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Propagation speed of travelling fronts in non local reaction–diffusion equations

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Abstract

The object of this paper is to provide variational formulas characterizing the speed of travelling front solutions of the following nonlocal diffusion equation:

$$\frac{\partial u}{\partial t} = J * u - u + f(u),$$

Where J is a dispersion kernel and f is any of the nonlinearities commonly used in various models ranging from combustion theory of ecology. In several situations, such as population dynamics, it is indeed natural to model the dispersion of a population using such operators. Furthermore, since travelling front solutions are expected to give the asymptotic behaviour in large time for solutions of the above equation, it is of the interest to characterize their speed. Our results, based on elementary techniques, generalize known results obtained for models involving local diffusion operators.

Keywords: Travelling front; Super and sub-solution; Maximum principle; Nonlocal reaction-diffusion equations; Propagation speed

1. Introduction

In this article, we are concerned with variational formulas characterizing the speed c of travelling fronts u arising in the study of a nonlocal reaction–diffusion model. More

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precisely, we study the solutions (u, c) of the following one dimensional integro-differential equation:

$$\begin{cases} Lu - cu' + f(u) = 0 & \text{on } \mathbb{R}, \\ u(x) \to 0 & \text{as } x \to -\infty, \\ u(x) \to 1 & \text{as } x \to +\infty, \end{cases} \tag{P}$$

where L is a nonlocal diffusion operator of the form

$$Lu = J \star u(x) - u(x) = \int_{\mathbb{R}} J(x-y)u(y)\,\mathrm{d}y - u(x), \tag{1.1}$$

with J an even positive kernel of mass one and f a given nonlinearity. Our results apply to more general operators of the form

$$Lu = au_{xx} + b(J \star u(x) - u(x)) + \mathrm{d}u_x - eu(x), \tag{1.2}$$

where $a, b, e \geqslant 0$, $(a, b) \neq (0, 0)$ and $d \in \mathbb{R}$. We will always assume in what follows that J satisfies the following:

$$J \in W^{1,1}(\mathbb{R}), \qquad J \geqslant 0, \qquad J(x) = J(-x) \quad \text{and} \quad \int_{\mathbb{R}} J = 1. \tag{H1}$$

The unknowns of this problem are the real number c, which represents the speed of the front, and u the profile of the front. The speed c can also be viewed as a nonlinear eigenvalue of the problem. Travelling-front solutions are expected to give the asymptotic behavior in large time for solutions of the following evolution problem (1.3), with say compactly supported initial data.

$$\frac{\partial u}{\partial t} = Lu + f(u). \tag{1.3}$$

It is therefore of interest to characterize the speed of these solutions. Such types of equation were derived in the early work of Kolmogorov–Petrovskii–Piscounov (KPP) (see [21]) on the spread of a gene . The dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x, y)u(y)\,\mathrm{d}y$, where $J(x, \cdot)$ is a probability density. Restricting to a one-dimensional setting and assuming that such a diffusion process depends only on the distance between two niches of the population, we end up with Eq. (1.3).

Eq. (1.3) also appears in the context of pattern formation in activator–inhibitor systems such as

$$\begin{cases} \frac{\partial u}{\partial t} - u_{xx} = f(u) - v, \\ -v_{xx} + v = u. \end{cases}$$

Observe that we can inverse the second equation. We can thus rewrite v in terms of u. Namely we have $v = J \star u$ with $J(x) = \frac{1}{2}\mathrm{e}^{-|x|}$, so that the system can be reformulated as

$$\frac{\partial u}{\partial t} = u_{xx} + J \star u - u + g(u) \quad \text{for } (x, t) \quad \in \mathbb{R} \times \mathbb{R}^+, \tag{1.4}$$

where $g(u) = f(u) + u$. For more information, see the excellent book of Murray [23] and [24]. In these two models the operator $\mathscr{A}(u) := J \star u - u$ represents the nonlocal diffusion of the species through its environment.

In this work, we study three types of nonlinearity *f* that we present below: $f \in C^1([0,1])$ and

- *Case A1*: *f* is of bistable type if for some $\rho > 0$, *f* satisfies
  - $f|_{(0,\rho)} < 0$ and $f|_{(\rho,1)} > 0$,
  - $f(0) = f(1) = 0$ and $f'(1) < 0$.
- *Case A2*: *f* is of ignition type if for some $\rho > 0$,
  - $f|_{(0,\rho)} \equiv 0$ and $f|_{(\rho,1)} > 0$,
  - $f(0) = f(1) = 0$ and $f'(1) < 0$.
- *Case B*: *f* is of monostable type if $f(0) = f(1) = 0, f|_{(0,1)} > 0$ and $f'(1) < 0$.

These three types of nonlinearities are commonly used in the literature to describe models of phase transition, nerve propagation, combustion, population dynamics and ecology : see [1,3,12–15,19,21,25,28–30].

Under some additional assumption on the kernel *J*, existence and in some cases uniqueness of travelling-wave solutions have been investigated by Bates et al. [2] and Chen [7] in the bistable case and completed by the work of one of the present authors [10], for the ignition case. The monostable case is the object of a forthcoming publication [11]. We summarize these results in the following theorem.

Theorem 1.1 (*Bates et al. [2]; Coville [10]; Coville and Dupaigne [11]*).

- *Let f be a nonlinearity of type* A1 *or* A2 *and assume that J satisfies* (H1) *and the following*:

$$\int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z < +\infty. \tag{H2}$$

  Then problem (P) *admits a solution* (u, c^*). *This solution is unique in the following sense*: *if* (v, c') *is another solution then* $c^* = c'$. *Moreover, if* $c^* \neq 0$ *then* $u(x) = v(x+\tau)$ *for a fixed* τ *and*
- *Let f be a monostable function and assume that J satisfies* (H1) *and the following integrability condition*

$$\forall \lambda > 0, \quad \int_{\mathbb{R}} J(z)\mathrm{e}^{\lambda z}\,\mathrm{d}z < +\infty. \tag{H3}$$

  Then there exists a minimal speed $c^* > 0$ *such that*
  - *If* $c \geqslant c^*$, *then problem* (P) *admits a solution* (u, c). *Moreover* $u' > 0$.
  - *If* $c < c^*$, *then problem* (P) *admits no solution* (u, c) *such that* $u' > 0$.

Remark 1.1. The condition (H3) can be weakened : it is enough to assume that (H3) holds for one given value of $\lambda > 0$.

Theorem 1.1 generalize known results for the standard reaction–diffusion problem below,

$$\frac{\partial u}{\partial t} = \Delta u + g(u) \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+. \tag{1.5}$$

This is due to the fact that the nonlocal operator L shares many properties of the Laplacian and in some limiting case reduces to it. Namely, assume J is compactly supported and let $J_\varepsilon(x) := \frac{1}{\varepsilon} J(\frac{1}{\varepsilon}x)$, with $\varepsilon > 0$ small. For $u \in C^2(\mathbb{R})$:

$$\begin{aligned} J_\varepsilon \star u - u &= \frac{1}{\varepsilon}\int J\left(\frac{1}{\varepsilon}y\right)(u(x-y)-u(x))\,\mathrm{d}y = \int J(z)(u(x-\varepsilon z)-u(x))\,\mathrm{d}z \\ &= -\varepsilon \int J(z)u'(x)z\,\mathrm{d}z + \frac{\varepsilon^2}{2}\int z^2 J(z)u''(x)\,\mathrm{d}z + o(\varepsilon^2) \\ &= C\varepsilon^2 u''(x) + o(\varepsilon^2), \end{aligned}$$

where we used the fact that J is even in the last equality. Hence,

$$Cu'' = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left[J_\varepsilon \star u - u\right].$$

The characterization by min–max formulas of the wave speed c in the context of (1.5) is well known. In one space dimension, Eq. (1.5) reduces to an ordinary differential equation and the speed of planar fronts satisfies the following min–max formulas.

Let $X = \{w \in C^2(\mathbb{R}) | w \text{ is increasing}, w(+\infty) = 1 \text{ and } w(-\infty) = 0\}$.

- For f of type A1 or A2, the speed c^* satisfies

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \left\{\frac{w'' + f(w)}{w'}\right\}, \tag{1.6}$$

$$c^* = \max_{w \in X} \inf_{x \in \mathbb{R}} \left\{\frac{w'' + f(w)}{w'}\right\}. \tag{1.7}$$

- For f of type B, the minimal speed c^* satisfies

$$c^* = \min_{w \in X} \sup_{x \in \mathbb{R}} \left\{\frac{w'' + f(w)}{w'}\right\}. \tag{1.8}$$

Min–max formulas for travelling fronts in systems of ODEs have been derived by Kan–On, Mischaikow–Hudson and Volpert, Volpert, Volpert, see [20,22,26,27]. Hamel [17] generalized min–max formulas to the setting of multidimensional travelling fronts in a cylinder. In the bistable case, Heinze et al. [18] have provided analogous variational formulas for quite general operators.

Under some extra assumption, an explicit formula for the minimal speed can be given: Kolmogorov et al. [21] proved in 1937 that the minimal speed c^* is given by

$$c^* = c^*_{KPP} = 2\sqrt{f'(0)},$$

if f is monostable and satisfies $f(s)/s \leqslant f'(0)$ for $s \in (0,1)$. This formula was recovered by Berestycki and Nirenberg [4] using a different approach. Weinberger in [28] generalized it to time-discrete models.

In the opposite situation, when f approaches a Dirac mass centered at one, Zeldovich and Frank–Kamenetskii (ZFK) [30] were able to give an asymptotic formula for the flame's front speed. In this case the speed is given by

$$c^* = c^*_{ZFK} \simeq \sqrt{\int_0^1 f(s)\,\mathrm{d}s}.$$

More recently, Berestycki et al. [5] have shown that the asymptotic speed formula c^*_{ZFK} holds for planar-front solutions of a system of ordinary differential equations. Other asymptotic formulas were derived in turbulent combustion: see Clavin and Williams [9]. Clavin [8] also explains the transition from c^*_{KPP} to c^*_{ZFK}.

In many cases, proofs rely deeply on shooting methods, phase plane analysis and good asymptotics. Since our equation is nonlocal, we cannot carry out most of these techniques. Nevertheless we can provide min–max formulas for the speed of travelling fronts of (1.3), analog to those above. Namely we have the following variational characterization of the wave-speed:

Theorem 1.2. *Let L be the operator given by* (1.1) *defined on $X := \{w \in C^1(\mathbb{R}) | w$ is increasing, $w(+\infty) = 1$ and $w(-\infty) = 0\}$.*

- *Assume* (H1) *and* (H2) *hold. For f of type* A1 *or* A2, *such that $\int_0^1 f(s)\,\mathrm{d}s \neq 0$ the speed c^* satisfies*

$$c^* = \min_{w\in X} \sup_{x\in\mathbb{R}} \left\{ \frac{Lw + f(w)}{w'} \right\}, \tag{1.9}$$

$$c^* = \max_{w\in X} \inf_{x\in\mathbb{R}} \left\{ \frac{Lw + f(w)}{w'} \right\}. \tag{1.10}$$

- *Assume* (H1) *and* (H3) *hold. For f of type B, the minimal speed c^* satisfies*

$$c^* = \min_{w\in X} \sup_{x\in\mathbb{R}} \left\{ \frac{Lw + f(w)}{w'} \right\}. \tag{1.11}$$

Remark 1.1. Theorem 1.2 remains valid for operators L given by (1.2), provided X is taken to be $X := \{w \in C^2(\mathbb{R}) | w$ is increasing, $w(+\infty) = 1$ and $w(-\infty) = 0\}$.

Remark 1.2. For f of type A2, $\int_0^1 f(s)\,\mathrm{d}s > 0$. For f of type A1, it may happen that $\int_0^1 f(s)\,\mathrm{d}s = 0$. In that case, it is known that $c^* = 0$, see [2].

The technique developed in this paper also applies to the traditional reaction–diffusion problem, thus providing an alternate proof of these formulas. In the KPP-like situation we were not able to give an exact explicit formula and only provide an upper bound in terms of a spectral formula. Namely, we show

$$c^*_{KPP} \leqslant \inf_{\lambda>0} \left\{ \frac{1}{\lambda} \left[\int_{\mathbb{R}} J(z) e^{\lambda z}\, dz - 1 + f'(0) \right] \right\}. \tag{1.12}$$

Recently, equality has been obtained by Carr and Chmaj [6] in the case where J has compact support and $f(u) = u(1-u)$.

Method and plan

The proof relies on two simple ideas:

- The construction of solutions via the method of sub and supersolutions.
- A comparison principle for sub and supersolutions of a bistable problem.

Though elementary in nature, the proof of these results requires a number of lemmas which we list and prove in Section 2. In Sections 3 and 4, we present the construction of a solution via the method of sub and supersolutions. Theorem 1.2 is the object of the Sections 5 and 6. Section 5 deals with the min–max formula in the bistable case while Section 6 is concerned with the monostable case and the proof of inequality (1.12).

2. Linear theory

We start this section with two maximum principles for integro-differential operators L defined on the real line by (1.1) or (1.2).

Theorem 2.1 (*Strong maximum principle*). *Let $u \in C^2(\mathbb{R})$ satisfy*

$$Lu \geqslant 0 \quad in \quad \mathbb{R} \quad (\textit{respectively}, \quad Lu \leqslant 0 \quad in\ \mathbb{R}).$$

Then u may not achieve a positive maximum (*resp. negative minimum*) *without being constant*.

This theorem immediately implies the following practical corollary:

Corollary 2.1. *Let $u \in C^2(\mathbb{R})$ satisfy*

$$\begin{cases} Lu \geqslant 0 & on\ \mathbb{R}, \\ u \to 0 & |x| \to +\infty. \end{cases}$$

Then

- *either $u < 0$,*
- *either $u \equiv 0$.*

Remark 2.1. *Similarly, if $Lu \leqslant 0$ then u is either positive or identically* 0.

The corollary is a straightforward consequence of the strong maximum principle.

Proof of Theorem 2.1. If $b=0$ then L is a standard elliptic operator, so we restrict ourselves to the case where $b>0$. We argue by contradiction. Assume that u is a nonconstant function and achieves a positive maximum somewhere, say at x_0.

Since u is a C^2 function, we have $eu(x_0)\leqslant 0$, $u'(x_0)=0$ and $u''(x_0)\leqslant 0$. Furthermore, since $\int_{\mathbb{R}} J(z)\,\mathrm{d}z=1$ and $u(y)-u(x_0)\leqslant 0$ for every y in $\mathbb{R}$, we have $J\star u(x_0)-u(x_0)=\int_{\mathbb{R}} J(x_0-y)(u(y)-u(x_0))\,\mathrm{d}y\leqslant 0$. Therefore, we have at the point x_0:

$$au''(x_0)+b(J\star u(x_0)-u(x_0))+\,\mathrm{d}u'(x_0)-eu(x_0)\leqslant 0 \tag{2.1}$$

and by our assumption

$$au''(x_0)+b(J\star u(x_0)-u(x_0))+\,\mathrm{d}u'(x_0)-eu(x_0)\geqslant 0. \tag{2.2}$$

These two equations imply that $eu(x_0)=0$, $au''(x_0)=0$ and

$$b(J\star u(x_0)-u(x_0))=b\left(\int J(x_0-y)(u(y)-u(x_0))\right)\mathrm{d}y=0. \tag{2.3}$$

By assumption, J is a smooth nonnegative function with $\overset{\circ}{\operatorname{supp}}(J)\not\equiv\emptyset$. Thus, we deduce from (2.3) that $u(y)=u(x_0)$ for all y in the set $x_0+\operatorname{supp}(J)$. If J is supported by $\mathbb{R}$ we obtain a contradiction immediately. If not, we can repeat the previous calculation for every y in $x_0+\operatorname{supp}(J)$, thus u is constant on the set $y+\operatorname{supp}(J)$ where y belongs to $x_0+\operatorname{supp}(J)$. By doing so infinitely many times, we cover all of $\mathbb{R}$ and thus end up with $u(y)=u(x_0)$ for all y in $\mathbb{R}$, which is a contradiction. □

Provided a is nonzero, we also have the following weak maximum principle:

Theorem 2.2 (*Weak maximum principle*). *Suppose $a>0$ and let $u\in H^1(\mathbb{R})$ satisfy the following inequality in the weak sense*:

$$Lu\geqslant 0 \quad on\ \mathbb{R}.$$

Then for any compact subset ω of $\mathbb{R}$,

$$\sup_{\omega} u\leqslant \sup_{\partial\omega} u^+.$$

Remark 2.1. A function $u\in H^1(\mathbb{R})$ satisfies $Lu\geqslant 0$ in the weak sense if for all nonnegative $\phi\in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} -au'\phi'+\mathrm{d}u'\phi-eu\phi+b(J\star u-u)\phi\geqslant 0.$$

We shall use the following easy corollary:

Corollary 2.2. *Let u satisfy the assumptions of Theorem 2.2, then u is nonpositive*.

Proof of Corollary 2.2. It is sufficient to show that for every positive δ, $u \leqslant \delta$. Now fix δ positive.

In one space dimension, $H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$, so u must go continuously to zero at infinity.

Whence there exists r_0 such that $|u| \leqslant \delta$ for every $|x| \geqslant r_0$. In particular $u^+(\pm r_0) \leqslant \delta$ and we may apply Theorem 2.2 with the compact set $\omega = [-r_0, r_0]$.

We end up with $u|_\omega \leqslant \sup\limits_{\partial\omega} u^+ \leqslant \delta$. Thus, $u \leqslant \delta$ on $\mathbb{R}$. □

Proof of Theorem 2.2. The proof follows that of Theorem 8.1 in Gilbarg and Trudinger's book [16]. For convenience of the reader, we provide its details. Let ω be a compact subset of $\mathbb{R}$. Assume by contradiction that

$$\sup_{\omega} u > \sup_{\partial\omega} u^+ = l.$$

Define a bilinear operator $\mathscr{L}$ on $H_0^1(\omega) \times H_0^1(\omega)$ by

$$\mathscr{L}(u, z) = \left(\int_\omega a u' z' - \mathrm{d} u' z - b(J * u - u) z + e u z \right) \mathrm{d}x. \tag{2.4}$$

By assumption, u satisfies $\mathscr{L}(u, z) \leqslant 0$ for all nonnegative $z \in H_0^1(\omega)$ i.e.

$$\int_\omega a u' z' - b(J * u - u) z \leqslant \int_\omega \mathrm{d} u' z - \int_\omega e u z. \tag{2.5}$$

Now let k be such that $\sup_\omega u > k \geqslant l$. The function $v := (u - k)^+$ is nontrivial and satisfies

$$v = \begin{cases} u - k & \textit{when } u > k, \\ 0 & \textit{otherwise}, \end{cases} \tag{2.6}$$

$$v' = \begin{cases} u' & \textit{when } u > k, \\ 0 & \textit{otherwise}, \end{cases} \tag{2.7}$$

so that $\Gamma := \operatorname{supp} v' \subset \{u > k\} \cap \operatorname{supp} u'$ and $v \in H_0^1(\omega)$. Also, since $e \geqslant 0$,

$$-\int_\omega e u v = -\int_\omega e v^2 - k \int_\omega e v \leqslant 0,$$

so that applying (2.5) with $z = v$ we obtain

$$\int_\omega a v' v' - b(J * u - u) v \leqslant C \int_\Gamma |v'| v. \tag{2.8}$$

Claim 2.1. $\int_\omega (J * u - u) v \leqslant 0$.

Proof. Extend v by

$$\tilde{v} = \begin{cases} v & \textit{in } \omega, \\ 0 & \textit{otherwise}. \end{cases}$$

Clearly $\int_\omega (J*u-u)v=\int_{\mathbb{R}}(J*u-u)\widetilde{v}$ and we only need to prove that $\int_{\mathbb{R}}(J*u-u)\widetilde{v}\leqslant 0$. Observe that for any constant α we have $J*\alpha-\alpha=0$ hence

$$\begin{aligned}\int_{\mathbb{R}}(J*u-u)\widetilde{v}(x)\,\mathrm{d}x &= \int_{\mathbb{R}}(J*(u-k)-(u-k))\widetilde{v}(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}} J*(u-k)\widetilde{v}(x)-\int_{\mathbb{R}}\widetilde{v}^2(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(u-k)(y)\widetilde{v}(x)\,\mathrm{d}y\,\mathrm{d}x-\int_{\mathbb{R}}\widetilde{v}^2(x)\,\mathrm{d}x.\end{aligned}$$

Since $(u-k)(y)\widetilde{v}(x)\leqslant (u-k)^+(y)\widetilde{v}(x)$ we have

$$\begin{aligned}&\int_{\mathbb{R}}(J*u-u)\widetilde{v}(x)\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(u-k)^+(y)\widetilde{v}(x)\,\mathrm{d}y\,\mathrm{d}x-\int_{\mathbb{R}}\widetilde{v}^2(x)\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)\widetilde{v}(y)\widetilde{v}(x)\,\mathrm{d}y\,\mathrm{d}x-\int_{\mathbb{R}}\widetilde{v}^2(x)\,\mathrm{d}x\\ &\quad\leqslant \tfrac{1}{2}\left(2\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)\widetilde{v}(y)\widetilde{v}(x)\,\mathrm{d}y\,\mathrm{d}x-\int_{\mathbb{R}}(\widetilde{v})^2(x)\,\mathrm{d}x-\int_{\mathbb{R}}\widetilde{v}^2(y)\,\mathrm{d}y\right)\\ &\quad\leqslant -\tfrac{1}{2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)[\widetilde{v}(y)^2-2\widetilde{v}(y)\widetilde{v}(x)+\widetilde{v}(x)^2]\,\mathrm{d}y\,\mathrm{d}x\right)\\ &\quad\leqslant -\tfrac{1}{2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)[\widetilde{v}(y)-\widetilde{v}(x)]^2\,\mathrm{d}y\,\mathrm{d}x\right)\leqslant 0\end{aligned}$$

and the claim is proved. □

From our claim we deduce the following inequality:

$$\begin{aligned}\int_\omega a(v')^2 &\leqslant C\int_\Gamma |v'|v\\ &\leqslant C\|v'\|_{L^2(\omega)}\|v\|_{L^2(\Gamma)}\end{aligned}$$

and end up with

$$\|v'\|_{L^2(\omega)}\leqslant C\|v\|_{L^2(\Gamma)}. \tag{2.9}$$

By the one-dimensional Sobolev embedding on compact subsets,

$$\|v\|_{L^\infty(\omega)}\leqslant C\|v'\|_{L^2(\omega)}. \tag{2.10}$$

Thus v is in $L^\infty(\omega)$ and since

$$\|v\|_{L^2(\Gamma)}\leqslant |\Gamma|^{1/2}\|v\|_{L^\infty(\omega)}, \tag{2.11}$$

we can combine the last three equations to obtain

$$\|v\|_{L^2(\Gamma)} \leqslant C|\Gamma|^{1/2}\|v\|_{L^2(\Gamma)}. \tag{2.12}$$

Therefore, we have

$$C^{-1/2} \leqslant |\Gamma| = |\text{supp } v'| \leqslant |\text{supp } u' \cap \{u > k\}|, \tag{2.13}$$

where C is a constant which only depends on $|\omega|$, a and $|d|$.

Since C is independent of k, one can let k go to $\sup_\omega u$. By the dominated convergence theorem, the right-hand side of (2.13) converges to $|\text{supp } u' \cap \{u = \sup_\omega u\}|$.

This implies that there exists a set of positive measure ω^+, where u takes its maximum value and u' is not identically zero. Since u is in H^1, $u' = 0$ a.e. on its level sets and we obtain a contradiction.

This ends the proof. $\square$

Next, we provide an elementary lemma to construct solutions to constant-coefficient linear equations of the form $Lu = f$.

Lemma 2.1. *Let $f \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and L defined by* (1.1) *or* (1.2). *Then there exists a unique solution $v \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ (additionally $v \in C^1(\mathbb{R})$ if $d \neq 0$, $v \in C^2(\mathbb{R})$ if $a > 0$) of*

$$\begin{cases} Lv = f & in\ \mathbb{R}, \\ v \to 0 & x \to -\infty, \\ v \to 0 & x \to +\infty. \end{cases} \tag{2.14}$$

Proof. We assume first that $(a, d) \neq (0, 0)$. Uniqueness follows from the maximum principle. Next, applying Fourier transform to (2.14), we obtain

$$(-a|\xi|^2 + b(\hat{J}(\xi) - 1) + \mathrm{i}d\xi - e)\hat{v} = \hat{F}.$$

Since $\|J\|_{L^1} = 1$, $|\hat{J}| \leqslant 1$ and since J is even, $\hat{J}$ is real-valued, so that

$$\begin{aligned} |-a|\xi|^2 + b(\hat{J}(\xi) - 1) + \mathrm{i}d\xi - e| &\geqslant |-a|\xi|^2 + b(\hat{J}(\xi) - 1) - e| \\ &= a|\xi|^2 + b(1 - \hat{J}(\xi)) + e \geqslant e > 0. \end{aligned}$$

If w is defined by

$$w := (-a|\xi|^2 + b(\hat{J}(\xi) - 1) + \mathrm{i}d\xi - e)^{-1}\hat{F}, \tag{2.15}$$

it follows that $w \in L^2(\mathbb{R})$ and that $v := \mathscr{F}^{-1}(w) \in L^2(\mathbb{R})$ solves (2.14) in the sense of distributions. By the dominated convergence theorem, $J * v \in C(\mathbb{R})$ and by elliptic regularity applied to the operator $\widetilde{L}v = Lv - J \star v$, v has the appropriate regularity for (2.14) to hold in the classical sense.

Also, since either a or d is nonzero, (2.15) implies that $\xi w \in L^2(\mathbb{R})$ so that $v \in H^1_0(\mathbb{R}) \subset C_0(\mathbb{R})$.

When $a = d = 0$, (2.14) can be rewritten as

$$v = \frac{1}{1 + e/b}(J \star v + f). \tag{2.16}$$

It follows easily from the dominated convergence theorem that $J \star v \in C_0(\mathbb{R})$ whenever $v \in C_0(\mathbb{R})$, so that the right-hand side of (2.16) is a (strict) contraction in $C_0(\mathbb{R})$ and admits a unique fixed point. The fact that $v \in L^2(\mathbb{R})$ can be obtained as above. □

3. Construction of a solution of (P)

In this section, we construct an increasing solution of problem (P) where L is defined by (1.1), using ordered sub and supersolutions.

Definition 3.1. w is a supersolution of (P) if $w \in C^2(\mathbb{R})$ and

$$\begin{cases} Lw - cw' + f(w) \leqslant 0 & \text{on } \mathbb{R}, \\ w \to 0 \quad \text{as } x \to -\infty, \\ w \to 1 \quad \text{as } x \to +\infty. \end{cases}$$

Subsolutions ψ are defined by reversing the above inequality.

Theorem 3.1. *Assume there exist two nonnegative smooth functions w and ψ such that w and ψ are respectively a super and a subsolution of (P), satisfying $\psi \leqslant w$. Assume further that w is increasing, $w \in L^2(\mathbb{R}^-)$ and $1 - w \in L^2(\mathbb{R}^+)$. Then, there exists a positive increasing solution u of (P).*

Remark 3.1. For $a > 0$, if L is defined by (1.2), since the weak maximum principle holds, the previous theorem remains true if w and ψ are only assumed to be weak super and subsolutions of (P).

Remark 3.2. Alternatively, the assumption of monotony and L^2 integrability on w can be dropped and replaced by the same assumption on ψ.

We break down the proof into two steps. In the first step we construct a sequence of functions starting from the supersolution. In the second we prove that this sequence has a subsequence which converges to a solution of (P).

Proof of 3.1.

3.1. Iteration procedure

Let w and ψ be nonnegative, respectively a super and a subsolution of (P). Also let $\lambda > 0$ be a parameter to be fixed later on. We claim that there exists a sequence of functions $\{u_n\}_{n \in \mathbb{N}}$ satisfying

$$u_0 = w \quad \text{and for } n \in \mathbb{N} \backslash \{0\},$$

$$\begin{cases} Lu_{n+1} - \lambda u_{n+1} - cu'_{n+1} = -f(u_n) - \lambda u_n & \text{in } \mathbb{R}, \\ u_{n+1} \to 0, & x \to -\infty, \\ u_{n+1} \to 1, & x \to +\infty. \end{cases} \tag{3.1}$$

We proceed as follows: let $g \in C_c^\infty(\mathbb{R})$ be a nonnegative function with $\|g\|_{L^1(\mathbb{R})} = 1$ and let $G(x) = \int_{-\infty}^x g(t)\,\mathrm{d}t$. Using the substitution $v_n = u_n - G$, (3.1) reduces to

$$\begin{cases} \widetilde{L}v_{n+1} - \lambda v_{n+1} = F(v_n, x) & \text{in } \mathbb{R}, \\ v_{n+1} \to 0, & x \to -\infty, \\ v_{n+1} \to 0, & x \to +\infty, \end{cases} \tag{3.2}$$

where $\widetilde{L}v = Lv - cv'$ and $F(v, x) = -f(v + G) - \lambda v - \widetilde{L}G$.

Using Lemma 2.1 and induction, in order to prove that v_n is well-defined, it is enough to show that $v_0 \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ and that $v \in L^2(\mathbb{R}) \cap C_0(\mathbb{R}) \Longrightarrow F(v, x) \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$.

On the one hand since $G(x) = 0$ whenever $-x \geqslant 1$ (and similarly $G(x) = 1$ for $x \geqslant 1$), it follows from the definition of w that $v_0 = w - G \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$.

On the other hand given $v \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$, since $f(0) = 0$,

$$|f(v + G)| \leqslant \|f'\|_\infty |v + G| \in L^2(\mathbb{R}^-) \quad \text{and} \quad \lim_{-\infty} f(v + G) = 0$$

and since $f(1) = 0$,

$$|f(v + G)| \leqslant \|f'\|_\infty |v + G - 1| \in L^2(\mathbb{R}^+) \quad \text{and} \quad \lim_{+\infty} f(v + G) = 0,$$

so that $f(v+G) \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$. Clearly $G', G'' \in L^2(\mathbb{R})$. Finally, the following lemma applied to $u = G$ shows that $J \star G - G \in L^2(\mathbb{R})$ and we can conclude that there exists a well-defined solution u_n of (3.1).

Lemma 3.1. Let $u \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then

$$\|J \star u - u\|_{L^2(\mathbb{R})} \leqslant C \|u'\|_{L^2(\mathbb{R})}.$$

Proof of Lemma 3.1. Using the fundamental theorem of calculus, we have that

$$J \star u(x) - u(x) = \int J(y)(u(x - y) - u(x))\,\mathrm{d}y = \int J(y)y \left(\int_0^1 u'(x - ty)\,\mathrm{d}t \right) \mathrm{d}y.$$

By the Cauchy–Schwartz inequality, it follows that

$$\begin{aligned} |J \star u(x) - u(x)|^2 &\leqslant \int_{\mathbb{R}} \int_0^1 J(y)|y|(u')^2(x - ty)\,\mathrm{d}t\,\mathrm{d}y \cdot \int_{\mathbb{R}} \int_0^1 J(y)|y|\,\mathrm{d}t\,\mathrm{d}y \\ &\leqslant C \int_{\mathbb{R}} \int_0^1 J(y)|y|(u')^2(x - ty)\,\mathrm{d}t\,\mathrm{d}y, \end{aligned}$$

hence

$$\begin{aligned} \|J \star u - u\|_{L^2(\mathbb{R})}^2 &\leqslant C \int_{\mathbb{R}} J(y)|y| \left(\int_0^1 \int_{\mathbb{R}} (u')^2(x - ty)\,\mathrm{d}x\,\mathrm{d}t \right) \mathrm{d}y \\ &\leqslant C \|u'\|_{L^2(\mathbb{R})}^2. \qquad \square \end{aligned}$$

3.2. Passing to the limit as $n \to \infty$

Since ψ and w are ordered functions i.e. $\psi \leqslant w$, it follows easily from induction and the maximum principle that for all $n \in \mathbb{N}\backslash\{0\}$,

$$\psi \leqslant u_n \leqslant w. \tag{3.3}$$

Also, if $\tau > 0$ and $z_n(x) = u_n(x+\tau) - u_n(x)$,we have

$$\begin{cases} Lz_{n+1} - \lambda z_{n+1} - cz'_{n+1} = -(f+\lambda)(u_n(x+\tau)) + (f+\lambda)(u_n(x)) & \text{in } \mathbb{R}, \\ z_{n+1} \to 0 \qquad |x| \to \infty. \end{cases} \tag{3.4}$$

Choosing $\lambda > 0$ so large that $-f - \lambda$ is nonincreasing, it follows from induction, the maximum principle and the fact that w is nondecreasing that for each $n \in \mathbb{N}$, $z_n \leqslant 0$ i.e.

$$x \to u_n(x) \text{is a nondecreasing function.} \tag{3.5}$$

Using (3.3), (3.5) and Helly's lemma, it follows that a subsequence of (u_n) converges pointwise to a nondecreasing function u satisfying

$$\psi \leqslant u \leqslant w.$$

By the dominated convergence theorem, $J \star u_n - u_n \to J \star u - u$. Rewriting (3.1) as

$$-cu'_{n+1} = u_{n+1} - J \star u_{n+1} - \lambda(u_n - u_{n+1}) - f(u_n), \tag{3.6}$$

observing that the right-hand side in the above equation is uniformly bounded, we conclude that $\{u_n\}$ is bounded e.g. in $C^1(\omega)$, where ω is an arbitrary bounded open subset of $\mathbb{R}$. Hence $u \in C^1(\mathbb{R})$ and by Helly's lemma,

$$u_n \to u \quad \text{uniformly in } \mathbb{R}. \tag{3.7}$$

Differentiating (3.6), we obtain similarly local C^2 bounds on u_n so that

$$u_n \to u \quad \text{in } C^2_{\text{loc}}. \tag{3.8}$$

Using (3.8), it is now a trivial matter to pass to the limit in the equation. Furthermore, since $\psi \leqslant u \leqslant w$, u has the desired limits at infinity of (P) and we have thus constructed an increasing solution u of (P).

4. L^2 estimates of solutions of (P)

Our goal in this section is to provide L^2 estimates of monotone solutions of problem (P). Since u is uniformly bounded, an easy computation from (P) shows that $\lim_{\pm\infty} u' = 0$.

Now we show that $u' \in L^2(\mathbb{R})$. Indeed, multiplying (P) by u' and integrating over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} \left(u' J \star u - \left(\frac{u^2}{2} \right)' \right) + c \int (u')^2 = \int f(u)u' = \int_0^1 f(s)\,\mathrm{d}s. \tag{4.1}$$

Since J is even,

$$\int_{\mathbb{R}} u' J \star u = [uJ \star u]_{-\infty}^{+\infty} - \int_{\mathbb{R}} uJ \star u' = 1 - \int_{\mathbb{R}} u' J \star u.$$

Hence, $\int_{\mathbb{R}}(u' J \star u - (u^2/2)') = 0$ and by (4.1), $u' \in L^2$.

Next, we show that $f(u) \in L^2(\mathbb{R})$. We need the following lemma:

Lemma 4.1. *Let u be a nondecreasing solution of (P). Then, $J \star u - u \in L^1(\mathbb{R})$. More precisely*,

$$\|J \star u - u\|_{L^1} \leqslant \int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z \quad \textit{and} \quad \int_{\mathbb{R}} (J \star u - u) = 0.$$

Proof. Clearly,

$$\int_{\mathbb{R}} |(J \star u - u)| \leqslant \int_{\mathbb{R}^2} J(x-y)|u(y) - u(x)|\,\mathrm{d}y\,\mathrm{d}x. \tag{4.2}$$

Since $u \in C^1(\mathbb{R})$,

$$|u(y) - u(x)| = |x-y| \int_0^1 u'(y + s(x-y))\,\mathrm{d}s.$$

Plug this equality in (4.2) to obtain

$$\begin{aligned}\int_{\mathbb{R}^2} & J(x-y)|u(y) - u(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} J(x-y)|x-y| \int_0^1 u'(x + s(y-x))\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{4.3}$$

Make the change of variables $z = x - y$, so that the right-hand side of (4.3) becomes

$$\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x. \tag{4.4}$$

As all terms [in (4.4)] are positive, we may apply Tonnelli's Theorem and permute the order of integration. We obtain

$$\begin{aligned}\int_{\mathbb{R}^2} J(z)|z| \int_0^1 u'(x - sz)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x &= \int_0^1 \int_{\mathbb{R}^2} J(z)|z|u'(x - sz)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s \\ &= \int_0^1 \int_{\mathbb{R}} J(z)|z|[u(+\infty) - u(-\infty)]\,\mathrm{d}z\,\mathrm{d}s \\ &= \int_{\mathbb{R}} J(z)|z|\,\mathrm{d}z < \infty.\end{aligned}$$

These last computations show that $J \star u - u$ is an integrable function and give a bound on its L^1 norm. Let us now compute $\int_{\mathbb{R}}(J \star u - u)\,\mathrm{d}x$. We have

$$\int_{\mathbb{R}} J \star u - u\,\mathrm{d}x = \int_{\mathbb{R}^2} J(x-y)(u(y) - u(x))\,\mathrm{d}y\,\mathrm{d}x.$$

Let $z = x - y$ so that

$$\int_{\mathbb{R}^2} J(z)(u(x-z) - u(x))\,\mathrm{d}z\,\mathrm{d}x = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))\,\mathrm{d}y\,\mathrm{d}z.$$

Make the change of variable $z \to -z$ in the left integral and obtain

$$I_1 := \int_{\mathbb{R}^2} J(z)(u(x+z) - u(x))\,\mathrm{d}z\,\mathrm{d}x = \int_{\mathbb{R}^2} J(z)(u(y) - u(y+z))\,\mathrm{d}y\,\mathrm{d}z =: I_2.$$

Fubini's theorem applied to the last integral shows that $I_1 = -I_2$, hence $I_1 = I_2 = 0$. □

Next, we integrate (P) over $[R, \infty)$, where $R > 0$ is chosen so large that $f(u(x)) > 0$ for $x > R$. We get

$$\int_R^\infty (J \star u - u) + cu(R) + \int_R^\infty f(u) = 0.$$

Using Lemma 4.1, we conclude that $f(u) \in L^1(R, \infty)$. Working similarly on $(-\infty, -R)$, it follows that $f(u) \in L^1(\mathbb{R})$. Using (P),(4.1) and Lemma 3.1,implies that $f(u) \in L^2(\mathbb{R})$.

We finally prove that $u \in L^2(\mathbb{R}^-)$ and $1-u \in L^2(\mathbb{R}^+)$. Using Lemma 2.1, we know that there exists $w \in L^2(\mathbb{R})$ such that $v := w+G$ (with $G \in C^\infty(\mathbb{R})$, $G \equiv 0$ in a neighbourhood of $-\infty$ and $G \equiv 1$ in a neighbourhood of $+\infty$) solves

$$\begin{cases} Lv - cv' + f(u) = 0, & \text{on } \mathbb{R}, \\ v \to 0 & \text{as } x \to -\infty, \\ v \to 1 & \text{as } x \to +\infty. \end{cases} \tag{4.5}$$

Since both u and v solve (4.5), it follows from the maximum principle that $u \equiv v$ i.e. u has the desired integrability.

5. Min–max formula: cases A1 and A2

In this section we prove the min–max formula for the asymptotic speed in the case where the nonlinearity is of bistable or ignition type. The proof relies on the construction of appropriate sub and supersolutions for the problem (P), and a uniqueness theorem which holds for solutions of (P) only when f is of bistable or ignition type.

We will prove the following:

Theorem 5.1. *Let $X = \{w \in C^1(\mathbb{R})|\ w$ increasing, $w(+\infty) = 1$ and $w(-\infty) = 0\}$, then the (unique) front speed is given by*

$$c^* = \min_{w\in X} \sup_{x\in\mathbb{R}} \left\{ \frac{Lw + f(w)}{w'} \right\}. \tag{5.1}$$

Proof of Theorem 5.1. Define c^1 by

$$c^1 = \min_{w\in X} \sup_{x\in\mathbb{R}} \left\{ \frac{Lw + f(w)}{w'} \right\}, \tag{5.2}$$

with X as above.

Then we just have to show that

$$c^* = c^1. \tag{5.3}$$

Since we know from the previous section that there exists an increasing solution (u^*, c^*) to (P), taking $w = u^*$ in the definition of c^1 yields

$$c^1 \leqslant c^*.$$

The main difficulty lies in the proof of the reverse inequality $c^1 \geqslant c^*$. We argue by contradiction and assume that $c^1 < c^*$. Let c be such that $c^1 \leqslant c < c^*$. From the definition of c^1, there exists a positive increasing function w which satisfies

$$\begin{cases} Lw - cw' + f(w) \leqslant 0 & \text{in } \mathbb{R}, \\ w \to 0, & x \to -\infty, \\ w \to 1, & x \to +\infty. \end{cases} \tag{5.4}$$

Since $c < c^*$, and $(u^*)' > 0$, u^* satisfies

$$\begin{cases} Lu^* - c(u^*)' + f(u^*) = (c^* - c)(u^*)' \geqslant 0 & \text{in } \mathbb{R}, \\ u^* \to 0, & x \to -\infty, \\ u^* \to 1, & x \to +\infty. \end{cases} \tag{5.5}$$

Observe that any translation of u^* and w are also respectively a sub and a supersolution of the same problem. Therefore, if we can order two translations of u^* and w, we will be done. Indeed, from the a priori estimates of Section 4 and Theorem 3.1, there would exist a positive solution of the following problem:

$$\begin{cases} Lu - cu' + f(u) = 0, & \text{in } \mathbb{R}, \\ u \to 0, & x \to -\infty, \\ u \to 1, & x \to +\infty, \end{cases} \tag{5.6}$$

which contradicts the uniqueness Theorem 1.1. □

The proof of Theorem 5.1 thus reduces to finding ordered translations of w and u^*. We claim the following.

Lemma 5.1. *There exists constants a and b such that* $w(s + a) \geqslant u^*(s + b)$.

Proof of Lemma 5.1. Without loss of generality, we may always assume $w(0) = u^*(0) = \theta/2$.

Now we define some quantities that we will use to construct sub and supersolutions. Let α positive, such that

$$f'(p) < -2\alpha \quad \text{whenever } |p - 1| < \alpha. \tag{5.7}$$

Let $\mu \in (0, \alpha/2)$ and define $a(s) = \mu e^{-\alpha s}$.
Choose $M > 0$ and $K > 0$ such that

$$w(\xi) - 1 < \frac{\alpha}{2} \quad \text{in } (M - 1, +\infty), \tag{5.8}$$

$$w'(\xi) > K \quad \text{in } [-1, M + 1]. \tag{5.9}$$

Define the following function:

$$b(s) = \frac{\mu\bar{\alpha}}{K}(1 - \mathrm{e}^{-\alpha s}),$$

where $\bar{\alpha} = 1 + \frac{\max\{f'(p):-1 \leqslant p \leqslant 2\}}{\alpha}$.
We will assume further that $\mu \leqslant \min\{\theta/2, K/\bar{\alpha}\}$.
We now define a sub and a supersolution as follows:

$$\widetilde{w}(\xi, s) = w(\xi + b(s)) + a(s), \tag{5.10}$$
$$\widetilde{u}(\xi, s) = u^*(\xi - \tau), \tag{5.11}$$

where $\tau > 0$ is taken so large that

$$w(\xi) + a(0) > u^*(\xi - \tau).$$

Let $z(\xi, s) = (\widetilde{w} - \widetilde{u})(\xi, s)$. z satisfies the next equations:

$$-\frac{\partial z}{\partial s} + Lz - cz_\xi \leqslant -a'(s) - w'(\xi, s)b'(s) + f(\widetilde{u}(\xi, s)) - f(\widetilde{w}(\xi, s) - a(s)), \tag{5.12}$$

$$z(\xi, 0) > 0 \quad \forall \xi \in \mathbb{R}, \tag{5.13}$$
$$z(\pm\infty, s) = a(s) \quad \forall s \in \mathbb{R}. \tag{5.14}$$

From (5.13, 5.14), by continuity, there exists $s_0 = \sup\{s > 0 | z(\xi, s) > 0\ \forall \xi \in \mathbb{R}\}$.

Claim 5.1. $s_0 = +\infty$.

Proof of Claim 5.1. We argue by contradiction. If not, $s_0 < +\infty$ and there exists $\xi_0 \in \mathbb{R}$ such that

$$0 = z(\xi_0, s_0) = \min_{\mathbb{R}} z(\xi, s_0). \tag{5.15}$$

Next, we use a kind of localization of minimum lemma. More precisely we claim

Claim 5.2. *Under the previous assumptions, we have* $\xi_0 > -1$.

Proof of Claim 5.2. Let $Z(\xi) = z(\xi, s_0)$, then Z satisfies:

$$LZ - cZ_\xi = f(\widetilde{u}(\xi, s_0)) - f(\widetilde{w}(\xi, s_0) - a(s_0)).$$

So at ξ_0 we have,

$$(J \star Z - Z)(\xi_0) > 0,$$

$$Z_\xi(\xi_0) = \widetilde{w}(\xi_0, s_0) - \widetilde{u}(\xi_0, s_0) = 0.$$

Thus, $f(\widetilde{u}(\xi_0, s_0)) - f(\widetilde{w}(\xi_0, s_0) - a(s_0)) > 0$, which implies $f(\widetilde{u}(\xi_0, s_0)) > 0$.

Recall that

$$\widetilde{u}(\xi_0, s_0) = \widetilde{w}(\xi_0, s_0),$$
$$\Rightarrow u^*(\xi_0 - \tau) = w(\xi_0 + b(s_0)) + a(s_0).$$

Thus,

$$u(\xi_0, s_0) = w(\xi_0 + b(s_0)) + a(s_0) > \theta$$
$$\Rightarrow w(\xi_0 + b(s_0)) > \theta - a(s_0) > \frac{\theta}{2}$$
$$\Rightarrow \xi_0 > w^{-1}(\frac{\theta}{2}) - b(s_0)$$
$$\Rightarrow \xi_0 > -1. \qquad \square$$

Remark 5.1. Claim 5.2 bounds from below the minimum of z.

Now, observe that, at (ξ_0, s_0), z satisfies:

$$-\frac{\partial z(\xi_0, s_0)}{\partial s} + Lz(\xi_0, s_0) - cz_\xi(\xi_0, s_0) \geqslant 0$$

and

$$-\frac{\partial z(\xi_0, s_0)}{\partial s} + [Lz - cz_\xi](\xi_0, s_0) \leqslant -a'(s_0) - w'(\xi_0, s_0)b'(s_0) + f(\widetilde{u}(\xi_0, s_0)) - f(\widetilde{w}(\xi_0, s_0) - a(s_0)).$$

So we end up with

$$Q = -a'(s_0) - w'(\xi_0, s_0)b'(s_0) + f(\widetilde{u}(\xi_0, s_0)) - f(\widetilde{w}(\xi_0, s_0) - a(s_0)) \geqslant 0.$$

Since at (ξ_0, s_0) we have,

$$\widetilde{u}(\xi_0, s_0) = \widetilde{w}(\xi_0, s_0)$$

and f is a smooth function, we can rewrite Q as

$$Q = \mu e^{-\alpha s_0} \left[\alpha - \frac{\alpha\bar{\alpha}}{K} w'(\xi_0 + b(s_0)) + f'(d) \right] \geqslant 0 \tag{5.16}$$

for some $d \in [\widetilde{w}(\xi_0, s_0) - a(s_0), \widetilde{w}(\xi_0, s_0)]$.

Now, from Claim 5.2, we are led to considering two cases:

1. *Case*: $\xi_0 \in [-1, M]$.
   Then, Q would satisfy

   $$0 > \mu e^{-\alpha s_0} \left[\alpha \left(1 - \frac{w'(\xi_0 + b(s_0))}{K} \right) - \frac{w'(\xi_0 + b(s_0))}{K} \max\{f'(p) \quad -1 \leqslant p \leqslant 2\} + f'(d) \right]$$

   which contradicts (5.16).

2. *Case*: $\xi_0 > M$.
   Then, Q would then verify

$$\mu e^{-\alpha s_0}\left[\alpha - \frac{\alpha\bar{\alpha}w'(\xi_0+b(s_0))}{K} + f'(d)\right] < \mu e^{-\alpha s_0}\left[-\alpha - \frac{\alpha\bar{\alpha}w'(\xi_0+b(s_0))}{K}\right] < 0,$$

   which also contradicts (5.16), thus proving Claim 5.1. □

From Claim 5.1, we have $z(\xi, s) \geqslant 0$ for all $(\xi, s) \in \mathbb{R} \times \mathbb{R}^+$. Let s go to infinity : we end up with $w(\xi - a) \geqslant u^*(\xi - b)$, where $a = \mu\bar{\alpha}/K$ and $b = \tau$. This ends the proof of Lemma 5.1.

6. Min–max formula: the monostable case

In this section we prove the min–max formula for the minimal speed in the case where the non linearity f is monostable. We are concerned with the following problem:

$$\begin{cases} Lu - cu' + f(u) = 0, & \text{in } \mathbb{R}, \\ u \to 0, & x \to -\infty, \\ u \to 1, & x \to +\infty, \end{cases} \tag{6.1}$$

where f is monostable and J has a fast decay near infinity. Uniqueness of solutions no longer holds in this situation. Nevertheless, the min–max formula still holds.

Theorem 6.1. *Let $X = \{w \in C^1(\mathbb{R}) | w(+\infty) = 1$ and $w(-\infty) = 0\}$, then we have*

$$c^* = \min_{w'>0,\ w\in X} \sup_{x\in\mathbb{R}} \left\{\frac{Lw + f(w)}{w'}\right\}. \tag{6.2}$$

Proof. We define c^1 as in the previous section:

$$c^1 = \min_{w'>0,\ w\in X} \sup_{x\in\mathbb{R}} \left\{\frac{Lw + f(w)}{w'}\right\}. \tag{6.3}$$

Then again we just have to show,

$$c^* = c^1. \tag{6.4}$$

As in the previous section, since we know from [10] that there exists an increasing solution of (6.1), for the speed c^*, we obviously have $c^1 \leqslant c^*$. The main difficulty again lies in the proof of $c^1 \geqslant c^*$. Before, showing $c^1 \geqslant c^*$, we will characterize the behavior of the speed of solutions of (6.1) when f is of ignition type.

Lemma 6.1. *Let f and g be two functions of type* A2, *such that $f \geqslant g$, $f \not\equiv g$, then the corresponding speeds c_f, c_g satisfy $c_f > c_g$.*

From this monotone charaterization of the speed, we easily obtain the following corollary:

Corollary 6.1. *There exists a sequence of approximations* $(f_n)_{n\in\mathbb{N}}$ *of f such that for each n,* f_n *is of type* A2 *and the corresponding speed* c_n *satisfies*

$$\lim_{n\to+\infty} c_n = c^*.$$

Proof of Corollary 6.1. Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 as n goes to infinity. And let χ_{δ_n} satisfy the following assumptions:

- $\chi_{\delta_n} \in C_0^\infty(\mathbb{R})$,
- $0 \leqslant \chi_{\delta_n} \leqslant 1$,
- $\chi_{\delta_n}(s) \equiv 0$ for $s \leqslant \delta_n$ and $\chi_{\delta_n}(s) \equiv 1$ for $s \geqslant 2\delta_n$,
- χ_{δ_n} is a monotone increasing sequence of function (i.e.$\chi_{\delta_n} \leqslant \chi_{\delta_p}$ for $p \geqslant n$).

Now define a new function $f_{\delta_n} = f\chi_{\delta_n}$. Since f_{δ_n} is of ignition type, there exists a unique travelling wave solution (u_n, c_n) of (6.5), cf. [7].

$$\begin{cases} Lu_n - c_n u_n' + f_{\delta_n}(u_n) = 0, & \text{in } \mathbb{R}, \\ u_n \to 0, & x \to -\infty, \\ u_n \to 1, & x \to +\infty. \end{cases} \tag{6.5}$$

By Lemma 6.1, $\{c_n\}$ is an increasing sequence. In fact $(c_n)_{n\in\mathbb{N}}$ is bounded by c^*:

Claim 6.1. $\forall n \in \mathbb{N} \quad c_n \leqslant c^*$.

Proof. We argue by contradiction. Then, there exists $c_n > c^*$. Since u_n is monotone increasing, u_n satisfies

$$\begin{cases} Lu_n - c^* u_n' + f_{\delta_n}(u_n) \geqslant 0 & \text{in } \mathbb{R}, \\ u_n \to 0 & \text{as } x \to -\infty, \\ u_n \to 1 & \text{as } x \to +\infty. \end{cases} \tag{6.6}$$

Therefore (u_n, c^*) is a subsolution of (6.5). Similarly, observe that $f \geqslant f_{\delta_n}$, therefore (u^*, c^*) satisfies

$$\begin{cases} Lu^* - c^*(u^*)' + f_{\delta_n}(u^*) \leqslant 0 & \text{in } \mathbb{R} \\ u^* \to 0 & \text{as } x \to -\infty, \\ u^* \to 1 & \text{as } x \to +\infty. \end{cases} \tag{6.7}$$

Therefore (u^*, c^*) is a supersolution of (6.5). Since f_{δ_n} is of type A2, we can apply Lemma 5.1 to get constants a and b such that $u^*(s+a) \geqslant u_n(s+b)$. Then, as in the previous section, we can apply Theorem 3.1, which implies the existence of a non trivial solution (u, c^*) to (6.5) which contradicts the uniqueness of the solution (u_n, c_n). This proves Claim 6.1. □

Since (c_n) is a bounded increasing sequence, it converges to a constant γ. From standard a priori estimates, there exists a subsequence still denoted (u_n) which converges to an increasing function $\bar{u}$ solution of (6.1).

Since $c^* = \inf\{c > 0|\ (6.1)$ has a positive increasing solution$\}$, we must have $\gamma = c^*$, which proves Corollary 6.1. □

Now, let us prove Lemma 6.1.

Proof of Lemma 6.1. Again, we argue by contradiction. Assume that $c_f < c_g$. Then, since they are increasing, u_f and u_g will be respectively a super and subsolution of

$$\begin{cases} Lw - c_g w' + f(w) \leqslant 0 & \text{in } \mathbb{R}, \\ w \to 0 & \text{as} \quad x \to -\infty, \\ w \to 1 & \text{as} \quad x \to +\infty. \end{cases} \tag{6.8}$$

Since *f* is of type A2, we can use Lemma 5.1 and Theorem 3.1 to get a non trivial solution (u, c_g) of (6.8), which violates the uniquess Theorem 1.1. The strict inequality follows by the same argument. □

We are now ready to prove the last inequality

$$c^1 \geqslant c^*. \tag{6.9}$$

Proof of inequality (6.9). We argue by contradiction, assuming that (6.9) is not true : there exists $c > 0$ such that $c_1 \leqslant c < c^*$. Therefore, by the definition of c^1, there exists a positive increasing function w such that

$$\begin{cases} Lw - cw' + f(w) \leqslant 0 & \text{in } \mathbb{R}, \\ w \to 0 & \text{as} \quad x \to -\infty, \\ w \to 1 & \text{as} \quad x \to +\infty. \end{cases} \tag{6.10}$$

Now, by Corollary 6.1, there exists $\delta_n > 0$ u_n increasing and $c_n > 0$ such that

$$\begin{cases} Lu_n - c_n u_n' + f_{\delta_n}(u_n) = 0 & \text{in } \mathbb{R}, \\ u_n \to 0 & \text{as} \quad x \to -\infty, \\ u_n \to 1 & \text{as} \quad x \to +\infty. \end{cases} \tag{6.11}$$

Therefore, if we replace c_n by c in (6.11), w and u_n become a super and a subsolution of the problem. We can then apply Lemma 5.1 and Theorem 3.1 to get a solution of (6.11) with speed c. But this contradicts the uniquess of the speed for problems with ignition nonlinearities. This ends the proof of the min–max formula in the monostable case. □

We can give a more precise bound for the minimal speed, if in addition to the common assumption that *f* is monostable, we assume further that $f'(0)s \geqslant f(s)$. This new assumption is known as the KPP assumption. When there are no integral terms, then it is known that $c^* = 2\sqrt{f'(0)}$, which can be also formulated as

$$c^* = \min_{\lambda > 0} \left\{ \frac{1}{\lambda}(\lambda^2 + f'(0)) \right\}.$$

We derive a similar formula when there is an integral term. Namely, we have

$$c^* \leqslant \min_{\lambda>0}\left\{\frac{1}{\lambda}\left(\int_{\mathbb{R}} J(z)\mathrm{e}^{\lambda z}\,\mathrm{d}z - 1 + f'(0)\right)\right\} = \gamma.$$

There are hints that in fact there is equality in the above equation, but we were not able to prove it. The proof relies on the same ideas: one assumes that the inequality is false then picks a constant $c \in (\gamma, c^*)$, finds good super and subsolutions for an ignition-type problem and concludes with the existence and uniqueness theorem. We omit the details of the proof and just present the construction of the super solution. A straight forward computation shows that exponential functions are eigenfunctions of the operator $Lw + f'(0)w := J \star w - w - cw' + f'(0)w$, i.e.($[L + f'(0)]\mathrm{e}^{\lambda x} = h(\lambda)\mathrm{e}^{\lambda x}$).

Therefore, since f is of KPP type,

$$L(\mathrm{e}^{\lambda x}) + f(\mathrm{e}^{\lambda x}) \leqslant h(\lambda)\mathrm{e}^{\lambda x}, \tag{6.12}$$

where $h(\lambda) = \int_{\mathbb{R}} J(z)\mathrm{e}^{\lambda z}\,\mathrm{d}z - 1 - c\lambda + f'(0)$.

Now use the definitions of γ and c to find some λ such that $h(\lambda) \leqslant 0$. Then argue as above : since there exists a supersolution of the monostable problem (6.1), and $c_n \to c^*$, we get a contradiction. $\square$

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