\Big[\int_0^t |\psi_s|^{p'} ds\Big], \end{aligned} \tag{4.4}$$

where we use Burkholder's inequality (with constant $c_{p'}$) in the first and Young's inequality in the last line. We conclude that the stochastic convolution $\int_0^\cdot S_{\cdot-s}\phi_s dW_s$ has a continuous version. Let us summarize the above observations.

Lemma 4.2 *Suppose that α, p', $\mathfrak{f}$, ϕ and ψ are as above. Then, the stochastic convolution*

$$t \mapsto \int_0^t S_{t-s}\phi_s dW_s$$

is well-defined and continuous. Furthermore, there exists a constant C depending on $\|S\|_{L(E)}$, α, p', T and $\mathfrak{f}$ such that for every $t \in [0,T]$

$$E\Big[\sup_{s\in[0,t]} \Big\|\int_0^s S_{s-r}\phi_r dW_r\Big\|_E^{p'}\Big] \leq C E\Big[\int_0^t |\psi_s|^{p'} ds\Big]. \tag{4.5}$$

Proof Except for (4.5), all claims were proved before. We now establish (4.5). Set $r \triangleq p'(\lambda-1)/(p'-1)$ and $q \triangleq p'/(p'-1)$. Using the factorization formula (4.3) for the first equality, (4.2) for the first inequality, and (4.4) for the final inequality, we obtain

$$\begin{aligned} &E\Big[\sup_{s\in[0,t]} \Big\|\int_0^s S_{s-r}\phi_r dW_r\Big\|_E^{p'}\Big] \\ &= \Big(\frac{\sin(\pi\alpha)}{\pi}\Big)^{p'} E\Big[\sup_{s\in[0,t]} \|R_\alpha Y(s)\|_E^{p'}\Big] \\ &\leq \Big(\frac{\sin(\pi\alpha)}{\pi}\Big)^{p'} \Big(\int_0^T s^r \|S_s\|^q_{L(E)} ds\Big)^{p'-1} E\Big[\int_0^t \|Y_s\|_E^{p'} ds\Big] \\ &\leq c_{p'} \Big(\frac{\sin(\pi\alpha)}{\pi}\Big)^{p'} \Big(\int_0^T s^r \|S_s\|^q_{L(E)} ds\Big)^{p'-1} \Big(\int_0^T \Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 ds\Big)^{p'/2} E\Big[\int_0^t |\psi_s|^{p'} ds\Big]. \end{aligned}$$

This completes the proof. □

It seems that there is no estimate of the type (4.5) in the monograph [9]. However, a related one can be found in its first edition, namely [8, Proposition 7.9].

Step 1: Definition of the Approximation Sequence. Let $0 < \alpha < 1/2$ and $p' > 1/\alpha$ be as in Sect. 2. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space (with right-continuous and complete filtration) which supports a standard cylindrical Brownian motion W and an $\mathscr{F}_0$-measurable random variable ξ_0 with distribution η. Take $n \in \mathbb{N}$ and define a process X^n as follows: $X_0^n \triangleq \xi_0 \mathbb{I}_{\|\xi_0\|_E \leq n}$ and for $k \in \mathbb{Z}_+$ and $k2^{-n} < t \leq (k+1)2^{-n}$ we define inductively

$$X_t^n \triangleq S_{t-k2^{-n}} X^n_{k2^{-n}} + \int_{k2^{-n}}^t S_{t-s}\mu(s, X^n_{k2^{-n}}, P^{X^n}_{k2^{-n}}) ds$$

$$+ \int_{k2^{-n}}^{t} S_{t-s}\sigma(s, X^n_{k2^{-n}}, P^{X^n}_{k2^{-n}})dW_s,$$

and

$$\nu^n_t \triangleq P^{X^n}_{k2^{-n}}, \quad \mu^n(t,\omega,\nu) \triangleq \mu(t,\omega(k2^{-n}),\nu), \quad \sigma^n(t,\omega,\nu) \triangleq \sigma(t,\omega(k2^{-n}),\nu), \quad (4.6)$$

where $(\omega,\nu) \in C(\mathbb{R}_+, E) \times M_c(E)$. At this point, recall the notation $P^X_t = P \circ X_t^{-1}$.

Let us explain the induction procedure in more detail: Suppose that $k \in \mathbb{Z}_+$ is such that X^n is well-defined on $[0, k2^{-n}]$ and

$$E\Big[\sup_{s\in[0,k2^{-n}]} \|X^n_s\|^{p'}_E\Big] < \infty.$$

Then, Lemma 4.2 and the linear growth assumption (A3) yield that X^n is also well-defined on $[0, (k+1)2^{-n}]$ and we also have

$$E\Big[\sup_{s\in[0,(k+1)2^{-n}]} \|X^n_s\|^{p'}_E\Big] < \infty.$$

The construction based on the factorization method yields that X^n has continuous paths. The following lemma collects our observations and further provides the dynamics of X^n.

Lemma 4.3 *The process X^n has a.s. continuous paths, for all $T > 0$ it holds that*

$$E\Big[\sup_{s\in[0,T]} \|X^n_s\|^{p'}\Big] < \infty,$$

and the dynamics of X^n are given by

$$X^n_t = S_t X^n_0 + \int_0^t S_{t-s}\mu^n(s, X^n, \nu^n_s)ds + \int_0^t S_{t-s}\sigma^n(s, X^n, \nu^n_s)dW_s, \quad t \in \mathbb{R}_+.$$

Proof It is only left to prove the formula for the dynamics. We use induction. Suppose that X^n has the claimed dynamics on $[0, k2^{-n}]$. Then, for $k2^{-n} < t \le (k+1)2^{-n}$ we obtain

$$\begin{aligned} X^n_t &= S_{t-k2^{-n}}\Big(S_{k2^{-n}}X^n_0 + \int_0^{k2^{-n}} S_{k2^{-n}-s}\mu^n(s, X^n, \nu^n_s)ds \\ &\quad + \int_0^{k2^{-n}} S_{k2^{-n}-s}\sigma^n(s, X^n, \nu^n_s)dW_s\Big) \\ &\quad + \int_{k2^{-n}}^t S_{t-s}\mu^n(s, X^n, \nu^n_s)ds + \int_{k2^{-n}}^t S_{t-s}\sigma^n(s, X^n, \nu^n_s)dW_s \\ &= S_t X^n_0 + \int_0^t S_{t-s}\mu^n(s, X^n, \nu^n_s)ds + \int_0^t S_{t-s}\sigma^n(s, X^n, \nu^n_s)dW_s. \end{aligned}$$

Consequently, the proof is complete. □

Step 2: Uniform moment bound. In this step we derive a moment estimate which is useful to establish tightness of the family $\{X^n : n \in \mathbb{N}\}$.

Lemma 4.4 *For every $T > 0$ and every bounded set $K \subset E$ we have*

$$\sup_{n\in\mathbb{N}} E\Big[\sup_{s\in[0,T]} \|X^n_s\|^{p'}_E \mathbb{I}_{X^n_0\in K}\Big] < \infty.$$

Proof Using Lemmata 4.2 and 4.3 together with the linear growth assumption (A3), we obtain

$$\begin{aligned}E\Big[\sup_{s\in[0,t]}\|X^n_s\|^{p'}_E\mathbb{I}_{X^n_0\in K}\Big] &\leq 3^{p'+1}\Big(\sup_{x\in K}\sup_{s\in[0,t]}\|S_sx\|^{p'}_E + E\Big[\Big(\int_0^t\|\mu^n(s,X^n,\nu^n_s)\|_E ds\Big)^{p'}\mathbb{I}_{X^n_0\in K}\Big]\\ &\quad + E\Big[\sup_{s\in[0,t]}\Big\|\int_0^t S_{t-s}\sigma^n(s,X^n,\nu^n_s)\mathbb{I}_{X^n_0\in K}dW_s\Big\|^{p'}_E\Big]\Big)\\ &\leq C\Big(1+E\Big[\Big(\int_0^t\Big(1+\sup_{r\in[0,s]}\|X^n_r\|_E\Big)ds\Big)^{p'}\mathbb{I}_{X^n_0\in K}\Big]\\ &\quad + E\Big[\int_0^t\Big(1+\sup_{r\in[0,s]}\|X^n_r\|_E\mathbb{I}_{X^n_0\in K}\Big)^{p'}ds\Big]\Big)\\ &\leq C\Big(1+\int_0^t E\Big[\sup_{r\in[0,s]}\|X^n_r\|^{p'}_E\mathbb{I}_{X^n_0\in K}\Big]ds\Big)\end{aligned}$$

for all $t\in[0,T]$. As $E[\sup_{r\in[0,T]}\|X^n_r\|^{p'}_E]<\infty$ by Lemma 4.3, the claim follows from Gronwall's lemma ([25, Lemma 4.4.15]). □

Step 3: Tightness of $\{X^n : n\in\mathbb{N}\}$. By the Arzelà–Ascoli characterization of tightness ([21, Theorem 23.4]), it suffices to prove that for every $T>0$ the family $\{X^n|_{[0,T]} : n\in\mathbb{N}\}$ is tight when seen as Borel probability measures on $C([0,T],E)$ endowed with the uniform topology. We adapt the compactness method from [15].

The equality (4.3) and Lemma 4.3 yield that

$$X^n = SX^n_0 + R_1\mu^n + \tfrac{\sin(\pi\alpha)}{\pi}R_\alpha Y^n, \tag{4.7}$$

where $\mu^n\equiv(s\mapsto\mu^n(s,X^n,\nu^n_s))$ and

$$Y^n_t \triangleq \int_0^t (t-s)^{-\alpha}S_{t-s}\sigma^n(s,X^n,\nu^n_s)dW_s,\quad t\in[0,T].$$

Fix $\varepsilon>0$. Since a.s. $X^n_0=\xi_0\mathbb{I}_{\|\xi_0\|_E\leq n}\to\xi_0$, we have $X^n_0\to\eta$ weakly, which implies that $\{X^n_0 : n\in\mathbb{N}\}$ is tight. Consequently, there exists a compact set $K\subset E$ such that

$$\sup_{n\in\mathbb{N}}P(X^n_0\notin K)\leq\frac{\varepsilon}{2}.$$

Now, we define

$$\begin{aligned}K_R\triangleq\Big\{&\omega\in C([0,T],E) : \omega = Sx_0 + R_1\psi + \tfrac{\sin(\pi\alpha)}{\pi}R_\alpha\phi,\\ &x_0\in K,\phi,\psi\in L^{p'}([0,T],E)\text{ with }\int_0^T\|\psi(s)\|^{p'}_E ds\leq R,\ \int_0^T\|\phi(s)\|^{p'}_E ds\leq R\Big\}.\end{aligned}$$

For every $t\in[0,T]$ the set $\{S_tx : x\in K\}$ is compact by the compactness of the semigroup (and the compactness of K for $t=0$). By [11, Lemma I.5.2], the map

$$[0,T]\times K\ni(t,x)\mapsto S_tx\in E$$

is uniformly continuous. Thus, the Arzelà–Ascoli theorem ([21, Theorem A.5.2]) yields that the set $\{Sx|_{[0,T]} : x\in K\}$ is relatively compact in $C([0,T],E)$, and we conclude from Lemma 4.1 that K_R is relatively compact in $C([0,T],E)$. Due to (4.7) and Chebyshev's inequality, we have

$$P(X^n|_{[0,T]}\in K_R)\geq 1-P(X^n_0\notin K)-P\Big(\int_0^T\|\mu^n(s,X^n,\nu^n_s)\|^{p'}_E\mathbb{I}_{X^n_0\in K}ds>R\Big)$$

$$
\begin{aligned}
&- P\Big(\int_0^T \|Y^n_s\|^{p'}_E \mathbb{I}_{X^n_0 \in K} ds > R\Big)\\
&\geq 1 - \frac{\varepsilon}{2} - \frac{1}{R}\Big(E\Big[\int_0^T \|\mu^n(s, X^n, \nu^n_s)\|^{p'}_E \mathbb{I}_{X^n_0 \in K} ds\Big]\\
&+ E\Big[\int_0^T \|Y^n_s\|^{p'}_E \mathbb{I}_{X^n_0 \in K} ds\Big]\Big).
\end{aligned}
$$

By virtue of (4.4), (A3) and Lemma 4.4, there exists an R independent of n such that

$$
P(X^n|_{[0,T]} \in K_R) \geq 1 - \varepsilon,
$$

which implies the tightness of the family $\{X^n|_{[0,T]} : n \in \mathbb{N}\}$. Consequently, the family $\{X^n : n \in \mathbb{N}\}$ is tight, too.

Step 4: The cylindrical martingale problem. By Step 3, we can extract a weakly convergent subsequence from the family $\{X^n : n \in \mathbb{N}\}$. For simplicity, we ignore this subsequence in our notation and assume that $X^n \to X$ weakly. With little abuse of notation, we write P^X_t for the law of X_t.

We now study the martingale property of a certain class of test processes. Take $g \in C^2_c(\mathbb{R})$, $f \in C_c(\mathbb{R})$ and $y^* \in D(A^*)$. The coordinate process on $C(\mathbb{R}_+, E)$ is denoted by X. We define

$$
\mathsf{Z} \triangleq f(\|\mathsf{X}_0\|_E)\Big(g(\langle \mathsf{X}, y^*\rangle_E) - g(\langle \mathsf{X}_0, y^*\rangle_E) - \int_0^\cdot \mathscr{L}(s)ds\Big),
$$

where

$$
\begin{aligned}
\mathscr{L}(s) \triangleq{}& \big(\langle \mathsf{X}_s, A^* y^*\rangle_E + \langle \mu(s, \mathsf{X}_s, P^X_s), y^*\rangle_E\big) g'(\langle \mathsf{X}_s, y^*\rangle_E)\\
&+ \tfrac{1}{2}\|\sigma^*(s, \mathsf{X}_s, P^X_s) y^*\|^2_H g''(\langle \mathsf{X}_s, y^*\rangle_E).
\end{aligned}
$$

In the following we show that Z is a $P \circ X^{-1}$-martingale (for the natural filtration of X).

For each $n \in \mathbb{N}$ we write $\sigma^{n,*}$ for the adjoint of σ^n and we define

$$
Z^n \triangleq f(\|X^n_0\|_E)\Big(g(\langle X^n, y^*\rangle_E) - g(\langle X^n_0, y^*\rangle_E) - \int_0^\cdot \mathscr{L}^n(s)ds\Big),
$$

where

$$
\begin{aligned}
\mathscr{L}^n(s) \triangleq{}& \big(\langle X^n_s, A^* y^*\rangle_E + \langle \mu^n(s, X^n, \nu^n_s), y^*\rangle_E\big) g'(\langle X^n_s, y^*\rangle_E)\\
&+ \tfrac{1}{2}\|\sigma^{n,*}(s, X^n, \nu^n_s) y^*\|^2_H g''(\langle X^n_s, y^*\rangle_E).
\end{aligned}
$$

Recall the mild dynamics of X^n as given in Lemma 4.3. Thanks to [28, Theorem 13], where we use the second part of (A3), we can pass to its analytically weak dynamics, i.e. we have

$$
\begin{aligned}
\langle X^n, y^*\rangle_E ={}& \langle X^n_0, y^*\rangle_E + \int_0^\cdot \big(\langle X^n_s, A^* y^*\rangle_E + \langle \mu^n(s, X^n, \nu^n_s), y^*\rangle_E\big)ds\\
&+ \int_0^\cdot \langle \sigma^{n,*}(s, X^n, \nu^n_s) y^*, dW_s\rangle_H.
\end{aligned}
$$

By virtue of these dynamics, Itô's formula yields that

$$
Z^n = f(\|X^n_0\|_E) \int_0^\cdot g'(\langle X^n_s, y^*\rangle_E)\langle \sigma^{n,*}(s, X^n, \nu^n_s) y^*, dW_s\rangle_H.
$$

In particular, Z^n is a local martingale. Let $m > 0$ be such that $f(x) = 0$ for $|x| \geq m$. We denote the quadratic variation process by $[\cdot, \cdot]$ and we deduce from (A3) that for all $T > 0$

$$\begin{aligned} E\big[[Z^n, Z^n]_T\big] &= E\Big[\int_0^T |f(\|X_0^n\|_E) g'(\langle X_s^n, y^*\rangle_E)|^2 \|\sigma^{n,*}(s, X^n, \nu_s^n) y^*\|_H^2 ds\Big] \\ &\leq C \int_0^T E\big[\|\sigma^n(s, X^n, \nu_s^n)\|^2_{L(H,E)} \mathbb{I}_{\|X_0^n\|_E \leq m}\big] ds \\ &\leq C \int_0^T E\big[\|\sigma^n(s, X^n, \nu_s^n)\|^{p'}_{L(H,E)} \mathbb{I}_{\|X_0^n\|_E \leq m}\big]^{2/p'} ds \\ &\leq C\Big(1 + \sup_{s \in [0,T]} E\big[\|X_s^n\|_E^{p'} \mathbb{I}_{\|X_0^n\|_E \leq m}\big]^{2/p'}\Big). \end{aligned} \tag{4.8}$$

Hence, Z^n is even a true martingale. Furthermore, using this estimate in combination with Burkholder's inequality, we obtain, again for all $T > 0$, that

$$\sup_{n \in \mathbb{N}} \sup_{s \in [0,T]} E\big[|Z_s^n|^2\big] \leq C\Big(1 + \sup_{n \in \mathbb{N}} \sup_{s \in [0,T]} E\big[\|X_s^n\|_E^{p'} \mathbb{I}_{\|X_0^n\| \leq m}\big]^{2/p'}\Big).$$

As the r.h.s. is finite thanks to Lemma 4.4, the family $\{Z_t^n : t \in [0, T], n \in \mathbb{N}\}$ is uniformly integrable for every $T > 0$. By assumption (A2), the map $\omega \mapsto \mathsf{Z}_t(\omega)$ is continuous (on $C(\mathbb{R}_+, E)$ endowed by the local uniform topology) for every $t > 0$. Thanks to [20, Proposition IX.1.12], we can conclude that Z is a martingale once we show that

$$P(|Z_t^n - \mathsf{Z}_t(X^n)| \geq \varepsilon) \to 0, \quad \varepsilon, t > 0. \tag{4.9}$$

The probability $P(|Z_t^n - \mathsf{Z}_t(X^n)| \geq \varepsilon)$ only depends on the law of X^n. Thus, to establish (4.9) we can use Skorokhod's coupling theorem and assume that X and $X^1, X^2, \dots$ are defined on the same probability space and that $X^n \to X$ almost surely (in the local uniform topology). Evidently, we have

$$|Z^n - \mathsf{Z}(X^n)| \leq \|f\|_\infty \int_0^{\cdot} |\mathscr{L}^n(s) - \mathscr{L}(X^n)(s)| ds.$$

In the following we show that a.s. for all $t > 0$

$$\int_0^t |\mathscr{L}^n(s) - \mathscr{L}(X^n)(s)| ds \to 0. \tag{4.10}$$

We have a.s. for every $t > 0$

$$\begin{aligned} \|X^n_{\lfloor t2^n \rfloor 2^{-n}} - X_t\|_E &\leq \|X^n_{\lfloor t2^n \rfloor 2^{-n}} - X_{\lfloor t2^n \rfloor 2^{-n}}\|_E + \|X_{\lfloor t2^n \rfloor 2^{-n}} - X_t\|_E \\ &\leq \sup_{s \in [0,t]} \|X_s^n - X_s\|_E + \|X_{\lfloor t2^n \rfloor 2^{-n}} - X_t\|_E \to 0. \end{aligned}$$

Hence, for every $\phi \in C_b(E)$ and $t > 0$, the dominated convergence theorem yields that

$$\int \phi(y) \nu_t^n(dy) = E\big[\phi(X^n_{\lfloor t2^n \rfloor 2^{-n}})\big] \to E\big[\phi(X_t)\big] = \int \phi(y) P_t^X(dy),$$

which implies that $\nu_t^n \to P_t^X$ weakly. Take $\omega, \omega^1, \omega^2, \dots \in C(\mathbb{R}_+, E)$ such that $\omega^n \to \omega$ in the local uniform topology and fix $t > 0$, $T > t$ and $\varepsilon > 0$. By the Arzelà–Ascoli theorem ([21, Theorem A.5.2]), there exists a compact set $K \subset E$ such that $\omega^n(s) \in K$ for all $s \in [0, T]$ and $n \in \mathbb{N}$, and

$$\sup_{n \in \mathbb{N}} \sup \big\{\|\omega^n(s) - \omega^n(r)\|_E : s, r \in [0, T], |s - r| \leq h\big\} \to 0 \text{ as } h \searrow 0. \tag{4.11}$$

Let d_c be a metric which induces the topology on $M_c(E)$, i.e. the topology of convergence in distribution. By (A2), the function $\langle \mu(t, \cdot, \cdot), y^* \rangle_E$ is uniformly continuous on the compact set $G \triangleq K \times \{P^X_t, \nu^1_t, \nu^2_t, \dots\}$ with respect to the product metric $(x, y) \times (\nu, \eta) \mapsto \|x - y\|_E + d_c(\nu, \eta)$. Thus, there exists a $\delta > 0$ such that

$$\begin{aligned} &(x, \nu), (y, \eta) \in G, \quad \|x - y\|_E + d_c(\nu, \eta) \le \delta \\ &\quad \Longrightarrow |\langle \mu(t, x, \nu) - \mu(t, y, \eta), y^* \rangle_E| \le \varepsilon. \end{aligned}$$

As $\nu^n_t \to P^X_t$ weakly, there exists an $N \in \mathbb{N}$ such that $d_c(\nu^n_t, P^X_t) \le \frac{\delta}{2}$ for all $n \ge N$. Furthermore, thanks to (4.11), there exists an $M \in \mathbb{N}$ such that for all $n \ge M$

$$\|\omega^n(\lfloor t2^n \rfloor 2^{-n}) - \omega^n(t)\|_E \le \tfrac{\delta}{2}.$$

Thus, for all $n \ge N \vee M$ we have

$$|\langle \mu(t, \omega^n(\lfloor t2^n \rfloor 2^{-n}), \nu^n_t) - \mu(t, \omega^n(t), P^X_t), y^* \rangle_E| \le \varepsilon.$$

Using that a.s. $X^n \to X$ in the local uniform topology, and recalling the definition of μ^n as given in (4.6), we conclude that a.s. for all $s > 0$

$$|\langle \mu^n(s, X^n, \nu^n_s) - \mu(s, X^n_s, P^X_s), y^* \rangle_E| \to 0.$$

By the same reasoning, using again (A2) and (4.6), we obtain that a.s. for all $s > 0$

$$|\|\sigma^{n,*}(s, X^n, \nu^n_s) y^*\|^2_H - \|\sigma^*(s, X^n_s, P^X_s) y^*\|^2_H| \to 0.$$

Consequently, as g' and g'' are bounded, we conclude that a.s. for all $s > 0$

$$|\mathscr{L}^n(s) - \mathscr{L}(X^n)(s)| \to 0. \tag{4.12}$$

Finally, it remains to deduce (4.10) from this observation and the dominated convergence theorem. By (A3), more precisely (2.1), we have for all $n \in \mathbb{N}$, $T > 0$ and $s \in [0, T]$

$$|\mathscr{L}^n(s) - \mathscr{L}(X^n)(s)| \le C\big(\|g'\|_\infty \|y^*\|_E + \tfrac{1}{2}\|g''\|_\infty \|y^*\|^2_E\big)\Big(1 + \sup_{m \in \mathbb{N}} \sup_{t \in [0,T]} \|X^m_t\|^2_E\Big), \tag{4.13}$$

where the r.h.s. is a.s. finite once again by the Arzelà–Ascoli theorem and the fact that $X^n \to X$ a.s. in the local uniform topology. Thus, we deduce from (4.12) and the dominated convergence theorem (applied to the Lebesgue integral) that (4.10) holds a.s. for all $t > 0$. This implies (4.9) and hence, we conclude that Z is a $P \circ X^{-1}$-martingale.

Step 5: Conclusion. We are in the position to complete the proof. Take $y^* \in D(A^*)$ and define

$$\begin{aligned} U_N &\triangleq \inf(t \in \mathbb{R}_+ \colon |\langle \mathsf{X}_t, y^* \rangle_E| \ge N), \\ S_N &\triangleq \begin{cases} 0, & \|X_0\|_E > N, \\ +\infty, & \|X_0\|_E \le N, \end{cases} \\ T_N &\triangleq U_N \wedge S_N, \end{aligned}$$

for $N \in \mathbb{N}$. Clearly, by the continuous paths of X, T_N is a stopping time for the filtration generated by X. Using Step 4 with $g \in C^2_c(\mathbb{R})$ such that $g(x) = x$ for all $|x| \le N$ and $f \in C_c(\mathbb{R})$ such that $f(x) = 1$ for all $|x| \le N$ yields that the process

$$\langle \mathsf{X}_{\cdot \wedge T_N}, y^* \rangle_E - \langle \mathsf{X}_0, y^* \rangle_E - \int_0^{\cdot \wedge T_N} \big(\langle \mathsf{X}_s, A^* y^* \rangle_E + \langle \mu(s, \mathsf{X}_s, P^X_s), y^* \rangle_E\big) ds$$

is a $P \circ X^{-1}$-martingale. Consequently, since $T_N \nearrow \infty$ as $N \to \infty$, the process

$$\langle \mathsf{X}, y^* \rangle_E - \langle \mathsf{X}_0, y^* \rangle_E - \int_0^{\cdot} \big(\langle \mathsf{X}_s, A^* y^* \rangle_E + \langle \mu(s, \mathsf{X}_s, P_s^X), y^* \rangle_E \big) ds$$

is a local $P \circ X^{-1}$-martingale. By virtue of the proof of [22, Proposition 5.4.6], using in addition the same argument with $g \in C_c^2(\mathbb{R})$ such that $g(x) = x^2$ for $|x| \le N$ yields that its quadratic variation process is given by $\int_0^{\cdot} \|\sigma^*(s, \mathsf{X}_s, P_s^X) y^*\|_H^2 ds$. Recall that $y^* \in D(A^*)$ was arbitrary. As $D(A^*)$ separates points, the representation theorem [29, Theorem 3.1] yields the existence of a standard cylindrical Brownian motion B defined on a standard extension of $(C(\mathbb{R}_+, E), \mathscr{B}(C(\mathbb{R}_+, E)), P \circ X^{-1})$ with the canonical filtration generated by X such that

$$\begin{aligned} &\langle \mathsf{X}, y^* \rangle_E - \langle \mathsf{X}_0, y^* \rangle_E - \int_0^{\cdot} \big(\langle \mathsf{X}_s, A^* y^* \rangle_E + \langle \mu(s, \mathsf{X}_s, P_s^X), y^* \rangle_E \big) ds \\ &= \int_0^{\cdot} \langle \sigma^*(s, \mathsf{X}_s, P_s^X) y^*, dB_s \rangle_H \end{aligned}$$

for all $y^* \in D(A^*)$.

Finally, noting that the initial value X_0 is distributed according to η under the probability measure $P \circ X^{-1}$, we conclude that X is an analytically weak solution process to the MKV SPDE with coefficients (A, μ, σ, η). By [28, Theorem 13] it is also a mild solution process and the existence of a martingale solution is proved. □

5 Proof of Theorem 2.8

Theorem 2.8 can be proved similar to Theorem 2.5. In the following, we outline the necessary changes, using the notation from Sect. 4. In case $\eta \in M_w^{p'}(E)$, Lemma 4.4 holds for $K = E$ and therefore, $\nu_t^n \to P_t^X$ in $M_w^{p^\circ}(E)$ by [5, Theorem 5.5] as $p^\circ < p'$. Let $X^1, X^2, \ldots$ be the approximation sequence as in Step 1 from the proof of Theorem 2.5 and suppose that X is a weak accumulation point. To keep the notation simple, assume that $X^n \to X$ weakly. Then,

$$E\Big[\sup_{s \in [0,T]} \|X_s\|_E^{p'} \Big] \le \liminf_{n \to \infty} E\Big[\sup_{s \in [0,T]} \|X_s^n\|_E^{p'} \Big] \le \sup_{n \in \mathbb{N}} E\Big[\sup_{s \in [0,T]} \|X_s^n\|_E^{p'} \Big] < \infty. \tag{5.1}$$

In Step 4 from the proof of Theorem 2.5, the processes Z and Z^n can be defined with $f \equiv 1$ and the estimate (4.8) should be replaced by

$$E\big[[Z^n, Z^n]_T \big] \le C\Big(1 + E\Big[\sup_{s \in [0,T]} \|X_s^n\|_E^{p'} \Big]^{2/p'} \Big),$$

where we use (2.7) from (A5). Here, the r.h.s. is finite as Lemma 4.4 holds for $K = E$. Recalling that $\nu_t^n \to P_t^X$ in $M_w^{p^\circ}(E)$ and taking the continuity condition (A4) into account, the convergence from (4.12) follows as in the proof of Theorem 2.8. Moreover, using again (2.7) from (A5) instead of (2.1) from (A3), and taking (5.1) into account, the inequality (4.13) can be replaced by

$$\begin{aligned} |\mathscr{L}^n(s) - \mathscr{L}(X^n)(s)| &\le C\big(\|g'\|_\infty \|y^*\|_E + \tfrac{1}{2} \|g''\|_\infty \|y^*\|_E^2 \big) \\ &\Big(1 + \sup_{m \in \mathbb{N}} E\Big[\sup_{t \in [0,T]} \|X_t^m\|^{p'} \Big]^{2/p'} + \sup_{m \in \mathbb{N}} \sup_{t \in [0,T]} \|X_t^m\|_E^2 \Big), \end{aligned}$$

where the expectation term is finite as Lemma 4.4 holds for $K = E$. By virtue of this bound, (4.9) follows from (4.12) and the dominated convergence theorem. Step 5 requires no modification, but we note that it is not necessary to introduce the stopping time S_N. Finally, we explain why $(t \mapsto P^X_t) \in C(\mathbb{R}_+, M^{p'}_w(E))$. Since a.s. $\|X_s\|^{p'}_E \to \|X_t\|^{p'}_E$ for $s \to t$ by the continuous paths of X, (5.1) and the dominated convergence theorem show that $(t \mapsto P^X_t) \in C(\mathbb{R}_+, M^{p'}_w(E))$. □

6 Proof of Theorem 2.11

The strategy of proof is borrowed from [13, Theorem 3.1]. Let $\gamma \in C(\mathbb{R}_+, M^p_w(E))$. We now define solutions to a certain class of classical SPDEs.

Definition 6.1 We call a triplet $(\mathbb{B}, W, X)$ a *martingale solution* to the SPDE with coefficients $(A, \mu, \sigma, \gamma, \eta)$ if $\mathbb{B}$ is a filtered probability space with right-continuous and complete filtration which supports a standard cylindrical Brownian motion W and a continuous E-valued adapted process X such that the following hold:

(i) $X_0 \sim \eta$.

(ii) Almost surely for all $t \in \mathbb{R}_+$

$$\int_0^t \|S_{t-s}\mu(s, X_s, \gamma(s))\|_E ds + \int_0^t \|S_{t-s}\sigma(s, X_s, \gamma(s))\|^2_{L_2(H,E)} ds < \infty.$$

(iii) Almost surely for all $t \in \mathbb{R}_+$

$$X_t = S_t X_0 + \int_0^t S_{t-s}\mu(s, X_s, \gamma(s))ds + \int_0^t S_{t-s}\sigma(s, X_s, \gamma(s))dW_s.$$

The proof of the following lemma is postponed to the end of this section.

Lemma 6.2 *For $i = 1, 2$, let $(\mathbb{B}^i, W^i, X^i)$ be a martingale solution to the SPDE with coefficients $(A, \mu, \sigma, \gamma^i, \eta)$ and let $u^i(t)$ be the law of X^i_t for $t \in \mathbb{R}_+$. Furthermore, assume that $u^i \in C(\mathbb{R}_+, M^p_w(E))$ for $i = 1, 2$. Then, for every $T > 0$ and $m > 0$ such that*

$$\max_{i=1,2} \sup_{s \in [0,T]} \|\gamma^i(s)\|_p \le m,$$

there exists a constant $C = C(p, T, S, m) > 0$ such that

$$|\mathsf{w}_p(u^1(s), u^2(s))|^p \le C \int_0^s |\kappa(\mathsf{w}_p(\gamma^1(r), \gamma^2(r)))|^p dr, \qquad s \in [0, T],$$

where $\kappa = \kappa_{T,m}$ is as in (U1) or (U2), respectively.

In the following we prove Theorem 2.11. For contradiction, let $(\mathbb{B}^i, W^i, X^i), i = 1, 2$, be two p-martingale solutions to the MKV SPDE (A, μ, σ, η) such that $P^1 \circ (X^1)^{-1} \neq P^2 \circ (X^2)^{-1}$. We define $u^i(t) \triangleq P^i \circ (X^i_t)^{-1}$ for $t \in \mathbb{R}_+$ and $i = 1, 2$. By definition of a p-martingale solution, we have $u^i \in C(\mathbb{R}_+, M^p_w(E))$ for $i = 1, 2$.

Lemma 6.3 $\mathfrak{s} \triangleq \inf(t \in \mathbb{R}_+ : u^1(t) \neq u^2(t)) < \infty$.

Proof For contradiction, assume that $\mathfrak{s} = \infty$, i.e. $u^1 = u^2 \equiv u$. Then, (B^i, W^i, X^i) are both martingale solutions to the SPDE $(A, \mu, \sigma, u, \eta)$. Thanks to Theorem A.1 (under $p > 1/\alpha$ and (U1)) or [9, Theorem 7.2] (under $p \ge 2$ and (U2)), this SPDE has a pathwise unique solution. By the Yamada–Watanabe theorem [28, Theorem 2], the SPDE also satisfies uniqueness in law. This contradicts $P^1 \circ (X^1)^{-1} \neq P^2 \circ (X^2)^{-1}$ and the claim follows. □

Remark 6.4 The conclusion of Lemma 6.3 might be compared to [12, Theorem 4.4.2] which shows that two solutions of certain (time-homogeneous) martingale problems have the same law already if they have the same one-dimensional distributions.

Lemma 6.2 yields that

$$|\mathsf{w}_p(u^1(s), u^2(s))|^p \le C \int_0^s |\kappa(\mathsf{w}_p(u^1(r), u^2(r)))|^p dr, \qquad s \in [0, \mathfrak{s}+1],$$

where κ depends on $\mathfrak{s}$, u^1 and u^2. Hence, recalling the properties of κ, Bihari's lemma ([26, Lemma 5.2.8]) implies that $u^1 = u^2$ on $[0, \mathfrak{s}+1]$. However, as this contradicts the definition of $\mathfrak{s}$, we can conclude that the MKV SPDE satisfies p-uniqueness in law.

Finally, let $(\mathbb{B}, W, X)$ and $(\mathbb{B}, W, Y)$ be two p-martingale solutions to the MKV SPDE (A, μ, σ, η). By the previous part of the proof, we know that X and Y have the same law. We write $\gamma(t) \triangleq P^X_t = P^Y_t$ for $t \in \mathbb{R}_+$. Now, $(\mathbb{B}, W, X)$ and $(\mathbb{B}, W, Y)$ both are martingale solutions to the SPDE $(A, \mu, \sigma, \gamma, \eta)$. Consequently, as this SPDE satisfies pathwise uniqueness by Theorem A.1 or [9, Theorem 7.2], we have a.s. $X = Y$. The proof of Theorem 2.11 is complete. □

It remains to prove Lemma 6.2.

Proof of Lemma 6.2 Thanks to Theorem A.1 or [9, Theorem 7.2], the SPDE can be realized on any driving system, and, thanks to the Yamada–Watanabe theorem [28, Theorem 2], it satisfies uniqueness in law. Consequently, we can w.l.o.g. assume that $(\mathbb{B}^1, W^1) = (\mathbb{B}^2, W^2) \equiv (\mathbb{B}, W)$. Take $T, m > 0$ such that $\sup_{s\in[0,T]} \|\gamma^i(s)\|_p \le m$. For $t \in [0, T]$ we have

$$\begin{aligned} E\big[\|X^1_s - X^2_s\|^p_E\big] &\le 2^{p-1} E\Big[\Big\| \int_0^s S_{s-r}(\mu(r, X^1_r, \gamma^1(r)) - \mu(r, X^2_r, \gamma^2(r)))dr \Big\|^p_E\Big] \\ &\quad + 2^{p-1} E\Big[\Big\| \int_0^s S_{s-r}(\sigma(r, X^1_r, \gamma^1(r)) - \sigma(r, X^2_r, \gamma^2(r)))dW_r \Big\|^p_E\Big]. \end{aligned}$$

We now estimate each of the latter terms separately, starting with the second term. In case $p > 1/\alpha$ and (U1) hold, using Lemma 4.2 yields that

$$\begin{aligned} &E\Big[\Big\| \int_0^t S_{t-r}(\sigma(r, X^1_r, \gamma^1(r)) - \sigma(r, X^2_r, \gamma^2(r)))dW_r \Big\|^p_E\Big] \\ &\le C E\Big[\int_0^t \big(\|X^1_r - X^2_r\|^p_E + |\kappa(\mathsf{w}_p(\gamma^2(r), \gamma^2(r)))|^p\big)dr\Big]. \end{aligned}$$

Using Burkholder's inequality instead of Lemma 4.2, the same inequality holds in case $p \ge 2$ and (U2) hold. Using Hölder's inequlity and (U1), we also get that

$$\begin{aligned} &E\Big[\Big\| \int_0^t S_{t-r}(\mu(r, X^1_r, \gamma^1(r)) - \mu(r, X^2_r, \gamma^2(r)))dr \Big\|^p_E\Big] \\ &\le E\Big[\Big(\int_0^t \|S_{t-r}(\mu(r, X^1_r, \gamma^1(r)) - \mu(r, X^2_r, \gamma^2(r)))\|_E dr\Big)^p\Big] \\ &\le E\Big[\Big(\int_0^t \mathfrak{g}(t-r)\big(\|X^1_r - X^2_r\|_E + \kappa(\mathsf{w}(\gamma^1(r), \gamma^2(r)))\big)dr\Big)^p\Big] \\ &\le \Big(\int_0^T [\mathfrak{g}(s)]^{p/(p-1)} ds\Big)^{p-1} E\Big[\int_0^t \big(\|X^1_r - X^2_r\|_E + \kappa(\mathsf{w}(\gamma^1(r), \gamma^2(r)))\big)^p dr\Big] \\ &\le C E\Big[\int_0^t \big(\|X^1_r - X^2_r\|^p_E + |\kappa(\mathsf{w}_p(\gamma^1(r), \gamma^2(r)))|^p\big)dr\Big], \end{aligned}$$

with $\mathfrak{g} = \mathfrak{g}_{T,m}$ as in (U1). A similar computation gives the inequality

$$E\Big[\Big\| \int_0^t S_{t-r}(\mu(r, X_r^1, \gamma^1(r)) - \mu(r, X_r^2, \gamma^2(r)))dr \Big\|_E^p\Big] \\ \leq CE\Big[\int_0^t \big(\|X_r^1 - X_r^2\|_E^p + |\kappa(\mathsf{w}_p(\gamma^1(r), \gamma^2(r)))|^p\big)dr \Big]$$

under (U2). Putting these estimates together, we obtain for all $t \in [0, T]$

$$E\big[\|X_t^1 - X_t^2\|_E^p\big] \leq C\Big(\int_0^t E\big[\|X_s^1 - X_s^2\|_E^p\big]ds + \int_0^t |\kappa(\mathsf{w}_p(\gamma^1(r), \gamma^2(r)))|^p dr \Big).$$

As $t \mapsto E[\|X_t^1 - X_t^2\|_E^p]$ is locally bounded, we deduce from Gronwall's lemma that

$$E\big[\|X_t^1 - X_t^2\|_E^p\big] \leq Ce^{CT} \int_0^t |\kappa(\mathsf{w}_p(\gamma^1(r), \gamma^2(r)))|^p dr$$

for all $t \in [0, T]$. Finally, the claim follows from the observation that

$$|\mathsf{w}_p(u^1(t), u^2(t))|^p \leq E\big[\|X_t^1 - X_t^2\|_E^p\big], \quad t \in [0, T].$$

The proof is complete. □

7 Proof of Theorem 3.4

Throughout this section we fix a finite time horizon $T > 0$. Except in the final step, all processes in the following are meant to be defined on the finite time interval $[0, T]$.
Step 1: Tightness in $M_c(M_c(C([0, T], E)))$. We adapt the argument from Step 3 in the proof of Theorem 2.5. In the first part of this step we establish a uniform moment bound. Recall that

$$L^N(x) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \quad x = (x_1, \dots, x_N) \in E^{\otimes N}.$$

By virtue of (A5), for all $0 < s, t \leq T$, $x = (x_1, \dots, x_N) \in E^{\otimes N}$ and $i = 1, \dots, N$ we have

$$\begin{aligned} \|\mu(t, x_i, L^N(x))\|_E^{p'} &\leq C\Big(1 + \|x_i\|_E^{p'} + \|L^N(x)\|_{p'}^{p'}\Big) \\ &= C\Big(1 + \|x_i\|_E^{p'} + \frac{1}{N}\sum_{j=1}^N \|x_j\|_E^{p'}\Big). \end{aligned} \tag{7.1}$$

Let

$$T_m \triangleq \inf\Big(t \in [0, T] \colon \frac{1}{N} \sum_{i=1}^N \|X_t^{N,i}\|_E^{p'} \geq m\Big), \quad m > 0.$$

By virtue of (A5) and Lemma 4.2, we obtain that

$$\begin{aligned}
&E\Big[\sup_{s\in[0,t\wedge T_m]}\Big\|\int_0^s S_{s-r}\sigma(r, X^{N,i}_r, \mathscr{X}^N_r)dW^i_r\Big\|^{p'}_E\Big] \\
&\quad\le E\Big[\sup_{s\in[0,t]}\Big\|\int_0^s S_{s-r}\sigma(r, X^{N,i}_{r\wedge T_m}, L^N(X^{N,1}_{r\wedge T_m}, \dots, X^{N,N}_{r\wedge T_m}))dW^i_r\Big\|^{p'}_E\Big] \qquad (7.2)\\
&\quad\le CE\Big[\int_0^t \Big(1 + \|X^{N,i}_{s\wedge T_m}\|^{p'}_E + \frac{1}{N}\sum_{j=1}^N \|X^{N,j}_{s\wedge T_m}\|^{p'}_E\Big)ds\Big].
\end{aligned}$$

Now, thanks to (7.1), (7.2) and the uniform moment bound on the initial values from (I), for all $t \in [0, T]$ we obtain

$$\begin{aligned}
E\Big[\sup_{s\in[0,t\wedge T_m]}\frac{1}{N}\sum_{i=1}^N \|X^{N,i}_s\|^{p'}_E\Big] &\le \frac{1}{N}\sum_{i=1}^N E\Big[\sup_{s\in[0,t\wedge T_m]}\|X^{N,i}_s\|^{p'}_E\Big]\\
&\le \frac{1}{N}\sum_{i=1}^N C\Big(1 + E\Big[\int_0^t \Big(\|X^{N,i}_{s\wedge T_m}\|^{p'}_E + \frac{1}{N}\sum_{j=1}^N \|X^{N,j}_{s\wedge T_m}\|^{p'}_E\Big)ds\Big]\Big)\\
&= C\Big(1 + 2E\Big[\int_0^t \frac{1}{N}\sum_{i=1}^N \|X^{N,i}_{s\wedge T_m}\|^{p'}_E ds\Big]\Big)\\
&\le C\Big(1 + \int_0^t E\Big[\sup_{r\in[0,s\wedge T_m]}\frac{1}{N}\sum_{i=1}^N \|X^{N,i}_r\|^{p'}_E\Big]ds\Big).
\end{aligned}$$

As, by definition of T_m, for all $t \in [0, T]$

$$\begin{aligned}
E\Big[\sup_{s\in[0,t\wedge T_m]}\frac{1}{N}\sum_{i=1}^N \|X^{N,i}_s\|^{p'}_E\Big] &\le \frac{1}{N}\sum_{i=1}^N E\big[\|X^{N,i}_0\|^{p'}_E\big] + m\\
&\le \sup_{n\in\mathbb{N}}\int \|\mathsf{X}^N_1(x)\|^{p'}_E \eta^n(dx) + m,
\end{aligned}$$

we deduce from Gronwall's lemma that

$$E\Big[\sup_{s\in[0,T\wedge T_m]}\frac{1}{N}\sum_{i=1}^N \|X^{N,i}_s\|^{p'}_E\Big] \le C.$$

Hence, letting $m \to \infty$ and using Fatou's lemma, we get that

$$E\Big[\sup_{s\in[0,T]}\|X^{N,1}_s\|^{p'}_E\Big] \le E\Big[\sup_{s\in[0,T]}\sum_{i=1}^N \|X^{N,i}_s\|^{p'}_E\Big] \le CN.$$

Next, using that $X^{N,i}$ and $X^{N,j}$ have the same law for all $i, j \le N$ by the symmetry part of (I) and the uniqueness part of (EUP), and the uniform moment bound on the initial values from (I), arguing as above, we get for all $t \in [0, T]$ that

$$E\Big[\sup_{s\in[0,t]}\|X^{N,1}_s\|^{p'}_E\Big] \le C\Big(1 + \int_0^t E\Big[\sup_{r\in[0,s]}\|X^{N,1}_r\|^{p'}_E\Big]ds\Big).$$

Thus, as $E\big[\sup_{s\in[0,T]}\|X_s^{N,1}\|_E^{p'}\big] < \infty$, we can apply Gronwall's lemma again and obtain that

$$\sup_{N\in\mathbb{N}} E\Big[\sup_{s\in[0,T]}\|X_s^{N,1}\|_E^{p'}\Big] \le C. \tag{7.3}$$

We are in the position to deduce tightness. Fix $\varepsilon > 0$. As the empirical distributions $\mathscr{X}_0^N$ converge weakly as $N \to \infty$ by assumption (I), [33, Proposition I.2.2] yields that the family $\{X_0^{N,1} : N \in \mathbb{N}\}$ is tight. Thus, there exists a compact set $K \subset E$ such that

$$\sup_{N\in\mathbb{N}} P\big(X_0^{N,1} \notin K\big) \le \frac{\varepsilon}{2}. \tag{7.4}$$

Recalling the notation from Step 0 in the proof of Theorem 2.5, (4.3) yields that

$$X^{N,1} = SX_0 + R_1\mu^N + \tfrac{\sin(\pi\alpha)}{\pi} R_\alpha Y^N,$$

where $\mu^N \equiv (s \mapsto \mu(s, X_s^{N,1}, \mathscr{X}_s^N))$ and

$$Y_t^N \triangleq \int_0^t (t-s)^{-\alpha} S_{t-s}\sigma(s, X^{N,1}, \mathscr{X}_s^N) dW_s^1, \quad t \in [0,T].$$

Now, we define

$$\begin{aligned} K_R \triangleq \Big\{\omega \in\ & C([0,T],E) : \omega = Sx_0 + R_1\psi + \tfrac{\sin(\pi\alpha)}{\pi} R_\alpha\phi, \\ & x_0 \in K, \phi, \psi \in L^{p'}([0,T],E) \text{ with } \int_0^T \|\psi(s)\|_E^{p'} ds \le R,\ \int_0^T \|\phi(s)\|_E^{p'} ds \le R\Big\}. \end{aligned}$$

Thanks to Lemma 4.1 and the compactness of the semigroup S, the set K_R is relatively compact in $C([0,T],E)$. Using (4.4), (A5) and the assumption that $X^{N,i}$ and $X^{N,j}$ have the same law for all $i, j \le N$, we estimate

$$E\Big[\int_0^T \|Y_s^N\|_E^{p'} ds\Big] \le C\Big(1 + \sup_{N\in\mathbb{N}}\sup_{s\in[0,T]} E\big[\|X_s^{N,1}\|_E^{p'}\big]\Big). \tag{7.5}$$

Similarly, thanks to (7.1), we obtain that

$$E\Big[\int_0^T \|\mu(s, X_s^{N,1}, \mathscr{X}_s^N)\|_E^{p'} ds\Big] \le C\Big(1 + \sup_{N\in\mathbb{N}}\sup_{s\in[0,T]} E\big[\|X_s^{N,1}\|_E^{p'}\big]\Big). \tag{7.6}$$

In summary, using Chebyshev's inequality and (7.4), (7.5) and (7.6), for every $\varepsilon > 0$ we can take $R = R(\varepsilon) > 0$ large enough such that

$$P\big(X^{N,1} \in K_R\big) \ge 1 - \varepsilon.$$

Consequently, the family $\{X^{N,1} : N \in \mathbb{N}\}$ is tight.

Step 2: Tightness in $M_c(M_w^{p^\circ}(C([0,T],E)))$. Let d_T be the uniform metric on $C([0,T],E)$, i.e.

$$d_T(\omega,\alpha) \triangleq \sup_{s\in[0,T]} \|\omega(s) - \alpha(s)\|_E, \quad \omega, \alpha \in C([0,T],E).$$

In the following we consider $\mathscr{X}^1, \mathscr{X}^2, \ldots$ as random variables with values in $M_w^{p^\circ}(C([0,T],E))$. Next, we show that $\{\mathscr{X}^N : N \in \mathbb{N}\}$ is tight.

Fix $\varepsilon > 0$ and define

$$a_n \triangleq n^{1/(p'-p^\circ)} 2^{n/(p'-p^\circ)}, \qquad b_n \triangleq \varepsilon n \Big/ \Big[\sup_{m \in \mathbb{N}} E\Big[\sup_{s \in [0,T]} \|X^{m,1}_s\|^{p'}_E \Big] \vee 1 \Big],$$

and

$$K_\varepsilon \triangleq \bigcap_{n \in \mathbb{N}} \Big\{ \nu \in M^{p^\circ}_w(C([0,T],E)) \colon \int d_T(\omega,0)^{p^\circ} \mathbb{I}_{d_T(\omega,0) \geq a_n} \nu(d\omega) < \frac{1}{b_n} \Big\}. \tag{7.7}$$

For every $N \in \mathbb{N}$ we obtain

$$\begin{aligned} P\big(\mathscr{X}^N \notin K_\varepsilon\big) &\leq \sum_{n=1}^\infty P\Big(\frac{1}{N} \sum_{i=1}^N d_T(X^{N,i},0)^{p^\circ} \mathbb{I}_{d_T(X^{N,i},0) \geq a_n} \geq \frac{1}{b_n} \Big) \\ &\leq \sum_{n=1}^\infty \frac{b_n}{N} \sum_{i=1}^N E\big[d_T(X^{N,i},0)^{p^\circ} \mathbb{I}_{d_T(X^{N,i},0) \geq a_n} \big] \\ &\leq \sum_{n=1}^\infty \frac{b_n}{a_n^{p'-p^\circ}} E\Big[\sup_{s \in [0,T]} \|X^{N,1}_s\|^{p'}_E \Big] \\ &\leq \varepsilon. \end{aligned}$$

By [33, Proposition I.2.2], Step 1 implies that the family $\{\mathscr{X}^N \colon N \in \mathbb{N}\}$, seen as random variables in $M_c(C([0,T],E))$, is tight. Consequently, as $\{\mathscr{X}^N \colon N \in \mathbb{N}\} \subset M^{p^\circ}_w(C([0,T],E))$, there exists a set $G_\varepsilon \subset M^{p^\circ}_w(C([0,T],E))$ which is relatively compact in $M_c(C([0,T],E))$ such that

$$\sup_{N \in \mathbb{N}} P\big(\mathscr{X}^N \notin G_\varepsilon\big) \leq \varepsilon.$$

Let K_ε be as in (7.7). Then, we have for all $N \in \mathbb{N}$

$$P\big(\mathscr{X}^N \notin G_\varepsilon \cap K_\varepsilon\big) \leq P\big(\mathscr{X}^N \notin G_\varepsilon\big) + P\big(\mathscr{X}^N \notin K_\varepsilon\big) \leq 2\varepsilon.$$

Using that the set $G_\varepsilon \cap K_\varepsilon$ is relatively compact in $M^{p^\circ}_w(C([0,T],E))$ by [5, Corollary 5.6], we can conclude that the family $\{\mathscr{X}^N \colon N \in \mathbb{N}\}$ is tight when seen as random variables with values in $M^{p^\circ}_w(C([0,T],E))$. From now on we assume that $\mathscr{X}^N \to \mathscr{X}$ in $M_c(M^{p^\circ}_w(C([0,T],E)))$, i.e that the laws of $\mathscr{X}^N$, which are considered as elements of $M_c(M^{p^\circ}_w(C([0,T],E)))$, converge weakly to the law of $\mathscr{X} \in M^{p^\circ}_w(C([0,T],E))$. We note that

$$\begin{aligned} E\Big[\int d_T(\omega,0)^{p'} \mathscr{X}(d\omega) \Big] &\leq \liminf_{N \to \infty} E\Big[\int d_T(\omega,0)^{p'} \mathscr{X}^N(d\omega) \Big] \\ &= \liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^N E\Big[\sup_{s \in [0,T]} \|X^{N,i}_s\|^{p'}_E \Big] \\ &= \liminf_{N \to \infty} E\Big[\sup_{s \in [0,T]} \|X^{N,1}_s\|^{p'}_E \Big] \\ &\leq \sup_{N \in \mathbb{N}} E\Big[\sup_{s \in [0,T]} \|X^{N,1}_s\|^{p'}_E \Big] < \infty. \end{aligned} \tag{7.8}$$

Thus, a.s. $\mathscr{X} \in M^{p'}_w(C([0,T],E)) \subset M^{p^\circ}_w(C([0,T],E))$.

Step 3: Convergence of test processes. Take $0 \leq s < t \leq T$, $t_1, \dots, t_m \in [0, s]$, $h_1, \dots, h_m \in C_b(E)$, $g \in C_c^2(\mathbb{R})$ and $y^* \in D(A^*)$. For $(\omega, \nu) \in C([0, T], E) \times M_w^{p^\circ}(C([0, T], E))$ we define

$$M_r(\omega, \nu) \triangleq g(\langle \omega(r), y^* \rangle_E) - g(\langle \omega(0), y^* \rangle_E) - \int_0^r \mathscr{L}_u(\omega, \nu) du, \quad r \in [0, T],$$

where

$$\begin{aligned} \mathscr{L}_u(\omega, \nu) \triangleq \big(\langle \omega(u), A^* y^* \rangle_E + \langle \mu(u, \omega(u), \nu \circ \mathsf{X}_u^{-1}), y^* \rangle_E\big) g'(\langle \omega(u), y^* \rangle_E) \\ + \tfrac{1}{2} \|\sigma^*(u, \omega(u), \nu \circ \mathsf{X}_u^{-1}) y^*\|_H^2 g''(\langle \omega(u), y^* \rangle_E), \end{aligned}$$

and

$$V(\omega, \nu) \triangleq \big(M_t(\omega, \nu) - M_s(\omega, \nu)\big) \prod_{i=1}^m h_i(\omega(t_i)).$$

For all $\nu \in M_w^{p^\circ}(C([0, T], E))$ and $k > 0$ we define

$$Z_k(\nu) \triangleq \int \big[(-k) \vee V(\omega, \nu) \wedge k\big] \nu(d\omega), \quad Z(\nu) \triangleq \liminf_{k \to \infty} Z_k(\nu).$$

If $\nu_n \to \nu$ in $M_w^{p^\circ}(C([0, T], E))$, then $(r \mapsto \nu_n \circ \mathsf{X}_r^{-1}) \to (r \mapsto \nu \circ \mathsf{X}_r^{-1})$ in $C([0, T], M_w^{p^\circ}(E))$, which follows from the inequality

$$\sup_{r \in [0, T]} \mathsf{w}_{p^\circ}(\nu_n \circ \mathsf{X}_r^{-1}, \nu \circ \mathsf{X}_r^{-1}) \leq \mathsf{w}_T^{p^\circ}(\nu_n, \nu),$$

where $\mathsf{w}_T^{p^\circ}$ is the p°-Wasserstein metric on $M_w^{p^\circ}(C([0, T], E))$. By the continuity assumption (C) and the dominated convergence theorem (which is applicable due to the local boundedeness part in (C)), V is continuous. Thus, [3, Theorem 8.10.61] yields that Z_k is continuous. In particular, Z is Borel measurable.

For all $(\omega, \nu) \in C([0, T], E) \times M_w^{p'}(C([0, T], E))$ we have

$$\big|V(\omega, \nu)\big|^{p'/2} \leq C\Big(1 + \int_s^t \big(\|\omega(r)\|_E^{p'} + \|\nu \circ \mathsf{X}_r^{-1}\|_{p'}^{p'}\big) dr\Big), \tag{7.9}$$

where we use (A5). Since a.s. $\mathscr{X}, \mathscr{X}^N \in M_w^{p'}(C([0, T], E))$, we have a.s.

$$\begin{aligned} Z(\mathscr{X}^N) &= \int V(\omega, \mathscr{X}^N) \mathscr{X}^N(d\omega) = \frac{1}{N} \sum_{i=1}^N V(X^{N,i}, \mathscr{X}^N), \\ Z(\mathscr{X}) &= \int V(\omega, \mathscr{X}) \mathscr{X}(d\omega). \end{aligned}$$

Our next aim is to show that a.s. $Z(\mathscr{X}) = 0$. Together with a monotone class argument, this implies that a.a. realizations of $\mathscr{X}$ are p'-solution measures to the MKV SPDE (A, μ, σ, η).

Lemma 7.1 *$E\big[|Z(\mathscr{X}^N)|\big] \to E\big[|Z(\mathscr{X})|\big]$ as $N \to \infty$.*

Proof The triangle inequality yields that

$$\begin{aligned} \big|E\big[|Z(\mathscr{X}^N)|\big] - E\big[|Z(\mathscr{X})|\big]\big| \leq \big|E\big[|Z(\mathscr{X}^N)|\big] - E\big[|Z_k(\mathscr{X}^N)|\big]\big| \\ + \big|E\big[|Z_k(\mathscr{X}^N)|\big] - E\big[|Z_k(\mathscr{X})|\big]\big| \\ + \big|E\big[|Z_k(\mathscr{X})|\big] - E\big[|Z(\mathscr{X})|\big]\big|. \end{aligned} \tag{7.10}$$

Using (7.9), we estimate

$$\begin{aligned} E\big[\big|Z(\mathscr{X}^N) - Z_k(\mathscr{X}^N)\big|\big] &\le \frac{1}{N}\sum_{i=1}^N E\big[\big|V(X^{N,i}, \mathscr{X}^N) - \big[(-k) \vee V(X^{N,i}, \mathscr{X}^N) \wedge k\big]\big|\big] \\ &\le \frac{1}{N}\sum_{i=1}^N E\big[\big|V(X^{N,i}, \mathscr{X}^N)\big|\mathbb{I}_{|V(X^{N,i}, \mathscr{X}^N)|>k}\big] \\ &\le \frac{1}{k^{p'/2-1}}\frac{1}{N}\sum_{i=1}^N E\big[\big|V(X^{N,i}, \mathscr{X}^N)\big|^{p'/2}\big] \\ &\le \frac{C}{k^{p'/2-1}}\Big(1 + \sup_{n\in\mathbb{N}}\sup_{r\in[0,t]} E\big[\|X^{n,1}_r\|^{p'}_E\big]\Big), \end{aligned}$$

where the constant C is independent of k and N. By virtue of (7.3), this bound shows that the first term on the r.h.s. of (7.10) converges to zero as $k \to \infty$ uniformly in N. A similar computation shows the same claim for the final term. Finally, the second term on the r.h.s. of (7.10) convergences to zero as $N \to \infty$ because $Z_k \in C_b(M^{p^\circ}_w(C([0, T], E)))$. The proof is complete. □

Lemma 7.2 $E\big[\big(Z(\mathscr{X}^N)\big)^2\big] \to 0$ *as* $N \to \infty$.

Proof We compute

$$\begin{aligned} E\big[\big(Z(\mathscr{X}^N)\big)^2\big] &= E\Big[\Big(\int V(\omega, \mathscr{X}^N)\mathscr{X}^N(d\omega)\Big)^2\Big] \\ &= \frac{1}{N^2}\sum_{i,j=1}^N E\big[V(X^{N,i}, \mathscr{X}^N)V(X^{N,j}, \mathscr{X}^N)\big]. \end{aligned}$$

Passing to the analytically weak formulation of $X^{N,i}$ and using Itô's formula, we get that

$$M(X^{N,i}, \mathscr{X}^N) = \int_0^\cdot g'(\langle X^{N,i}_s, y^*\rangle_E)\langle\sigma^*(s, X^{N,i}_s, \mathscr{X}^N_t)y^*, dW^i_s\rangle_H,$$

see Step 4 in the proof of Theorem 2.5 for more details. We obtain

$$\begin{aligned} [M(X^{N,i}, \mathscr{X}^N), M(X^{N,i}, \mathscr{X}^N)] &= \int_0^\cdot \big(g'(\langle X^{N,i}_s, y^*\rangle_E)\big)^2\|\sigma^*(s, X^{N,i}_s, \mathscr{X}^N_s)y^*\|^2_H ds \\ &\le \|g'\|^2_\infty\|y^*\|^2_E \int_0^\cdot \|\sigma(s, X^{N,i}_s, \mathscr{X}^N_s)\|^2_{L(H,E)} ds, \end{aligned}$$

where $[\cdot, \cdot]$ denotes the quadratic variation process. Taking expectation and using (A5), we further get

$$E\big[[M(X^{N,i}, \mathscr{X}^N), M(X^{N,i}, \mathscr{X}^N)]_T\big] \le C\Big(1 + E\Big[\sup_{r\in[0,T]}\|X^{N,1}_r\|^{p'}\Big]^{2/p'}\Big) < \infty. \tag{7.11}$$

Consequently, $M(X^{N,i}, \mathscr{X}^N)$ is a square-integrable martingale. Therefore, we obtain that

$$\begin{aligned} &E\Big[M_t(X^{N,i}, \mathscr{X}^N)M_s(X^{N,j}, \mathscr{X}^N)\prod_{k=1}^N h_k(X^{N,i}_{t_k})h_k(X^{N,j}_{t_k})\Big] \\ &\quad = E\Big[M_s(X^{N,i}, \mathscr{X}^N)M_s(X^{N,j}, \mathscr{X}^N)\prod_{k=1}^N h_k(X^{N,i}_{t_k})h_k(X^{N,j}_{t_k})\Big]. \end{aligned} \tag{7.12}$$

For $i \neq j$, as W^i and W^j are independent, [7, Proposition A.3] yields that

$$\Big[\int_0^\cdot \langle \sigma^*(t, X_t^{N,i}, \mathscr{X}_t^N)y^*, dW_t^i\rangle_H, \int_0^\cdot \langle \sigma^*(t, X_t^{N,j}, \mathscr{X}_t^N)y^*, dW_t^j\rangle_H\Big] = 0,$$

which implies that

$$[M(X^{N,i}, \mathscr{X}^N), M(X^{N,j}, \mathscr{X}^N)] = 0.$$

Consequently, the product $M(X^{N,i}, \mathscr{X}^N)M(X^{N,j}, \mathscr{X}^N)$ is a martingale for $i \neq j$ (see [20, Proposition I.4.50]). We therefore obtain for $i \neq j$ that

$$E\Big[M_t(X^{N,i}, \mathscr{X}^N)M_t(X^{N,j}, \mathscr{X}^N)\prod_{k=1}^N h_k(X_{t_k}^{N,i})h_k(X_{t_k}^{N,j})\Big]$$
$$= E\Big[M_s(X^{N,i}, \mathscr{X}^N)M_s(X^{N,j}, \mathscr{X}^N)\prod_{k=1}^N h_k(X_{t_k}^{N,i})h_k(X_{t_k}^{N,j})\Big].$$

Together with (7.12), we deduce that

$$E\big[V(X^{N,i}, \mathscr{X}^N)V(X^{N,j}, \mathscr{X}^N)\big] = 0, \qquad i \neq j.$$

In summary, using also (7.11) and Burkholder's inequality, we obtain

$$\frac{1}{N^2}\sum_{i,j=1}^N E\big[V(X^{N,i}, \mathscr{X}^N)V(X^{N,j}, \mathscr{X}^N)\big] = \frac{1}{N^2}\sum_{i=1}^N E\big[\big(V(X^{N,i}, \mathscr{X}^N)\big)^2\big]$$
$$\leq \frac{C}{N}\Big(1 + \sup_{n\in\mathbb{N}} E\Big[\sup_{r\in[0,t]} \|X_r^{n,1}\|^{p'}\Big]^{2/p'}\Big).$$

As the final term converges to zero as $N \to \infty$, the claim of the lemma follows. □

Combining Lemmas 7.1 and 7.2, we obtain that

$$E\big[|Z(\mathscr{X})|\big] = \lim_{N\to\infty} E\big[|Z(\mathscr{X}^N)|\big] \leq \lim_{N\to\infty} E\big[\big(Z(\mathscr{X}^N)\big)^2\big]^{1/2} = 0,$$

which implies that a.s. $Z(\mathscr{X}) = 0$.

Step 4: Identifying the limit. We are now in the position to identify the limit $\mathscr{X}$. However, to use our assumption (UL) we first have to adjust our setting to the infinite time horizon. By Step 1 and [21, Theorem 23.4], the family $\{\mathscr{X}^N : N \in \mathbb{N}\}$ is tight when considered as random variables in $M_c(C(\mathbb{R}_+, E))$. Let $\mathscr{X}^*$ be an accumulation point of $\{\mathscr{X}^N : N \in \mathbb{N}\}$. In the following we show that a.s. $\mathscr{X}^* = \mathscr{X}^0$ where $\mathscr{X}^0$ denotes the unique law of a p'-solution process of the MVK SPDE (A, μ, σ, η), see assumption (UL). Together with the tightness in $M_c(M_w^{p^\circ}(C([0, T], E)))$, this then implies that $\mathscr{X}^N|_{[0,T]} \to \mathscr{X}^0|_{[0,T]}$ in probabilty when seen as random variables in $M_w^{p^\circ}(C([0, T], E))$. The final claim of the theorem will then follow from Vitali's theorem.

Lemma 7.3 *There exists a countable set $D \subset D(A^*)$ such that for every $y^* \in D(A^*)$ there exists a sequence $y_1^*, y_2^*, \dots \in D$ with $y_n^* \to y^*$ and $A^*y_n^* \to A^*y^*$.*

Proof As E is assumed to be separable, there exists a countable dense subset F. As A generates a C_0-semigroup, there exists a $\lambda > 0$ in the resolvent set of A and hence also in the resolvent set of A^*, see [30, Theorem 1.5.3, Lemma 1.10.2]. Now, set

$$D \triangleq \big\{(\lambda - A^*)^{-1}x : x \in F\big\} \subset D(A^*).$$

Obviously, D is countable. Let $y^* \in D(A^*)$ and set $x \triangleq \lambda y^* - A^* y^* = (\lambda - A^*) y^*$. As F is dense, there exists a sequence $x_1, x_2, \dots \in F$ such that $x_n \to x$. Now, set $y_n^* \triangleq (\lambda - A^*)^{-1} x_n \in D$ for $n \in \mathbb{N}$. We have $y_n^* \to (\lambda - A^*)^{-1} x = y^*$ as $(\lambda - A^*)^{-1} \in L(E)$. Furthermore, $-A^* y_n^* = (\lambda - A^*) y_n^* - \lambda y_n^* = x_n - \lambda y_n^* \to x - \lambda y^* = -A^* y^*$. This shows the claim. □

Let $\mathscr{C} \subset C_b(E)$ be a countable set which is measure determining ([12, Proposition 3.4.2]). Let $\mathscr{G} \triangleq \{g_1^n, g_2^n : n \in \mathbb{N}\}$, where $g_1^n, g_2^n \in C_c^2(\mathbb{R})$ are such that $g_1^n(x) = x$ and $g_2^n(x) = x^2$ for $|x| \leq n$.

We realize $\mathscr{X}^*$ on a probability space $(\Omega, \mathscr{F}, P)$. Let G be the set of all $\omega \in \Omega$ such that $\mathscr{X}^*|_{[0,M]}(\omega) \in M_w^{p'}(C([0, M], E))$ for all $M \in \mathbb{N}$, $\mathscr{X}_0^*(\omega) = \eta$ and $Z(\mathscr{X}^*(\omega)) = 0$ for all $s, t \in \mathbb{Q}_+$, $s < t$, $y^* \in D$, $g \in \mathscr{G}$, $m \in \mathbb{N}$, $t_1, \dots, t_m \in [0, s] \cap \mathbb{Q}_+$ and $h_1, \dots, h_m \in \mathscr{C}$.

Lemma 7.4 $P(G) = 1$.

Proof Notice that a.s. $\mathscr{X}_0^* = \eta$ thanks to assumption (I). Fix $M \in \mathbb{N}$ and suppose that $\mathscr{X}^{N_m} \to \mathscr{X}^*$ in $M_c(M_c(C(\mathbb{R}_+, E)))$ as $m \to \infty$. Recalling (7.8), we get a.s. $\mathscr{X}^*|_{[0,M]} \in M_w^{p'}(C([0, M], E))$. By Step 2, there exists a subsequence of $\{\mathscr{X}^{N_m}|_{[0,M]} : m \in \mathbb{N}\}$ which converges to a limit $\mathscr{X}^\circ$ in $M_c(M_w^{p^\circ}(C([0, M], E)))$ and we have $\mathscr{X}^*|_{[0,M]} = \mathscr{X}^\circ$ in law. Now, for Z defined with $T = M$, Step 3 yields that a.s. $Z(\mathscr{X}^\circ) = 0$. As $\mathscr{X}^*|_{[0,M]} = \mathscr{X}^\circ$ in law, it follows that a.s. $Z(\mathscr{X}^*) = 0$. In summary, G is the intersection of countably many full sets and therefore a full set by itself. This is the claim. □

Take $\omega \in G$. For every $y^* \in D$, a monotone class argument and similar considerations as in Step 5 of the proof for Theorem 2.5 show that

$$\langle \mathsf{X}, y^* \rangle_E - \langle \mathsf{X}_0, y^* \rangle_E - \int_0^\cdot \big(\langle \mathsf{X}_s, A^* y^* \rangle_E + \langle \mu(s, \mathsf{X}_s, \mathscr{X}_s^*(\omega)), y^* \rangle_E \big) ds$$

is a local $\mathscr{X}^*(\omega)$-martingale (on the space $C(\mathbb{R}_+, E)$ endowed with the natural filtration generated by the coordinate process X) with quadratic variation process $\int_0^\cdot \|\sigma^*(s, \mathsf{X}_s, \mathscr{X}_s^*(\omega)) y^*\|_H^2 ds$. Using Lemma 7.3 and the fact that ucp (uniformly on compacts in probability) limits of continuous local martingales are again continuous local martingales (see [6, Lemma B.11]), we can conclude that the same holds for all $y^* \in D(A^*)$. As $D(A^*)$ separates points, the representation theorem [29, Theorem 3.1] and the equivalence of the weak and mild formulation as given by [28, Theorem 13] yield that $\mathscr{X}^*(\omega)$ is a p'-solution measure of the MKV SPDE (A, μ, σ, η).

Consequently, by assumption (UL), a.s. $\mathscr{X}^* = \mathscr{X}^0$. Using that $\{\mathscr{X}^N|_{[0,T]} : N \in \mathbb{N}\}$ is tight in $M_c(M_w^{p^\circ}(C([0, T], E)))$ by Step 3, we conclude that $\mathscr{X}^N|_{[0,T]} \to \mathscr{X}^0|_{[0,T]}$ in $M_c(M_w^{p^\circ}(C([0, T], E)))$ as $N \to \infty$. As $\mathscr{X}^0$ is deterministic, we also get $\mathsf{w}_T^{p^\circ}(\mathscr{X}^N, \mathscr{X}^0) \to 0$ in probability, where $\mathsf{w}_T^{p^\circ}$ denotes the p°-Wasserstein metric on $M_w^{p^\circ}(C([0, T], E))$.

Finally, we show that the family $\{|\mathsf{w}_T^{p^\circ}(\mathscr{X}^N, \mathscr{X}^0)|^{p^\circ} : N \in \mathbb{N}\}$ is uniformly integrable. In this case, (3.3) follows from Vitali's theorem. We estimate

$$\begin{aligned} E\Big[\big(|\mathsf{w}_T^{p^\circ}(\mathscr{X}^N, \delta_0)|^{p^\circ}\big)^{(p'/p^\circ)}\Big] &= E\Big[\Big(\frac{1}{N}\sum_{i=1}^N \sup_{s \in [0,T]} \|X_s^{N,i}\|_E^{p^\circ}\Big)^{p'/p^\circ}\Big] \\ &\leq E\Big[\frac{1}{N}\sum_{i=1}^N \sup_{s \in [0,T]} \|X_s^{N,i}\|_E^{p'}\Big] \end{aligned}$$

$$= E\Big[\sup_{s\in[0,T]} \|X_s^{N,1}\|_E^{p'}\Big],$$

where we use Hölder's inequality in the second line. Since $p'/p^\circ > 1$ and $\mathscr{X}^0 \in M_w^{p'}(C([0,T],E))$, (7.3) yields that the family $\{|\mathsf{w}_T^{p^\circ}(\mathscr{X}^N, \mathscr{X}^0)|^{p^\circ} : N \in \mathbb{N}\}$ is uniformly integrable. Consequently, (3.3) holds. Finally, the chaotic property follows from [33, Proposition I.2.2]. □

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Appendix A: An Existence and Uniqueness Result for classical SPDEs

Let $\mu\colon \mathbb{R}_+ \times E \to E$ and $\sigma\colon \mathbb{R}_+ \times E \to L(H,E)$ be Borel functions, and take a constant $0 < \alpha < 1/2$. The following theorem should be compared to [8, Theorem 7.6]. Its proof follows the standard path but for completeness we outline the argument.

Theorem A.1 *Assume that for every $T > 0$ there exist Borel functions $\mathfrak{f} = \mathfrak{f}_T\colon (0,T] \to [0,\infty]$ and $\mathfrak{g} = \mathfrak{g}_T\colon (0,T] \to [0,\infty]$ such that*

$$\int_0^T \Big(\Big[\frac{\mathfrak{f}(s)}{s^\alpha}\Big]^2 + \big[\mathfrak{g}(s)\big]^{p/(p-1)}\Big) ds < \infty,$$

and

$$\begin{aligned}
\|S_t(\sigma(s,x) - \sigma(s,y))\|_{L_2(H,E)} &\le \mathfrak{f}(t)\|x-y\|_E, \\
\|S_t(\mu(s,x) - \mu(s,y))\|_E &\le \mathfrak{g}(t)\|x-y\|_E, \\
\|S_t\sigma(s,x)\|_{L_2(H,E)} &\le \mathfrak{f}(t)(1+\|x\|_E), \\
\|S_t\mu(s,x)\|_E &\le \mathfrak{g}(t)(1+\|x\|_E),
\end{aligned}$$

for all $0 < t, s \le T$ and $x, y \in E$. Then, for any $\eta \in M_c(\eta)$, on any driving system $(\mathbb{B}, W)$ there exists a unique, up to indistinguishability, continuous mild solution process X to the SPDE

$$dX_t = AX_t dt + \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 \sim \eta.$$

Moreover, for every $p > 1/\alpha$, $T > 0$ and $\eta \in M_w^p(E)$,

$$E\Big[\sup_{s\in[0,T]} \|X_s\|_E^p\Big] < \infty.$$

Here, a mild solution is meant to be defined in the usual sense, i.e. similar to Definition 6.1 without the coefficient γ.

Sketch of Proof We start by proving the second part of the theorem. Let $p > 1/\alpha$ and assume that $\eta \in M^p_w(E)$. Take a completed filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ which supports a cylindrical Brownian motion W and an $\mathscr{F}_0$-measurable ξ such that $\xi \sim \eta$. Moreover, let $\mathscr{H}^p$ be the space of continuous E-valued processes $Y = (Y_t)_{t\geq 0}$ such that

$$E\Big[\sup_{s\in[0,T]} \|Y_s\|^p_E\Big] < \infty, \quad \forall T > 0.$$

Define a map $I \colon \mathscr{H}^p \to \mathscr{H}^p$ by

$$I(Y)(t) \triangleq S_t\xi + \int_0^t S_{t-s}\mu(s, Y_s)ds + \int_0^t S_{t-s}\sigma(s, Y_s)dW_s, \quad t \in \mathbb{R}_+.$$

Let us elaborate in more detail that $I(\mathscr{H}^p) \subset \mathscr{H}^p$. First of all,

$$E\Big[\sup_{s\in[0,T]} \|I(Y)(s)\|^p_E\Big] < \infty, \quad \forall T > 0,$$

follows from the linear growth assumptions, Lemma 4.2 and the estimate

$$\begin{aligned} E\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S_{t-s}\mu(s, Y_s)ds \Big\|^p_E\Big] &\leq E\Big[\sup_{t\in[0,T]} \Big(\int_0^t \mathfrak{g}(t-s)(1 + \|Y_s\|_E)ds\Big)^p\Big] \\ &\leq E\Big[\Big(\int_0^T [\mathfrak{g}(s)]^{p/(p-1)}ds\Big)^{p-1} \int_0^T (1 + \|Y_s\|_E)^p ds\Big] \\ &\leq \Big(\int_0^T [\mathfrak{g}(s)]^{p/(p-1)}ds\Big)^{p-1} 2^{p+1}T\Big(1 + E\Big[\sup_{s\in[0,T]} \|Y_s\|^p_E\Big]\Big), \end{aligned}$$

which uses Hölder's inequality. While the first two terms in the definition of $I(Y)$ are clearly continuous ([26, Lemma 6.2.9]), the last integral is continuous thanks to Lemma 4.2. This shows that $I(\mathscr{H}^p) \subset \mathscr{H}^p$. Next, for $Y, Z \in \mathscr{H}^p$ and $T > 0$ set

$$\Phi_T(Y, Z) \triangleq E\Big[\sup_{s\in[0,T]} \|Y_s - Z_s\|^p_E\Big].$$

For every $T > 0$, using the Lipschitz hypothesis and similar arguments as above, we obtain

$$\Phi_T(I(Y), I(Z)) \leq C \int_0^T E\big[\|Y_s - Z_s\|^p_E\big]ds \leq C \int_0^T \Phi_s(Y, Z)ds, \tag{A.1}$$

where the constant depends on T, $\mathfrak{g} = \mathfrak{g}_T$ and $\mathfrak{f} = \mathfrak{f}_T$. Define now inductively a sequence $X^0, X^1, X^2, \ldots$ such that $X^0 \triangleq S\xi$ and $X^n \triangleq I(X^{n-1})$ for $n = 1, 2, \ldots$. It follows from (A.1) and induction that

$$\Phi_T(X^{n-1}, X^n) \leq \frac{C^n T^n}{n!}\Phi_T(X^0, X^1), \quad n = 1, 2, \ldots, \quad T > 0.$$

Consequently, by a Borel–Cantelli argument ([22, Theorem 5.2.9]), we deduce that the sequence $X^1, X^2, \ldots$ converges a.s. in the local uniform topology to a continuous process X. Furthermore, we obtain that

$$\begin{aligned} E\Big[\sup_{n\in\mathbb{N}}\sup_{s\in[0,T]} \|X^n_s - \xi\|^p_E\Big] &\leq \sum_{n=1}^\infty n^{p+1} E\Big[\sup_{s\in[0,T]} \|X^n_s - X^{n-1}_s\|^p_E\Big] \\ &\leq \sum_{n=1}^\infty n^{p+1}\frac{C^n T^n}{n!}\Phi_T(X^0, X^1) < \infty. \end{aligned}$$

Thus, $X \in \mathscr{H}^p$ and the dominated convergence theorem together with (A.1) yield that

$$\Phi_T(I(X), X) = \lim_{n\to\infty} \Phi_T(I(X), X^n) \leq \lim_{n\to\infty} C \int_0^T \Phi_s(X, X^n)ds = 0.$$

We conclude that a.s. $I(X) = X$, which shows that X is a continuous mild solution process. Furthermore, (A.1) and Gronwall's lemma yield uniqueness up to indistinguishability.

Finally, let us comment on the general case where $\eta \in M_c(E)$. For $m = 1, 2, \dots,$ let X^m be a solution as constructed above for the initial value $\xi^m \triangleq \xi \mathbb{I}_{\|\xi\|_E \leq m}$. Then, a.s. $X^n = X^{n+1}$ on $\{\|\xi\|_E \leq n\}, n = 1, 2, \dots$. Consequently, $\lim_{n\to\infty} X^n$ is a.s. well-defined and a continuous mild solution. To prove the uniqueness statement, let Y be a solution process on the same driving system as $X^1, X^2, \dots$. For every $m \in \mathbb{N}$, notice that

$$\bar{X}^m \triangleq \begin{cases} Y, & \text{on } \{\|\xi\|_E \leq m\}, \\ X^m, & \text{otherwise}, \end{cases}$$

is a solution to the SPDE with initial value $\xi \mathbb{I}_{\|\xi\|_E \leq m}$. Hence, by the above uniqueness statement, we get that a.s. $\bar{X}^m = X^m$ and therefore a.s. $X^m = Y$ on $\{\|\xi\|_E \leq m\}$. This yields the claimed uniqueness. □

Appendix B: Proof of Theorem 2.12

By virtue of Theorem 2.11, it suffices to prove the existence of a p-solution process on any given driving system $(\mathbb{B}, W)$. We use a classical argument based on a fixed point theorem (see [5, Theorem 4.21] for the argument in finite dimensions with finite time horizon). In the following we show existence on a finite time interval $[0, T]$ with a random initial value $\xi_0 \sim \eta$. The existence of a global solution follows from the local result by pasting. Let $\gamma \in C([0, T], M^p_w(E))$. By Theorem A.1 (in case $p > 1/\alpha$ and (L1) holds) and [16, Theorem 3.3] (in case $p \geq 2$, (L2) holds and S is a generalized contraction), there exists a continuous mild solution process $X^{(\gamma)}$ to the SPDE

$$dX_t = AX_t dt + \mu(t, X_t, \gamma_t)dt + \sigma(t, X_t, \gamma_t)dW_t, \quad X_0 = \xi_0,$$

such that

$$E\Big[\sup_{s\in[0,T]} \|X^{(\gamma)}_s\|^p_E\Big] < \infty.$$

Furthermore, by the Yamada–Watanabe theorem [28, Theorem 2], the law of $X^{(\gamma)}$ is fully characterized by A, μ, σ, η and γ. We now define a map $\Phi\colon C([0, T], M^p_w(E)) \to C([0, T], M^p_w(E))$ by

$$\Phi(\gamma)(t) = P \circ (X^{(\gamma)}_t)^{-1}, \quad t \in [0, T].$$

Let $\gamma, \gamma' \in C([0, T], M^p_w(E))$. As in the proof of Lemma 6.2, we obtain the estimate

$$E\big[\|X^{(\gamma)}_t - X^{(\gamma')}_t\|^p_E\big] \leq C\Big(\int_0^t E\big[\|X^{(\gamma)}_s - X^{(\gamma')}_s\|^p_E\big]ds + \int_0^t |\mathsf{w}_p(\gamma(s), \gamma'(s))|^p ds\Big), \quad t \in [0, T].$$

Thus, Gronwall's lemma yields that

$$|\mathsf{w}_p(\Phi(\gamma)(t), \Phi(\gamma')(t))|^p \leq E\big[\|X^{(\gamma)}_t - X^{(\gamma')}_t\|^p_E\big] \leq C \int_0^t |\mathsf{w}_p(\gamma(s), \gamma'(s))|^p ds.$$

Using Induction we get for every $k \in \mathbb{N}$ that

$$\sup_{s\in[0,T]} |\mathsf{w}_p(\Phi^k(\gamma)(s), \Phi^k(\gamma')(s))|^p \le C^k \int_0^T \frac{(T-s)^{(k-1)}}{(k-1)!} |\mathsf{w}_p(\gamma(s), \gamma'(s))|^p ds$$
$$\le \frac{C^k T^k}{k!} \sup_{s\in[0,T]} |\mathsf{w}_p(\gamma(s), \gamma'(s))|^p.$$

Thus, there exists an $N \in \mathbb{N}$ such that Φ^k is a contraction on the Polish space $C([0,T], M_w^p(E))$ for all $k \ge N$. Thanks to the theorem in [4], this yields that Φ has a fixed point and consequently, restricted to the time interval $[0,T]$, that there exists a solution process to the MKV SPDE with coefficients (A, μ, σ, η).

Finally, the existence of a global solution process, i.e. a solution process defined on the infinite time interval $\mathbb{R}_+$, follows by pasting: Set $X^0 \triangleq \xi_0$. For $n \in \mathbb{N}$, let X^n be a solution process with coefficients $\mu(\cdot + n - 1, \cdot, \cdot), \sigma(\cdot + n - 1, \cdot, \cdot)$, initial value X^{n-1}_{n-1} and driving noise $W^n \triangleq W_{\cdot+n-1} - W_{n-1}$. Finally, define

$$X_t \triangleq \sum_{k=1}^\infty X^k_{t-(k-1)} \mathbb{I}_{k-1\le t<k}, \qquad t \in \mathbb{R}_+.$$

For $k < t \le k+1$ we compute that

$$\begin{aligned} X_t &= S_{t-k} X^k_k + \int_0^{t-k} S_{t-k-s}\mu(s+k, X^{k+1}_s, P^{X^{k+1}}_s) ds \\ &\quad + \int_0^{t-k} S_{t-k-s}\sigma(s+k, X^{k+1}_s, P^{X^{k+1}}_s) dW^{k+1}_s \\ &= S_{t-k} X^k_k + \int_k^t S_{t-s}\mu(s, X^{k+1}_{s-k}, P^{X^{k+1}}_{s-k}) ds + \int_k^t S_{t-s}\sigma(s, X^{k+1}_{s-k}, P^{X^{k+1}}_{s-k}) dW_s \\ &= S_{t-k} X^k_k + \int_k^t S_{t-s}\mu(s, X_s, P^X_s) ds + \int_k^t S_{t-s}\sigma(s, X_s, P^X_s) dW_s. \end{aligned}$$

Thus, by induction, X is a solution process to the MKV SPDE with coefficients (A, μ, σ, η). □

Appendix C: Proof of Proposition 3.5

Fix $N \in \mathbb{N}$ and consider the (separable) Hilbert spaces $\widetilde{E} \triangleq \bigoplus_{i=1}^N E$ and $\widetilde{H} \triangleq \bigoplus_{i=1}^N H$. Here, recall that $\bigoplus_{i=1}^N$ denotes the Hilbert space direct sum. Moreover, for $t \in \mathbb{R}_+$ and $e = (e^1, \dots, e^N) \in \widetilde{E}$ we set

$$L(e) \triangleq \frac{1}{N} \sum_{i=1}^N \delta_{e^i},$$

$$\widetilde{\mu}(t, e) \triangleq \bigoplus_{i=1}^N \mu(t, e^i, L(e)) \in \widetilde{E},$$

$$\widetilde{\sigma}(t, e) \triangleq \bigoplus_{i=1}^N \sigma(t, e^i, L(e)) \in L(\widetilde{H}, \widetilde{E}).$$

It is not hard to check that the process $\widetilde{W} \triangleq \bigoplus_{i=1}^{N} W^i$ is a standard cylindrical Brownian motion. The system of SPDEs associated to the processes $X^{N,1}, \ldots, X^{N,N}$ can now be written as

$$d\widetilde{X}_t = \widetilde{A}\widetilde{X}_t dt + \widetilde{\mu}(t, \widetilde{X}_t)dt + \widetilde{\sigma}(t, \widetilde{X}_t)d\widetilde{W}_t, \quad \widetilde{A} \triangleq \bigoplus_{i=1}^{N} A,$$

where $\widetilde{A}$ generates the C_0-semigroup $\widetilde{S} \triangleq \bigoplus_{i=1}^{N} S$ on $\widetilde{E}$. Thus, by virtue of Theorem A.1, the claim of the proposition follows in case $\widetilde{\mu}$ and $\widetilde{\sigma}$ satisfy suitable Lipschitz and linear growth conditions, which we check in the following. Take $T > 0$ and let $\mathfrak{f} = \mathfrak{f}_T$ be as in (L1). Then, for all $0 < t, s \le T$ and $e = (e^1, \ldots, e^N) \in \widetilde{E}$ we get

$$\begin{aligned}
\|\widetilde{S}_t\widetilde{\sigma}(s, e, L(e))\|^2_{L_2(\widetilde{H}, \widetilde{E})} &= \sum_{i=1}^{N} \|S_t\sigma(s, e^i, L(e))\|^2_{L_2(E,H)} \\
&\le \big[\mathfrak{f}(t)\big]^2 \sum_{i=1}^{N} C\big(1 + \|e^i\|^2_E + \|L(e)\|^2_p\big) \\
&\le \big[\mathfrak{f}(t)\big]^2 CN\big(1 + \|e\|^2_{\widetilde{E}}\big).
\end{aligned}$$

Similarly, we obtain for all $0 < t, s \le T$ and $e = (e^1, \ldots, e^N)$, $f = (f^1, \ldots, f^N) \in \widetilde{E}$ that

$$\begin{aligned}
&\|\widetilde{S}_t(\widetilde{\sigma}(s, e, L(e)) - \widetilde{\sigma}(t, f, L(f)))\|^2_{L_2(\widetilde{H}, s\widetilde{E})} \\
&\quad \le \big[\mathfrak{f}(t)\big]^2\big(C\|e - f\|^2_{\widetilde{E}} + CN|\mathsf{w}_p(L(e), L(f))|^2\big).
\end{aligned}$$

We now estimate $\mathsf{w}_p(L(e), L(f))$. Set

$$F \triangleq \frac{1}{N}\sum_{i=1}^{N} \delta_{(e^i, f^i)}.$$

Then, $F(dx \times E) = L(e)(dx)$, $F(E \times dx) = L(f)(dx)$ and

$$\Big(\iint \|x - y\|^p_E F(dx, dy)\Big)^{1/p} = \Big(\frac{1}{N}\sum_{i=1}^{N} \|e^i - f^i\|^p_E\Big)^{1/p} \le \frac{1}{N^{1/p}}\|e - f\|_{\widetilde{E}}.$$

Hence, we obtain

$$|\mathsf{w}_p(L(e), L(f))|^2 \le \|e - f\|^2_{\widetilde{E}},$$

and finally,

$$\|\widetilde{S}_t(\widetilde{\sigma}(s, e, L(e)) - \widetilde{\sigma}(t, f, L(f)))\|^2_{L_2(\widetilde{H}, \widetilde{E})} \le \big[\mathfrak{f}(t)\big]^2 CN\|e - f\|^2_{\widetilde{E}}.$$

We conclude that the coefficient $\widetilde{\sigma}$ satisfies the linear growth and Lipschitz conditions from Theorem A.1. Similar computations show the same for the coefficient $\widetilde{\mu}$. We omit the remaining details. □

Appendix D: Proof of Theorem 3.6

We borrow the main idea from the proof of [24, Theorem 3.3]. By virtue of Theorem 2.12, let Y^i be a p-solution process to the MKV SPDE (A, μ, σ, η) on the driving system $(\mathbb{B}, W^i)$

with initial value ξ_0^i. Take $T > 0$ and denote by $\mathscr{X}_t^0$ the projection of $\mathscr{X}^0$ to the time t value. By virtue of the proof of Lemma 6.2, using Lemma 4.2 (in case $p > 1/\alpha$ and (L1) holds) and part (b) of [16, Lemma 3.3] (in case $p \geq 2$, (L2) holds and S is a generalized contraction), for all $t \in [0, T]$ we obtain that

$$\begin{aligned} E\Big[\sup_{s\in[0,t]} \|X_s^{N,i} - Y_s^i\|_E^p\Big] &\leq CE\Big[\int_0^t \big(\|X_s^{N,i} - Y_s^i\|_E^p + |\mathsf{w}_p(\mathscr{X}_s^N, \mathscr{X}_s^0)|^p\big)ds\Big] \\ &\leq CE\Big[\int_0^t \Big(\sup_{r\in[0,s]} \|X_r^{N,i} - Y_r^i\|_E^p + |\mathsf{w}_p(\mathscr{X}_s^N, \mathscr{X}_s^0)|^p\Big)ds\Big]. \end{aligned}$$

Thus, Gronwall's lemma yields that

$$E\Big[\sup_{s\in[0,t]} \|X_s^{N,i} - Y_s^i\|_E^p\Big] \leq C\int_0^t E\big[|\mathsf{w}_p(\mathscr{X}_s^N, \mathscr{X}_s^0)|^p\big]ds, \quad t \in [0, T]. \tag{D.1}$$

We set

$$\mathscr{Y}^N \triangleq \frac{1}{N}\sum_{i=1}^N \delta_{Y^i}.$$

Using the coupling $\frac{1}{N}\sum_{i=1}^N \delta_{(X^{N,i}, Y^i)}$, we obtain that

$$|\mathsf{w}_t^p(\mathscr{X}^N, \mathscr{Y}^N)|^p \leq \frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} \|X_s^{N,i} - Y_s^i\|_E^p.$$

Hence, for all $t \in [0, T]$

$$\begin{aligned} E\big[|\mathsf{w}_t^p(\mathscr{X}^N, \mathscr{X}^0)|^p\big] &\leq C\big(E\big[|\mathsf{w}_t^p(\mathscr{X}^N, \mathscr{Y}^N)|^p\big] + E\big[|\mathsf{w}_t^p(\mathscr{Y}^N, \mathscr{X}^0)|^p\big]\big) \\ &\leq C\Big(\int_0^t E\big[|\mathsf{w}_p(\mathscr{X}_s^N, \mathscr{X}_s^0)|^p\big]ds + E\big[|\mathsf{w}_t^p(\mathscr{Y}^N, \mathscr{X}^0)|^p\big]\Big) \\ &\leq C\Big(\int_0^t E\big[|\mathsf{w}_s^p(\mathscr{X}^N, \mathscr{X}^0)|^p\big]ds + E\big[|\mathsf{w}_t^p(\mathscr{Y}^N, \mathscr{X}^0)|^p\big]\Big). \end{aligned}$$

Using Gronwall's lemma once again, we conclude that

$$E\big[|\mathsf{w}_T^p(\mathscr{X}^N, \mathscr{X}^0)|^p\big] \leq CE\big[|\mathsf{w}_T^p(\mathscr{Y}^N, \mathscr{X}^0)|^p\big].$$

The r.h.s. converges to zero as $N \to \infty$ by [24, Corollary 2.14], as $W^1, W^2, \dots$ and the initial values are i.i.d. and so are $Y^1, Y^2, \dots$. Thus, (3.5) is proved.

Let us now show the second claim, namely (3.4). We deduce from (3.5) and (D.1) that

$$E\Big[\max_{i=1,\dots,k} \sup_{s\in[0,T]} \|X_s^{N,i} - Y_s^i\|_E^p\Big] \leq CkE\big[|\mathsf{w}_T^p(\mathscr{X}^N, \mathscr{X}^0)|^p\big] \to 0$$

as $N \to \infty$. This immediately implies (3.4). The proof is complete. □

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