Open access • Journal Article • DOI:10.1063/1.1693161

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Published on: 01 Jan 1970 - Physics of Fluids (American Institute of PhysicsAIP)

Topics: Relativistic electron beam, Beam (structure), Electron, Magnetic field and Plasma

Title
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Journal
Physics of Fluids, 13(7)

ISSN
00319171

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Publication Date
1970

DOI
10.1063/1.1693161

Peer reviewed

Propagation of High Current Relativistic Electron Beams*

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(Received 25 August 1969; final manuscript received 4 February 1970)

Theoretical self-consistent relativistic electron beam models are developed which allow the propagation of relativistic electron fluxes in excess of the Alfvén–Lawson critical-current limit for a fully neutralized beam. Development of a simple, fully relativistic, self-consistent equilibrium is described which can carry arbitrarily large currents at or near complete electrostatic neutralization. A discussion of a model for magnetic neutralization is presented wherein it is shown that large numbers of electrons from a background plasma are counterstreaming slowly within the beam so that the net current density in the system, and therefore, the magnetic field, is nearly zero. A solution of an initial-value problem for a beam–plasma system is given which indicates that magnetic neutralization can be expected to occur for plasma densities that are large compared with beam densities. It is found that the application of a strong axial magnetic field to a uniform beam allows propagation regardless of the magnitude of the beam current. Some comparisons are made with recent experimental data.

I. INTRODUCTION

Theoretical interest in relativistic electron beams began with Bennett's paper[1] in which he pointed out that electrostatically neutralized high current, electron streams can be magnetically self-focusing. Alfvén[2] was motivated to consider charged particle beams in order to explain certain observations concerning cosmic rays. He derived an upper limit to the possible current of cosmic rays that can propagate through space in a given direction. His model was a cylindrically symmetric, monoenergetic, uniform current density stream of identical particles, and he assumed that the ionized matter in interstellar space would insure electrical neutralization. The current limit, I_A, which Alfvén derived is due to the pinch forces of the self-magnetic field of the beam, and is of order given by

$$I_A \simeq 17\,000\beta\gamma \text{ A}, \tag{1}$$

where β is the particle stream velocity divided by the velocity of light, and $\gamma = (1 - \beta^2)^{-1/2}$. Qualitatively, it is easy to see how this limit comes about. The uniform current density assumption implies a magnetic field within the beam proportional to radius, and electrostatic neutralization implies that the energy is a constant. Therefore, we are able to integrate the equations of motion to obtain the particle trajectories shown in Fig. 1. (They are drawn for particles without angular momentum.) If the net current included within the maximum radial position of a particle is small compared with I_A, its motion is approximately sinusoidal, as shown by trajectory a in Fig. 1. As the included current increases, the trajectory passes through the beam axis at a greater angle (trajectory b) until at an included current of

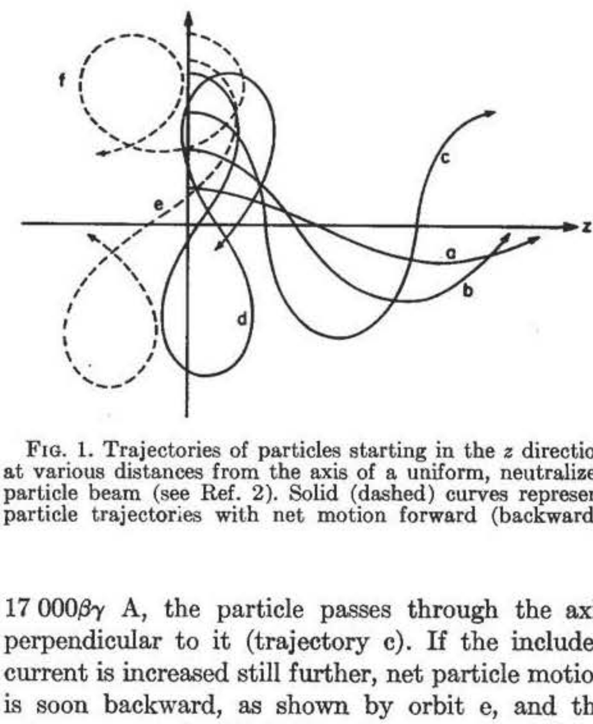


FIG. 1. Trajectories of particles starting in the z direction at various distances from the axis of a uniform, neutralized particle beam (see Ref. 2). Solid (dashed) curves represent particle trajectories with net motion forward (backward).

$17\,000\beta\gamma$ A, the particle passes through the axis perpendicular to it (trajectory c). If the included current is increased still further, net particle motion is soon backward, as shown by orbit e, and the extreme case of orbit f. Therefore, we cannot have currents in excess of about I_A under the above assumptions. It should be noted that this limit is independent of any physical dimensions. The beam current can be written

$$I \equiv Ne\beta c \equiv 17\,000\nu\beta, \tag{2}$$

where ν is the number of electrons per classical electron radius ($r_0 = 2.82 \times 10^{-15}$ m) of beam length,

$$\nu \equiv \frac{Ne^2}{4\pi\epsilon_0 mc^2} \equiv Nr_0; \tag{3}$$

current in this model is limited to $\nu \lesssim \gamma$. The velocity of light is c, e is the magnitude of the electron charge, m is the electron rest mass, and ϵ_0 is the permittivity of free space.

Lawson[3] also considered the uniform beam model in treating both partially and fully electrostatically neutralized electron beams. He arrived at a current limit of I_A for a fully neutralized beam by arguments similar to Alfvén's as well as by simply requiring that a beam electron Larmor radius in the maximum self-field of the beam be of the same order as the beam radius. For an arbitrary fractional electrostatic neutralization f, Lawson obtained a current limit of $17\,000\beta^3\gamma/(\beta^2 + f - 1)$. In principle, then, arbitrarily large currents could be carried by a uniform beam if one carefully adjusted f to be $1 - \beta^2$, or at least within the range given by

$$1 - \beta^2 - \frac{\gamma\beta^2}{2\nu} > f > 1 - \beta^2 + \frac{\gamma\beta^2}{2\nu}, \qquad (4)$$

a balancing act which is difficult to do experimentally if ν/γ is to be large compared with one.

Led by Martin and his co-workers of the United Kingdom Atomic Energy Authority, a high-voltage pulse technology has recently been developed which is capable of the production of short ($\leq 10^{-7}$ sec) bursts of relativistic electrons with currents in excess of I_A.[4-10] The two most striking experimental results to date are the following: (1) At low ambient pressure ($\lesssim$ 0.01 Torr) in the beam drift region, Graybill, Uglam, and Nablo were unable to propagate beams with more current than about $\frac{1}{2}I_A$.[11] (2) At higher ambient pressures ($\gtrsim$ 0.1 Torr), Yonas and Spence[12] and Andrews *et al.*[13] have propagated currents well over I_A.

The first observation fits in well with the current limits for the neutralized beams of Alfvén and Lawson. The second result, however, led us to the development of theoretical, self-consistent beam models which allow the propagation of relativistic electron fluxes in excess of I_A. In Sec. II, we present and develop a simple, fully relativistic self-consistent equilibrium which can carry arbitrarily large currents when near or at complete electrostatic neutralization. A second possible way to propagate arbitrarily large currents is if the beam is magnetically neutralized as well as electrostatically neutralized. By magnetic neutralization, we mean that large numbers of electrons from a background plasma are counterstreaming slowly within the beam so that the net current density in the system, and, therefore, the magnetic field, is nearly zero. There would then be no fields acting on the particles of the beam and (ignoring the obvious problem of instabilities) they would propagate in nearly straight lines. In current limit terminology, the limit would be

$$17\,000\beta^3\gamma/[\beta^2(1 - f_m) - (1 - f)],$$

where f_m is the fractional magnetic, or current, neutralization. This mode of beam propagation has been proposed by Yonas and Spence, Andrews *et al.*, and others,[14] as the mechanism responsible for the second experimental result given above, and experimental verification of this has been obtained by these workers. In Sec. III, we solve an initial value problem for a beam–plasma system which indicates that magnetic neutralization as described above can be expected to occur for plasma densities large compared to beam densities. In Sec. IV, we apply a strong axial magnetic field to a uniform beam and find that it will then be able to propagate regardless of the beam current.

II. NONUNIFORM BEAM EQUILIBRIUM

Here we wish to consider a fully relativistic equilibrium electron beam solution to the Vlasov equation,

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\cdot\frac{\partial f_e}{\partial \mathbf{x}} - e(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_e}{\partial \mathbf{p}} = 0, \qquad (5)$$

and the relevant Maxwell's equations,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j}, \qquad (6)$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}. \qquad (7)$$

The electric field and magnetic induction are $\mathbf{E}$ and $\mathbf{B}$, respectively, f_e is the electron distribution function, $\mathbf{v}$, $\mathbf{p}$, and $-e$ are the electron velocity, momentum, and charge, respectively, and $\mathbf{j}$ and ρ are the current and charge densities. We are using mks units so that $(\mu_0\epsilon_0)^{-1/2} = c$, the free space velocity of light. The beam is infinitely long and without variation in the z direction, cylindrically symmetric, and confined to a finite radius b. We also assume an immobile positive ion background which partially or fully neutralizes the electron beam charge density. There are no external fields.

The constants of motion for an electron in the assumed beam are the Hamiltonian H, the canonical axial momentum P_z, and the angular momentum p_θ, which are given by

$$H = \gamma(r)mc^2 - e\Phi(r)$$
$$= c\{m^2c^2 + p_\perp^2 + [P_z + eA_z(r)]^2\}^{1/2} - e\Phi(r), \quad (8)$$

$$P_z = \gamma m v_z - eA_z \equiv p_z - eA_z, \tag{9}$$

$$p_\theta = \gamma m r^2 \omega = \gamma m(x v_y - y v_x). \tag{10}$$

$\Phi(r)$ and $A_z(r)$ are the electrostatic and magnetic potentials, which are functions only of r, the radial position. The electron mass is m, ω is its angular velocity, and p_z and $p_\perp$ are the parallel and perpendicular (relative to the z axis) ordinary momenta of the electron, respectively. Any function of these constants of the motion is a solution to Eq. (5), so we choose the particularly simple, but interesting case of monoenergetic electrons having the same axial canonical momentum,

$$f_e(\mathbf{x}, \mathbf{p}) \equiv f_e(r, \mathbf{p})$$
$$= \frac{n_e(0)c^2}{2\pi\epsilon_e}\, \delta(H - \epsilon_e)\, \delta(P_z - \gamma_0 m V_z). \tag{11}$$

Defining $A_z(r = 0) = 0 = \Phi(r = 0)$, from Eqs. (8) and (9) we have that γ_0 and V_z are the values of γ and v_z for an electron at $r = 0$, and that $\epsilon_e = \gamma_0 mc^2$. The first two moments of this distribution function are

$$n_e(\mathbf{x}) \equiv n_e(r) = \int d\mathbf{p}\, f_e(r, \mathbf{p})$$
$$\equiv \int_{-\infty}^{\infty} dp_z \int_0^\infty p_\perp\, dp_\perp \int_0^{2\pi} d\theta\, f_e(r, \mathbf{p}), \tag{12}$$

$$n_e\langle v_z\rangle = \int_{-\infty}^{\infty} dp_z \int_0^\infty p_\perp\, dp_\perp \int_0^{2\pi} d\theta\, f_e(r, \mathbf{p})\, \frac{p_z}{\gamma m}. \tag{13}$$

(Since f_e and γ are even functions of $p_\perp$, $n_e\langle v_x\rangle = 0$ and $n_e\langle v_y\rangle = 0$.) Because of the δ function in P_z and using Eqs. (8) and (9) these can be rewritten, after p_z and θ integrations,

$$n_e(r) = \frac{n_e(0)c^2}{\epsilon_e} \int_1^\infty u\, du (mc\alpha)^2\, \delta(mc^2\alpha u - e\Phi - \epsilon_e), \tag{14}$$

$$n_e\langle v_z\rangle = \frac{n_e(0)c^2}{\epsilon_e} \int_1^\infty du(\gamma_0 m V_z + eA_z) mc^2\alpha \cdot \delta(mc^2\alpha u - e\Phi - \epsilon_e), \tag{15}$$

where

$$\alpha^2 \equiv 1 + \frac{(\gamma_0 m V_z + eA_z)^2}{m^2c^2}, \tag{16}$$

$$u \equiv \left(1 + \frac{p_\perp^2}{m^2c^2\alpha^2}\right)^{1/2}. \tag{17}$$

The δ functions in these integrations are zero over the whole range of the u integration unless $e\Phi + \epsilon_e \geq mc^2\alpha$. Therefore, we obtain

$$n_e(r) = \begin{cases} n_e(0)\left[1 + \dfrac{e\Phi(r)}{\epsilon_e}\right], & r \leq b, \\ 0, & r > b, \end{cases} \tag{18}$$

$$n_e\langle v_z\rangle = \begin{cases} \dfrac{n_e(0)c^2}{\epsilon_e}[\gamma_0 m V_z + eA_z(r)], & r \leq b, \\ 0, & r > b, \end{cases} \tag{19}$$

where b is defined by

$$\frac{\epsilon_e + e\Phi(b)}{mc^2\alpha(b)} = 1. \tag{20}$$

We can now obtain the self-fields of the beam assuming that the background ions provide a charge neutralization fraction f, $0 \leq f \leq 1$. The potential equations obtained from Eqs. (6) and (7) are

$$\nabla\cdot\mathbf{E} \equiv -\nabla^2\Phi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = \begin{cases} \dfrac{e(f-1)}{\epsilon_0} n_e(0)\left[1 + \dfrac{e\Phi}{\epsilon_e}\right], & r \leq b, \\ 0, & r > b, \end{cases} \tag{21}$$

$$(\nabla\times\mathbf{B})_z \equiv [\nabla\times(\nabla\times\mathbf{A})]_z = -\nabla^2 A_z = \begin{cases} -\dfrac{n_e(0)e}{\epsilon_0\epsilon_e}(\gamma_0 m V_z + eA_z), & r \leq b, \\ 0, & r > b. \end{cases} \tag{22}$$

Let

$$\frac{1}{L_e^2} = \frac{n_e(0)e^2}{\epsilon_0\epsilon_e}, \tag{23}$$

$$\Phi_0 = -\frac{n_e(0)L_e^2 e}{\epsilon_0} = -\frac{\epsilon_e}{e}, \tag{24}$$

$$A_0 = -\frac{n_e(0)\gamma_0 m V_z e L_e^2}{\epsilon_0\epsilon_e} = -\frac{\gamma_0 m V_z}{e}. \tag{25}$$

Then, Eqs. (21) and (22) for $r \leq b$ assume the form of modified Bessel equations of order zero for dependent variables $\Phi - \Phi_0$ and $A_z - A_0$. Taking the solution for which $\Phi(0) = 0 = A_z(0)$ there results

$$\Phi(r) = \begin{cases} \Phi_0\left[1 - I_0\left(\frac{r}{L_e}(1-f)^{1/2}\right)\right], & r \le b, \\ \Phi_0\left[1 - I_0\left(\frac{b}{L_e}(1-f)^{1/2}\right)\right] - \frac{b}{L_e}(1-f)^{1/2}\Phi_0 I_1\left[\frac{b}{L_e}(1-f)^{1/2}\right]\ln\frac{r}{b}, & r > b, \end{cases} \tag{26}$$

$$A_z(r) = \begin{cases} A_0\left[1 - I_0\left(\frac{r}{L_e}\right)\right], & r \le b, \\ A_0\left[1 - I_0\left(\frac{b}{L_e}\right)\right] - \frac{b}{L_e} A_0 I_1\left(\frac{b}{L_e}\right)\ln\frac{r}{b}, & r > b, \end{cases} \tag{27}$$

where I_m is the modified Bessel function of the first kind and order m. The solutions for $r > b$ were obtained by integrating from $r = b$. For the electric and magnetic fields these give

$$E_r(r) = \begin{cases} \Phi_0 \frac{(1-f)^{1/2}}{L_e} I_1\left(\frac{r}{L_e}(1-f)^{1/2}\right), & r \le b, \\ \Phi_0 \frac{b(1-f)^{1/2}}{rL_e} I_1\left(\frac{b}{L_e}(1-f)^{1/2}\right), & r > b, \end{cases} \tag{28}$$

$$B_\theta(r) = \begin{cases} \frac{A_0}{L_e} I_1\left(\frac{r}{L_e}\right), & r \le b, \\ \frac{A_0 b}{rL_e} I_1\left(\frac{b}{L_e}\right), & r > b. \end{cases} \tag{29}$$

Combining Eqs. (26) and (27) with Eqs. (8), (18), and (19), we obtain

$$\frac{n_e(r)}{n_e(0)} = I_0\left(\frac{r}{L_e}(1-f)^{1/2}\right) = \frac{\gamma(r)}{\gamma_0}, \tag{30}$$

$$v_z(r) = V_z \frac{I_0(r/L_e)}{I_0[(r/L_e)(1-f)^{1/2}]}. \tag{31}$$

Thus, since $I_0(0) = 1$, if the beam is neutralized ($f = 1$), the density is uniform, as is γ. However, the axial velocity distribution (and, therefore, j_z) are far from uniform for $r/L_e \gg 1$, because[15]

$$I_n(x) \sim \frac{e^x}{(2\pi x)^{1/2}}\left(1 - \frac{4n^2-1}{8x} + \cdots\right) \quad \text{for} \quad x \gg 1. \tag{32}$$

Since $v_z(b)$ is limited by c, this means that $V_z \ll c$.

The circumstance under which $r/L_e \gg 1$ is that the total current, I, being carried by the beam be large compared with I_A, as we shall now show. From Eq. (29) and Ampere's law,

$$I = \frac{1}{\mu_0}\oint \mathbf{B}\cdot d\mathbf{l}\,\bigg|_{r=b} = -2\pi\frac{\epsilon_0 mc^3}{e}\gamma_0\beta_z\frac{b}{L_e}I_1\left(\frac{b}{L_e}\right). \tag{33}$$

Defining I_A for this nonuniform beam analogously to one of Lawson's derivations of it,[3] we take I_A as the current for which a beam electron Larmor radius R_L, in the maximum beam self-magnetic field, is half the beam radius:

$$R_L = \frac{\gamma(b)mv_z(b)}{eB_\theta(b)} = \frac{b}{2}. \tag{34}$$

Using Eqs. (29)–(31), we readily obtain

$$I_A = -\frac{4\pi\epsilon_0 mc^3}{e}\gamma_0\beta_z(b) \simeq -17\,000\beta_z(b)\gamma_0, \tag{35}$$

where $\beta_z(b) = v_z(b)/c$. Therefore,

$$\frac{I}{I_A} = \frac{1}{2}\frac{b}{L_e}\frac{I_1(b/L_e)}{I_0(b/L_e)} \sim \frac{1}{2}\frac{b}{L_e} - \frac{1}{4}, \tag{36}$$

the asymptotic form being for $b/L_e \gg 1$. This says that arbitrarily large current can be carried within a given radius, b, so long as the sheath thickness, L_e, is sufficiently small compared with b—so long as the beam can be created with sufficiently high density. Note that L_e is the usual collisionless skin depth, c/ω_{pb}, where

$$\omega_{pb} = \left(\frac{n_e(0)e^2}{\gamma_0\epsilon_0 m}\right)^{1/2} \tag{37}$$

is the electron plasma frequency of the beam for all r [on account of Eq. (30)]. Hence, if $I \gg I_A$, E_r and B_θ drop off nearly exponentially inside of $r = b$, becoming small compared with their maxima inside the depth c/ω_{pb} from $r = b$. This means that the particles which start out at $r = b$ with $v_z = V_z I_0(b/L_e)$ (for $f = 1$), leave the region of high magnetic field before they have had a chance to turn around, as they could in the "uniform beam" case, Fig. 1.

Even though all of the above equations are valid for any f we find that it is not possible for the equilibrium to exist unless there is a certain minimum of neutralization. If we restrict the maximum energy that an electron can gain in the electric field to $[\gamma(b) - 1]mc^2$, then we may write

$$\gamma_0 = \frac{\gamma(b)}{I_0[(b/L_e)(1-f)^{1/2}]} \geq 1. \tag{38}$$

This implies that

$$f \geq 1 - \left(\frac{L_e}{b} I_0^{-1}[\gamma(b)]\right)^2, \tag{39}$$

where the symbol $I_0^{-1}[\gamma(b)]$ means the argument, χ, for which $I_0(\chi) = \gamma(b)$. Therefore, unless $\gamma(b)$ is unlimited, if $b/L_e \gg 1$, f will be limited to very near 1. For example, if $\gamma(b) = 6$ ($\simeq$2.5 MV of kinetic energy) and $I = 200\,000$ A, then at minimum f ($\gamma_0 = 1$), $I/I_A \simeq 12$, corresponding to $b/L_e = 24.5$, and a minimum f of about 0.98! If we wish to put no limit on $\gamma(b)$ the consequences are impractical—$f = 0.5$ and $\gamma_0 = 1$ for $b/L_e = 24.5$ gives $\gamma(b) \simeq 2 \times 10^6$. Thus, we are reasonably justified in sticking to $f = 1$ in most of what follows.

Since $f = 1$ implies a uniform electron density within the beam, the quantity ν defined in Eq. (3) is again useful. Here, it is

$$\nu = \frac{n_e(0)\pi b^2 e^2}{4\pi\epsilon_0 mc^2} = \frac{\gamma_0 b^2}{4L_e^2}. \tag{40}$$

This implies that

$$\left(\frac{\nu}{\gamma_0}\right)^{1/2} = \frac{b}{2L_e} \simeq \frac{I}{I_A} \tag{41}$$

for a high current beam. Since I, γ_0, and b are usually experimentally measurable, ν, ω_{pb}, and c/ω_{pb} could be calculated and compared with other measurements, such as density measurements, or characteristic lengths for fields. Another experimentally measurable quantity is the propagation velocity for the bulk of an electron beam.[12,13] For $f = 1$, $I \equiv 17\,000\nu\bar{\beta}$ can be used to define such an average velocity $\bar{\beta}c$ if the current is known. Using (41) and (35), together with $\beta_z^2(b)\gamma_0^2 = \gamma_0^2 - 1$, we obtain for $I \gg I_A$

$$\bar{\beta} \simeq \frac{17\,000}{I}\frac{\gamma_0^2 - 1}{\gamma_0}. \tag{42}$$

Thus, if $I \simeq 10^5$ A and $\gamma_0 \simeq 2$, $\bar{\beta} \simeq \frac{1}{4}$, compared with $\beta_z(b) \simeq 0.86$. Finally, for $f = 1$,

$$\gamma_0^2 = \frac{1}{1 - \beta_\perp^2 - \beta_z^2} = \frac{1}{1 - \beta_z^2 I_0^2(b/L_e)}, \tag{43}$$

where $\beta_\perp c$ is the radial velocity of an electron at $r = 0$. Consequently,

$$\beta_\perp^2 = \beta_z^2\left[I_0^2\left(\frac{b}{L_e}\right) - 1\right], \tag{44}$$

so that for a high current beam, $\beta_\perp^2 \gg \beta_z^2$.

So far we have learned a great deal about this equilibrium without knowing anything about the details of electron motion. And it is clear that the self-consistent fields given by Eqs. (28) and (29) are such that electron orbits will not easily be obtained from the equations of motion. In fact, however, it is possible to obtain an orbit integral for quite a general equilibrium ($\partial/\partial t = 0$), infinite cylindrically symmetric ($\partial/\partial\phi = 0$) beam with no axial variation ($\partial/\partial z = 0$) using the three constants of the motion given by Eqs. (8)–(10). We may even have an axial magnetic field, via a vector potential component A_θ, if it is a function only of r. In this case Eq. (10) becomes

$$p_\theta = \gamma mr^2\omega - eA_\theta(r)r. \tag{45}$$

We merely have to solve

$$\gamma mc^2 = c\left[m^2c^2 + p_r^2 + \left(\frac{p_\theta}{r} + eA_\theta(r)\right)^2 + [p_z + eA_z(r)]^2\right]^{1/2} \tag{46}$$

for the radial momentum, p_r. If a subscript a implies the quantity evaluated at some initial time, t_a, we obtain

$$\frac{dr}{dt} = \pm\frac{1}{\{1 - (e/\gamma_a mc^2)[\Phi_a - \Phi(r)]\}} \cdot\Bigg\{v_{ra}^2 + \frac{r_a^2\omega_a^2}{r^2}(r^2 - r_a^2) + \frac{2e}{\gamma_a m}\left[\left(A_\theta(r) - \frac{r_a}{r}A_{\theta a}\right)\frac{r_a^2\omega_a}{r} + v_{za}[A_{za} - A_z(r)] - [\Phi_a - \Phi(r)]\right] - \left(\frac{e}{\gamma_a m}\right)^2\left[\left(A_\theta(r) - \frac{r_a}{r}A_{\theta a}\right)^2 + [A_{za} - A_z(r)]^2 - \frac{1}{c^2}[\Phi_a - \Phi(r)]^2\right]\Bigg\}^{1/2}. \tag{47}$$

We have taken $p_r \equiv \gamma m v_r \equiv \gamma m(dr/dt)$ in this equation. We, therefore, obtain the quadrature

$$t(r) - t_a = \pm\int_{r_a}^{r} \frac{dt}{dr'}\,dr'. \qquad (48)$$

The turning points of the radial motion are the zeros of dr/dt, so that Eq. (48) is defined only if r is between these turning points. The sign to be taken is $+$ $(-)$ according to whether r is greater (less) than r_a. The potentials A_z and Φ must, of course, be the self-consistent ones for the beam. For the present beam, using Φ and A_z from Eqs. (26) and (27), $A_\theta \equiv 0$, and the constants of the motion, one can obtain[16]

$$t(r) - t_a = \pm\int_{r_a}^{r} \frac{r'\,dr' I_0[(r'/L_e)(1-f)^{1/2}]}{(c^2 r'^2\{I_0^2[(r'/L_e)(1-f)^{1/2}] - (1/\gamma_0^2) - \beta_z^2 I_0^2(r'/L_e)\} - [p_\theta^2/(\gamma_0 m)])^{1/2}}. \qquad (49)$$

The remaining components of the motion can be put in integral form similarly,

$$z(r) - z_a = \int_{t_a}^{t} v_z(t')\,dt' = \int_{r_a}^{r} v_z(r')\frac{dt}{dr'}\,dr', \qquad (50)$$

$$\theta(r) - \theta_a = \int_{t_a}^{t} \omega(t')\,dt' = \int_{r_a}^{r} \omega(r')\frac{dt}{dr'}\,dr', \qquad (51)$$

$$\omega(r') = \frac{I_0[(r_a/L_e)(1-f)^{1/2}]}{I_0[(r'/L_e)(1-f)^{1/2}]}\frac{r_a^2\omega_a}{r'^2} \qquad (52)$$

is readily obtained from the constancy of p_θ together with Eq. (30). Hence, formally, we have the electron orbits for all allowed values of p_θ. Useful results, of course, require numerical computation or approximations.

We can also formally determine the distribution of angular momentum, $F(p_\theta)$, for the electrons in this beam model. By definition

$$F(p_\theta) = \int d\mathbf{x}\,d\mathbf{p}\,f_e(\mathbf{x},\mathbf{p})\,\delta[p_\theta - (xp_y - yp_x)]. \qquad (53)$$

With the substitutions $x = r\cos\phi$, $y = r\sin\phi$, $p_x = p_\perp\cos\theta$ and $p_y = p_\perp\sin\theta$, and using

$$\int_0^{2\pi} d\theta\,\delta[p_\theta - p_\perp r\sin(\theta-\phi)] = \frac{2}{(p_\perp^2 r^2 - p_\theta^2)^{1/2}}, \qquad (54)$$

$$F(p_\theta) = \frac{2n_e(0)}{\gamma_0 m}\int_{r_1}^{r_2} \frac{r\,dr I_0[(r/I_e)(1-f)^{1/2}]}{(r^2c^2\{I_0^2[(r/L_e)(1-f)^{1/2}] - (1/\gamma_0^2) - \beta_z^2 I_0^2(r/L_e)\} - [p_\theta^2/(\gamma_0 m)^2])^{1/2}}. \qquad (55)$$

r_1 and r_2 are the inner and outer turning points of a particle with angular momentum p_θ. From this we can see that p_θ^2 can be anything from 0 to $p_{\theta\,\max}^2$, the maximum of the function

$$h(r) \equiv (\gamma_0 mcr)^2\left[I_0^2\left(\frac{r}{L_e}(1-f)^{1/2}\right) - \frac{1}{\gamma_0^2} - \beta_z^2 I_0^2\left(\frac{r}{L_e}\right)\right]. \qquad (56)$$

For large currents and $f = 1$ it is reasonable to obtain the radius, R, at which a particle with $p_{\theta\max}$ circulates. By writing $\partial h(r)/\partial r = 0$ we obtain

$$1 - \frac{1}{\gamma_0^2} - \beta_z^2 I_0^2\left(\frac{R}{L_e}\right) = \beta_z^2\frac{R}{L_e}I_0\left(\frac{R}{L_e}\right)I_1\left(\frac{R}{L_e}\right). \qquad (57)$$

Using

$$\beta_z^2 = \frac{\beta_z^2(b)}{I_0^2(b/L_e)} = \frac{[1-(1/\gamma_0^2)]}{I_0^2(b/L_e)},$$

the asymptotic expansions for I_0 and I_1 [Eq. (32)] give

$$\frac{R}{L_e} \simeq \frac{1}{2}\frac{b}{L_e} - \frac{1}{4}\ln\frac{b}{L_e} + \frac{1}{2}\left[\frac{1}{4}\left(\frac{2b}{L_e} - \ln\frac{b}{L_e}\right)^2 - \frac{3}{2}\right]^{1/2}$$

$$\simeq \frac{b}{L_e} - \frac{1}{2}\ln\frac{b}{L_e}. \qquad (58)$$

For example, if $\gamma_0 = 2$ and $I = 10^5$ A, then $I/I_A = 3.4$ so that $b/L_e = 7.3$, and $R/L_e \simeq 6.3$. In this case, then, a particle with $p_{\theta\max}$ will have a (constant) z velocity of about $\frac{1}{3}c$, compared with $\beta(b) \simeq 0.865$ and $\bar{\beta} \simeq \frac{1}{4}$ found above.

Returning now to $F(p_\theta)$, comparison of Eqs. (49) and (55) reveals that if $\tau(p_\theta)$ is the time it takes for a particle with angular momentum p_θ to go from its outer turning point—r_1 to r_2—then

$$F(p_\theta) = \frac{2n_e(0)}{\gamma_0 m}\tau(p_\theta). \qquad (59)$$

The current being carried by particles with angular momentum between p_θ and $p_\theta + dp_\theta$ may be written

$$dI(p_\theta) = -eF(p_\theta)\frac{Z(p_\theta)}{\tau(p_\theta)}\,dp_\theta = -\frac{2n_e(0)}{\gamma_0 m}Z(p_\theta), \qquad (60)$$

where $Z(p_\theta)$ is the distance a particle with angular momentum, p_θ, travels between turning points. From Eq. (50), $Z(p_\theta)$ is given by

$$Z(p_\theta) = V_z \int_{r_1}^{r_2} \frac{rI_0(r/L_e)\,dr}{(c^2r^2\{I_0^2[(r/L_e)(1-f)^{1/2}] - (1/\gamma_0^2) - \beta_z^2 I_0^2(r/L_e)\} - p_\theta^2/\gamma_0^2 m^2)^{1/2}}. \tag{61}$$

For a low current beam, we will see that $Z(p_\theta)$ is a constant. However, for $I \gg I_A$, it is apparent from a numerical computation to follow that the higher p_θ particles contribute more to the current [have a greater $Z(p_\theta)$] than do the low p_θ particles.

So far we have mainly discussed $I \gg I_A$. We now take a look at the low current limit. This requires that $b \ll L_e$. Since $I_0(x) \simeq 1 + (\frac{1}{2}x)^2$ and $I_1(x) \approx \frac{1}{2}x$ for small x, it follows that $n_e(r) \approx n_e(0)$, $\gamma(r) \approx \gamma_0$, $v_z(r) \approx V_\nu$, and

$$\Phi(r) \simeq \frac{n_e(0)e(1-f)r^2}{4\epsilon_0}, \qquad r \le b, \tag{62}$$

$$A_z(r) \simeq \frac{n_e(0)eV_z r^2}{4\epsilon_0 c^2}, \qquad r \le b, \tag{63}$$

$$E_r \simeq -\frac{n_e(0)e(1-f)r}{2\epsilon_0}, \qquad r \le b, \tag{64}$$

$$B_\theta \simeq -\frac{n_e(0)eV_z r}{2\epsilon_0 c^2}, \qquad r \le b. \tag{65}$$

All of these are characteristic of the uniform beam. Consequently, it emerges as the self-consistent, fully relativistic solution to the Vlasov equation for a low current beam, $I \ll I_A$. This was obtained by Mjolsness,[17] and nonrelativistically by Longmire[18] with some nonuniform effects. For $f = 1$, Eq. (57) reveals that in the uniform beam, the particle with $p_{\theta\,\max}^2$ circulates at $b/(2)^{1/2}$. In the case of uniform v_z and γ, and $f = 1$, the perpendicular energy available from A_z at radius r gives

$$\nu mc^2\beta_z^2\left(1 - \frac{r^2}{b^2}\right) = \frac{p_r^2}{2\gamma_0 m} + \frac{p_\theta^2}{2\gamma_0 m r^2}. \tag{66}$$

Hence

$$p_{\theta\,\max}^2 = \frac{\nu\gamma_0 m^2c^2b^2\beta_z^2}{2}, \tag{67}$$

and for all p_θ^2 up to $p_{\theta\,\max}^2$, r_1 and r_2 are given by

$$r_{2,1}^2 = \frac{b^2}{2}\left[1 \pm \left(1 - \frac{p_\theta^2}{p_{\theta\,\max}^2}\right)^{1/2}\right]. \tag{68}$$

Solving Eq. (66) for $p_r^2 \equiv \gamma_0^2 m^2 (dr/dt)^2$ easily gives the orbit integral

$$t(r) = -\frac{b}{[2(\nu/\gamma_0)]^{1/2}V_z}\int_{r_2}^{r}\frac{r\,dr}{(r^2 - r_1^2)^{1/2}(r_2^2 - r^2)^{1/2}}, \qquad r_1 < r < r_2. \tag{69}$$

This gives an arcsin, and the orbits obtained by Lawson[3] and others[17,19] result. Under the assumptions for which Eq. (69) is valid, we obtain for $|p_\theta| < p_{\theta\max}$

$$\tau(p_\theta) = \frac{\pi b}{2[2(\nu/\gamma_0)]^{1/2}V_z} = \frac{Z(p_\theta)}{V_z} \tag{70}$$

and

$$F(p_\theta) = \begin{cases} \dfrac{N}{2p_{\theta\,\max}}, & |p_\theta| < p_{\theta\,\max} \\ 0 & \text{otherwise}, \end{cases} \tag{71}$$

and the beam is indeed, uniform.

Let us now return to high current beams and look at some numerical results. In Fig. 2 we plot $h(r)$ for beam parameters appropriate to the beam of Andrews *et al.*, $\gamma_0 = 2$, $I = 10^5$ A, and $f = 1$. We observe that p_θ^2 can be anything from 0 to about $101(mcL_e)^2\gamma_0/4\nu$. For any allowed p_θ, r_1 and r_2 can be obtained from the graph. A particle with $p_{\theta\,\max}$ circulates at $R/L_e \simeq 6.27$, from the numerical work, compared to 6.3 found from Eq. (58). In Fig. 3 we plot $z(r)$, $r(t)$, and $z(t)$ for several p_θ values, and find that the higher angular momentum particles go somewhat farther in the z direction between radial turning points. From Eq. (60), they, therefore, contribute more to the current. Note, however, from Fig. 3(b) and Eq. (59), that there are more low angular momentum particles.

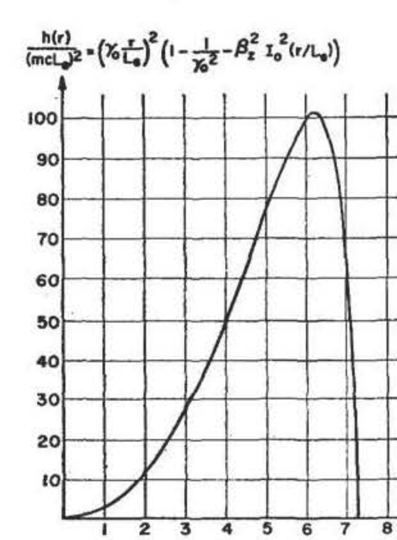
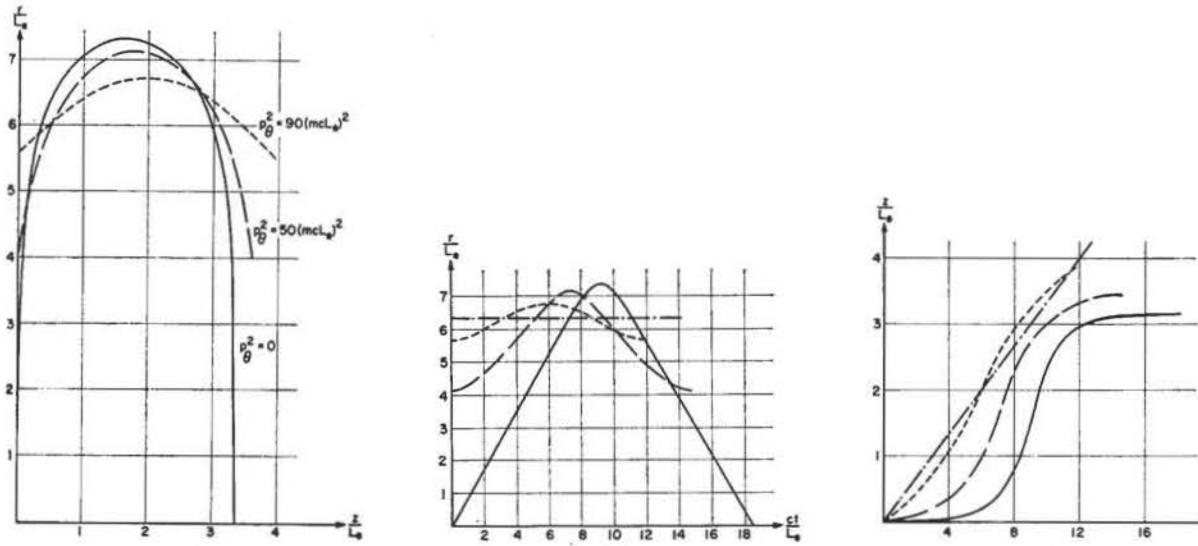


FIG. 2. $h(r)$ for $f = 1$, $\gamma_0 = 2$, $I = 10^5$ A ($\beta_z = 0.00386$, $b/L_e = 7.3$).



(a) (b) (c)

FIG. 3. (a) Particle radial position vs axial distance traveled by electrons having $p_\theta^2 = 0$, $50(mcL_e)^2$, and $90(mcL_e)^2$. (Plot runs from r_1 to r_2 and back.) (b) r/L_e vs ct/L_e for electrons with $p_\theta^2 = 0$, $50(mcL_e)^2$, $90(mcL_e)^2$, and $p_\theta^2{}_{\max} \approx 101(mcL_e)^2$. — $p_\theta^2 = 0$ (plotted from $r = 0$ to $r = b$ and back to $r = 0$); -- $p_\theta^2 = 50(mcL_e)^2$ (plotted from r_1 to r_2 and back to r_1); — — $p_\theta^2 = 90(mcL_e)^2$ (plotted from r_1 to r_2 and back to r_1); -·- $p_\theta^2 = p_\theta^2{}_{\max}$ (r is a constant). (c) z/L_e vs ct/L_e for electrons with $p_\theta^2 = 0$, $(50mcL_e)^2$, $90(mcL_e)^2$, and $p_\theta^2{}_{\max} \approx 101(mcL_e)^2$. — $p_\theta^2 = 0$ (plotted from $r = 0$ to $r = b$ and back to $r = 0$); -- $p_\theta^2 = 50$ $(mcL_e)^2$ (plotted from r_1 to r_2 and back to r_1); -- $p_\theta^2 = 90(mcL_e)^2$ (plotted from r_1 to r_2 and back to r_1); -·- $p_\theta^2 = p_\theta^2{}_{\max}$ ($z \approx ct/3$).

We now suppose that this beam could be set up with $f = 1$ in a "drift tube" with a perfectly conducting wall at $a \geq b$. Then, the sum of the magnetic field energy per meter of beam inside and outside the beam, U, is

$$U = \gamma_0^2 K \beta_z^2 \frac{b}{L_e} \cdot \left\{ I_1\left(\frac{b}{L_e}\right) I_0\left(\frac{b}{L_e}\right) - \frac{b}{2L_e}\left[I_0^2\left(\frac{b}{L_e}\right) - I_1^2\left(\frac{b}{L_e}\right)\right]\right\} + \gamma_0^2 K \left(\frac{b}{L_e}\right)^2 \beta_z^2 I_1^2\left(\frac{b}{L_e}\right) \ln\frac{a}{b}. \qquad (72)$$

In this equation, $K = \pi\epsilon_0(mc^2/e)^2 = 7.28$ J/m. Suppose the beam source is able to supply W J/m (e.g., 2 kJ in a 10 m long beam is 200 J/m). Let α_w be defined by

$$W \equiv \alpha_w \gamma_0^2 K \beta_z^2 I_0^2\left(\frac{b}{L}\right) \equiv \alpha_w K(\gamma_0^2 - 1). \qquad (73)$$

Part of W is in particle kinetic energy, U_p, and the rest is in the magnetic field. Let α_p and α_f be the field and particle contributions to α_w. Then

$$U_p = \int_0^b (\gamma_0 - 1) mc^2 n_e(r) 2\pi r\, dr = (\gamma_0 - 1)\gamma_0 K \frac{b^2}{L_e^2} \equiv \alpha_p(\gamma_0^2 - 1)K. \qquad (74)$$

The minimum magnetic field energy occurs when $a = b$, for which $\alpha_f \equiv \alpha_{\min}$:

$$\alpha_{\min} = \frac{b}{L_e}\left[\frac{I_1(b/L_e)}{I_0(b/L_e)} - \frac{b}{2L_e}\left(1 - \frac{I_1^2(b/L_e)}{I_0^2(b/L_e)}\right)\right]. \qquad (75)$$

For the case considered above, $\gamma_0 = 2$, $I = 10^5$ A ($b/L_e = 7.3$), we obtain $\alpha_p = 35.5K$ and $\alpha_{\min} = 3.13K$, for a total minimum necessary energy of 845 J/m. Note that this beam is a very efficient user of energy—most of it is in particle energy if $a = b$. By contrast, for a uniform beam

$$\frac{U}{U_p} = \frac{I}{17\,000}\frac{(\gamma_0 + 1)^{1/2}}{4\gamma_0}. \qquad (76)$$

Therefore, if it could exist at 10^5 A and $\gamma_0 = 2$, the uniform beam would have more energy tied up in fields than in particle motion. Suppose $a > b$, but $(a - b)/b \ll 1$. Then expanding the logarithm in Eq. (72), we obtain

$$\frac{a - b}{b} \simeq \alpha_f \left(\frac{L_e}{b}\right)^2 \frac{I_0^2(b/L_e)}{I_1^2(b/L_e)} - \frac{L_e}{b}\frac{I_0(b/L_e)}{I_1(b/L_e)} - \frac{b}{2L_e}\left(\frac{I_0^2(b/L_e)}{I_1^2(b/L_e)} - 1\right). \qquad (77)$$

The asymptotic expansions for I_0 and I_1 in both (75) and (77) enable us to write

$$\frac{a - b}{b} \approx \left(\frac{L_e}{b}\right)^2\left[1 + \frac{L_e}{b} + \left(\frac{L_e}{b}\right)^2\right](\alpha_f - \alpha_{\min}) \approx \frac{1}{4}\frac{I_A^2}{I^2}(\alpha_f - \alpha_{\min}). \qquad (78)$$

Then for $b/L_e = 7.3$, and $(a - b)/b = 0.1$, $\alpha_f = 7.7$. Thus, a tube radius 10% greater than the beam radius results in more than double the field energy being required. Up until this point, we have had no way to fix the beam radius given the beam current and electron energy; I/I_A merely fixes b/L_e. However, if the beam is in a drift tube of known radius, a, Eq. (78) enables us to fix b given the source energy. For example, if $W = 1000$ J/m, $\gamma_0 = 2$ and $I = 10^5$ A, then 155 J/m are available for fields outside of $r = b$. This uniquely determines b at about 0.85 A. It also implies that the beam would be hindered from pinching to a smaller radius than 0.85 A by lack of sufficient energy. In fact, this tendency against pinching would be stronger, the higher the current, as can be seen by the second approximate equality in Eq. (78).

Although there are many beam models which can carry arbitrarily large currents (for example, Bennett's[1] and Benford's[20]), the one we have been considering is particularly interesting in that it is monoenergetic, and it is confined to a finite radius. Both of these are characteristic to some extent of most high current beam experiments to date. In addition, the current density is confined to a shell near the edge of the beam, and Bradley and Ingraham have observed high current beams which exhibit this characteristic.[21]

More generally, we could superimpose beams such as we have considered with different values of P_z. An example is the electron distribution function

$$f_e = \frac{c^2}{2\pi\epsilon_e}\,\delta(H - \epsilon_e)[n_1(0)\,\delta(P_z - \gamma_0 m V_1) + n_2(0)\,\delta(P_z + \gamma_0 m V_2)], \qquad (79)$$

with $n_1(0) \ll n_2(0)$ and V_1 near c. The result would be a fast core carrying current below I_A, and a very slowly moving "halo" carrying most of the current, in which particles without angular momentum would be traveling backward over part of their orbits, much like trajectory d in Fig. 1.

III. MAGNETIC NEUTRALIZATION

We now take up the notion of magnetic neutralization of an electron beam by a background plasma. We will develop a model in this chapter which indicates that cancellation of the beam current by large numbers of slowly counterstreaming electrons from a background plasma can be expected to occur. We assume the existence of a three-dimensionally infinite, uniform, charge neutral, field free plasma consisting of mobile electrons and immobile ions. An electron beam is assumed to be moving through the plasma with velocity $\mathbf{v}_0$, the magnitude of which is large compared with the thermal velocity of the background plasma. At initial time, $t = 0$, the beam extends from $z' = -\infty$ to $z' = 0$ (a primed coordinate indicating the laboratory frame of reference) along the z' axis, and it is neither electrostatically nor magnetically neutralized. We require that the effect of the beam on the background plasma be small so that linear perturbation theory is valid, and then we consider the perturbed plasma motion in detail. Our results will, therefore, be valid for plasma density large compared with the beam density. The motion of the beam is assumed to be unaffected by the interaction. (We ignore the obvious problem of the two stream instability because the experiments which we are attempting to explain do not seem to be dominated by it.)

We solve this problem here with cold plasma two mass approximation relativistic fluid equations. This method enables us to extract the essential physics with a minimum of algebraic complication. It can be shown that a kinetic treatment with a two mass approximation Maxwellian gives the same result in the cold plasma limit.[16] In the present treatment, we will see that the use of the two mass approximation involves dropping terms of order v_e^2/c^2, where v_e is the plasma electron thermal velocity and c is the velocity of light. Therefore, retaining the pressure term in the momentum conservation equation would be inconsistent for beam velocities near c.

We attack this problem in the rest frame of the beam, in which plasma is streaming by the beam with velocity $-v_0\hat{e}_z$. (An unprimed coordinate is a beam-at-rest frame coordinate.) In this frame, the beam stretches from $z = -\infty$ to $z = 0$ for all time and produces no magnetic field. We derive our fluid equations from the Vlasov equation with a phenomenological relaxation term,

$$\frac{\partial F(\mathbf{p})}{\partial t} + \mathbf{v}\cdot\frac{\partial F(\mathbf{p})}{\partial \mathbf{x}} - e(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial F(\mathbf{p})}{\partial \mathbf{p}} = -\frac{[F(\mathbf{p}) - f_0(\mathbf{p})] - (\delta n/n)f_0(\mathbf{p})}{\tau}. \qquad (80)$$

The relaxation term, much like that in the Krook–Bhatnagar–Gross equation,[22] is constructed to conserve particles locally, since δn, the perturbed plasma number density due to the beam–plasma interaction, is related to the "total" and unperturbed plasma electron mementum distribution functions $F(\mathbf{p})$ and $f_0(\mathbf{p})$, respectively, by

$$\delta n = n\int [F(\mathbf{p}) - f_0(\mathbf{p})]\, d\mathbf{p}. \qquad (81)$$

The unperturbed plasma density is n, and τ is a phenomenological relaxation time. The first two moments of Eq. (80) are, assuming a cold plasma,

$$\frac{\partial N}{\partial t} + \nabla \cdot NV = 0, \tag{82}$$

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right)\mathbf{P} = -e(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{\mathbf{P} - \mathbf{p}_0}{\tau}. \tag{83}$$

$N(\mathbf{x}, t)$, $\mathbf{V}(\mathbf{x}, t)$, and $\mathbf{P}(\mathbf{x}, t)$ are the electron "fluid" density, velocity, and momentum, respectively, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The electron charge is $-e$. At $t = 0$, the electron fluid quantities N, $\mathbf{V}$, $\mathbf{P}$, have their unperturbed values n, $-v_0\hat{e}_z$, and $\mathbf{p}_0 = -\gamma_0 m v_0 \hat{e}_z$, respectively. The electron rest mass is m and $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. After $t = 0$, these quantities suffer perturbations due to the interaction with the beam so that $N = n + \delta n$, $\mathbf{V} = -v_0\hat{e}_z + \delta\mathbf{v}$, and $\mathbf{P} = -\gamma_0 m v_0 \hat{e}_z + \delta\mathbf{p}$. There are no applied fields, so $\mathbf{E}$ and $\mathbf{B}$ have only perturbation contributions $\delta\mathbf{E}$, and $\delta\mathbf{B}$, and only $\delta\mathbf{E}$ exists at $t = 0$ in the beam-at-rest frame. The linearized fluid equations for the perturbed quantities are, therefore,

$$\frac{\partial\, \delta n}{\partial t} + n\nabla \cdot \delta\mathbf{v} - v_0 \frac{\partial}{\partial z}\, \delta n = 0, \tag{84}$$

$$\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial z}\right) \delta\mathbf{p} = -e(\delta\mathbf{E} - v_0\hat{e}_z \times \delta\mathbf{B}) - \frac{\delta\mathbf{p}}{\tau}. \tag{85}$$

We also have Maxwell's equations for the field quantities in terms of the plasma quantities:

$$\nabla \cdot \delta\mathbf{E} = -\frac{\delta n e}{\epsilon_0} - \frac{n_b e}{\epsilon_0}, \tag{86}$$

$$\nabla \times \delta\mathbf{B} = \mu_0\, \delta\mathbf{j} + \frac{1}{c^2}\frac{\partial\, \delta\mathbf{E}}{\partial t}, \tag{87}$$

$$\nabla \times \delta\mathbf{E} = -\frac{\partial\, \delta\mathbf{B}}{\partial t}, \tag{88}$$

where $\delta\mathbf{j}$ is the background plasma current density due to the interaction. In order to close this set of equations, we need a relationship between $\delta\mathbf{v}$ and $\delta\mathbf{p}$. The "total" quantities N, $\mathbf{V}$, and $\mathbf{P}$ and the perturbed quantities δn, $\delta\mathbf{v}$, and $\delta\mathbf{p}$ are defined in terms of $F(\mathbf{p})$ by

$$N = \int F(\mathbf{p})\, d\mathbf{p} = n + \delta n, \tag{89}$$

$$N\mathbf{V} = \int \mathbf{v}F(\mathbf{p})\, d\mathbf{p}, \qquad N\,\delta\mathbf{v} = \int (\mathbf{v} + v_0\hat{e}_z)F(\mathbf{p})\, d\mathbf{p}, \tag{90}$$

$$N\mathbf{P} = \int \mathbf{p}F(\mathbf{p})\, d\mathbf{p}, \qquad N\,\delta\mathbf{p} = \int (\mathbf{p} - \mathbf{p}_0)F(\mathbf{p})\, d\mathbf{p}, \tag{91}$$

where $\mathbf{v}$ and $\mathbf{p}$ are the velocity and momentum of an individual electron. Thus,

$$N\mathbf{V} = N\,\delta\mathbf{v} - Nv_0\hat{e}_z. \tag{92}$$

But $\mathbf{p} = \gamma m\mathbf{v}$ for each electron, where

$$\gamma = (1 + p^2/m^2c^2)^{+1/2}.$$

A Taylor expansion of γ under the assumption $|\mathbf{p} - \mathbf{p}_0| \ll mc$ enables us to obtain

$$N\mathbf{V} = \int \frac{\mathbf{p}}{\gamma m} F(\mathbf{p})\, d\mathbf{p} = \frac{N}{\gamma_0 m}\left(\mathbf{p}_0 + \delta\mathbf{p} - \frac{v_0^2\, \delta p_z \hat{e}_z}{c^2}\right) + O\left(\int \frac{(\mathbf{p} - \mathbf{p}_0)^2}{m^2c^2} F(\mathbf{p})\, d\mathbf{p}\right). \tag{93}$$

Therefore,

$$\delta\mathbf{v} \simeq \frac{\delta\mathbf{p}}{\gamma_0 m} - \frac{v_0^2}{c^2}\frac{\delta p_z \hat{e}_z}{\gamma_0 m} = \frac{\delta p_x\hat{e}_x + \delta p_y\hat{e}_y}{\gamma_0 m} + \frac{\delta p_z\hat{e}_z}{\gamma_0^3 m} \tag{94}$$

which is a statement of the two mass approximation. The terms we have dropped from Eq. (93) to get (94) are of the same order as the pressure term in the momentum equation would have been had we kept it.

The Fourier–Laplace transform, defined by the operator

$$\int_0^\infty dt \exp(-st) \int_{-\infty}^{\infty} d\mathbf{x} \exp(-i\mathbf{k}\cdot\mathbf{x}),$$

of the linearized fluid equations and the last two of Maxwell's equations are

$$(s - ik_z v_0)\, \delta n + i\mathbf{k} \cdot \delta\mathbf{v} n = 0, \tag{95}$$

$$\left(s - ik_z v_0 + \frac{1}{\tau}\right) \delta\mathbf{p} = -e(\delta\mathbf{E} - v_0\hat{e}_z \times \delta\mathbf{B}), \tag{96}$$

$$-s\, \delta\mathbf{B} = i\mathbf{k} \times \delta\mathbf{E}, \tag{97}$$

$$i\mathbf{k} \times \delta\mathbf{B} = \mu_0\, \delta\mathbf{j} + \frac{s\,\delta\mathbf{E}}{c^2} - \frac{1}{c^2}\, \delta\mathbf{E}(\mathbf{k}, t = 0). \tag{98}$$

The Laplace and Fourier transform variables are s and $\mathbf{k}$, respectively, where $k_z = \mathbf{k} \cdot \hat{e}_z$. δn, $\delta\mathbf{v}$, $\delta\mathbf{p}$, $\delta\mathbf{E}$, and $\delta\mathbf{B}$ are now all functions of $\mathbf{k}$ and s, and we have used $\delta n(t = 0) = \delta\mathbf{p}(t = 0) = \delta\mathbf{B}(t = 0) = 0$. The plasma response current $\delta\mathbf{j}$ is related to δn, $\delta\mathbf{v}$, and $\delta\mathbf{E}$ by

$$\delta\mathbf{j} = -e\delta(N\mathbf{V}) = -e(n\delta\mathbf{v} - v_0\hat{e}_z\, \delta n) \equiv \boldsymbol{\sigma} \cdot \delta\mathbf{E}, \tag{99}$$

where $\boldsymbol{\sigma}$ is the response "conductivity" tensor. Finally, from the Fourier transform of Eq. (86) evaluated at $t = 0$,

$$\delta\mathbf{E}(\mathbf{k}, t = 0) = -\frac{i\mathbf{k}\rho_b(\mathbf{k})}{\epsilon_0 k^2}. \tag{100}$$

ρ_b, the charge density of the beam, is the only charge density at $t = 0$. By our assumptions, it is not a function of time in the beam-at-rest frame. The magnitude of $\mathbf{k}$ is k.

Equations (94)–(100) constitute a closed set of equations, and in the Appendix, we solve them. The results for $\delta\mathbf{j}$ and $\delta\mathbf{E}$ in Fourier–Laplace transform space are

$$\delta j_1 = \frac{i\rho_b(\mathbf{k})\omega_p^2}{kD_s} \cdot\left[\left(1 - \frac{v_0^2k_z^2}{c^2k^2}\right)\left(k^2 + \frac{s^2}{c^2} + \frac{\omega_p^2(s - ik_zv_0)}{c^2(s - ik_zv_0 + 1/\tau)}\right) - \frac{\omega_p^2k_\perp^2v_0^2(s - ik_zv_0)}{k^2c^4(s - ik_zv_0 + 1/\tau)}\right], \tag{101a}$$

$$\delta j_2 = -\frac{\omega_p^2k_\perp v_0\rho_b(\mathbf{k})}{ks\ D_s}\left(1 + \frac{ik_zv_0s}{k^2c^2}\right)\left(k^2 + \frac{s^2}{c^2}\right), \tag{101b}$$

$$\delta j_3 = 0 = \delta E_3, \tag{101c}$$

$$\delta E_1 = -\frac{i\rho_b(k)}{s\epsilon_0k\ D_s} \cdot\left[\left(k^2 + \frac{s^2}{c^2}\right)\left(s + \frac{1}{\tau} - ik_zv_0\right)(ik_zv_0 - s) - \frac{\omega_p^2}{c^2}(s - ik_zv_0)^2\left(1 - \frac{k_\perp^2v_0^2[1 + (s^2/k^2c^2)]}{(s - ik_zv_0)^2}\right)\right], \tag{101d}$$

$$\delta E_2 = \frac{\omega_p^2k_\perp v_0}{kc^2\epsilon_0\ D_s}\left(1 + \frac{ik_zv_0s}{k^2c^2}\right)\rho_b(\mathbf{k}), \tag{101e}$$

where

$$D_s = \left[\left(s + \frac{1}{\tau} - ik_zv_0\right)(ik_zv_0 - s) - \omega_p^2\left(1 - \frac{v_0^2}{c^2}\frac{k_z^2}{k^2}\right)\left(k^2 + \frac{s^2}{c^2}\right) - \frac{\omega_p^2}{c^2}(s - ik_zv_0)^2\left(1 - \frac{k_\perp^2v_0^2[1 + (s^2/k^2c^2)]}{(s - ik_zv_0)^2}\right) - \frac{\omega_p^4(s - ik_zv_0)}{\gamma_0^2c^2(s + 1/\tau - ik_zv_0)}\right]. \tag{102}$$

The 1, 2, and 3 directions are defined by

$$\hat{e}_1 = \frac{\mathbf{k}}{k}, \qquad \hat{e}_3 = \frac{\mathbf{k}\times\hat{e}_z}{k_\perp}, \qquad \hat{e}_2 = \hat{e}_3\times\hat{e}_1, \tag{103}$$

and $k_\perp$ is the component of $\mathbf{k}$ perpendicular to $\hat{e}_z$. $\delta\mathbf{B}$ is easily obtained from Eqs. (101) and (97), giving only $\delta B_3 \neq 0$. The perturbed charge density, $\delta\rho = -e\ \delta n$, is most easily obtained from Eq. (101a) through the Fourier–Laplace transform of the charge continuity equation,

$$s\ \delta\rho + ik\ \delta j_1 = 0. \tag{104}$$

We are interested in the behavior of our beam plasma system after the initial transients (and presumably the effects of our artificial initial conditions) have died down. Therefore, we take advantage of the final-value theorem of Laplace transform theory, which enables us to write for any quantity, δQ,

$$\lim_{t\to\infty} \delta Q(\mathbf{k}, t) = \lim_{s\to 0} \delta Q(\mathbf{k}, s). \tag{105}$$

To obtain the spatial variation of the quantity, we must invert the Fourier transform:

$$\delta Q(\mathbf{x}, t\to\infty) = \int \frac{d^3k\ \exp(i\mathbf{k}\cdot\mathbf{x})}{(2\pi)^3}\left[\lim_{t\to\infty}\delta Q(\mathbf{k})\right]. \tag{106}$$

For ρ_b, we choose a uniform beam of radius b and electron density n_b:

$$\rho_b(\mathbf{x}) = \begin{cases} -n_be, & z \leq 0,\ r \leq b, \\ 0, & \text{otherwise}. \end{cases} \tag{107}$$

The Fourier inversions of the quantities $\delta\mathbf{j}$, $\delta\mathbf{E}$, $\delta\mathbf{B}$, and $\delta\rho$ for this $\rho_b(\mathbf{x})$ are obtained in the Appendix. These results, in the beam-at-rest frame, valid for a weakly collisional plasma ($\omega_p\tau \gg 1$) are, for $z < 0$:

$$\delta j_r = n_bev_0\frac{\gamma_0\omega_pb}{v_0}\sin\frac{\omega_pz}{\gamma_0v_0}\exp\left(\frac{z}{2v_0\tau}\right)\begin{Bmatrix} I_1\left(\dfrac{\omega_pr}{v_0}\right)K_1\left(\dfrac{\omega_pb}{v_0}\right) \\ I_1\left(\dfrac{\omega_pb}{v_0}\right)K_1\left(\dfrac{\omega_pr}{v_0}\right) \end{Bmatrix}, \tag{108}$$

$$\delta j_z = -n_bev_0\left[\gamma_0^2\frac{\omega_pb}{c}\begin{Bmatrix} I_0\left(\dfrac{\omega_pr}{c}\right)K_1\left(\dfrac{\omega_pb}{c}\right) \\ -I_1\left(\dfrac{\omega_pb}{c}\right)K_0\left(\dfrac{\omega_pr}{c}\right) \end{Bmatrix} - \cos\frac{\omega_pz}{\gamma_0v_0}\exp\left(\frac{z}{2v_0\tau}\right)\gamma_0^2\frac{\omega_pb}{v_0}\begin{Bmatrix} I_0\left(\dfrac{\omega_pr}{v_0}\right)K_1\left(\dfrac{\omega_pb}{v_0}\right) \\ -I_1\left(\dfrac{\omega_pb}{v_0}\right)K_0\left(\dfrac{\omega_pr}{v_0}\right) \end{Bmatrix}\right], \tag{109}$$

$$\delta B_\theta = -\mu_0 n_b e v_0 \left[\gamma_0^2 b \begin{Bmatrix} I_1\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ I_1\left(\frac{\omega_p b}{c}\right) K_1\left(\frac{\omega_p r}{c}\right) \end{Bmatrix} - \gamma_0^2 b \cos\frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \begin{Bmatrix} I_1\left(\frac{\omega_p r}{v_0}\right) K_1\left(\frac{\omega_p b}{v_0}\right) \\ -I_1\left(\frac{\omega_p b}{v_0}\right) K_1\left(\frac{\omega_p r}{v_0}\right) \end{Bmatrix} \right], \tag{110}$$

$$\delta E_r = \frac{n_b e b}{\epsilon_0} \left[\gamma_0^2 \frac{v_0^2}{c^2} \begin{Bmatrix} I_1\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ I_1\left(\frac{\omega_p b}{c}\right) K_1\left(\frac{\omega_p r}{c}\right) \end{Bmatrix} - \gamma_0^2 \cos\frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \begin{Bmatrix} I_1\left(\frac{\omega_p r}{v_0}\right) K_1\left(\frac{\omega_p b}{v_0}\right) \\ I_1\left(\frac{\omega_p b}{v_0}\right) K_1\left(\frac{\omega_p r}{v_0}\right) \end{Bmatrix} \right], \tag{111}$$

$$\delta E_z = -\frac{n_b e b}{\epsilon_0} \gamma_0 \sin\frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \begin{Bmatrix} \frac{v_0}{\omega_p b}\left[1 - \frac{\omega_p b}{v_0} I_0\left(\frac{\omega_p r}{v_0}\right) K_1\left(\frac{\omega_p b}{v_0}\right)\right] \\ I_1\left(\frac{\omega_p b}{v_0}\right) K_0\left(\frac{\omega_p r}{v_0}\right) \end{Bmatrix}, \tag{112}$$

$$\delta\rho = n_b e \left[\begin{Bmatrix} 1 - \cos\frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \\ 0 \end{Bmatrix} + (\gamma_0^2 - 1) \begin{Bmatrix} \frac{\omega_p b}{c} I_0\left(\frac{\omega_p r}{c}\right) K_1\left(\frac{\omega_p b}{c}\right) \\ -\frac{\omega_p b}{c} I_1\left(\frac{\omega_p b}{c}\right) K_0\left(\frac{\omega_p r}{c}\right) \end{Bmatrix} \right.$$
$$\left. - (\gamma_0^2 - 1) \frac{\omega_p b}{v_0} \cos\frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \begin{Bmatrix} I_0\left(\frac{\omega_p r}{v_0}\right) K_1\left(\frac{\omega_p b}{v_0}\right) \\ -I_1\left(\frac{\omega_p b}{v_0}\right) K_0\left(\frac{\omega_p r}{v_0}\right) \end{Bmatrix} \right]. \tag{113}$$

The upper (lower) line is for $r < b$ ($r > b$), and the quantities, δQ, have arguments ($\mathbf{x}$, $t \to \infty$). All of these quantities are zero for $z > 0$. I_m and K_m are modified Bessel functions of the first and second kind, respectively, and order m. In these results, a contribution to each perturbed quantity of order $\exp(-\omega_p|z|/c)$, which is, therefore, significant only within c/ω_p of $z = 0$, has been dropped. (Note that $\omega_p\tau \gg 1$ implies $c/\omega_p \ll 2v_0\tau$ for v_0 near c.) This contribution is discussed in the Appendix. In addition, as mentioned in the Appendix, we have also dropped the collisional damping of the induced plasma current, which was pointed out to us by Lee and Sudan,[23] as it is unimportant in the parameter regime of particular interest to us.

Let us now look at a few of the characteristics of this solution in the beam-at-rest frame. Firstly, we note that several of the perturbed quantities are discontinuous across $r = b$. This is due to the discontinuous beam model and the cold plasma assumption. {Retaining the strongly damped terms—O [$\exp(-\omega_p |z|/c)$]—would result in all quantities being continuous through $z = 0$.} We can easily calculate that the net axial current in the entire beam–plasma system is 0. For $\omega_p b/c \gg 1$, the current density is confined to a sheath of thickness c/ω_p around $r = b$ since[15]

$$I_{0,1}\left(\frac{\omega_p r}{c}\right) K_{0,1}\left(\frac{\omega_p b}{c}\right) \sim \frac{c \exp[-(\omega_p/c)\,|b - r|]}{2\omega_p (rb)^{1/2}}. \tag{114}$$

The same thing can be said about the rest of the quantities except for $\delta\rho$ and δE_z. We are, therefore, led to the following physical interpretation: The electron "fluid" flowing in toward the beam from the right does not know the beam is there until it reaches $z = 0$ (actually $z \sim c/\omega_p$ had we not dropped the strongly damped term). Suddenly encountering the beam, the electron fluid expands within the beam (that is, the density decreases as plasma electrons are thrown out of the beam) in an attempt to neutralize the bulk of the beam charge density. A standing wave is set up as a result of this; this wave is simply a damped plasma oscillation in the laboratory frame. When the electron fluid oscillation has been damped ($|z| > 2v_0\tau$), the bulk of the beam charge density has been neutralized by a net ion density of n_b having been left behind [the first term for $r < b$ in Eq. (113)]. The excess charge has been carried off to infinity since the

$$2\pi \int_0^\infty (\delta\rho + \rho_b) r\, dr$$

is zero for $|z| > 2v_0\tau$. The ions that have been left behind for $|z| > 2v_0\tau$ must be contributing a current density of $-n_b e v_0$ to δj_z, the magnitude of which is certainly small compared to this everywhere but within c/ω_p of $r = b$. The canceling current is a result of a net acceleration in the $-z$ direction of the electron fluid within the beam. From the Appendix, Eq. (A22),

$$\delta v_z = -\frac{n_b}{n} v_0 \left\{ \left[\left\{ \begin{matrix} 1 - \cos \dfrac{\omega_p z}{\gamma_0 v_0} \exp\left(\dfrac{z}{2v_0\tau}\right) \\ 0 \end{matrix} \right\} - \frac{\omega_p b}{c} \left\{ \begin{matrix} I_0\left(\dfrac{\omega_p r}{c}\right) K_1\left(\dfrac{\omega_p b}{c}\right) \\ -I_1\left(\dfrac{\omega_p b}{c}\right) K_0\left(\dfrac{\omega_p r}{c}\right) \end{matrix} \right\} \right. \right.$$
$$\left. \left. + \cos \frac{\omega_p z}{\gamma_0 v_0} \exp\left(\frac{z}{2v_0\tau}\right) \left\{ \begin{matrix} \dfrac{\omega_p b}{v_0} I_0\left(\dfrac{\omega_p r}{v_0}\right) K_1\left(\dfrac{\omega_p b}{v_0}\right) \\ -\dfrac{\omega_p b}{v_0} I_1\left(\dfrac{\omega_p b}{v_0}\right) K_0\left(\dfrac{\omega_p r}{v_0}\right) \end{matrix} \right\} \right] \right\}, \qquad z < 0. \tag{115}$$

As before, the upper (lower) line is $r < b$ ($r > b$), and $\delta v_z = 0$ for $z > 0$. [In this, we have dropped the same contributions as in Eqs. (108)–(113).] It is clear that the electron current due to the first pair of braces in δv_z for $r < b$ is exactly that required to cancel the same term in $\delta\rho$, leaving only the sheath current density in δj_z. From Eq. (A20) it is clear that δE_z is responsible for the acceleration of the electron fluid. Finally, we can see that unless $n_b \ll n$, δv_z and $\delta\rho$ will not be small compared to the unperturbed quantities in the electron plasma. Since we have used linear perturbation theory to obtain our solution, we must have $|\delta v_z| \ll v_0$ and $|\delta\rho| \ll ne$. Therefore, our solution can be valid only for $n_b \ll n$.

We now transform the complete solution into the laboratory frame of reference using the appropriate relativistic transformation for each quantity. With primes denoting laboratory frame quantities, for $z' - v_0 t' < 0$, and large t', there results

$$\delta j_r' \simeq n_b' e v_0 \sin \frac{\omega_p'(z' - v_0 t')}{v_0} \exp\left(\frac{z' - v_0 t'}{2v_0\tau'}\right) \left\{ \begin{matrix} \dfrac{\omega_p' b}{v_0} I_1\left(\dfrac{\omega_p' r'}{v_0}\right) K_1\left(\dfrac{\omega_p' b}{v_0}\right) \\ \dfrac{\omega_p' b}{v_0} I_1\left(\dfrac{\omega_p' b}{v_0}\right) K_1\left(\dfrac{\omega_p' r'}{v_0}\right) \end{matrix} \right\}, \tag{116}$$

$$\delta j_z' + \left\{ \begin{matrix} -n_b' e v_0 \\ 0 \end{matrix} \right\} \simeq -n_b' e v_0 \left[\left\{ \begin{matrix} \dfrac{\omega_p' b}{c} I_0\left(\dfrac{\omega_p' r'}{c}\right) K_1\left(\dfrac{\omega_p' b}{c}\right) \\ -\dfrac{\omega_p' b}{c} I_1\left(\dfrac{\omega_p' b}{c}\right) K_0\left(\dfrac{\omega_p' r'}{c}\right) \end{matrix} \right\} \right.$$
$$\left. + \cos \frac{\omega_p'(z' - v_0 t')}{v_0} \exp\left(\frac{z' - v_0 t'}{2v_0\tau'}\right) \left\{ \begin{matrix} \left[1 - \dfrac{\omega_p' b}{v_0} I_0\left(\dfrac{\omega_p' r'}{v_0}\right) K_1\left(\dfrac{\omega_p' b}{v_0}\right)\right] \\ \dfrac{\omega_p' b}{v_0} I_1\left(\dfrac{\omega_p' b}{v_0}\right) K_0\left(\dfrac{\omega_p' r'}{v_0}\right) \end{matrix} \right\} \right], \tag{117}$$

$$\delta B_\theta' \simeq -\mu_0 n_b' e v_0 b \left\{ \begin{matrix} I_1\left(\dfrac{\omega_p' r'}{c}\right) K_1\left(\dfrac{\omega_p' b}{c}\right) \\ I_1\left(\dfrac{\omega_p' b}{c}\right) K_1\left(\dfrac{\omega_p' r'}{c}\right) \end{matrix} \right\}, \tag{118}$$

$$\delta E_r' \simeq -\frac{n_b' e b}{\epsilon_0} \cos \frac{\omega_p'(z' - v_0 t')}{v_0} \exp\left(\frac{z' - v_0 t'}{2v_0\tau'}\right) \left\{ \begin{matrix} I_1\left(\dfrac{\omega_p' r'}{v_0}\right) K_1\left(\dfrac{\omega_p' b}{v_0}\right) \\ I_1\left(\dfrac{\omega_p' b}{v_0}\right) K_1\left(\dfrac{\omega_p' r'}{v_0}\right) \end{matrix} \right\}, \tag{119}$$

$$\delta E_z' \simeq -\frac{n_b' e b}{\epsilon_0} \sin \frac{\omega_p'(z'-v_0t')}{v_0} \exp\left(\frac{z'-v_0t}{2v_0\tau'}\right) \begin{cases} \dfrac{v_0}{\omega_p' b}\left[1-\dfrac{\omega_p' b}{v_0} I_0\left(\dfrac{\omega_p' r'}{v_0}\right) K_1\left(\dfrac{\omega_p' b}{v_0}\right)\right] \\ I_1\left(\dfrac{\omega_p' b}{v_0}\right) K_0\left(\dfrac{\omega_p' r'}{v_0}\right) \end{cases}, \tag{120}$$

$$\delta\rho' + \begin{Bmatrix} -n_b' e \\ 0 \end{Bmatrix} \simeq -n_b' e \begin{cases} \cos \dfrac{\omega_p'(z'-v_0t')}{v_0} \exp\left(\dfrac{z'-v_0t'}{2v_0\tau'}\right) \\ 0 \end{cases}. \tag{121}$$

The upper (lower) line is again for $r < b$ ($r > b$). Note that to $\delta\rho'$ and $\delta j_z'$ above, we have added the beam charge and current densities; therefore, the exhibited quantities are the *net* charge and current densities. Since contributions from the strongly damped terms have been dropped, all of the above quantities are zero for $z' - v_0t' > 0$.

In this frame it is clear that the net charge is zero away from the front of the beam, and that at a fixed $z' < v_0t'$, we have a simple damped plasma oscillation. As in the beam frame, for $\omega_p'b/c \gg 1$, the net current density is confined to a sheath of thickness c/ω_p'. Therefore, if a beam electron has left this sheath before it has gained much perpendicular energy—$|v_z| \gg |v_\perp|$—then we are justified in having said that the beam is unaffected by the interaction. We would, therefore, want the Larmor radius R_L of a beam electron in the maximum magnetic field to be large compared with c/ω_p'. This is

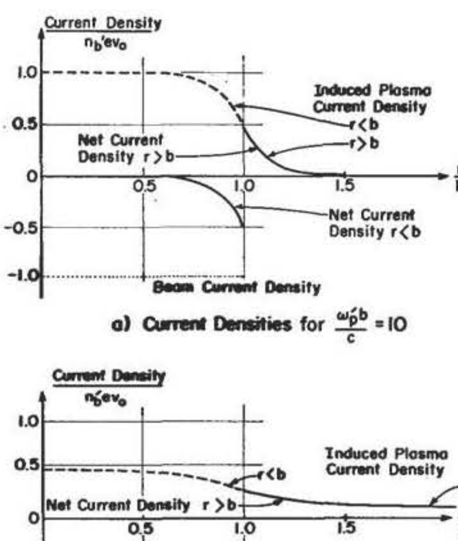


FIG. 4. Current densities relative to $n_b'ev_0$ for $z' \gg 2v_0\tau'$.

nearly equivalent to the original $n_b \ll n$ requirement:

$$R_L = \frac{\gamma_0 m v_0}{e\,\delta B_\theta(r=b)} \cong \frac{\gamma_0 m v_0}{e\mu_0 n_b' e v_0 b}\frac{2\omega_p' b}{c} \gg \frac{c}{\omega_p'} \tag{122}$$

or

$$n_b' \ll \frac{\omega_p'^2 2m\gamma_0}{\mu_0 c^2 e^2} = 2\gamma_0 n'. \tag{123}$$

Approximately the same inequality results from consideration of time scales. For the plasma to be able to charge and current neutralize the system before the beam has expanded significantly, we must have $\omega_p'^{-1} \ll \omega_{pb}'^{-1}$, ω_{pb}' being the beam electron plasma frequency. Except for the factor 2, relation (123) results. Thus, an arbitrarily large total beam current can be propagated in this model as long as the beam radius is sufficiently large so that $n_b' \ll n'$. As an indication of how the space dependence of the net current density changes with $\omega_p'b/c$, in Fig. 4, we plot the ratio of $\delta j_z'$ to $n_b'ev_0$ for $\omega_p'b/c = 1$ and 10.

If $b \ll c/\omega_p'$, the total current within $r' = b$ for $|z'| > 2v_0\tau'$ is[15]

$$-2\pi b^2 n_b' e v_0 I_1\left(\frac{\omega_p' b}{c}\right) K_1\left(\frac{\omega_p' b}{c}\right) \simeq -n_b' e v_0 \pi b^2, \tag{124}$$

which is the full beam current. The same magnitude net current with the opposite sign is flowing outside $r' = b$. Hence, in this limit, all of the return current being supplied by the plasma is outside the beam, and no magnetic neutralization occurs.

For the beam of Andrews *et al.*, a current of 10^5 A at $\gamma_0 = 2$ in a radius of 5 cm gives $n_b' \simeq 2\times10^{11}/\text{cm}^3$. An ambient pressure of 0.5 Torr implies about $2\times10^{16}/\text{cm}^3$ neutral density. The plasma density at a point after a length L of beam has passed is approximately given by

$$n' \simeq n_b' n_0 \sigma_I L, \tag{125}$$

where n_0 is the neutral density and σ_I is the appropriate effective ionization cross section. From $\sigma_I \simeq 2\times10^{-18}$ cm^2,[24]

$$n' \simeq 4 n_b' L \tag{126}$$

(L in meters). A pulse duration of 50 nsec implies a beam length of order 15 m. Therefore, for purposes of our theory, we assume $n' \gtrsim 2 \times 10^{12}$ ($\gg n_b'$), so that $\omega_p' > 8 \times 10^{10}$/sec, $c/\omega_p' < 0.4$ cm and $b\omega_p'/c > 12$. Estimating τ' from the formula given by Rose and Clark,[25] we obtain about 1.5×10^{-9} sec, so that $\omega_p'\tau' \gg 1$ and $2v_0\tau' \sim 1$ m. From these numbers, we see that the case of interest is $b\omega_p'/c \gg 1$, in which a great deal of current neutralization is to be expected. [The length characteristic of the collisional damping of the return current is estimated[23] to be of order $(b\omega_p'/c)^2 2v_0\tau' \gg 2v_0\tau'$. Therefore, collisional damping should not dominate the current neutralization over most of the beam length in the high current beam propagation experiments to date.]

IV. LONGITUDINAL GUIDE FIELD

In this section we take up the problem of a uniform electron beam of radius b, infinite in the axial (z) direction, in the presence of a uniform, axial magnetic induction $\mathbf{B}_0 = B_0\hat{e}_z$. We will find that if B_0 is much larger than the self-magnetic induction of the beam, the electron beam can be expected to propagate.

For this problem, Cartesian coordinates (x, y, z) prove to be the most convenient. Therefore, we express the self-fields of the beam, Eqs. (64) and (65), as

$$E_x = -\frac{Ne(1-f)x}{2\pi\epsilon_0 b^2}, \quad E_y = -\frac{Ne(1-f)y}{2\pi\epsilon_0 b^2}, \tag{127}$$

$$B_x = \frac{NeVy}{2\pi\epsilon_0 c^2 b^2}, \qquad B_y = -\frac{NeVx}{2\pi\epsilon_0 c^2 b^2}, \tag{128}$$

where V is the beam propagation velocity and N is again the number of electrons per meter of beam. We intend to use these fields, together with $\mathbf{B}_0$, in

$$\frac{d}{dt}\gamma m\mathbf{v} = -e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{129}$$

under the assumptions which make a uniform beam self-consistent:

$$|v_\perp| \ll V, \tag{130a}$$

$$|v_z - V| \ll V, \tag{130b}$$

$$\gamma = (1-\beta^2)^{-1/2} \simeq \text{const.} \tag{130c}$$

$\beta = V/c$ is assumed near one.

Under the above assumptions the equations of motion for a beam electron reduce approximately to

$$\gamma m\ddot{x} \simeq \frac{Ne^2[(1-f)-\beta^2]x}{2\pi\epsilon_0 b^2} - e\dot{y}B_0, \tag{131}$$

$$\gamma m\ddot{y} \simeq \frac{Ne^2[(1-f)-\beta^2]y}{2\pi\epsilon_0 b^2} + e\dot{x}B_0, \tag{132}$$

$$\gamma^3 m\ddot{z} \simeq 0. \tag{133}$$

Dropped terms are of order $v_\perp/c$ or $|v_z - V|/c$. A dot represents a time derivative in these equations. Defining

$$\Omega_0 = \frac{eB_0}{\gamma m}, \tag{134a}$$

$$\Omega^2 = \frac{Ne^2}{\gamma 2\pi\epsilon_0 m b^2}(1-f-\beta^2) \equiv \frac{2\nu c^2}{\gamma b^2}(1-f-\beta^2), \tag{134b}$$

$$\zeta = x + iy, \tag{134c}$$

and multiplying Eq. (132) by i and adding it to Eq. (131), results in

$$\ddot{\zeta} = -\Omega^2\zeta + i\Omega_0\dot{\zeta}. \tag{135}$$

Looking for solutions of the form A exp ($i\omega t$), we obtain

$$\omega^2 - \Omega_0\omega - \Omega^2 = 0. \tag{136}$$

The two roots of this equation are

$$\omega_\pm = \frac{\Omega_0}{2}\left\{1 \pm \left[1 + \left(\frac{2\Omega}{\Omega_0}\right)^2\right]^{1/2}\right\}. \tag{137}$$

If we take initial conditions $\zeta(t=0) = \zeta_0$ and $\dot{\zeta}(t=0) = 0$, then

$$\zeta(t) = \zeta_0\left(\frac{\omega_-}{\omega_- - \omega_+}\exp(i\omega_+ t) + \frac{\omega_+}{\omega_+ - \omega_-}\exp(i\omega_- t)\right). \tag{138}$$

We can find $x(t)$ and $y(t)$ from the real and imaginary parts of $\zeta(t)$.

Consider the limit $\Omega_0^2 \gg 4|\Omega^2|$. In this case,

$$\omega_+ \simeq \Omega_0\left[1 + \left(\frac{\Omega}{\Omega_0}\right)^2\right], \quad \omega_- \approx -\frac{\Omega^2}{\Omega_0} \tag{139}$$

and we obtain

$$\zeta(t) \simeq \zeta_0\left[\left(\frac{\Omega}{\Omega_0}\right)^2 \exp(i\Omega_0 t) + \left(1 - \frac{\Omega^2}{\Omega_0^2}\right)\exp\left(-i\frac{\Omega^2}{\Omega_0}t\right)\right]. \tag{140}$$

Hence, in the large axial field limit we have the sum of two rotations—a high-frequency gyration with radius $|\zeta_0|\ (\Omega/\Omega_0)^2 \ll |\zeta_0|$ about a guiding center which slowly rotates around the beam axis with radius nearly $|\zeta_0|$. These are, of course, superimposed on the "uniform" motion in the z direction with velocity V.

The assumption that $|v_\perp| \ll V$, taking the worst case for $|\xi|$, yields the requirement

$$B_0 \gg \frac{|B_{\max}|}{\beta^2}(1 - \beta^2 - f), \qquad (141)$$

Where $|B_{\max}|$ is the maximum value of the self-magnetic field of the beam. This is much stronger than the inequality $\Omega_0^2 \gg 4|\Omega^2|$ for a high current beam. For $f = 1$, the relation is $B_0 \gg |B_{\max}|$ and we have a practical limit for neutralized beam current in this model since arbitrarily large guide fields are expensive. However, in principle, by applying a large enough guide field, arbitrarily large currents could be propagated without the occurrence of catastrophic pinching due to the self-field of the beam.

In this discussion, we have assumed that the axial magnetic induction is uniform. However, the analysis should be applicable so long as the change in the guide field over a gyroradius is small.

It also should be noticed that the perpendicular motion equation, (140), contains only Ω^2. This means that a change in the sign of $1 - f - \beta^2$ only changes the direction of the slow rotation. One can, therefore, apply it to a totally unneutralized beam ($f = 0$) as well as to a neutralized beam ($f = 1$). Applicability extends to magnetically neutralized beams as well with the substitution of $1 - f - \beta^2(1 - f_m)$ for $1 - f - \beta^2$ in Ω^2. In this case, of course, less guide field is required since $|B_{\max}|$ decreases.

V. DISCUSSION

In the three preceding sections, we presented three models of relativistic electron beams which allow the propagation of arbitrarily large currents within a finite radius. The three models avoid the catastrophic self-magnetic pinch exhibited by electrostatically neutralized high current uniform beams by three distinct physical mechanisms. The fully relativistic self-consistent equilibrium does it by concentrating the current density near the edge of the beam so that beam electrons have left high field regions before they have a chance to turn around on themselves. The initial value problem solution suggests that a beam propagating into a high-density background plasma will avoid the self-pinch problem by inducing plasma currents which cancel out the self-magnetic field of the beam. Finally, adding the strong axial guide field to the uniform beam solves the problem by limiting radial excursions by a beam electron to small ones in the form of a rotation about the guiding center of the electron whose radial position is approximately constant.

All three of our models serve to explain some experimental observations made to date on propagating high current relativistic electron beams. Experiments with an axial guide field being done by Bzura, Andrews, and Fleischmann[26] with the guide field related to the maximum beam self-field by $|B_0| > |B_{\max}|$, at various ambient pressures, indicate that the guide field does help with beam propagation. Relatively slow beam propagation velocities observed both by Andrews *et al.*,[13] and by Yonas and Spence,[12] and the beams with a shell current density observed occasionally by Bradley and Ingraham, seem to point to the nonuniform equilibrium. Finally, magnetic field measurements made on high ν/γ beams injected into drift regions at pressures above 0.1 Torr indicate that partial magnetic neutralization takes place,[12,13] as predicted by the model of Sec. III. However, none of our models is adequate to explain all phenomena observed even in a single experiment. Except perhaps those with magnetic guide fields, experiments to date have not been performed in such a way that we should expect complete explanation by one of our physical principles. For example, no attempt has been made to start a high current beam off with a shell current density into a background plasma very nearly equal to the beam density. Nor has a systematic attempt been made to study beams propagating into high-density, quiescent plasmas. Instead, experimental groups usually inject beams into neutral gas, and they have found that for significant beam propagation, ambient pressures of above 0.1 Torr are necessary. As previously mentioned [Eq. (126)], this means that the background plasma density, n', is continually building up during an experiment according to $n' \approx 4n_b'L$ (at an ambient pressure of about 0.5 Torr of air and with L in meters), where n_b' is the beam density. Thus, we can expect $f > 1$ after a half-meter of beam has passed and $n' \simeq 20n_b'$ after about 5 m. With such a rapid build-up of plasma in these experiments the model of Sec. II can at best be a state through which the beam–plasma–neutral gas system passes early in the interaction, on its way to becoming at least partially magnetically neutralized, as observed experimentally. This would also account for the relatively slow propagation velocities observed for the beam front.[12,13] Possibly the lingering effects of having the current density concentrated in a shell early in the pulse is the explanation of the Bradley and Ingraham observations.[21]

ACKNOWLEDGMENT

This work was supported in part by the Office of

Naval Research under Contract No. N 00014-67-A-0077-0003.

APPENDIX

Here we wish to take up the solution of Eqs. (94)–(100) of Sec. III, and the inverse Fourier transformation of the elements of that solution. In order to facilitate this solution, we introduce the Cartesian coordinate system defined in Eq. 103:

$$\hat{e}_1 = \frac{\mathbf{k}}{k}, \qquad \hat{e}_3 = \frac{\mathbf{k}\times\hat{e}_z}{k_\perp}, \qquad \hat{e}_2 = \hat{e}_3\times\hat{e}_1, \tag{A1}$$

where $k_\perp$ is the component of $\mathbf{k}$ perpendicular to $\hat{e}_z$, and k is the magnitude of $\mathbf{k}$. We, therefore, have

$$\hat{e}_z\cdot\hat{e}_1 = \frac{k_z}{k}, \qquad \hat{e}_z\cdot\hat{e}_2 = \frac{k_\perp}{k}, \qquad \hat{e}_z\cdot\hat{e}_3 = 0. \tag{A2}$$

Equations (94), (95), and (99) taken together give the components of $\boldsymbol{\delta}\mathbf{j}$ in terms of those of $\boldsymbol{\delta}\mathbf{p}$, and Eqs. (96) and (97) combine to give $\boldsymbol{\delta}\mathbf{p}$ in terms of $\boldsymbol{\delta}\mathbf{E}$. Putting the resulting two expressions together and using $\boldsymbol{\delta}\mathbf{j} \equiv \boldsymbol{\sigma}\cdot\boldsymbol{\delta}\mathbf{E}$ result in the following components for $\boldsymbol{\sigma}$:

$$\sigma_{11} = \frac{\epsilon_0\omega_p^2 s[1 - (v_0^2/c^2)(k_z^2/k^2)]}{(s + 1/\tau - ik_zv_0)(s - ik_zv_0)}, \tag{A3a}$$

$$\sigma_{12} = \sigma_{21} = \frac{\epsilon_0\omega_p^2 ik_\perp v_0[1 + (ik_zv_0s/k^2c^2)]}{(s - ik_zv_0 + 1/\tau)(s - ik_zv_0)}, \tag{A3b}$$

$$\sigma_{22} = \frac{\epsilon_0\omega_p^2(s - ik_zv_0)}{s(s - ik_zv_0 + 1/\tau)}\left(1 - \frac{k_\perp^2v_0^2[1 + (s^2/k^2c^2)]}{(s - ik_zv_0)^2}\right), \tag{A3c}$$

$$\sigma_{33} = \frac{\epsilon_0\omega_p^2(s - ik_zv_0)}{s(s - ik_zv_0 + 1/\tau)}, \tag{A3d}$$

$$\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0. \tag{A3e}$$

The square of the electron plasma frequency is

$$\omega_p^2 \equiv \frac{ne^2}{\gamma_0 m\epsilon_0}. \tag{A4}$$

The wave equation for $\boldsymbol{\delta}\mathbf{E}$ with the source term is obtained by combining Eqs. (97), (98), and (100):

$$\mathsf{Y}\cdot\boldsymbol{\delta}\mathbf{E} \equiv \left[\left(k^2 + \frac{s^2}{c^2}\right)\mathsf{1} - k^2\hat{e}_1\hat{e}_1 + \frac{s}{\epsilon_0c^2}\boldsymbol{\sigma}\right]\cdot\boldsymbol{\delta}\mathbf{E} = -\frac{is\rho_b(\mathbf{k})\hat{e}_1}{\epsilon_0c^2k} \equiv S_1\hat{e}_1, \tag{A5}$$

where S_1 and Y are as defined in this equation, and $\mathsf{1}$ is the unit dyadic. Considering Eqs. (A3), (A5) is equivalent to

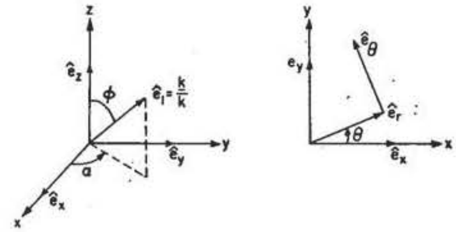


FIG. 5. The relationship among unit vectors.

$$\begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{pmatrix}\begin{pmatrix} \delta E_1 \\ \delta E_2 \\ \delta E_3 \end{pmatrix} = \begin{pmatrix} S_1 \\ 0 \\ 0 \end{pmatrix}, \tag{A6}$$

which is readily solved for the components of $\boldsymbol{\delta}\mathbf{E}$:

$$\delta E_1 = \frac{Y_{22}S_1}{Y_{11}Y_{22} - Y_{12}^2}, \tag{A7a}$$

$$\delta E_2 = -\frac{Y_{12}S_1}{Y_{11}Y_{22} - Y_{12}^2}, \tag{A7b}$$

$$\delta E_3 = 0. \tag{A7c}$$

$\boldsymbol{\delta}\mathbf{j} = \boldsymbol{\sigma}\cdot\boldsymbol{\delta}\mathbf{E}$ then gives for the components of $\boldsymbol{\delta}\mathbf{j}$,

$$\delta j_1 = \frac{(\sigma_{11}Y_{22} - \sigma_{12}Y_{12})S_1}{Y_{11}Y_{22} - Y_{12}^2}, \tag{A8a}$$

$$\delta j_2 = \frac{(\sigma_{12}Y_{22} - \sigma_{22}Y_{12})S_1}{Y_{11}Y_{22} - Y_{12}^2}, \tag{A8b}$$

$$\delta j_3 = 0. \tag{A8c}$$

These quantities are given explicitly in Sec. III, Eqs. (101).

We now wish to take the time asymptotic forms of $\boldsymbol{\delta}\mathbf{j}$, $\boldsymbol{\delta}\mathbf{E}$, $\boldsymbol{\delta}\mathbf{B}$, and $\delta\rho$, obtained from Eqs. (101) using Eq. (105), and invert the Fourier transforms in order to obtain the spatial dependence of these quantities. From Fig. 5 and Eq. (A1), we obtain the coordinate relations

$$\hat{e}_1 = \hat{e}_r\frac{k_\perp}{k}\cos(\alpha - \theta) + \hat{e}_\theta\frac{k_\perp}{k}\sin(\alpha - \theta) + \hat{e}_z\frac{k_z}{k}, \tag{A9a}$$

$$\hat{e}_2 = -\hat{e}_r\frac{k_z}{k}\cos(\alpha - \theta) - \hat{e}_\theta\frac{k_z}{k}\sin(\alpha - \theta) + \hat{e}_z\frac{k_\perp}{k}, \tag{A9b}$$

$$\hat{e}_3 = \hat{e}_r\sin(\alpha - \theta) - \hat{e}_\theta\cos(\alpha - \theta). \tag{A9c}$$

Then from Eq. (106), the spatial dependence of any component of $\boldsymbol{\delta}\mathbf{j}$, $\boldsymbol{\delta}\mathbf{E}$, and $\boldsymbol{\delta}\mathbf{B}$, as well as $\delta\rho$, is given by

$$\delta Q(\mathbf{x}, t\to\infty) = \int_0^\infty \frac{k_\perp\,dk_\perp}{(2\pi)^3}\int_{-\infty}^{\infty} dk_z\int_0^{2\pi} dx \cdot\exp\left[ik_\perp r\cos(\alpha - \theta) + ik_zz\right]\delta Q(\mathbf{k}, t\to\infty). \tag{A10}$$

The uniform beam charge density [Eq. (107)] has the Fourier transform

$$\rho_b(\mathbf{k}) = -2\pi n_b eb \frac{J_1(k_\perp b)}{k_\perp} \int_{-\infty}^{0} dz_0 \exp(-ik_z z_0), \tag{A11}$$

where J_m is the Bessel function of the first kind and order m. Therefore, after performing the α integration, we obtain

$$0 = \delta j_\theta(\mathbf{x}, t\to\infty) = \delta E_\theta(\mathbf{x}, t\to\infty) = \delta B_r(\mathbf{x}, t\to\infty), \tag{A12a}$$

$$\delta j_r = -n_b eb\omega_p^2 v_0 \int_0^\infty \frac{k_\perp\, dk_\perp J_1(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{0} dz_0 \int_{-\infty}^{\infty} dk_z \frac{ik_z \exp[ik_z(z-z_0)]}{D_0}, \tag{A12b}$$

$$\delta j_z = n_b eb\omega_p^2 v_0 \int_0^\infty \frac{k_\perp^2 J_0(k_\perp r)J_1(k_\perp b)\, dk_\perp}{2\pi} \int_{-\infty}^{0} dz_0 \int_{-\infty}^{\infty} dk_z \frac{\exp[ik_z(z-z_0)]}{D_0}, \tag{A12c}$$

$$\delta E_r = -\frac{n_b ebv_0^2}{\epsilon_0} \int_0^\infty \frac{k_\perp\, dk_\perp J_1(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{0} dz_0 \int_{-\infty}^{\infty} dk_z \frac{\exp[ik_z(z-z_0)]\{k_z^2[1+(i/k_z v_0\tau)]+(\omega_p^2/c^2)\}}{D_0}, \tag{A12d}$$

$$\delta E_z = \frac{n_b ebv_0^2}{\epsilon_0} \int_0^\infty \frac{dk_\perp J_0(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{0} dz_0 \int_{-\infty}^{\infty} dk_z \frac{\exp[ik_z(z-z_0)]ik_z\{k_z^2[1+(i/k_z v_0\tau)]+(\omega_p^2/c^2)\}}{D_0}, \tag{A12e}$$

$$\delta B_\theta = \frac{n_b ebv_0\omega_p^2}{\epsilon_0 c^2} \int_0^\infty \frac{k_\perp\, dk_\perp}{2\pi} J_1(k_\perp r)J_1(k_\perp b) \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{\infty} \frac{\exp[ik_z(z-z_0)]}{D_0} dk_z, \tag{A12f}$$

$$\delta\rho = -\frac{n_b eb\omega_p^2}{\gamma_0^2} \int_0^\infty \frac{dk_\perp J_0(k_\perp r)J_1(k_\perp b)}{2\pi} \int_{-\infty}^{0} dz_0 \int_{-\infty}^{\infty} dk_z \exp[ik_z(z-z_0)] \frac{1}{D_0}\left(k_z^2 + \frac{\omega_p^2}{c^2[1+(i/k_z v_0\tau)]} + \gamma_0^2 k_\perp^2\right). \tag{A12g}$$

D_0, obtained by setting $s = 0$ in Eq. (102), can be factored exactly, the result being

$$D_0 = \frac{v_0^2}{1+(i/k_z v_0\tau)}(k_z - k_1)(k_z - k_2) \cdot \left[k^2\left(1+\frac{i}{k_z v_0 \tau}\right) + \frac{\omega_p^2}{c^2}\right], \tag{A13}$$

where

$$k_{1,2} = -\frac{i}{2v_0\tau} \pm \left(\frac{\omega_p^2}{\gamma_0^2 v_0^2} - \frac{1}{4v_0^2\tau^2}\right)^{1/2}. \tag{A14}$$

The arguments of all of these functions are $(\mathbf{x}, t\to\infty)$. Since the k_z integrals required in δj_r and δE_z are the z derivatives of those required in δj_z and δE_r, respectively, we have only three different k_z integrations to do. They are easily done by contour integration and the residue theorem. Restricting ourselves to a weakly collisional plasma, we stipulate

$$\omega_p\tau \gg 1 \tag{A15}$$

and drop terms of order $1/\omega_p\tau$ compared with 1. However, in k_1 and k_2 we must keep the imaginary parts in order to properly locate these poles and to damp the resultant residues. In this case, D_0 has roots at k_3 and k_4 given by

$$k_{3,4} \cong \pm i\left(k_\perp^2 + \frac{\omega_p^2}{c^2}\right)^{1/2} \tag{A16}$$

to go with k_1 and k_2. Since we are concerned only with current neutralization, we have dropped a pole due to finite "collision" time which, as pointed out to us by Lee and Sudan,[23] results in the slow decay of the current neutralization. However, this effect occurs on a length scale greater than beam lengths in the high current beam experiments to date. (See the discussion at the end of Sec. III.) For $z - z_0 > 0$, we must complete the contour in the upper half k_z plane in order to have convergence on the "infinite circle." This contour includes only the pole at $k_z = k_3$. For $z - z_0 < 0$, the contour must be completed in the lower half k_z plane. This contour encloses the three poles, $k_z = k_1, k_2, k_4$. Dropping terms of order $1/\omega_p\tau$, a typical one of the k_z integrations is

$$\int_{-\infty}^{\infty} dk_z \frac{\exp[ik_z(z-z_0)]}{D_0} \simeq -\frac{\pi \exp\{-[k_\perp^2+(\omega_p^2/c^2)]^{1/2}(z-z_0)\}}{[k_\perp^2+(\omega_p^2/c^2)]^{1/2}[k_\perp^2+(\omega_p^2/v_0^2)]}, \quad z-z_0>0$$

$$\simeq \frac{2\pi}{v_0^2}\left(\frac{\gamma_0 v_0}{\omega_p}\frac{\exp[(z-z_0)/2v_0\tau]\sin(\omega_p/\gamma_0 v_0)(z-z_0)}{k_\perp^2+(\omega_p^2/v_0^2)} - \frac{\exp\{[k_\perp^2+(\omega_p^2/c^2)]^{1/2}(z-z_0)\}}{2[k_\perp^2+(\omega_p^2/c^2)]^{1/2}[k_\perp^2+(\omega_p^2/v_0^2)]}\right), \quad z-z_0<0. \tag{A17}$$

The z_0 integrals can now be done using standard forms (for example, from Dwight[27]), taking care to change the integrand for $z < 0$ at $z = z_0$ as required. Dropping terms of order $1/\omega_p\tau$, we obtain

$$\delta j_r = n_b e v_0 \frac{b\omega_p^2}{v_0^2} \begin{cases} \frac{1}{2}\frac{\partial}{\partial z}\Gamma_1^1(-z), & z > 0, \\ \frac{\gamma_0 v_0}{\omega_p}\sin\frac{\omega_p z}{\gamma_0 v_0}\exp\left(\frac{z}{2v_0\tau}\right)\left(\Psi_1^1 - \frac{1}{2}\frac{\partial}{\partial z}\Gamma_1^1(z)\right), & z < 0, \end{cases} \tag{A18a}$$

$$\delta j_z = -n_b e v_0 \frac{b\omega_p^2}{v_0^2} \begin{cases} \frac{1}{2}\Gamma_0^2(-z), & z > 0 \\ \frac{\gamma_0^2 v_0^2}{\omega_p^2}\left[1 - \cos\left(\frac{\omega_p z}{\gamma_0 v_0}\right)\exp\left(\frac{z}{2v_0\tau}\right)\right]\Psi_0^2 + \Sigma_0^2 - \frac{1}{2}\Gamma_0^2(z), & z < 0, \end{cases} \tag{A18b}$$

$$\delta B_\theta = -\frac{n_b e v_0 b\omega_p^2}{\epsilon_0 c^2 v_0^2} \begin{cases} \frac{1}{2}\Gamma_1^1(-z), & z > 0 \\ \frac{\gamma_0^2 v_0^2}{\omega_p^2}\left[1 - \cos\left(\frac{\omega_p z}{\gamma_0 v_0}\right)\exp\left(\frac{z}{2v_0\tau}\right)\right]\Psi_1^1 + \Sigma_1^1 - \frac{1}{2}\Gamma_1^1(z), & z < 0, \end{cases} \tag{A18c}$$

$$\delta E_r = -\frac{n_b e b}{\epsilon_0} \begin{cases} \frac{1}{2}\Gamma_1^3(-z), & z > 0 \\ -\gamma_0^2\left[1 - \cos\left(\frac{\omega_p z}{\gamma_0 v_0}\right)\exp\left(\frac{z}{2v_0\tau}\right)\right]\Psi_1^1 + \Sigma_1^3 - \frac{1}{2}\Gamma_1^3(z), & z < 0, \end{cases} \tag{A18d}$$

$$\delta E_z = -\frac{n_b e b}{\epsilon_0} \begin{cases} -\frac{1}{2}\frac{\partial}{\partial z}\Gamma_0^2(-z), & z > 0 \\ \frac{\gamma_0\omega_p}{v_0}\sin\left(\frac{\omega_p z}{\gamma_0 v_0}\right)\exp\left(\frac{z}{2v_0\tau}\right)\Psi_0^0 + \frac{1}{2}\frac{\partial}{\partial z}\Gamma_0^2(z), & z < 0, \end{cases} \tag{A18e}$$

$$\delta \rho = n_b e b \begin{cases} \frac{\omega_p^2}{2c^2}\Gamma_0^2(-z), & z > 0 \\ \left[1 - \cos\frac{\omega_p z}{\gamma_0 v_0}\exp\left(\frac{z}{2v_0\tau}\right)\right]\left(\frac{\omega_p^2}{v_0^2}\Psi_0^0 + \gamma_0^2\Psi_0^2\right) + \frac{\omega_p^2}{c^2}\left[\Sigma_0^2 - \frac{1}{2}\Gamma_0^2(z)\right], & z < 0. \end{cases} \tag{A18f}$$

Γ_n^ℓ, Σ_n^ℓ, and Ψ_n^ℓ are the $k_\perp$ integrals defined by

$$\Gamma_n^\ell(\pm z) \equiv \Gamma_n^\ell(r; b; \pm z) \equiv \int_0^\infty \frac{k_\perp^\ell\, dk_\perp J_n(k_\perp r) J_1(k_\perp b)}{[k_\perp^2 + (\omega_p^2/c^2)][k_\perp^2 + (\omega_p^2/v_0^2)]} \cdot \exp\left[\mp\left(k_\perp^2 + \frac{\omega_p^2}{c^2}\right)^{1/2} z\right], \tag{A19a}$$

$$\Sigma_n^\ell \equiv \Sigma_n^\ell(r; b) \equiv \int_0^\infty \frac{k_\perp^\ell\, dk_\perp J_n(k_\perp r) J_1(k_\perp b)}{[k_\perp^2 + (\omega_p^2/v_0^2)][k_\perp^2 + (\omega_p^2/v_0^2)]}, \tag{A19b}$$

$$\Psi_n^\ell \equiv \Psi_n^\ell(r; b) \equiv \int_0^\infty \frac{k_\perp^\ell\, dk_\perp J_n(k_\perp r) J_1(k_\perp b)}{[k_\perp^2 + (\omega_p^2/v_0^2)]}. \tag{A19c}$$

The integrals Σ_n^ℓ and Ψ_n^ℓ can be done exactly by a contour integration method described by Watson.[28] Note that since

$$\Gamma_n^\ell(r; b; 0) = \Sigma_n^\ell(r; b)$$

and

$$\left(\frac{\partial}{\partial z}\right)\Gamma(-z)\bigg|_{z=0} = \left(\frac{\partial}{\partial z}\right)\Gamma(z)\bigg|_{z=0},$$

all of the plasma response quantities are continuous through $z = 0$. It can be shown[16] that each Γ_n^ℓ is of order $\exp(-\omega_p|z|/c)$. Therefore, for $|z| > c/\omega_p$, the contribution to the response functions from the Γ_n^ℓ's will be negligibly small. Since $\omega_p\tau \gg 1$ implies $2v_0\tau \gg c/\omega_p$, we drop the Γ_n^ℓ contributions as they add little to the physics of the problem. However, we do lose continuity of our plasma response functions through $z = 0$ in the process. The results to this level of approximation are presented in Sec. III, Eqs. (108)–(113).

Finally, we wish to calculate δv_z. From Eqs. (94) and (96)

$$\delta v_z = -\frac{e}{\gamma_0^3 m}\frac{\delta E_z}{(s - ik_z v_0 + 1/\tau)}. \tag{A20}$$

This gives the time asymptotic limit

$$\delta v_z = -\frac{e}{\gamma_0^3 m}\frac{v_0\rho_b(\mathbf{k})}{\epsilon_0\, D_0}\left(k_z^2 + \frac{\omega_p^2}{c^2[1 + (i/k_z v_0\tau)]}\right). \tag{A21}$$

Comparing δv_z with δE_z and δE_z in Eqs. (A12), and dropping $O(1/\omega_p\tau)$, we obtain

$$\delta v_z = \begin{cases} \dfrac{n_b e b}{\epsilon_0} \dfrac{e}{\gamma_0^3 m v_0} \left\{ \Sigma_0^2 - \tfrac{1}{2}\Gamma_0^2 - \gamma_0^2 \Psi_0^0 \left[1 - \cos \dfrac{\omega_p z}{\gamma_0 v_0} \exp\left(\dfrac{z}{2 v_0 \tau}\right) \right] \right\}, & z < 0 \\ \dfrac{n_b e^2 b}{2\epsilon_0 \gamma_0^3 m v_0} \Gamma_0^2, & z > 0. \end{cases} \tag{A22}$$

* A portion of this paper is taken from a thesis submitted by D. A. Hammer to Cornell University in partial fulfillment of the requirements for the Ph.D. degree.

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