AMERICAN INSTITUTE OF ELECTRICAL ENGINEERS.

March 22nd, 1899.

The 133rd meeting of the INSTITUTE was held this date at 12 West 31st street and was called to order by President Kennelly at 8:25 P. M.

THE PRESIDENT :—The Secretary will read the announcements for the evening.

THE SECRETARY :—At the meeting of Council this afternoon the following associate members were elected.

Name.	Address.	Endorsed by
GREGG, TOM HOWARD	Supt. Electrical Construction, U.S. Light House Board, Tompkinsville, S. I., N. Y., residence, New Brighton, S. I.	Leroy Clarke, Jr. O. R. Roberson. J. D. Bishop.
HORN, HAROLD J.	Electrical Engineer, John A. Roebling's Sons' Co., residence, 36 W. State St., Trenton, N. J.	J. H. Klinck. H. S. Webb. W. S. Franklin.
JOHNSON, HOWARD S.	Engineer and Sales Agent, Morgan-Gardner Electric Co., residence, 70 Jefferson Ave, Columbus, O.	Fred'k Bedell. Harris J. Ryan. H. S. Rogers.
MILLER, HERBERT S.	Electrical Engineer, Diehl Mfg. Co., residence, 1025 E. Jersey St., Elizabeth, N. J.	E. H. Bennett. Philip Diehl. Ralph W. Pope.
POMEROY, WILLIAM D.	Electrician, Akron Electric Mfg. Co., 1106 So. Main St., Akron, O.	H. J. Ryan. Chas. S. Brown. F. W. Phisterer.
WHITTED, THOS. BYRD	Electrical Tester, The General Electric Co., residence, 211 State St., Schenectady, N. Y.	A. L. Rohrer C. P. Steinmetz. Theo. Stebbins.

Total 6.

The Council, in accordance with the Constitution, canvassed the returns from nominations and selected the following Council nominees for the coming election.

FOR PRESIDENT :—Dr. Arthur E. Kennelly.

For Vice-Presidents:—J. W. Lieb, Jr., Charles F. Scott, and L. B. Stillwell.

For Managers:—C. O. Mailloux, S. Dana Greene, C. S. Bradley, W. D. Weaver; and in place of W. F. C. Hasson of San Francisco, who has resigned on account of removing to the Hawaiian Islands, Dr. F. A. C. Perrine has been appointed by Council to fill out the unexpired term of one year.

For Treasurer:—George A. Hamilton.

For Secretary:—Ralph W. Pope.

The following Local Honorary Secretaries were appointed:

For Great Britain, H. F. Parshall, London; for Australasia, J. S. Fitzmaurice, Sydney, N. S. W.; for Canada, Prof. R. B. Owens, Montreal.

The President:—The business of the evening will be the consideration of a paper by Prof. Pupin on the "Propagation of Long Electric Waves." We have the pleasure of Prof. Pupin's presence and will ask him to come forward and present the paper.

A paper presented at the 133d Meeting of the American Institute of Electrical Engineers, New York, March 22, 1899. President Kennelly in the Chair.

PROPAGATION OF LONG ELECTRICAL WAVES.

BY M. I. PUPIN.

INTRODUCTION.

This paper describes an experimental method of investigating the propagation of long electrical waves and discusses the mathematical theory bearing upon the same.

The study of the propagation of electrical waves received a powerful impulse by Hertz's discovery of a method of producing waves the length of which could be conveniently measured within the space of a laboratory. The oscillations which emit such waves are of very high frequency, in the vicinity of a thousand million vibrations per second.

In telephony, telegraphy, and long-distance transmission of power, oscillations of only several hundred vibrations per second, or even less than one hundred are employed. The waves accompanying these slow vibrations are hundreds of miles long. It seems, therefore, a hopeless task to undertake to devise an experimental method which will do for these excessively long waves what the Hertzian method has done for short waves, the so-called Hertzian waves. This explains the singular fact that whereas there is an extensive mass of experimental facts which throw much light upon the mathematical theory of the Hertzian waves *there is to-day scarcely a single experiment which can throw any light upon the mathematical theory of long electrical waves.* The experiments described in Section III. of this paper are, therefore, the first experiments of this kind on record.

It appears at first sight as if there should be no difference between the mathematical theory of short waves and that of long waves, and that whatever throws light upon one should illuminate the other also. But this difference does exist, and it is due to the fact that the Hertzian waves are waves emitted by free oscillations, whereas the long waves employed in telegraphy, telephony, and long-distance transmission of power, are due to forced electrical oscillations. The one theory deals, therefore, with free, and the other principally with forced electrical oscillations. Besides, these long waves proceed generally from a terminal apparatus of large impedance and the principal object in transmitting them is to have them absorbed in a receiving apparatus of large impedance. The question : How much of the energy transmitted at one end is received at the other end ? is the principal question in the mathematical theory of long waves. The experimental researches which have done so much for our clear understanding of the propagation of the Hertzian waves can, therefore, help us but little in the advancement of our knowledge of the mathematical theory of propagation of electrical energy for telegraphy, telephony, and long-distance transmission of power.

The shortness of the wave-length makes the Hertzian oscillations manageable, the excessive wave-length makes, apparently, the experimental investigation of the propagation of slowly alternating electrical vibrations a practical impossibility. But does a long period necessarily mean a long wave ? The wave-length of sodium light, for instance, is shorter in glass than it is in vacuum, because light travels more slowly in glass than it does in vacuum. If we could increase the index of refraction of glass to anything we please, we could correspondingly diminish the wave-length. It is all a question of velocity of propagation. Now the simplest manner of viewing this velocity is that devised by Fresnel. He constructed over the same base in the boundary surface between the two media under consideration two cylinders parallel to the ray, one cylinder extending into the vacuum and the other into the glass. Let the heights of these two cylinders be each equal to the velocity of propagation in the two media, then whatever radiant energy was in one of the cylinders at any given moment will be in the other after the lapse of one second. The velocity of propagation is, therefore, proportional to the amount of energy which the medium stores up per unit length of the rectilinear path of the ray, when a given stress is propagated through it. If in place of glass we interposed in the path of the ray a substance which could store up one million times as much energy per unit length of rectilinear path as the vacuum can when the same ray is propagated through them then we should have the velocity and therefore the wave-length also one million times smaller in this medium than in vacuum.

This very thing can be done in the case of electrical waves. Consider a coil represented in Fig. 6.

It consists of a certain number of layers of copper wire wound in the following way:—After winding a layer of wire, a sheet of tinfoil is wrapped around this layer; the next layer is then wound and again a sheet of tinfoil wrapped, and so on. The tinfoil layers are connected in series to each other and then grounded through *c*. Everything is adjusted in such a way that the coil when completed has the same coefficient of self-induction, the same capacity, and the same resistance, as a first-class telephone wire ten miles in length. The distance between the faces of the coil is three inches. Such a coil is capable of storing up as much of the energy of a given electric wave as a long-distance telephone wire 10 miles in length can, hence interposing such a coil in the path of an electric wave will make the wave advance through a distance of three inches only in the same time during which it would pass over ten miles on the telephone wire. Connecting 24 such coils in series we have a loop which is in *every particular* equivalent to a loop of long-distance telephone wire 240 miles in length. An electrical wave will be propagated along it in just the same way as along the telephone line, with no other modification except that which a ray of light experiences in passing from a vacuum to a denser medium, and that is, a smaller velocity and therefore a shorter wave-length. A wave-length of, roughly, 140 miles corresonds to a frequency of about 1,000 periods per second when the wave advances along the telephone line now in use between New York and Chicago. Now 140 miles of a telephone air line correspond to 14 coils and therefore the same wave advancing through the coils would develop its whole wave-length within these 14 coils. The wave takes a spiral path. The axis of the spiral equals the length of the 14 coils, that is three and a-half feet. The rectilinear velocity of the wave and therefore its rectilinear wave-length have been reduced over two hundred thousand times.

Such a *slow-speed conductor* is an exact representation of a medium possessing an excessively high index of refraction and offers a new and convenient method of producing short waves even for very long periods of oscillation. It brings, therefore, the phenomena of propagation of long electrical waves within the reach of laboratory investigation. Such, briefly stated, is the new experimental method which forms the subject of this paper. The matter is discussed fully in Section III.

The scientific interest attached to experimental investigations of this kind needs no further commentary. Their practical utility will be evident when one considers that very many practical problems in telegraphy, telephony, and long-distance transmission of power depend on experimental investigations of this sort. Muirhead's artificial cables have helped much to advance the art and the science of submarine telegraphy; the slow-speed conductor described in this paper will, it is hoped, do for land lines as much as Muirhead's artificial cables have done for submarine cables.

The slow-speed conductor just described was constructed over four years ago in the electro-mechanical laboratory of Columbia University. It was a home-made affair and although adjusted with the greatest care it developed a certain objectionable feature which those skilled in the art of manufacturing condensers could have foretold with certainty. Its capacity and leakage constant varied considerably and it could not stand a high voltage, not higher than 300 volts. To overcome this difficulty a new form of slow-speed conductor represented in Figs. 10 and 11 was constructed. This form is called a *loaded conductor*. In how far the mathematical theory of electrical propagation given in Sect. I is applicable to such a conductor had to be shown. This is done in Section II, where two arrangements are discussed and it is shown that a conductor of this kind consisting of 400 sections, each section having the same coefficient of self-induction, the same capacity, and the same resistance as a long-distance telephone wire of 2½ miles in length is equivalent to a loop of such a wire of 1,000 miles in length for all frequencies which are of any importance in telegraphy, telephony, and long-distance transmission of power by machinery designed to generate electromotive forces of frequencies which are now generally employed. This part of the mathematical theory contained in this paper is believed to be new. The other part, contained in Section I, is, of course, not altogether new. That which is considered novel and important should be stated here briefly, for the purpose of elucidating beforehand the plan of this somewhat lengthy essay.

The most essential elements in the mathematical theory of electrical wave propagation are contained in the answers to the following two questions:—

First Question.—What variation does the wave energy undergo during its propagation from the transmitting to the receiving apparatus?

The mathematical theory given in Section I. of this paper answers this question by constructing the mean electro kinetic energy curve for two most important, and, at the same time, most general cases. In the first case the effect of the transmitting apparatus alone is considered, in the second case the effects of both the transmitting and the receiving apparatus are taken into consideration. In the first case the mean electro-kinetic energy curve (Fig. 2) consists of the superposition of a simple harmonic upon a catenary, in the second case Fig. 3 this curve consists of the superposition of a double harmonic upon a double catenary. The mechanical illustration of this result is extremely simple and seems to have escaped the notice of previous mathematical investigations. In the first case the curve can be illustrated by the forced vibration of a heavy string which is stretched by a certain tension between two points on the same horizontal line. In the second case the mean electro-kinetic energy curve finds a striking illustration in the forced vibration of a heavy string stretched by a certain tension between two points on the same horizontal line and carrying a weight at its middle point. An experimental investigation described in Section III. led to the conclusion that this is one of the most striking features of the propagation of long electrical waves and the theory in Section I. as formulated in such a way as to give a strong emphasis to this interesting feature. This is one of the elements which is considered important and novel in the mathematical theory of Section I.

Second Question.—What are the means which the theory suggests for measuring the wave-length and the velocity of propagation of long electrical waves which accompany forced electrical oscillations?

The mathematical theory given in Section I. answers this question. It shows that having plotted the mean electro-kinetic energy curve by measurements which involve the use of an ordinary ammeter or voltmeter the wave-length can be determined from this curve by measuring the distance between two sharply defined minima. From the wave-length and the known period the velocity of propagation can be calculated. This experimental method is essentially the same as the one which Hertz employed for rapid oscillations. It could not conveniently be applied to ordinary telegraph and telephone lines, but applied to a "slow-speed conductor" it enables us to measure the quantities just mentioned with as high a degree of accuracy as may be desired, in fact the method becomes with such a conductor more direct than, and at least as accurate as, the Hertzian method, provided, of course, that one has an accurately constructed slow-speed conductor at his disposal. This is the second element which is considered important and novel in the mathematical theory of Section I.

There are two more motives which influenced the formulation of the mathematical theory of Section I. and which should now be mentioned. The less important one will be mentioned first. It is clear that equation (6) of this section is the most comprehensive mathematical statement of this theory. It is the general solution of the equation of propagation. From it the forced as well as the free oscillations are deduced in this paper. This general solution was stated in that particular form, because the general solution of the differential equations of electrical oscillations on a "slow-speed loaded conductor" discussed in Section II, is of the same form, so that a comparison of the two cases can be readily made.

The second motive concerns what may be called the physical aspect of the mathematical theory of wave propagation along conducting wires. Most of the mathematical investigations dealing with this subject are purely symbolic. Mr. Oliver Heaviside has done much to introduce the living language of physics in place of the sign language of mathematical analysis. But Mr. Heaviside's English is often much clearer than his Arithmetic, such at any rate seems to be the general impression, so that much remains yet to be done even after Mr. Heaviside's most brilliant epoch of intense activity and radical reforms in the field of long wave propagation. That which remains to be done is not so much on the purely mathematical side of it, for that is pretty well understood now, and has been so ever since the time of Lagrange and Fourier. It is the physical side of the theory which needs cultivation. The time seems to be ripe for looking upon the problems of electrical wave propagation somewhat in the same manner in which the physical theory of light views the phenomena of radiation, reflection, interference, and absorption. According to this view the transmitting apparatus is a source of radiation, the receiving apparatus is a boundary of secondary radiation due to reflection of the wave energy which arrives there; the wave on the line conductor is an interference wave, the components of which are the direct wave from the transmitting end and the reflected wave of the receiving end. The power absorbed by the receiving end is equal to the difference of the wave energy which arrives there and the energy which is reflected per unit of time. Then again there is energy absorbed all along the line which interferes with the efficiency of transmission. *To reduce this absorption to a minimum without increasing the cost of the line beyond prohibitory limits is the ultima thule of long-distance electrical transmission engineering.* This problem contains the most essential point in the whole theory of electrical wave propagation for telephony, telegraphy, and other purposes. A mere mathematical solution of the equation of propagation does not shed much light upon this side of our theory; a careful physical consideration of the matter will supply the deficiency. Thus, the power absorbed in any element of the line depends upon the angle of lag between the current, and the potential gradient or electromotive intensity in that element. Such is the physical view of propagation of light through absorbing media. This angle depends upon the ratio of reactance to resistance of the element and we have at once the *simple rule that an efficient transmission requires a line in which the reactance per unit length should be large in comparison to the resistance.* In other words the power factor of the line should be as small as possible. The ideal line acts like a perfectly transparent medium. At every point of such a medium the electric force and the magnetic force differ in phase by a quarter period. The introduction of these elements into the theory of Section I. forms another novel feature of this section, and this introduction seems to simplify both the mathematical form and the physical aspect of the theory very much.

To bring this theory within the reach of those who mostly need it, and that is telegraph and telephone engineers, is one of the principal aims of this paper. Hence its somewhat didactic form.

SECTION I. [1]

Electrical Oscillations on a Linear Conductor of Uniformly Distributed Capacity, Self-Induction, and Resistance.

The conductor is a loop of wire A B (Fig. 1). At one point of the loop is a transmitting apparatus A, at the diametrically opposite point is a receiving apparatus B. The distance between A and B is l, equal to one-half the length of the whole loop. The distance of any element ds from A is denoted by s.

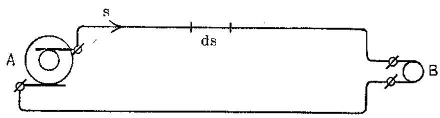

Fig. 1.

General Solution of the Problem.

§ 1. Let L, R, C, be the coefficient of self-induction, the ohmic resistance, and the capacity, respectively, per unit length of the line. Let y be the current and V the potential at any element ds, then by putting the sum of reactions in ds equal to zero, in accordance with the law of equality of action and reaction, we obtain

$$\left(L\frac{dy}{dt} + Ry + \frac{\delta V}{\delta s} \right) ds = 0 \tag{1}$$

An observation should now be made which is usually overlooked. In forming this equation the dissipative reactions set up in the neighboring conductors have been neglected. The inaccuracy thus introduced is small for air lines. In the case of submarine cables the errors arising from this may be considerable.

If x be the displacement current, then

$$x = C\frac{dV}{dt} = -\frac{\delta y}{\delta s} \tag{2}$$

From (1) and (2) we obtain the equation of propagation :—

$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} = \frac{1}{C}\frac{\delta^2 y}{\delta s^2} \tag{3}$$

The current is propagated in form of a plane wave. The

1. This section read before the *American Mathematical Society*, February meeting, 1896.

velocity of propagation v, neglecting the effect of the resistance R, is given by

$$v = \frac{1}{\sqrt{L C}}$$

Arbitrary conditions:—The equation of propagation was deduced from (1). This last equation is a mathematical expression of the law of equality of action and reaction at any point of the line where the uniformity of the line is not disturbed by the interposition of transmitting or receiving apparatus. But at points where such apparatus exists the mathematical expression for the law of equality of action and reaction is different from (1) and has to be determined from physical considerations in each particular case. At such points the equation of propagation will be modified. There are evidently as many of these subsidiary equations as there are points of discontinuity on the line. They are said to express the *boundary conditions* at these points. The mathematical function for the current y will have to satisfy not only (3) but also every one of the boundary equations. The conductor discussed here has two such points, A and B. The arbitrary conditions entering into our problem will be completely specified if we know the manner in which the electromotive force generated at A is impressed upon the line and if, in addition, the constants of the circuit in the transmitting and in the receiving apparatus are given. This will be done now.

Generator at A impresses an electromotive force

$$e = f(t)$$

where $f(t)$ is some analytical function of t.

The electro-magnetic constants of the circuit at A and B are as follows:—

L_0, R_0, C_0, and L_1, R_1, C_1, are the effective co-efficient of self-induction, the effective resistance, and the capacity of the transmitting and of the receiving apparatus, respectively.

This fixes the arbitrary conditions and we can proceed now to deduce the equations, which express the boundary conditions.

Let V_0 be the potential at $s = 0$

" V_{2l} " " " " $s = 2l$

Let $\left\{ \begin{matrix} V_l \\ V'_l \end{matrix} \right.$ " " " " $s = l$

Where V_l is the potential at that terminal of the receiving apparatus which is nearest to point $s = 0$ and V'_l is the potential at the other terminal.

Let P_0 be the potential difference in condenser C_0.
Let P_1 " " " " " " C_1.

Stating the law of equality of action and reaction for the terminals of A and B we obtain the following two equations expressing the so-called boundary conditions :—

$$\left.\begin{array}{l}\left[L_0 \dfrac{dy}{dt} + R_0 y + P_0 + V_0 - V_{2l}\right]_{s=0} = f(t) \\ \\ \left[L_1 \dfrac{dy}{dt} + R_1 y + P_1 + V'_l - V_l\right]_{s=l} = 0\end{array}\right\} \quad (4)$$

It should be observed here that we infer from purely physical considerations that the potential V is discontinuous at $s = 0$ and $s = l$. In a symmetrical system, like the one before us, the discontinuity amounts to this:

$$V_0 = - V_{2l}$$
$$V_l = - V'_l$$

In other words

$$V = - V'$$

where V is the potential at any point between $s = 0$ and $s = l$
and V' " " " " " " " $s = l$ and $s = 2l$

It is also a matter of purely physical considerations which leads us to assume that y is a function which is continuous all along the line.

The physical meaning of the problem suggests the following solution :

$$y = (K_1 \cos m\xi + K_2 \sin m\xi)\, e^{k_1 t} \quad (5)$$

where $\xi = l - s$ and the origin of co-ordinates is thus transferred to the point where the receiving apparatus is located.

This will satisfy (3) provided that

$$- m^2 = k_1 C (k_1 L + R)$$

Solution (4) contains three arbitrary constants. K_1, K_2, and k_1, as many as the number of existing arbitrary conditions.

These constants are selected so as to satisfy (4). This is done by inserting the value of y from (5) into (4) and determining K_1 K_2 and k_1 so as to satisfy this equation. It is evident that $k = k_1$.

To calculate K_1 and K_2 introduce the following abbreviations:—

$$\lambda_0 = L_0 + \frac{1}{k^2 C_0}$$

$$\lambda_1 = L_1 + \frac{1}{k^2 C_1}$$

$$h_0 = k\, C\,(k\,\lambda_0 + R_0)$$

$$h_1 = k\, C\,(k\,\lambda_1 + R_1)$$

$$D_0 = k\, C\, E.$$

The following values for K_1 and K_2 are obtained from the two boundary equations:—

$$K_1 = \frac{2\,m\,D_0}{F}$$

$$K_2 = \frac{h_1\,D_0}{F}$$

where

$$F = (h_0\,h_1 - 4\,m^2)\sin m\,l + 2\,m\,(h_0 + h_1)\cos m\,l$$

and equation (4) can now be written

$$y = [\,2\,m\cos m\,\xi + h_1 \sin m\,\xi\,]\,\frac{D_0\,e^{kt}}{F} \tag{6}$$

This is the most general solution of our problem. It includes both forced and free oscillations.

Forced Harmonic Oscillations.

§. 2 Harmonic oscillations maintained by the action of an alternator impressing a simple harmonic E.M.F. upon the line are of universal interest and will be considered here. They are employed in experimental investigations and in industrial arts. In this case the impressed E.M.F. is the real part of $E e^{ipt}$ and the current will be the real part of y in (6). We shall have now

$$-m^2 = -(a + i\beta)^2 = i p C (i p L + R)$$

$$\therefore \; a = \sqrt{\frac{p C}{2} [\sqrt{p^2 L^2 + R^2} + p L]}$$

$$\beta = \sqrt{\frac{p C}{2} [\sqrt{p^2 L^2 + R^2} - p L]}$$

Three distinct cases arise which will be discussed in turn. It is well to state here that the discussion will be conducted in all cases in accordance with the following programme :—

First, we shall inquire how the available energy varies during its propagation between the transmitting and the receiving end; *secondly*, what is the wave-length and the velocity of propagation; and *thirdly*, does the theory indicate a practicable method of measuring the wave-length and the velocity of propagation. These are evidently the essential elements which enter into the description of wave propagation.

First case.—The impedance of the transmitting and of the receiving apparatus is negligibly small:—This is the simplest case and is generally considered in elementary treatises.

We have

$$h_0 = h_1 = 0$$

Hence

$$F = -4 m^2 \sin m l$$

$$y = -\frac{i p C E \cos m \xi \, e^{ipt}}{2 m \sin m l}$$

Remembering that

$$\cos m \xi = \tfrac{1}{2} [(e^{\beta\xi} + e^{-\beta\xi}) \cos a \xi - i (e^{\beta\xi} - e^{-\beta\xi}) \sin a \xi]$$

$$\sin m \xi = \tfrac{1}{2} [(e^{\beta\xi} + e^{-\beta\xi}) \sin a \xi + i (e^{\beta\xi} - e^{-\beta\xi}) \cos a \xi]$$

We shall have for the real part of y the following :—

$$\eta = \frac{\frac{1}{2} E \sqrt{p C} [(e^{\beta\xi} - e^{-\beta\xi}) \sin a\xi \cos(pt - \varphi - \psi) - (e^{\beta\xi} + e^{-\beta\xi}) \cos a\xi \sin(pt - \varphi - \psi)]}{\sqrt[4]{p^2 L^2 + R^2} \sqrt{e^{2\beta l} + e^{-2\beta l} - 2\cos 2 a l}}$$

$$= A\,[e^{\beta\xi}\sin(p\,t - \psi - a\,\xi) + e^{-\beta\xi}\sin(p\,t - \psi + a\,\xi)] \qquad (7)$$

where

$$\tan\psi = \frac{e^{-\beta l}\sin(a\,l - \varphi) + e^{\beta l}\sin(a\,l + \varphi)}{e^{-\beta l}\cos(a\,l - \varphi) - e^{\beta l}\cos(a\,l + \varphi)}$$

$$\tan\varphi = \frac{a}{\beta}$$

Let $\lambda =$ wave-length, then evidently

$$\lambda = \frac{2\,\pi}{a}$$

If T is the period of the impressed E.M.F., then denoting by v the velocity of propagation we shall have

$$v\,T = \lambda = \frac{2\,\pi}{a}$$

On account of this relation a should be called the "*velocity constant.*" If we could measure λ we could calculate v. On this point more will be said in the discussion of the next case.

An experimental exploration of the current along the line would necessarily measure the mean square of the current. There are no instruments which indicate the instantaneous value of a variable current or potential. Besides, this mean square measures the mean value of the available electro-kinetic energy at the point under consideration. Hence it is a most important quantity and its introduction into the propagation theory seems to simplify the apparent complexity of this branch of electro-mechanics.

Let $M(\eta^2)$ denote the mean square of the current at any point on the line, then since

$$M(\eta^2) = \frac{2}{T}\int_0^{\frac{T}{2}} \eta^2\,dt$$

we shall obtain from (7)

$$M(\eta^2) = \frac{1}{8}\frac{E^2 p\,C\,[e^{2\beta\xi} + e^{-\beta\xi} + 2\cos 2a\xi]}{\sqrt{p^2L^2 + R^2}\;[e^{2\beta l} + e^{-2\beta l} - 2\cos 2\,al]} \qquad (8)$$

The physical meaning of this formula will be discussed in connection with the corresponding expression which will be obtained in the next case.

Second case:—The impedance of the receiving apparatus, only, is negligibly small.

In this case

$$h_1 = 0$$

$$F = -4\, m^2 \sin m\, l + 2\, m\, h_0 \cos m\, l$$

The current is equal to the real part of

$$y = \frac{D\, e^{ipt} \cos m\, \xi}{-2\, m \sin m\, l + h_0 \cos m\, l} \tag{9}$$

It is evident that

$$\frac{D}{-2\, m \sin m\, l + h_0 \cos m\, l}$$

measures the initial amplitude of the wave but does not affect its subsequent variation during its propagation from the transmitting apparatus along the line. Since this variation is the real object of our study, it is superfluous to perform here the actual calculations of the initial amplitude in terms of α, β, l, and h_0. Those interested in the design and installation of telegraph and telephone lines will have no difficulty in performing this task. Much confusion is avoided by keeping these somewhat lengthy and tedious calculations out of the main body of the mathematical analysis of wave propagation. They are not essential and should not be allowed to obscure the view of those elements of the theory which are of fundamental importance.

The amplitude can be written

$$\frac{D}{P + i\, Q} = \frac{D\, e^{-i\psi}}{\sqrt{P^2 + Q^2}}$$

Hence (9) assumes the form

$$y = \frac{D\, e^{i(pt-\psi)} \cos m\, \xi}{\sqrt{P^2 + Q^2}}$$

$$= A\,[(e^{\beta\xi} + e^{-\beta\xi})\sin(pt-\psi)\cos\alpha\xi + (e^{\beta\xi} - e^{-\beta\xi})\cos(pt-\psi)\sin\alpha\xi + iX]$$

Hence the real part of y will be

$$\eta = A[e^{\beta\xi}\sin(pt-\psi-\alpha\xi)+e^{-\beta\xi}\sin(pt-\psi+\alpha\xi)] \tag{10}$$

From this the potential V is easily deduced. Since

$$-\frac{\delta\eta}{\delta s} = \frac{\delta\eta}{\delta\xi} = C\frac{dV}{dt}$$

we shall have

$$V = A_1[e^{\beta\xi}\cos(pt-\psi-\varphi-\alpha\xi)-e^{-\beta\xi}\cos(pt-\psi-\varphi+\alpha\xi)] \tag{11}$$

$$V_1 = -V$$

where

$$A_1 = \frac{A\sqrt{\alpha^2+\beta^2}}{p\ C} = \frac{A\sqrt[4]{p^2L^2+R^2}}{\sqrt{p\ C}}$$

$$\tan\varphi = \frac{\alpha}{\beta}$$

The displacement current x plays a very important part in telephony owing to the facility with which it will produce cross-talk and thus make itself objectionable. Several devices have been tried in telephony to get rid of this source of annoyance.[1]

The expression for x is easily obtained from the relation.

$$x = C\frac{dV}{dt} \tag{12}$$

The equations of the mean square curves are now easily obtained.

$$\left.\begin{aligned} M(\eta^2) &= \frac{A^2}{2}(e^{2\beta\xi}+e^{-2\beta\xi}+2\cos 2\alpha\xi) \\ M(V^2) &= \frac{A_1^2}{2}(e^{2\beta\xi}+e^{-2\beta\xi}-2\cos 2\alpha\xi) \\ M(x^2) &= \frac{p^2C^2A_1^2}{2}(e^{2\beta\xi}+e^{-2\beta\xi}-2\cos 2\alpha\xi) \end{aligned}\right\} \tag{13}$$

Discussion of the equations:—It is evident that the mathe-

1. See J. J. Carty, TRANSACTIONS, vol. viii, p. 100, 1891.

matical relations deduced for the second case are of the same form as those obtained for the first case. The effect, therefore, of the transmitting apparatus upon the wave is to modify its initial amplitude, only, and nothing else. It is sufficient, therefore, to discuss the physical meaning of the results of the second case.

The current wave, equation (10):—It can be decomposed into two components η_1 and η_2; thus,

$$\eta = \eta_1 + \eta_2$$

$$\eta_1 = A\, e^{\beta\xi} \sin\,(p\,t - \psi - a\,\xi)$$

$$\eta_2 = A\, e^{-\beta\xi} \sin\,(p\,t - \psi + a\,\xi)$$

Each of these components is a progressive wave.

η_2 is maximum at $\xi = -l$, and minimum at $\xi = l$

η_1 " " " $\xi = +l$, " " " $\xi = -l$

The first wave starts from one pole of the transmitting alternator and describes a right handed motion around the loop. The second wave starts at the other pole of the alternator and travels in the opposite direction. The waves have the same initial amplitude, the same velocity, and they become attenuated at the same rate. The distribution of the wave around the loop is perfectly symmetrical. The resultant current wave η is an interference wave. On account of attenuation the interference is not capable of producing a stationary wave, because when the two interfering waves meet they have unequal amplitudes.

When the resistance per unit length is made small in comparison to the reactance and the line is sufficiently short, the attenuation constant β becomes so small that

$$e^{\beta\xi} = e^{-\beta\xi} = 1$$

and in that case

$$\eta = \eta_1 + \eta_2 = A\,[\sin\,(p\,t - \psi - a\,\xi) + \sin\,(p\,t - \psi + a\,\xi)\,]$$

$$= 2\,A\,\cos\,a\,\xi\,\sin\,(p\,t - \psi)$$

that is, a stationary wave is formed.

But even if the line is long, provided that R is sufficiently small in comparison to pL, as in the case of efficient long-distance telephone lines, the two waves will be nearly equal for quite a distance on each side of the middle point of the loop where in general the receiving apparatus is located. Hence in the vicinity of this point the resultant current wave approximates very nearly the form of an interference wave. This fact manifests itself in an interesting manner and will be brought out presently in connection with the discussion of a method which this theory suggests for measuring experimentally the wave-length and the velocity of long waves.

The potential and the displacement current waves, equations (11) *and* (12) :—They are just like the current wave, interference waves, and the remarks just made with reference to the current wave apply to them also. An interesting relation between these waves and the current wave deserves a careful attention. It is the phase-difference φ. This angle measures the attenuation, as will be seen presently.

Efficiency of transmission :—Equation (1) can be written

$$L\frac{d\eta}{dt} + R\eta = \frac{\delta V}{\delta\xi}$$

The quantity $\frac{\delta V}{\delta\xi}$ should, therefore, be called *the electromotive intensity*. Its value is easily obtained from (11.) Thus

$$\frac{\delta V}{\delta\xi} = \frac{A(\alpha^2+\beta^2)}{p\,C}\left[e^{\beta\xi}\sin(pt-\psi-\alpha\xi+\theta)+e^{-\beta\xi}\sin(pt-\psi+\alpha\xi+\theta)\right] \tag{14}$$

where

$$\theta = -2\varphi + \frac{\pi}{2}$$

The angle θ is the angle of lag between the current and the electromotive intensity as can be seen by comparing (10) and (14). It will be shown now that this angle of lag plays the same part here as the angle of lag between the impressed electromotive force and the current in ordinary alternating current circuits.

Consider the equation

$$W = R\,M(\eta^2) = M\left(\eta \times \frac{\delta V}{\delta\xi}\right)$$

That is to say, the mean value of the work per second done by the electro-motive intensity equals the mean value of the rate of dissipation per unit length of the line. This dissipation causes the attenuation of the wave and thus diminishes the efficiency of transmission. A small value of R will evidently prevent it. But that this efficiency is not a question of ohmic resistance, only, will be seen from the following consideration:

$$\cos\theta = \sin 2\varphi = 2\sin\varphi\cos\varphi = \frac{R}{\sqrt{p^2L^2+R^2}}$$

or

$$\tan\theta = \frac{pL}{R}$$

Now let

$$M\left[\left(\frac{\delta V}{\delta\xi}\right)^2\right] = \frac{A_v^2}{2}$$

$$M(\eta^2) = \frac{A_\eta^2}{2}$$

Then since

$$M\left[\left(\frac{\delta V}{\delta\xi}\right)^2\right] = (p^2L^2+R^2)\,M(\eta^2)$$

we shall have

$$\frac{1}{2}\,\frac{R}{\sqrt{p^2L^2+R^2}}\,A_v\,A_\eta = R\,M(\eta^2)$$

$$\therefore\ \frac{1}{2}A_v\,A_\eta\cos\theta = R\,M(\eta^2)$$

For efficient transmission we must have, therefore, a large angle of lag between the current and the electromotive intensity. The quantity $\frac{pL}{R}$, that is the ratio of reactance to resistance, is the most essential element, and not R alone, in questions of efficiency of this kind. Employing the terminology which has been generally adopted among electrical engineers, we have the following simple rule:—*The power factor of the line must be as small as possible.* The physical reason for this is not far to seek. A large angle of lag between the electromotive intensity and the current, means the same thing here as it does in ordinary circuits, and that is, it means a large self-induction reaction in comparison to the dissipative resistance reaction, and this again means a large amount of energy stored up in comparison to the energy dissipated. This stored up energy is returned to the generator in the case of ordinary circuits and propagated in the case of long lines. The consideration of the angle of lag θ or, what is the same thing, *the power factor of the line*, enables us therefore to view the wave propagation in the same simple light in which we view the energy transfer in ordinary alternating current circuits. But it should be observed that *the power factor of the line* is not the same thing as the power factor of an ordinary alternating current circuit In wave propagation of electrical energy, the power factor of the line measures the power consumed on the line only; the power absorbed in the receiving apparatus is measured by another power factor.

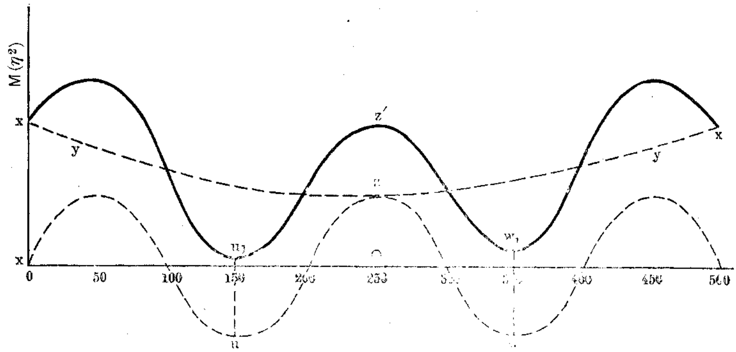

FIG. 2.

The mean square curves, equation (13) :—Plotting the curves by taking these mean squares for ordinates and ξ for the abscissa we obtain an extremely simple representation of the variation of the mean electro-kinetic energy during its propagation along the line. The thick line $x\ u_1\ z'\ w^1\ x$ diagram Fig. 2 represents the curve of $M\ (\eta^2)$. It consists of the superposition of two curves, the catenary $x\ y\ z\ y\ x$ and the simple harmonic $x\ u\ z\ w$. The wave-length of this simple harmonic is one-half as long as the wave-length of the progressive wave, that is

$$u\ w = \frac{\lambda}{2}$$

Let $u_1\ w_1$ be two points on the $M\ (\eta^2)$ curve such that $u\ w = u_1\ w_1$, then these points will be shown to be important points in the experimental determination of the wave-length.

Concerning an experimental method of measuring the wave-length and the velocity of propagation of long electrical waves: —Many serious attempts were made long before the time of Hertz to measure the velocity of propagation of an electrical disturbance in a linear conductor. The most notable among these were the experiments of Wallaston and Fizeau. The cause of their failure has been discussed before, and needs no further commentary here. Hertz was the first to conceive the idea of determining this velocity by measuring experimentally the wave-length of a harmonic disturbance of known periodicity, and in order to obtain a sufficiently short wave-length he made the period sufficiently short. The solution of the problem of producing powerful harmonic electrical oscillations of very high frequency and therefore short wave-length forms the foundation of his classical experiments.

It will be pointed out now that the Hertzian method of measuring the velocity of propagation is applicable to forced electrical oscillations of long period; the difficulties involved in it will be brought out and a way of avoiding them will be discussed more fully in Section III. of this paper.

Consider as an illustration the following example:

$$l = 250 \text{ miles},$$
$$L = .005 \text{ henry},$$
$$R = 1 \text{ ohm},$$
$$C = .01 \text{ microfarad},$$
$$p = 6000.$$

Such a line represents very nearly a long-distance telephone line of 500 miles in length such as in use now between New York and Chicago. Fig. 2 is a $M(\eta^2)$ curve answering this example.

In this case we have approximately

$$\beta = .000706$$
$$v = 1.42 \times 10^5 \text{ miles, roughly.}$$
$$\frac{\lambda}{2} = 142 \text{ miles}, \quad \text{"}$$

Hence

$$z\, w_1 = z\, u_1 = 71 \text{ miles}$$

At w_1 we have $\xi = 71$, hence

$$e^{2\beta\xi} = e^{142 \times .000706} = e^{.1}$$

This shows that the catenary is very flat in the vicinity of the origin o, the most distant point on the loop, and therefore the points u_1 and w_1 will be very near the minima points nearest to o. The distance between the minima is, therefore, equal to a half wave-length corresponding to frequency 500. Hence if we plot the M (η^2) curve for a telephone line such as specified above, and determine the distance between the two furthest minima, this will give the half wave-length. Since the period is known, the velocity can be calculated. This method of determining experimentally the velocity of propagation is the same as the one devised by Hertz. It is evident, however, that in its practical execution it would offer many almost insurmountable difficulties, the chief among them being the excessively long wave-length and the consequent necessity of distributing the points of observation over long distances. But a simple consideration will show that a long period does not necessarily mean a long wave-length in the case of propagation along conductors. We have in this case

$$\lambda = T v = \frac{T}{\sqrt{L C}}$$

For a frequency of

$$T = \frac{1}{500}$$

we have

$$\lambda = 284$$

miles approximately.

If the surrounding medium had a million times the permeability and specific inductive capacity as the ordinary atmosphere we should have for the same conductor

$$\lambda = 1.5 \text{ foot, about,}$$

that is about the same wave-length as Hertz obtained for his very high frequencies. The velocity of propagation is a matter of the amount of energy per unit length of the path of the wave when a given current and potential are propagated along that path. We can make that amount anything we please, and thus modify the velocity of propagation and the wave-length in any desirable manner as will be shown in Sect. III.

Third case. The impedances of both the receiving and the transmitting apparatus are taken into account:—This is the most general case. Solution (6) in its complete form must be employed here. The current is the real part of

$$y = (2\, m \cos m\, \xi + h_1 \sin m\, \xi)\, \frac{D_0\, e^{ip}}{F}$$

The separation of the real and imaginary parts of this expression can be done as follows:—

$$\frac{D_0\, e^{ipt}}{F} = i\, A\, e^{i(pt-\psi)}$$

It can be shown that

$$2\, m \cos m\xi + h_0 \sin m\xi = Xe^{\beta\xi} \cos(\alpha\xi+\varepsilon) + Ye^{-\beta\xi} \cos(\alpha\xi - \delta)$$

$$- i\,[X e^{\beta\xi} \sin(\alpha\xi+\varepsilon) - Ye^{-\beta\xi} \sin(\alpha\xi - \delta)\,]$$

Hence the real part of y will be

$$\eta = -A[Xe^{\beta\xi} \cos(pt-\psi-\varepsilon-\alpha\xi) + Ye^{-\beta\xi} \cos(p\,t-\psi-\delta+\alpha\xi)] \quad (15)$$

The potential V is easily obtained from

$$\frac{\delta\eta}{\delta\xi} = C\,\frac{dV}{dt}$$

$$V = A_1[-Xe^{\beta\xi} \sin(pt-\psi-\varphi-\varepsilon-\alpha\xi) + Ye^{-\beta\xi} \sin(p\,t-\psi-\varphi-\delta+\alpha\xi)] \quad (16)$$

$$V' = -V,\; A_1 = \frac{A\sqrt{\alpha^2+\beta^2}}{p\,C}$$

$$\frac{\delta V}{\delta\xi} = A_1\sqrt{\alpha^2+\beta^2}[Xe^{\beta\xi} \cos(p\,t-\psi-\varepsilon-\alpha\xi+\theta) +$$

$$+ Ye^{-\beta\xi} \cos(p\,t-\psi-\delta+\alpha\xi+\theta)] \quad (16\text{ a})$$

where

$$\theta = \frac{\pi}{2} - 2\varphi$$

We have here as in the preceding case

$$\tan\varphi = \frac{\alpha}{\beta} \;\therefore\; \tan\theta = \frac{p\,L}{R}$$

where, as before, θ is the angle of lag between the potential gradient and the current, and *cos θ is the power factor of the line.*

$$\left.\begin{aligned} M(\eta^2) &= \frac{A^2}{2}\left[X^2 e^{2\beta\xi} + Y^2 e^{-2\beta\xi} + 2XY\cos(2\alpha\xi - \varepsilon + \delta)\right] \\ M(V^2) &= \frac{A_1^2}{2}\left[X^2 e^{2\beta\xi} + Y^2 e^{-2\beta\xi} - 2XY\cos(2\alpha\xi + \varepsilon - \delta)\right] \end{aligned}\right\} \quad (17)$$

The quantities A, A_1 X, Y, ε, δ can be calculated when required. The calculation is excessively long and tedious, and has been omitted on that account. The questions proposed in this investigation can be answered without a knowledge of the numerical values of these quantities.

Physical interpretation of the third case:—Little can be said here which has not already been mentioned in connection with the preceding case. *The current* and *the potential waves* are interference waves. The interfering components are two in number, just as in the preceding case.

$$\eta = \eta_1 + \eta_2$$

where

$$\eta_1 = -A\, Y e^{-\beta\xi} \cos(pt - \psi - \delta + \alpha\xi)$$

$$\eta_2 = +A\, X\, e^{\beta\xi} \cos(pt - \psi - \varepsilon - \alpha\xi + \pi)$$

η_1 is maximum at $\xi = -l$ and minimum at $\xi = 0$

η_2 " " " $\xi = 0$ " " " $\xi = -l$

Hence η_1 is the direct or the incident wave proceeding from the machine at the transmitting end, and η_2 is a reflected wave, the reflection taking place at the apparatus of the receiving end. In this respect, then, this case differs from the preceding one, there being no reflection when there is no receiving apparatus. The presence of the receiving apparatus acts as a source of secondary radiation. The reflected waves coming from the receiving apparatus may be called the counter-current waves produced by the counter-electromotive force due to the reaction at the receiving end. The transmitting apparatus acts like a source of light, and the receiving apparatus acts like a reflecting surface. But here is a distinction which deserves to be mentioned in this place. Light waves proceed in straight lines, whereas these electric waves bend around corners with perfect ease; they follow the conducting wire. This difference is due principally to the fact that the current waves discussed here are conduction current waves, whereas the waves of light are waves of displacement currents of very short wave-length, and such waves will not bend easily around corners. Displacement current waves impinging upon a conductor will, of course, produce conduction currents and hence cross-talk in telephony, and there is every reason to believe that even ordinary light-waves when falling upon a conductor produce conduction current-waves which will bend around corners, if by the time they have reached a corner they have not been attenuated out of existence.

Comparing the potential gradient wave (16 a) to the current-wave it will be seen that the same angle of lag θ between the electromotive intensity or potential gradient and the current appears. It measures here in the same way as it did in the preceding case, the efficiency of transmission and on account of the same physical reasons. A line possessing perfect efficiency is like a perfectly transparent medium In such a medium the electric force and the magnetic are, at every point of a luminous wave, in quadrature.

The importance of this angle of lag can be shown here by inquiring how much energy per second passes at any point $-\xi$ toward the receiving apparatus.

Let $\frac{W}{2}$ = total power of transmitting apparatus radiated on one-half of the loop.

H = rate of heat generated on the line between the points $-l$ and $-\xi$.

It can be easily shown that

$$\frac{W}{2}=\frac{A\ A_1}{2}[(Y^2e^{2\beta l}-X^2e^{-2\beta l})\sin\varphi+2\,X\,Y\sin(\alpha l+\varepsilon-\delta)\cos\varphi]$$

$$H=\frac{W}{2}-\frac{A A_1}{2}[(Y^2e^{2\beta\xi}-X^2e^{-2\beta\xi})\sin\varphi+2XY\sin(\alpha\xi+\varepsilon-\delta)\cos\varphi]$$

The energy per second radiating from any point $-\xi$ toward the receiving apparatus is

$$\frac{W}{2}-H=E=\frac{A A_1}{2}[(Y^2e^{2\beta\xi}-X^2e^{-\beta\xi})\sin\varphi+2XY\sin(\alpha\xi+\varepsilon-\delta)\cos\varphi]$$

$$= E_1\sin\varphi -- E_2\cos\varphi$$

Where E_1 and E_2 are positive quantities. Hence the greater φ the greater will be E.

Passing now to the consideration of the mean square values of η and V we obtain additional illustrations of the physical fact that long-distance transmission of power along conducting wires is a process of transference by means of wave radiation following the same laws as every other kind of wave radiation. Equations (17) express simply a geometrical relation between the incident and the reflected waves of current and potential, and their resultants. They state that these resultants are found by the ordinary rules of compounding waves or vectors in general. Now it happens, fortunately, that the mathematical expression for these resultants admits of a very simple geometrical construction, and of a suggestive mechanical illustration.

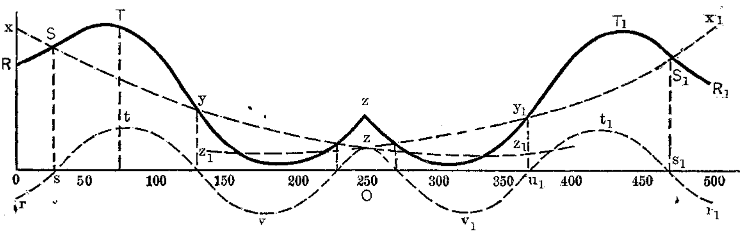

FIG. 3.

In Fig. 3 the thick line $R\ S\ T\ Z\ T_1\ S_1\ R_1$ represents the $M(\eta^2)$ curve on a 500-mile telephone line mentioned above. Each half consists of the superposition of a simple harmonic $r\ t\ u\ v\ z$ upon a portion of a catenary $x\ S\ y\ z$. It should be observed that the lowest point of the catenary of which $x\ S\ y\ z$ is a portion is to the right of z. That is to say, the double catenary $x\ S\ y\ z\ y_1\ s_1\ x_1$ is the catenary which we obtain by hanging a weight at the middle point of a heavy string which is suspended at its terminals. So that the effect of the receiving apparatus upon the $M(\eta^2)$ curve may be described broadly by stating that this curve can be represented by the forced vibrations of a heavy string, the extremities of which are held by a certain tension at two points on the same horizontal line and carrying a weight at its middle point. The density of the string, the stretching tension, and the weight at its middle point, correspond to the coefficient of self-induction and the capacity per unit length of the wire, and to the self-induction of the receiving apparatus, respectively.

An interesting analogy will be mentioned here. If the self-induction of the receiving apparatus is balanced by a capacity, and the resistance is negligibly small, the cusp of Fig. 3 disappears and in place of it we have the same formation as in Fig. 2. This case corresponds to a weight suspended by means of a spiral spring to the middle of the string, the weight and the elasticity of the spring being adjusted so that their period equals the period of the impressed force.

Note.—Another way of looking upon the effect of the line reactance upon the efficiency of transmission is to consider the attenuation constant β. When the reactance per unit length of line is large in comparison to resistance then for all frequencies

$$\beta = \tfrac{1}{2} R\, C\, v,$$

where v is the velocity of propagation and it is given by

$$v = \frac{1}{\sqrt{L C}}$$

This relation takes place here of the so-called $K R$ law.

The reactance diminishes the speed of propagation, but enables the line to transmit all frequencies (within certain limits which are of importance in telephony) with the same velocity and the same attenuation. It makes the line what Mr. Heaviside calls "distortionless."

Free Oscillations on a Linear Conductor of Uniformly Distributed Self-Induction, Resistance, and Capacity.

§ 2. Free oscillations on a conductor of this kind are readily calculated for a few special cases. Equation (6) is a general solution for free oscillations also, provided, however, that m has such a value as to make

$$F = 0, \text{ since } D_0 = 0$$

that is, we must have

$$(h_0\, h_1 - 4\, m^2) \sin m\, l + 2\, m\, (h_0 + h_1) \cos m\, l = 0 \qquad (19)$$

but, of course, in this case

$$h_0 = k\, C\ (k\, \lambda_0 + R_0)$$
$$h_1 = k\, C\ (k\, \lambda_1 + R_1)$$
$$-\, m^2 = k\, C\, (k\, L + R)$$

Equation (19) is a transcendental equation and can be be solved in a few simple cases.

Case 1. *The transmitting and the receiving apparatus are not present.*

In this case

$$h_0 = h_1 = 0$$

Equation (19) reduces to

$$\sin m\, l = 0$$

$$\therefore m = \frac{s\,\pi}{l}$$

where s can have any integer value from 1 to ∞. The periods of free oscillations are calculated from the equation

$$- m^2 = k^2 L\, C + k\, R\, C = - \frac{s^2\,\pi^2}{l^2}$$

$$\therefore k = - \frac{R}{2\,L} \pm \sqrt{-1}\sqrt{\frac{1}{L\,C}\frac{s^2\,\pi^2}{l^2} - \frac{R^2}{4\,L^2}}$$

$$= - \frac{R}{2\,L} \pm i\,k_s$$

There are therefore an infinite number of periods which are harmonically related to each other unless the damping factor $\frac{R}{2\,L}$ is not sufficiently small in comparison to $\frac{\pi^2}{l^2}\frac{1}{LC}$.

The most general solution of this case can be written

$$y = e^{-\frac{R}{2L}t}\ {}^{s}\sum_{1}^{\infty} A_s \cos\frac{s_1\pi}{l}\xi \cos(k_s\, t - \varepsilon_s) \tag{20}$$

Case 2.—Transmitting apparatus is not present and in place of the receiving apparatus there is a break in the wire.

In this case $h_0 = o$, $h_1 = \infty$

Equation (19) reduces to

$$\cos m\, l = 0$$

$$\therefore m = \frac{2\,s + 1}{l}\,\frac{\pi}{2}$$

$$k = - \frac{R}{2L} \pm i\sqrt{\frac{1}{L\,C}\left(\frac{2\,s+1}{l}\,\frac{\pi}{2}\right)^2 - \frac{R^2}{4L^2}}$$

$$= - \frac{R}{2\,L} \pm i\,k_{2s+1}$$

$$y = e^{-\frac{Rt}{2L}}\ {}^{s}\sum_{0}^{\infty} A_{2s+1} \sin\frac{2\,s+1}{l}\,\frac{\pi}{2} \cos(k_{2s+1} - \varepsilon_{2s+1}) \tag{21}$$

The damping factor is the same for all frequencies, hence the color of the complex harmonic vibration remains unchanged during the whole epoch while the vibrations last. The dying out sound of a bell is a striking illustration of this interesting relation.

Whenever the circuit is made or opened, we shall have in addition to the forced vibrations, free vibrations also. The lower harmonics of these free vibrations will be quite within the ordinary frequencies, especially on long lines. There is no doubt that these free vibrations interfered considerably with the successful working of harmonic telegraphy. The discussion of a method of preventing the development of free vibrations of any period is reserved for a future occasion.

SECTION II.[1]

OSCILLATIONS ON A LOADED CONDUCTOR.

Introduction:—This part of the paper discusses the forced and the free electrical oscillations in a loaded conductor.

FIRST ARRANGEMENT.

The conductor consists of $2n$ equal coils, $L_1 \ldots . L_n$ (Fig. 4) connected in series, so as to form a closed loop. At one point A of this loop is an alternator, at the diametrically opposite point is a receiving apparatus B. At equal distances $(n - 1)$ equal condensers,

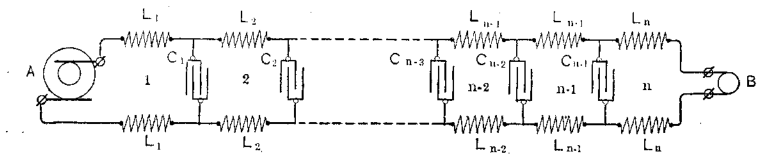

FIG. 4.

ers, $C_1 \ldots . C_{n-1}$, form bridges across the loop. The whole loop is thus divided into n component circuits, 1, 2 . . . . n. The component circuits except the first and the nth are equal to each other in the sense that they have equal resistance, capacity, and self-induction. The first and the nth circuit differ from the other circuits on account of the presence of alternator A in the former, and that of the receiving apparatus B in the latter circuit.

It is evident that in the limit when n becomes infinitely large, this conductor becomes an ordinary telegraph or telephone line with uniformly distributed resistance, capacity, and self-induction. The practical question arises now, under what conditions will a conductor loaded in this manner become equivalent with sufficient approximation to a uniform telegraph line when n is not infinitely large? This problem does not seem to have been solved before. Professor Blakesley in his book on alternating currents devotes considerable attention to the discussion of a similar problem, but his efforts do not appear to throw any light upon this matter.

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1. This section read before the *American Mathematical Society*, March meeting, 1899.

In its main features this problem is similar to that which Lagrange solved in his "Mecanique Analytique, sec. partie, sect. VI.," the problem, namely, of the free vibrations of a string, fixed at its two ends, and loaded at equidistant points by equal weights. But it is much more general than that of Lagrange, in the first place because frictional resistances are taken into consideration, and secondly, forced as well as free oscillations are considered.

Specification of the constants and of the currents:—

Let L_0 and L_1 = co-efficient of self-induction of A and B respectively.

Let R_0 and R_1 = Ohmic resistance of A and B, respectively.

Let C_0 and C_1 = Capacity of A and B, respectively.

Let L, R, C, be the corresponding quantities of the coils and condensers in the several component circuits.

Let the real part of $E e^{ipt}$ be the E.M.F. impressed by the alternator A.

Let $x_1 \ldots . x_n$ be the currents in the n component circuits.

Let $P_1 \ldots . P_{n-1}$ be the differences of potential in the line condensers.

Let P_0 and P' be the differences of potential in the condensers in A and B, respectively.

Let $\xi_1 \ldots . \xi_{n-1}$ be the condenser currents.

We shall have

$$\left.\begin{aligned} \xi_1 &= C\frac{d P_1}{dt}, \qquad \xi_2 = C\frac{d P_2}{dt}, \text{ etc.} \\ \xi_1 &= x_1 - x_2, \qquad \xi_2 = x_2 - x_3, \text{ etc.} \end{aligned}\right\} \quad (1)$$

A. Forced Oscillations.

Stating the law of equality of action and reaction for each component circuit we obtain the following n differential equations:—

$$\left.\begin{aligned} (L_0 + 2\,L)\frac{d x_1}{dt} + (R_0 + 2\,R)\,x_1 + P_1 + P_0 &= E\,e^{ipt} \\ 2\,L\frac{d x_2}{dt} + 2\,R\,x_2 + P_2 - P_1 &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 2\,L\frac{d x_{n-1}}{dt} + 2\,R\,x_{n-1} + P_{n-1} - P_{n-2} &= 0, \\ (L_1 + 2\,L)\frac{d x_n}{dt} + (R_1 + 2\,R)\,x_n - P_{n-1} + P' &= 0. \end{aligned}\right\} \quad (2)$$

When the steady state has been reached, the currents will be just like the impressed E.M.F., simple harmonics of the time t, that is

$$\begin{aligned} x_1 &= A_1 e^{ipt}, \\ x_2 &= A_2 e^{ipt}, \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{3}$$

where $A_1, A_2, \ldots$ are complex quantities.

From (3) follows that for the differential co-efficients in (2) we can substitute currents, because

$$\begin{aligned} \frac{d x_m}{dt} &= i p x_m \\ \frac{d^2 x_m}{dt^2} &= -p^2 x_m \end{aligned} \tag{4}$$

Hence differentiating each member of (2) and substituting from (4) and (1) we obtain

$$\begin{aligned} 2C(-p^2 L + ip R) x_1 + \xi_1 - 0 &= ip C E e^{ipt} \\ &\quad + C(p^2 \lambda_0 - ip R_0) x_1 \\ 2C(-p^2 L + ip R) x_2 + \xi_2 - \xi_1 &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ 2C(-p^2 L + ip R) x_{n-1} + \xi_{n-1} - \xi_{n-2} &= 0 \\ 2C(-p^2 L + ip R) x_n + 0 - \xi_{n-1} &= C(p^2 \lambda_1 - ip R_1) x_n \end{aligned}$$

where

$$\lambda_0 = L_0 - \frac{1}{p^2 C_0} \qquad \lambda_1 = L_1 - \frac{1}{p^2 C_1}$$

Introducing the following abbreviations:—

$$\begin{aligned} h &= 2C(-p^2 L + ip R) \\ D &= ip C E e^{ipt} + C(p^2 \lambda_0 - ip R_0) = D_0 - h_0 x_1 \\ h_1 &= -p^2 \lambda_1 + ip R_1 \end{aligned}$$

we obtain

$$\left.\begin{aligned} h x_1 + \xi_1 - 0 &= D \\ h x_2 + \xi_2 - \xi_1 &= 0 \\ \ldots\ldots\ldots\ldots & \ldots \\ h x_{n-1} + \xi_{n-1} - \xi_{n-2} &= 0 \\ h x_n + 0 - \xi_{n-1} &= -h_1 x_n \end{aligned}\right\} \tag{5}$$

Another form is obtained by substituting for ξ_1 ξ_2 .. as follows:—

$$\left.\begin{array}{rl} (h_0 + h + 1)\,x_1 \;+ 0 \;- x_2 &= D_0 \\ (h + 2)\,x_2 \;- x_1 \;- x_3 &= 0 \\ \dots\dots\dots\dots\dots\dots & \\ (h + 2)\,x_{n-1} - x_{n-2} - x_n &= 0 \\ (h_1 + h + 1)\,x_n \;+ 0 \;- x_{n-1} &= 0 \end{array}\right\} \quad (6)$$

Two methods of solving these equations present themselves, the direct method and the indirect one.

The direct method:—In this method the system (6) forms the starting point. Consider the following determinant:—

$$\begin{vmatrix} h_0+h+1, & -1, & 0, & 0, & \dots & & \\ -1, & (h+2), & -1, & 0, & \dots & & \\ 0, & -1, & h+2, & -1, & 0 \dots & & \\ \dots & & & & & & \\ 0, & 0, & 0, & 0, & 0, \dots & -1, & h+2, -1 \\ 0, & 0, & 0, & 0, & 0, \dots & 0, & -1, (h_1+h+1) \end{vmatrix} = \varDelta$$

Let now $\varDelta_m$ stand for the minor of that term in the *m*th column which contains h, then

$$x_m = \frac{\varDelta_m}{\varDelta} D_0 \qquad (7)$$

If $\varDelta_m$ and $\varDelta$ could be evaluated by the ordinary rules of expanding a determinant then (7) would give the solution of the problem for forced oscillations. The free oscillations could then be readily obtained from it. But the direct expansion of $\varDelta_m$ and $\varDelta$ seems to be a matter of much difficulty, so that (7) is merely a symbolic solution of no actual value. The direct method leads, therefore, to no effective result. In his investigation of the problem referred to above, Lagrange did not adopt the direct method. The indirect method employed in this paper is different from that employed by him, and it had to be devised as Lagrange's method does not seem to be applicable here, because, as already stated, the two problems, though similar in their general features, are essentially different.

The indirect method:—This method may be called the method of successive eliminations.

The starting point is system (5). Adding the n equations we obtain

$$h\,(x_1 + x_2 + \ldots + x_{n-1} + x_n) = D - h_1\,x_n \tag{8}$$

The indirect method can now be readily explained. It consists in successively eliminating from the left-hand member of (8) the currents $x_n \ldots x_2$ and thus obtaining in place of (8) an equation containing in its left-hand member x_1 as the only unknown quantity. From this equation the complete solution of the problem can then be easily obtained.

Elimination of x_n.

First step.

$$\begin{aligned}
x_1 &= x_1 \\
x_2 &= x_1 - \xi_1 \\
x_3 &= x_1 - \xi_1 - \xi_2 \\
x_4 &= x_1 - \xi_1 - \xi_2 - \xi_3 \\
&\ldots\ldots\ldots\ldots \\
x_{n-1} &= x_1 - \xi_1 - \xi_2 - \xi_3 - \ldots - \xi_{n-3} - \xi_{n-2} \\
x_n &= x_1 - \xi_1 - \xi_2 - \xi_3 - \ldots - \xi_{n-3} - \xi_{n-2} - \xi_{n-1}
\end{aligned}$$

$$\therefore h\,(x_1 + \ldots + x_n) = n\,h\,x_1 - h\,[\,(n-1)\,\xi_1 + (n-2)\,\xi_2 + \ldots + 2\,\xi_{n-2} + \xi_{n-1}] \tag{9}$$

Second step.

$$\begin{aligned}
\xi_1 &= D - h\,x_1 \\
\xi_2 &= D - h\,(x_1 + x_2) \\
\xi_3 &= D - h\,(x_1 + x_2 + x_3) \\
&\ldots\ldots\ldots\ldots \\
\xi_{n-1} &= D - h\,(x_1 + x_2 + \ldots + x_{n-1})
\end{aligned}$$

$$\therefore (n-1)\,\xi_1 + (n-2)\,\xi_2 + \ldots + 2\,\xi_{n-2} + \xi_{n-1} = \frac{n\,(n-1)}{2}\,D - \frac{h'}{2}\,[n(n-1)\,x_1 + (n-1)\,(n-2)\,x_2 + \ldots] \tag{10}$$

By means of (9) and (10) equation (8) transforms into

$$n\,h\,x_1 + \frac{h^2}{2}[n(n-1)x_1 + (n-1)(n-2)x_2 + \ldots + 2\times 1\,x_{n-2} + x_{n-1}]$$

$$= D\left[1 + \frac{n(n-1)}{2}h\right] - h_1 x_n \quad (11)$$

In place of (8) we have (11) and in the left-hand member of this equation, x_n does not appear.

To eliminate the remaining variables $x_{n-1}, \ldots x_2$ we have to repeat the same operations through which we have just passed during the elimination of x_n. In each elimination the same two steps just shown have to be made. It seems, therefore, superfluous to go into any further details here. Before giving the final result it is well to observe here, that the following theorem can be employed with advantage in performing the summations which occur in each elimination.

Theorem :—

Let $S = 1.2.3 \ldots\ldots (s+1) + 2.3 \ldots\ldots (s+2) + \ldots\ldots$

$$+ (n-s)(n-s+1) \ldots\ldots n,$$

then

$$S = \frac{(n-s)(n-s+1)\ldots\ldots n\,(n+1)}{s+2}$$

It can be proved as follows:

$$(n-s)(n-s+1)\ldots\ldots n = \frac{1}{s+2}\,[(n-s)\ldots\ldots n\,(n+1)$$

$$-(n-s-1)\ldots\ldots n] = \frac{1}{s+2}\,[U_{n-s} - U_{n-s-1}]$$

It is evident that

$$S = \frac{1}{s+2}\,[U_{n-s} - U_{n-s-1} + U_{n-s-1} - U_{n-s-2}$$

$$+ \ldots + U_2 - U_1 + U_1 - U_0] = \frac{1}{s+2}\,U_{n-s}$$

since $U_0 = 0$

The final result of the eliminations indicated above is

$$x_1\left[n\,h+\frac{(n+1)\,n\,(n-1)}{3\,!}h^2+\frac{(n+2)(n+1)\,n\,(n-1)(n-2)}{5\,!}h^3+\,.\,.\right]=$$

$$=D\left[1+\frac{n(n-1)}{2\,!}h+\frac{(n+1)\,n\,(n-1)(n-2)}{4\,!}h^2+\,.\,.\,.\,.\right]$$

$$-\,h_1\,x_n\quad .\,.\,.\,.\;(12)$$

Equation (12) takes place of equation (8). Its left-hand member contains the variable x_1 only. The ultimate object of the successive eliminations has, therefore, been reached. Our problem now can be readily solved. The last equation can be much simplified by the following substitution:

$$h=-\,4\,\sin^2\theta$$

Consider the mth term of the left-hand member of (12), namely:

$$x_1\frac{(n+m-1)\;(n+m-2)\;.\,.\,.\,.\;n\;.\,.\,.\,.\;(n-m+2)\;(n-m+1)}{(2\,m-1)\,!}\,h^m$$

This term becomes

$$\frac{x_1\,(-\,1)^m\,[(2\,n)^2-2^2\,(m-1)^2]\;[(2\,n)^2-2^2\,(m-2)^2]\,.\,.\,.\,.\,2\,n}{(2\,m-1)\,!}$$

$$\times\;2\sin\theta\,\sin^{2\,m-1}\theta$$

or if we put $2\,n=\nu$, then the left-hand member of (12) becomes

$$-x_1\,\nu\,\sin\theta\left[\sin\theta-\frac{\nu^2-2^2}{3\,!}\sin^3\theta+\frac{(\nu^2-2^2)\,(\nu^2-4^2)}{5\,!}\sin^5\theta-\,.\,.\,.\,.\,.\right]$$

The series in parenthesis is well known.[1] The expression can be written in the following concise form:

$$-\,\frac{2\sin\theta\,\sin 2\,n\,\theta}{\cos\theta}\,x_1$$

It may be shown now in a similar manner that the right-hand member of (12) can be written

$$\frac{D\cos\,(2\,n-1)\,\theta}{\cos\theta}-h_1\,x_n$$

1. Todhunter Trigonometry, p. 230.

Equation (12) can, therefore, be written

$$x_1 = \frac{-D\cos(2n-1)\theta + h_1 x_n \cos\theta}{2\sin\theta\sin n\theta} \qquad \ldots.(13)$$

The remaining currents can now be easily calculated. Thus

$$x_2 = (h+1)x_1 - D$$

$$= (2\cos 2\theta - 1)x_1 - D$$

$$= \frac{h_1 x_n \cos 3\theta - D\cos(2n-3)\theta}{2\sin\theta\sin 2n\theta}$$

$$x_3 = \frac{h_1 x_n \cos 5\theta - D\cos(2n-5)\theta}{2\sin\theta\sin 2n\theta}$$

It is evident that in general

$$x_m = \frac{h_1 x_n \cos(2m-1)\theta - D\cos[2(n-m)+1]\theta}{2\sin\theta\sin 2n\theta} \qquad (14)$$

This expression for x_m is still incomplete as it contains two unknown quantities, namely x_n and x_1. The last one is contained in $D = D_0 - h_0 x_1$. These two have now to be eliminated.

Let $m = n$, then

$$x_n = \frac{h_1 x_n \cos(2n-1)\theta - (D_0 - h_0 x_1)\cos\theta}{2\sin\theta\sin 2n\theta}$$

From which

$$x_n = \frac{(D_0 - h_0 x_1)\cos\theta}{h_1\cos(2n-1)\theta - 2\sin\theta\sin 2n\theta}$$

$$= \frac{(D_0 - h_0 x_1)\cos\theta}{(h_1 - 1)\cos(2n-1)\theta + \cos(2n+1)\theta}$$

$$= \frac{(D_0 - h_0 x_1)\cos\theta}{A} \qquad \ldots. \qquad (15)$$

From (14) we obtain by substituting $D_0 - h_0 x_1$ for D the following value for x_1

$$x_1 = \frac{D_0\cos(2n-1)\theta - h_1 x_n\cos\theta}{(h_0 - 1)\cos(2n-1)\theta + \cos(2n+1)\theta}$$

$$= \frac{D_0\cos(2n-1)\theta - h_1 x_n\cos\theta}{B} \qquad (16)$$

Combining (15) and (16) we obtain

$$x_1 = \frac{[A \cos (2\, n - 1)\, \theta - h_1 \cos\, \theta]\, D_0}{A\, B - h_1\, h_0 \cos^2 \theta} \tag{17}$$

$$x_n = - \frac{D_0 \sin 2\, \theta \sin 2\, n\, \theta}{A\, B - h_0\, h_1 \cos^2 \theta} \tag{18}$$

The most convenient way of finding the value of any other current, say x_m, is to go back to (6) and find x_2 from the first equation by inserting the value of x_1 from (17). We find

$$x_2 = \frac{[A \cos (2\, n - 3)\, \theta - h_1 \cos \theta \cos 3\, \theta]\, D_0}{A\, B - h_0\, h_1 \cos^2 \theta}$$

The general formula can now be easily guessed. It is

$$\left.\begin{aligned} x_m &= \frac{[A \cos (2n - 2m + 1)\, \theta - h_1 \cos \theta \cos (2m - 1)\, \theta]\, D_0}{A\, B - h_0\, h_1 \cos^2 \theta.} \\ &= \frac{[2 \sin\theta \cos(2n - 2m + 1)\theta + h_1 \sin(2n - 2m + 2)\theta] D_0}{h_0 h_1 \sin(2n - 2)\theta - 4 \sin^2\theta \sin 2n\theta + 2 \sin\theta (h_0 + h_1) \cos(2n - 1)\theta} \end{aligned}\right\} \tag{19}$$

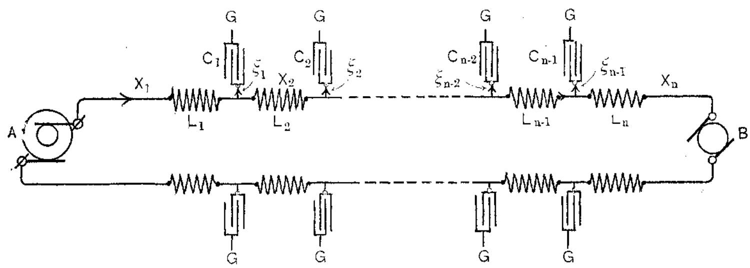


FIG. 5.

This value of x_m satisfies all the conditions, which forced oscillations have to satisfy. It is therefore the complete solution of our problem for forced oscillations, taking into account the reactions of the transmitting and the receiving apparatus. The angle θ is a complex angle, so that the real part of (19) is the actual current in circuit m.

SECOND ARRANGEMENT.

In this arrangement there are 2 $(n-1)$ condensers each of capacity C, all connected to ground as shown in Fig 5. In place of (5) and (6) we shall have here the following differential equations :—

$$\left.\begin{aligned}
&h\,x_1+\xi_1-0 &&=\tfrac{1}{2}(D_0-h_0\,x_1)\\
&h\,x_2+\xi_2-\xi_1 &&=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots &&\\
&h\,x_m+\xi_m-\xi_{m-1} &&=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots &&\\
&h\,x_n+0-\xi_{n-1} &&=-\tfrac{1}{2}h_1\,x_n
\end{aligned}\right\}\quad(20)$$

or

$$\left.\begin{aligned}
&(h+1)\,x_1-0-x_2 &&=\tfrac{1}{2}(D_0-h_0\,x_1)\\
&(h+2)\,x_2-x_1-x_3 &&=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots &&\\
&(h+2)\,x_m-x_{m-1}-x_{m+1} &&=0\\
&(h+1)\,x_n-x_{n-1}-0 &&=-\tfrac{1}{2}h_1\,x_n
\end{aligned}\right\}\quad(21)$$

Solution (19) will therefore hold for these equations also, provided that in it we multiply h_0, h_1, and D_0 by $\frac{1}{2}$ and also remember that in this case

$$-4\sin^2\theta=(-p^2\,L\,C+i\,p\,R\,C)=h$$

Note:—Solution (19) can also be obtained by a short cut. Equation (6) of Section I. of this paper suggests the follow. ing as a trial solution:—

$$x_m=K_1\cos 2\,(n-m)\,\theta+K_2\sin 2\,(n-m)\,\theta.$$

It will be found that this satisfies all equations in (6) and (21) except the first and the last, provided that

$$h=-4\sin^2\theta.$$

Now by giving K_1 and K_2 their proper values the first and the last equation may also be satisfied. Having thus determined K_1 and K_2 we obtain (19). Considerable calculation can thus be saved. The longer and more tedious method was selected because it offers a convenient means of discussing certain cases of wave propagation, not considered in this paper, which cannot very well be attacked directly by the infinitesimal method. This matter is reserved for a future occasion.

B. Free Oscillations.

Equation (19) holds true for free as well as forced oscillations. But since in the case of free oscillations $D_0 = 0$ it follows that the denominator of (19) must vanish to prevent the vanishing of all the currents. We shall have, therefore, in this case

$$h_0 h_1 \sin(2n-2)\theta - 4\sin^2\theta \sin 2n\theta + 2\sin\theta(h_0+h_1)\cos(2n-1)\theta = 0 \quad (22)$$

From this equation θ has to be determined. A solution can be obtained for a small number of problems. The two most important will be considered here.

First. The transmitting and the receiving apparatus are not present. In this case

$$h_0 = h_1 = 0,$$

$$x_m = B \cos(2n - 2m + 1)\theta \quad (23)$$

It is found from (22) that (23) is actually the solution of the differential equations (6) for $h_0 = h_1 = D_0 = 0$, provided that

$$\theta = \frac{s\pi}{2n}$$

where s may be any integer from 1 to 2 n.

Hence the most general solution will be

$$x_m = \sum_{s=1}^{2n} B_s \cos(2n - 2m + 1)\frac{s\pi}{2n} \quad (24)$$

But it should be observed now that x_m is a periodic function of the time, that is

$$x_m = A_m e^{pt}$$

Hence in (24) each amplitude B contains the time factor e^{pt} that is

$$B_s = A_s e^{p_s t}$$

The constant p_s which measures the period of the free oscillation is determined from the relation

$$h = -4 \sin^2\theta.$$

In the case of free oscillations

$$h = p^2 L C + p R C,^* \qquad \theta = \frac{s\pi}{2n}$$

Hence

$$p_s^2 L C + p_s R C = -4 \sin^2 \frac{s\pi}{2n} \tag{25}$$

This enables us to determine p_s. Before solving this equation it is desirable to make the following substitution :

Let L', C', R' be the total co-efficient of self-induction, capacity, and resistance, respectively, of one-half of the conductor, then

$$L = \frac{L'}{n}{}^*$$

$$C = \frac{C'}{n}$$

$$R = \frac{R'}{C}$$

Let l denote the half-length of a uniform wire having λ, r, c, for co-efficient of self-induction, resistance and capacity per unit length, and let

$$l\lambda = L', \qquad l r = R', \qquad l c = C'$$

then

$$L = \frac{l\lambda}{n}, \qquad C = \frac{l c}{n}, \qquad R = \frac{l r}{n}.$$

* This will apply to second arrangement, Fig. 5, but not to the first arrangement. The calculation for first arrangement is slightly different and can be easily made.

From (23) we obtain

$$\frac{l^2}{n^2}(p^2_s \lambda c + p_s r) = -4 \sin^2 \frac{s \pi}{2 n}$$

$$\therefore p_s = -\frac{r}{2\lambda} \pm \sqrt{-1}\sqrt{\frac{1}{\lambda c}\frac{4 n^2}{l^2} \sin^2 \frac{s \pi}{2n} - \frac{r^2}{4\lambda^2}}$$

$$p_s = -\frac{r}{2 \lambda} \pm i k_s$$

Equation (22) becomes now

$$x_m = e^{-\frac{r t}{2\lambda}} \sum_{s=1}^{2n} A_s \cos (2n-2m+1) \frac{s \pi}{2n} \cos (k_s t - \varepsilon_s) \qquad (26)$$

Second case:—The transmitting apparatus is not present, and in place of the receiving apparatus there is a break in the line at B.

In this case $h_0 = 0$, $h_1 = \infty$. Equation (19) gives

$$x_m = B \sin (2 n - m + 2) \theta$$

provided that

$$\cos (2 n - 1) \theta = 0$$

$$\text{or } \theta = \frac{2 s + 1}{2 n - 1} \frac{\pi}{2}$$

We shall have, therefore,

$$p_{2s+1} = -\frac{r}{2 \lambda} \pm i \sqrt{\frac{1}{\lambda c} \frac{4 n^2}{l^2} \sin^2 \frac{2 s + 1}{2 n - 1} \frac{\pi}{2} - \frac{r^2}{4 \lambda^2}}$$

$$= -\frac{r}{2 \lambda} \pm i k_{2s+1}$$

$$\therefore x_m = e^{-\frac{r t}{2\lambda}} \sum_{s=0}^{2n} A_{2s+1} \sin (2n-2m+2) \frac{2s+1}{\theta n-1} \frac{\pi}{2} \cos (k_{2s+1} t - \varepsilon_{2s+1}) \qquad (27)$$

The question arises now, under what conditions will a loaded conductor of this kind become approximately equivalent to a uniform wire?

Let

$$L = .0125$$
$$C = .025$$
$$R = 2.5$$
$$p = 3000$$

that is the frequency is about 500 *p.p.s.*

It will be found that since

$$-4 \sin^2 \theta = -p^2 L C + i p R C = -(a + i \beta)^2 = -4 \mu^2$$

or

$$\sin \theta = \tfrac{1}{2} (a + i \beta) = \mu$$
$$= .026 + .0014\, i$$
$$\theta = \mu \text{ very nearly.}$$

If n coils have the same co-efficient of self-induction, the same capacity and the same resistance as a uniform wire of length l, and m coils correspond to a length s, then

$$m : n :: s : l$$

$$\therefore m = \frac{ns}{l}$$

and

$$2\ (n-m)+1\ \theta = 2n \left(1 - \frac{s}{l} + \frac{1}{2n} \right) \mu$$

Again

$$-p^2 L C + i p R C = \frac{l^2}{n^2} (-p^2 \lambda c + i p r c) = -\frac{l^2}{n^2} m^2$$

where m has the same meaning as in section I.

$$\therefore \theta = \mu = \frac{l}{n} \frac{m}{2}$$

$$\therefore 2n \left(1 - \frac{s}{l} + \frac{1}{2n} \right) \mu = \left(l - s + \frac{l}{2n} \right) m$$

$$= (l - s)\, m$$

when n is large

In the apparatus described in Sect. III, $n = 200$ hence above expression reduces to

$$2\,n\left(1-\frac{s}{l}+\frac{1}{400}\right)\mu=(l-s)\,m=m\,\xi$$

When this value of θ is substituted in (19), this equation reduces to equation (6) of Section I., which shows that under the conditions just described, the loaded conductor described here becomes equivalent to a uniform wire. Experiments performed upon the loaded conductor described in Section III, verify this theoretical conclusion. Up to about 1000 p.p.s. the loaded conductor (having for L, C, R, of each section the values given above) behaves, very nearly, like a uniform slow-speed conductor.

SECTION III.

EXPERIMENTS WITH SLOW-SPEED CONDUCTORS.

Slow-Speeed Conductor with uniformly distributed capacity, self-induction and resistance.—This conductor consists of a number of coils, generally twenty-four, joined in series. The construction of each coil is represented in Fig. 6. A number of

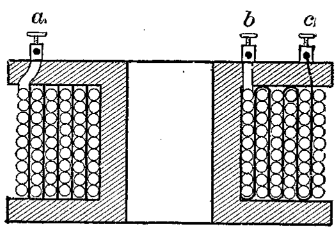


FIG. 6.

layers of No. 20 wire are wound upon a wooden spool; the height of the layers is three inches, the diameter of the inside layer is seven inches, the number of layers is eight. Each layer when wound is covered with a sheet of paraffined paper, then a sheet of tinfoil is wrapped and covered with a sheet of paraffined paper, and then the next layer is wound. The same operation is repeated after each layer of wire. The tinfoil sheets are all connected in series and to the binding post c. The thick vertical lines between the layers represent the tinfoil. The spools are carefully turned and everything is done to secure the equality of the coils. Experimental measurements of the electrical constants of the coils showed that the coils were equal to each other to within less than one per cent. But it should be observed that this equality can be verified experimentally when the coils are well heated up so as to expel the moisture between the layers; moisture prevents an accurate measurement of capacity on account of excessive leakage. The heating was done electrically. By repeated trials the result aimed at was finally obtained, namely, that each coil should have the following constants:—

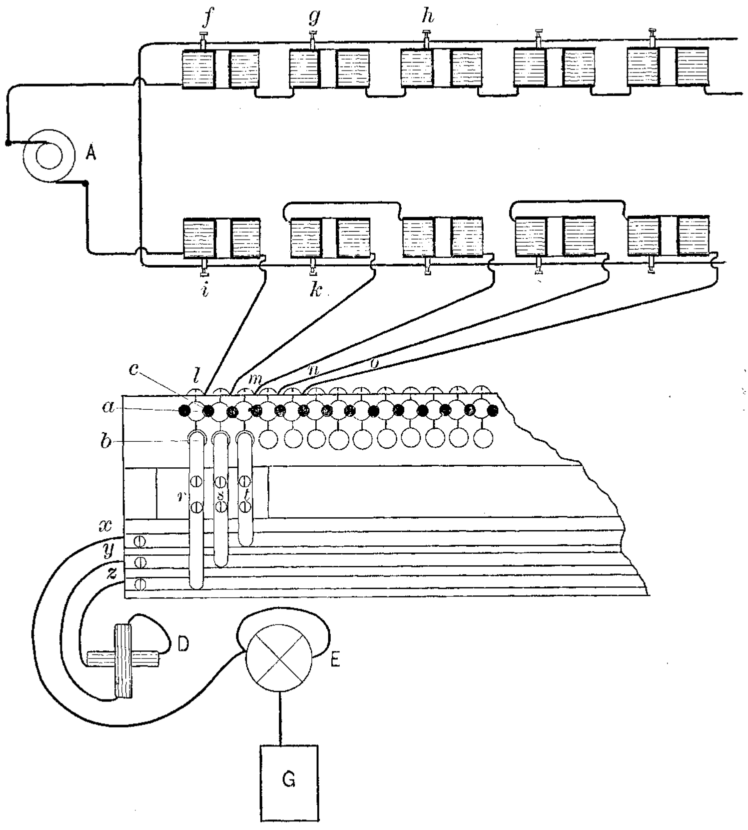


FIG. 7.

$$L = .05 \text{ henry}$$

$$C = .1 \text{ microfarad}$$

$$R = 10 \text{ ohms.}$$

That is to say, each coil was equivalent to about ten miles of telephone wire now in use between New York and Chicago. The leakage was more than on ordinary telegraph lines under fair conditions. It cannot be avoided in coils of this kind, but as long as the impressed E.M.F. is not above 300 volts it does not interfere very seriously with the successful operation on conductors of this kind.

The coils were connected in series as represented in Figs. 7 and 7[a]. In Fig. 7. the exciting machine is A. The binding posts *f g* to *i k* . . . lead to the tinfoils. They are all connected together and then grounded. The coils are not connected to each other directly but through a switchboard *a b x y z*, the lower part of Fig. 7. This switchboard is constructed as follows:—Into a well-seasoned board of oak X Y Z Fig. 7[a] and *a b x y z* Fig. 7 are driven two rows of circular brass plugs *a*. . . *c* . . . Fig 7 or *b c d* . . . *h i k* . . . Fig. 7[a]. These are connected by rods driven in from binding screws *s t u* . . Fig. 7[a]. The upper row of plugs

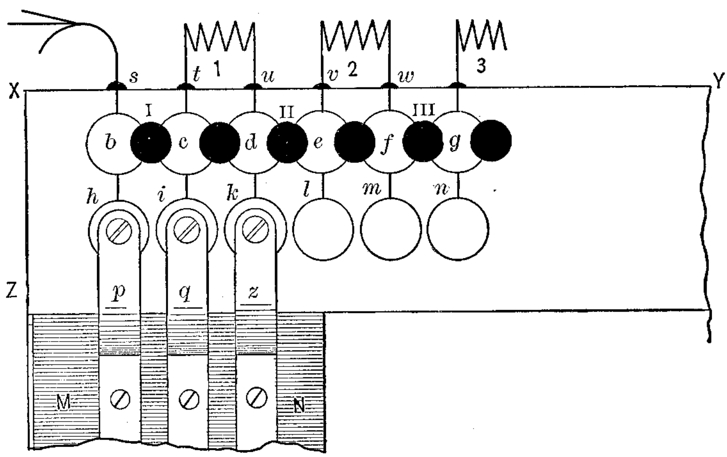


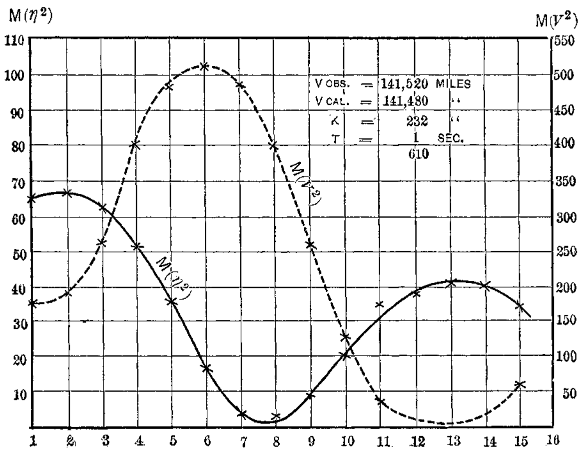
FIG. 7a.

has holes I, II, III, represented by black circles in (Fig. 7[a]) into which plungers, ordinary condenser plugs, are inserted; thus a plunger in I (Fig. 7[a]) connects plugs *b* and *c*. The connection of the coils to the brass plugs is represented correctly in Fig. 7[a] (but not in Fig. 7 owing to a mistake of the draughtsman). A wire *a* leads from alternator A to the first plug *b*. The terminals of the coils 1, 2, 3, etc., are connected as represented to binding screws *t*, *u*, *v*, *w*, . . . Suppose now that the connecting plungers are in I, II, III, etc., the coils are then connected together and the circuit is established. To measure the current and the potential at various points of this circuit proceed as follows:—Three long brass bars *x y z* (Fig. 7) running parallel to the rows of brass plugs can be connected to any three consecutive plugs by means of three brass strips $z\ s\ t$ (Fig. 7) or $p\ q\ z$ (Fig. 7[a]). The two upper bars $x\ y$ Fig. 7 are connected to a Siemens electro-dynamometer D (reading down to .02 ampere), the lower is connected to a Thomson multicellular voltmeter E (reading up to 250 volts). Suppose now that a reading is to be taken at the point between coils 1 and 2. That would correspond to a point 10 miles outside of the sending station on the long-distance telephone wire mentioned above. The brass strips $p\ q\ z$ (Fig. 7[a]) which are connected to a wooden slide M N (Fig. 7[a]) are moved to the right until strip p reaches plug k; at the same time q will be on l and z will be on m. The connecting plunger in II is then removed and the current made to pass through the electro-dynamometer, D, at the same time the voltmeter E is connected to point III. The two readings give the mean square of current at II and the R.M.S. of potential at III. The readings from which curves in Fig. 8 and Fig. 9 were plotted were taken in this manner.

The alternators which supplied the impressed E.M.F. were two small machines, each having four separate armatures and four fields. The four fields rotated on the same shafts. In this manner any frequency between about 25 P.P.S. and 750 P.P.S. could be obtained. The E.M.F.'s generated were not simple harmonics; the effect of the higher harmonics (the fifth was predominant, but not strong) was weakened by tuning. Well known precautions were taken to keep the speed and excitation constant. The electromotive forces employed ranged between 60 and 234 volts. The length of the equivalent line operated upon was usually 240 miles. The number of observations made ran into hundreds. The two recorded in Fig. 8 and Fig. 9 are among the best. The most serious source of inaccuracy was found to be the variation of the speed and excitation of the alternators. The slow-speed conductor gave no serious trouble, provided that it was kept reasonably warm by passing through it from time to time a strong current for several minutes.

The mean square curves, Fig. 8 and Fig. 9.—Fig. 8 represents the mean square curves of current (full line) and of potential (dotted line) when there was no receiving apparatus present. The impressed E.M.F. was 234 volts and the frequency 610 P.P.S. There were 24 coils in series, hence an equivalent of 240 miles of long-distance telephone wire mentioned above. The current curve represents a little over one-half of the theoretical current curve in Fig. 2. The curve of mean square of potential is not given in Fig. 2.

The abscissæ 1, 2, 3, . . . measure the number of the coil in front of which the reading was taken. Thus abscissa 1 means that the current reading was taken when the slide had the position indicated in Fig. 7[a] and plug I was removed. Hence this reading represents the value of current between the machine and the slow-speed conductor. Voltmeter reading taken in this position of the slide represents, of course, reading 2. To get the voltmeter reading 1 the slide *M N* had to be moved one peg to the left and the connecting plunger left in I. There were 24 coils in series, hence reading 13 representing the reading between coils 12 and 13 gives the reading at the middle point of the loop. Here



p. p. s. 610

M (η^2)	M(V^2)	COIL
63	178	1
67	190	2
64	256	3
53	400	4
37	484	5
18	506	6
4	480	7
3.5	400	8
9.5	256	9
20	134	10
35	40	11
37		12
41		13
41	60	14
34		15

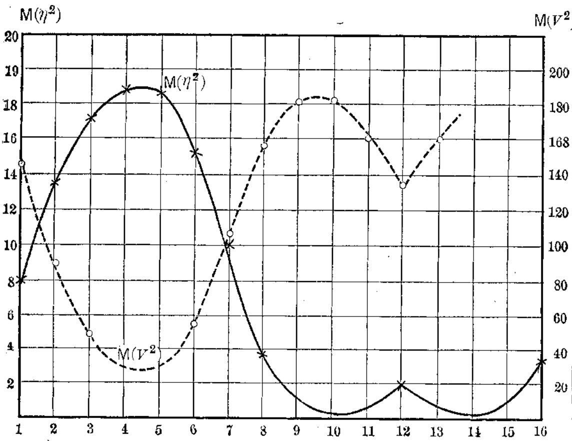
FIG. 8.

the *M* (η^2) is a maximum and *M* (V^2) is zero, as required by theory. The voltmeter did not read below sixty volts, hence no *M* (V^2) readings could be taken in the immediate vicinity of the middle point. But the lowest readings were carefully determined on each side of this point, and the course of the *M* (V^2) curve in this vicinity is thus fixed. The column headed *M* (η^2) in the table on the right of Fig. 8 gives the electro-dynamometer readings just as they were read off the instrument. The figures in column headed *M* (V^2) represent the first three figures of the voltmeter readings squared. These curves agree remarkably well with the curves given by theory, in fact much better than one would expect from the apparent complexity and the apparent multiplicity of the apparatus employed. But it should be observed that the actual experimental operations involved after everything has been once set up are extremely simple and capable of great precision. The maximum of the middle point of the M (η^2) curve is not exactly at 13 which shows that the two sides of the slow-speed conductor were not perfectly symmetrical.

The determination of wave-length and of the velocity of propagation.—The distance between the maximum in the vicinity of reading 13 and the minimum in the vicinity of reading 8 represents a quarter wave-length. Now this distance is 29 divisions, and since each division represents 2 miles it follows that the wave length

$$\lambda = 232 \text{ miles.}$$

$$v = \lambda\, T = 232 \times 610 = 141520 \text{ miles.}$$



COILS	M(η^2)	M(V^2)
1	8	144
2	13.5	90
3	17	46
4	18.5	0
5	18.5	0
6	15	56
7	10	104
8	3.5	156
9	0	182
10	0	182
11	0	161
12	2	137
13	0	164
14	0	
15	0	
16	3	

Fig. 9.

Calculating v from the formula

$$v = \frac{2\pi}{T}\,\frac{1}{a}$$

we obtain

$$v = 141480 \text{ miles.}$$

The agreement between the observed and the calculated value is so remarkable that one is much tempted to attribute it to luck rather than to good management. That such is actually the case is admitted here frankly, but it should be observed, in justice to the method, that a large series of curves obtained with different frequencies and under different conditions support the belief that with careful precautions and a reasonably well made and carefully nursed slow-speed conductor this method is actually capable of determining λ and v with much accuracy, in fact a greater accuracy than is usually obtained in wave-length determinations of the Hertzian waves. The measurement of T introduces the largest error, but this error will not cause a disagreement between v observed and v calculated since the same T is used in both cases.

The curves in Fig. 9 represent the mean square values of the current and of the potential when a coil with a coefficient of self-induction of one henry was placed in the middle. There were 22 coils in series; the frequency and electromotive force were the same as in the preceding case. The cusp predicted by theory (see Fig. 3) occurs therefore at reading twelve, that is, between the eleventh and twelfth coil. A comparison between Fig. 9 and Fig. 3 shows a very satisfactory agreement between

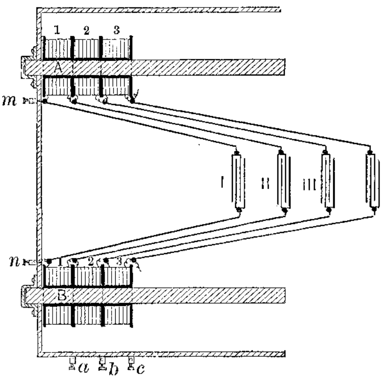


FIG. 10.

theory and experiment. The emormous sag of the current in the vicinity of the receiving apparatus is due to the high frequency and the large self-induction, and therefore large reactance of the receiving coil. The power delivered to this coil is not small in spite of the diminished current, for it will be seen from the $M(V^2)$ curve that the potential is high in the vicinity of the coil where the current is small.

The slow-speed conductor just described cannot stand high voltage and, besides, its leakage is large unless handled with much care. Another type of slow-speed conductor which does not have these objectionable features is represented in Fig. 10 and Fig.11. It consists of a large number of equal spools 1 2 3 . . . . connected in series. Fig. 10 represents a part of two rows of these spools, each row mounted on the same tube, one for each row. These tubes are marked A and B. From each wire connecting two consecutive coils runs a wire to a binding post leading to a section of a condenser; these sections are denoted by I, II, III, etc., in Fig. 10. There are as many of these sections as there are coils, and they are all equal to each other. Each coil has as nearly as possible the following constants

$$L = .0125\ H.$$
$$R = 2.5 \text{ ohms.}$$

The capacity of each condenser section is .025 microfarad. Each coil with its condenser section is equal to 2½ miles of telephone wire mentioned above. Two rows of 40 coils each with corresponding condenser sections are mounted together and enclosed in a dust-tight glass case. Such case represents a loop of 200 miles

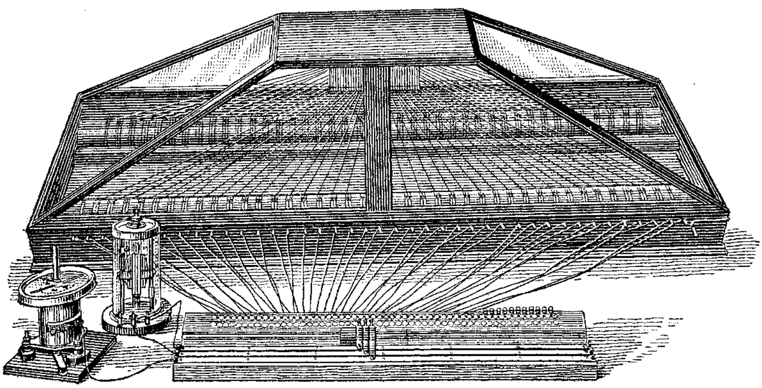


FIG. 11

of long-distance telephone wire. The electro-mechanical laboratory of Columbia University has five such cases and these represent together a loop of a thousand miles. Fig. 11 is taken from a photograph of one of these cases. The two rows of coils are seen near the bottom of the case. The square box near the top of the case is the condenser box with the condenser sections. The wires running radially from this box are the wires connecting the condenser sections to the coils. The condenser sections and the coils can be connected in two ways, both of which were discussed in Section II and illustrated by Fig. 4 and Fig. 5 of that section. In the first arrangement, Fig. 4, half as many condenser sections are required as in the second and, therefore, the capacity per unit length of equivalent wire will be smaller. The theory given in Section III states that up to about 1,000 P.P.S. such a loaded conductor will be equivalent to a uniform air line. Experiment confirms the theory, for it shows that $M(\eta^2)$ and $M(V^2)$ curves obtained with such a conductor are the same as those given in Fig. 8 and Fig. 9, at any rate up to 750 P.P.S.

This loaded conductor offers advantages of more exact and more solid and durable construction. The spools can be wound so accurately that the difference in self-induction and resistance between them is exceedingly small. The condenser sections are made of tinfoil and selected mica and adjusted carefully. Two consecutive condenser sections will differ from each other in capacity by as much as even three per cent., but then the average capacity of forty such sections will be very nearly equal to the average capacity of the consecutive forty sections. It is this average capacity which determines the wave-length and the velocity of propagation. The capacity of these condensers remains practically constant. They can stand 3500 volts with impunity.

Laboratory for Electro-Mechanics,
Columbia University, New York, March, 1899.