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Propagators for a Scalar Field in a Homogeneous Expanding Universe. I

——*Case of the Friedmann Universes*——

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In view of a recent interest in the quantum field-theoretical creation of particles in a big-bang universe (which, via the problem how their vacuum state should be defined, will be connected with their propagators whose structure depends also on that of the universe), our previous formulae for bi-scalar Green's functions corresponding to a massless scalar field in the radiation- and matter-dominated stages of the Friedmann universe with flat 3-space are extended in a classical level. One is to derive the formulae for a massive scalar field in the same universe, and another lies in deriving the ones applicable to the respective stages of a closed universe with spherical topology. As an application, we discuss a massless scalar field (e.g., photons or gravitons defined suitably) and its physical property in the cases where its source distribution is spatially uniform and where that is of a delta-singularity. It is shown that the energy-momentum tensor in the first case is formally the same as that for a perfect fluid whose sound velocity relative to the light velocity is unity, while the tensor in the second case leads naturally to Robertson's formula for the apparent luminosity of a receding galaxy. The behavior of photons or gravitons generated from a turbulent medium in an early universe is also dealt with.

§ 1. Introduction

As shown by DeWitt and Brehme,[1] the propagation of any massless scalar or vector wave in a curved space-time is different from that in the Minkowski space-time, because the relevant Green's function does not generally vanish inside the light cone. However, even in the simplest case of an isotropic expanding universe with flat 3-space, their procedure for deriving the Green's function is practically infeasible owing to the difficulty in fiinding out required expression for a time-like or space-like geodesic interval in the universe. The infeasibility is more serious in the case of an isotropic expanding universe with closed 3-space, because of its topological nature. In spite of this, by another procedure, one of the authors (H.N.) and Kimura[2a] derived the bi-scalar Green's function in the Friedmann universe with flat 3-space, and the one in a closed universe was later dealt with[2d] to study the problem of Mach's principle in the Brans-Dicke cosmology.[3]

Now, the bi-scalar Green's function stemmed from our investigation (on the basis of Arnowitt, Deser and Misner's canonical formalism[4] to general relativity) of the quantum nature of gravitons (whose constituent field is a massless scalar field extracted suitably from the transverse-traceless part of the metric tensor) in the background Friedmann universe. After that, Parker[5] studied the quantization of a massive scalar or spinor field in any isotropic expanding universe with flat 3-space by laying emphasis on a possibility of particle creation in the universe. On the other hand, Zel'dovich and Starobinski[6] studied the particle creation in the Kasner universe (adopted as representing an early, highly anisotropic and vacuum stage of the universe) to see how the reaction of the produced particles leads to isotropization of the universe. Related works are now going on in various aspects,[7] e.g., Parker and Fulling's work on the avoidance of initial singularity of the big-bang universe via a quantum theoretical violation of the energy conditions entering into the singularity theorems.[8]

On the other hand, it is well-known that various propagators play a singnificant role in the usual quantum field theory. Contrary to this, there are almost no mention about the propagators in the above works. Clutton-Brock[9] has recently referred to them in his work on the homogenization of an early universe via quantum effect of hadrons, but those propagators have implicitly been assumed to be of the same structure as the ones in the Minkowski space-time, in contradiction to DeWitt and Brehme's result.[1] Such a disregard of the propagators will mainly be due to the situation that the vacuum state cannot be defined uniquely in the cosmological space-time,[2c] which leads to the violation of the well-known relation between the vacuum expectation value of field operators and the related propagator in the usual quantum field theory.

In spite of the above situation, it will be useful to study how the mass term in the Lagrangian density and the topological nature of a background universe have influence upon the structure of the bi-scalar Green's functions derived previously. Such a work and the related one (to be made in the next paper) in the case of an anisotropic universe may be an inevitable step necessary to look into the problem how those Green's functions can be connected with the concept of vacuum state, in relation to which the particle creation in the respective universes should be discussed.

In § 2, after summarizing the dynamical behavior of the Friedmann universes with flat, open and closed 3-spaces, the Lagrangian approach to a massless or massive scalar field in these universes is dealt with. In § 3 general formulae for various Green's functions of the scalar field are given in a heuristic manner. Sections 4 and 5 are devoted to the derivation of those bi-scalar functions in the universe with flat 3-space and in the one with closed 3-space, respectively. In § 6, as an application of the Green's functions thus derived, a massless scalar field generated from two kinds of sources, i.e., the spatially uniform distribution of sources and the point source, is dealt with. In § 7 what property has the scalar field obtained in § 6 is made clear, by examining its energy-momentum tensor. In § 8 the behavior of photons or gravitons to be generated from a turbulent medium in an early universe is discussed.

§ 2. Lagrangian formalism of a scalar field in the Friedmann universes

As is well-known, the metric for an isotropic universe is of the form

$$ds^2 = -g_{ij}dx^i dx^j = a^2(\eta)(d\eta^2 - d\sigma^2) \tag{2·1}$$

with

$$d\sigma^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta = \begin{cases} d\chi^2 + \sin^2\chi d\Omega^2, & (\varepsilon=1) \\ d\chi^2 + \chi^2 d\Omega^2, & (\varepsilon=0) \\ d\chi^2 + \mathrm{sh}^2\chi d\Omega^2, & (\varepsilon=-1) \end{cases} \tag{2·2}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, and $(-g)^{1/2} = a^4(\eta)\gamma^{1/2}$ with $\gamma^{1/2} \equiv \{\det(\gamma_{\alpha\beta})\}^{1/2} = (\sin^2\chi, \chi^2, \mathrm{sh}^2\chi)\sin\theta$. Moreover, the cosmic time t is defined by

$$dt = a(\eta)d\eta\,. \tag{2·3}$$

If we denote the background density and pressure of the universe by ρ and p, respectively, we have $\rho \propto a^{-4}$ or a^{-3} according as its substratum is dominated by radiation ($p/\rho = 1/3$) or matter ($p/\rho = 0$). Then the temporal behavior of the scale factor $a(\eta)$ is given as follows.

(i) The stage dominated by radiation:

$$\begin{cases} a = a_0 \sin\eta\,, & t = a_0(1-\cos\eta)\,, & (\varepsilon=1) \\ a = a_0\eta\,, & t = \frac{1}{2}a_0\eta^2, & (\varepsilon=0) \\ a = a_0\,\mathrm{sh}\,\eta\,, & t = a_0(\mathrm{ch}\,\eta - 1)\,, & (\varepsilon=-1) \end{cases} \tag{2·4}$$

where a_0 is positive constant.

(ii) The stage dominated by matter:

$$\begin{cases} a = a_1(1-\cos\eta)\,, & t = a_1(\eta - \sin\eta)\,, & (\varepsilon=1) \\ a = \frac{1}{2}a_1\eta^2, & t = \frac{1}{6}a_1\eta^3, & (\varepsilon=0) \\ a = a_1(\mathrm{ch}\,\eta - 1)\,, & t = a_1(\mathrm{sh}\,\eta - \eta)\,, & (\varepsilon=-1) \end{cases} \tag{2·5}$$

where a_1 is a positive constant different from a_0; these parameters can be settled by various astronomical observations. While the spatial part of the universes with $\varepsilon=0$ and -1 is infinite, the universe with $\varepsilon=1$ has a closed 3-space, so that we shall henceforth assume that it has a spherical topology, i.e., $(\eta, \chi, \theta, \varphi) \equiv (\eta, \chi + 2\pi, \theta, \varphi)$.

Lagrangian formalism for a scalar field

Let us take the Lagrangian density of a scalar field with mass m in the universe under consideration as follows:

$$\mathcal{L} = \tfrac{1}{2}(-g)^{1/2}\{g^{ij}\phi_{,i}\phi_{,j} + (m^2 + \lambda R/6)\phi^2\}, \tag{2·6}$$ *)

*) The system of units such as $c = \hbar = 1$ is used.

where $R \equiv g^{ij} R_{ij}$ is the scalar curvature and $\lambda = 1$ or 0 according as we postulate the conformal invariance (when $m=0$)[6] of $\mathcal{L}$ or not.[5] [It will be useful to point out that, if we extract a massless scalar field from $a^{-2}(\eta) g^{TT}_{\alpha\beta}/\sqrt{2}$[2a] or $a^{-1}(\eta) A_\alpha{}^T$[2b] ($g^{TT}_{\alpha\beta}$ and $A_\alpha{}^T$ stand for the transverse-traceless part of the metric perturbation $\delta g_{\alpha\beta}$ and the transverse part of the electro-magnetic potential A_α) in the universe with $\varepsilon=0$, these scalar fields obey the above $\mathcal{L}$ with $m=0$ in the cases $\lambda=0$ (for gravitons) and $\lambda=1$ (for photons), respectively.] For simplicity and owing to the situation that $R=0$ at least in the radiation dominated stage of the universes, let us temporarily consider the case $\lambda=0$. Then there arises the following field equation:

$$(\Box - m^2)\phi \equiv a^{-2}\{-\partial_\eta{}^2 - 2(a'/a)\partial_\eta + \nabla^2 - m^2 a^2\}\phi = 0\,, \tag{2·7}$$

where $\nabla^2 \equiv \gamma^{-1/2}\partial_\alpha(\gamma^{1/2}\gamma^{\alpha\beta}\partial_\beta)$ and $a' \equiv da/d\eta$.

The energy-momentum tensor of the scalar field is given by

$$T_i{}^j \equiv (-g)^{-1/2}\left\{\phi_{,i}\,\frac{\partial\mathcal{L}}{\partial(\phi_{,j})} - \delta_i{}^j\mathcal{L}\right\} = \phi_{,i}\phi^{,j} - \frac{1}{2}\,\delta_i{}^j(\phi_{,k}\phi^{,k} + m^2\phi^2)\,, \tag{2·8}$$

where $\phi^{,i} \equiv g^{ij}\phi_{,j}$. By virtue of Eq. (2·7), it follows from Eq. (2·8) that

$$T^j_{i;j} = (\Box\phi - m^2\phi)\,\phi_{,i} = 0 \tag{2·9}$$

showing the conservation of T_i^j.

At this stage, we shall mainly be concerned with the Friedmann universe with spherical 3-space and the one with flat 3-space, the latter of which may be expressed more conveniently by Cartesian coordinates $\boldsymbol{x}=(x, y, z)$ or $\gamma_{\alpha\beta}=\delta_{\alpha\beta}$. To solve Eq. (2·7), let us put

$$\phi = a^{-1}(\eta)\int f_k(\eta)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}\,, \qquad (\varepsilon=0)$$

$$\phi = a^{-1}(\eta)\sum_{n,l,m} f_n(\eta)\,\frac{B_n{}^l(\chi)}{\sin\chi}\,P_l{}^m(\cos\theta)\begin{pmatrix}\cos\\ \sin\end{pmatrix} m\varphi\,, \quad (\varepsilon=1) \tag{2·10}$$

where $P_l{}^m$ is the associated Legendre function and $B_n{}^l$ is Infeld and Schild's function[2d] obeying the differential equation

$$\{d^2/d\chi^2 + n^2 - l(l+1)/\sin^2\chi\}\, B_n{}^l = 0\,, \tag{2·11}$$

in which we must have $n=1, 2, 3, \cdots$ and $l=0, 1, 2, \cdots, (n-1)$ for the adopted topology. On inserting Eq. (1·10) into Eq. (2·7), we obtain

$$\begin{cases} \{d^2/d\eta^2 + k^2 + (m^2 - R/6)\,a^2\} f_k = 0\,, & (\varepsilon=0) \\ \{d^2/d\eta^2 + n^2 + (m^2 - R/6)\,a^2\} f_n = 0\,, & (\varepsilon=1) \end{cases} \tag{2·12}$$

where $R=6a^{-2}(a''/a+\varepsilon)$ is the scalar curvature; the R-dependent terms disappear if we adopt $\lambda=1$ (in place of $\lambda=0$) in Eq. (2·6). Here let us denote two particular solutions of Eq. (2·12) for f_k or f_n by (a_k, b_k) or (a_n, b_n) as follows:

$$\begin{cases} a_p(\eta)=1\,, & a_p{}'(\eta)=0\,, \\ b_p(\eta)=0\,, & b_p{}'(\eta)=1 \end{cases} \text{ at } \eta=\eta_*\,, \tag{2·13}$$

where $p=k$ or n, and η_* stands for a special epoch. Then, by the use of Eq. (2·12), it is easily seen that

$$a_p(\eta)\,b_p{}'(\eta)-a_p{}'(\eta)\,b_p(\eta)=1 \text{ at any epoch.} \tag{2·14}$$

Remark The counterparts of Eqs. (2·10) and (2·11) in the universe with $\varepsilon=-1$ are obtained from those when $\varepsilon=1$ by replacing $\sin\chi$ and Σ_n with $\mathrm{sh}\,\chi$ and $\int_0^\infty dn$, respectively.

§ 3. General formulae for various Green's functions

For simplicity, let us denote two events in any Friedmann universe by $x=(\eta, \boldsymbol{x})$ and $x'=(\eta', \boldsymbol{x}')$, so that $\boldsymbol{x}$ in the case $\varepsilon=1$ is an abbreviation of (χ, θ, φ). Then we can define the following 2-point function $G(x;x')=-G(x';x)$ satisfying the homogeneous wave equation (2·7):

$$G(x;x')=\frac{1}{(2\pi)^3}\{a(\eta)\,a(\eta')\}^{-1}\int d\boldsymbol{k}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\{a_k(\eta)\,b_k(\eta')-a_k(\eta')\,b_k(\eta)\} \tag{3·1·1}$$

in the case $\varepsilon=0$ (where $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$), and

$$G(x;x')=\frac{1}{(2\pi)^2}\{a(\eta)\,a(\eta')\}^{-1}\sum_n n\frac{\sin(nu)}{\sin u}\{a_n(\eta)\,b_n(\eta')-a_n(\eta')\,b_n(\eta)\} \tag{3·1·2}$$

in the case $\varepsilon=1$ (where $\cos u\equiv\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos\Theta$ and $\cos\Theta\equiv\cos\theta\cos\theta' +\sin\theta\sin\theta'\cos(\varphi-\varphi')$), which satisfy the boundary conditions

$$(\varepsilon=0):\ \begin{cases} G(x;x')=0\,, \\ \partial_\eta G(x;x')=-a^{-2}(\eta)\,\delta(\boldsymbol{x}-\boldsymbol{x}') \end{cases} \text{ at } \eta=\eta' \tag{3·2·1}$$

and

$$(\varepsilon=1):\ \begin{cases} G(x;x')=0\,, \\ \partial_\eta G(x;x')=-a^{-2}(\eta)\,\{\bar\gamma(\boldsymbol{x},\boldsymbol{x}')\}^{-1/2}\delta(\boldsymbol{x},\boldsymbol{x}') \end{cases} \text{ at } \eta=\eta'. \tag{3·2·2}$$

In Eq. (3·2·2), $\bar\gamma(\boldsymbol{x},\boldsymbol{x}')\equiv\{\gamma(\boldsymbol{x})\,\gamma(\boldsymbol{x}')\}^{1/2}$ and the 3-dimensional delta function is defined by[2d)]

$$\delta(\boldsymbol{x},\boldsymbol{x}')\equiv(2\pi)^{-2}\{\bar\gamma(\boldsymbol{x},\boldsymbol{x}')\}^{1/2}\sum_n n\frac{\sin(nu)}{\sin u}. \tag{3·3}$$

After the above preparation, let us further introduce the following two-point functions:

$$\begin{cases} G_{\mathrm{ret}}(x;x')=-\theta(\eta-\eta')\,G(x;x'), \\ G_{\mathrm{adv}}(x;x')=\theta(\eta'-\eta)\,G(x;x'), \\ \overline{G}(x;x')=\tfrac{1}{2}\{G_{\mathrm{ret}}(x;x')+G_{\mathrm{adv}}(x;x')\}, \end{cases} \tag{3·4}$$

where $\theta(\eta-\eta')=1-\theta(\eta'-\eta)$ is the step function. By making use of Eqs. (3·1) $\sim$(3·3), we can verify that any one of the three quantities defined by Eq. (3·4). e.g., $G_{\rm ret}(x;x')$, satisfies the inhomogeneous wave equation

$$(\Box-m^2)G_{\rm ret}(x;x')=-\{\bar{g}(x,x')\}^{-1/2}\delta^4(x,x'),\qquad(3\cdot5)$$

where $\bar{g}(x,x')\equiv\{g(x)g(x')\}^{1/2}=\{a(\eta)a(\eta')\}^4\bar{\gamma}(\boldsymbol{x},\boldsymbol{x}')$ ($\bar{\gamma}=1$ if $\varepsilon=0$) is a bi-scalar density introduced by DeWitt and Brehme[1] and $\delta^4(x,x')=\delta(\eta-\eta')\delta(\boldsymbol{x}-\boldsymbol{x}')$ or $\delta(\eta-\eta')\delta(\boldsymbol{x},\boldsymbol{x}')$ stands for the 4-dimensional delta function. Since the right-hand side of Eq. (3·5) is a bi-scalar of the required singularity and $(\Box-m^2)$ on the left-hand side is the covariant Klein-Gordon operator, we may call $G_{\rm ret}(x;x')$, $G_{\rm adv}(x;x')$ and $\bar{G}(x;x')$ bi-scalar retarded, advanced and symmetric Green's functions, respectively, because of their proper surviving regions.

In what follows, let us rewrite $G(x;x')$, $G_{\rm ret}(x;x')$, $G_{\rm adv}(x;x')$ and $\bar{G}(x;x')$ for a massless ($m=0$) field as $D(x;x')$, $D_{\rm ret}(x;x')$, $D_{\rm adv}(x;x')$ and $\bar{D}(x;x')$, respectively. Then it is easily seen that the formulae (3·1) for $G(x;x')$ are formally equivalent to their counterparts[2a),2d)] for $D(x;x')$.

Remark The expression for $G(x;x')$ in the universe with $\varepsilon=-1$ is obtained from Eq. (3·1·2) by replacing $\sin(nu)/\sin u$ and $\sum_n$ with ${\rm sh}(nu)/{\rm sh}\,u$ and $\int_0^\infty dn$, respectively, where

$${\rm ch}\,u\equiv{\rm ch}\,\chi\,{\rm ch}\,\chi'-{\rm sh}\,\chi\,{\rm sh}\,\chi'\cos\Theta\,.$$

§ 4. The bi-scalar function $G(x;x')$ in the universe with $\varepsilon=0$

By making use of Eqs. (2·4) and (2·5), we can reduce the first of Eq. (2·12) to

$$(p/\rho=1/3):\ \{d^2/d\eta^2+k^2+(ma_0)^2\eta^2\}\begin{pmatrix}a_k\\ b_k\end{pmatrix}=0\qquad(4\cdot1)$$

and

$$(p/\rho=0):\ \{d^2/d\eta^2+k^2-2(1-\lambda)/\eta^2+(\tfrac{1}{2}ma_1)^2\eta^4\}\begin{pmatrix}a_k\\ b_k\end{pmatrix}=0\,,\qquad(4\cdot2)$$

where a_k and b_k should satisfy the initial conditions (2·13), and λ in Eq. (4·2) is the parameter appearing in Eq. (2·6).

(1) *The case of a massless* ($m=0$) *field*:

In this case, it is an easy matter to solve Eqs. (4·1) and (4·2) without any approximation. A substitution of the solutions in Eq. (3·1·1) for $D(x;x')$ in place of $G(x;x')$ and the subsequent $d\boldsymbol{k}$-integration lead to

$$(p/\rho=1/3):\ D(x;x')=-\frac{\varepsilon(\xi)}{4\pi}\{a(\eta)a(\eta')\}^{-1}\frac{\delta(|\xi|-r)}{r}\qquad(4\cdot3)$$

and

$$(p/\rho=0):\ D(x;x') = -\frac{\varepsilon(\xi)}{4\pi}\{a(\eta)a(\eta')\}^{-1}\left\{\frac{\delta(|\xi|-r)}{r}\right.$$

$$\left. +(1-\lambda)\theta(|\xi|-r)/(\eta\eta')\right\}, \tag{4·4}$$

as already shown,[2a)] where $\varepsilon(\xi)=\theta(\xi)-\theta(-\xi)$ and $\xi\equiv\eta-\eta'$.

(2) *The case of a massive field*:

In this case, it is very difficult to solve exactly Eqs. (4·1) and (4·2) because of the coupling between m and $a(\eta)$, except when $k=0$. If $k=0$ in particular, we can solve these equations in the form of $\eta^{1/2}Z_{1/4}(\frac{1}{2}ma_0\eta^2)$ and $\eta^{3/2}Z_\nu(ma_1\eta^3/6)$ $(\nu^2\equiv(5-8\lambda)/36)$, respectively. Then, for instance, we have

$$(p/\rho=1/3):\ a_0(\eta)b_0(\eta')-a_0(\eta')b_0(\eta)=(\pi/4)(\eta\eta')^{1/2}\{J_{1/4}(cz)Y_{1/4}(cz')$$

$$-J_{1/4}(cz')Y_{1/4}(cz)\}, \tag{4·5}$$

where $c\equiv\frac{1}{2}ma_0\eta_*{}^2$ and $z\equiv(\eta/\eta_*)^2$, $z'\equiv(\eta'/\eta_*)^2$.

However, the temporal variation of mass terms in Eqs. (4·1) and (4·2) is rather slow, so that it will be permissible to perform the following approximation: After solving these equations by regarding $m^2a^2(\eta)$ as m^2 and deriving from Eq. (3·1·1) the expression for $G(x;x')$ (the first step), we recover the mass-increasing effect by replacing m appearing in $G(x;x')$ thus derived with $\mu(\eta,\eta')\equiv m\{a(\eta)\cdot a(\eta')\}^{1/2}$ (the second step). For illustration, we shall consider only the radiation dominated stage. Then, as the first-step solution, we have

$$(p/\rho=1/3):\ a_k(\eta)b_k(\eta')-a_k(\eta')b_k(\eta)=-\sin\{k_0(\eta-\eta')\}/k_0 \tag{4·6}$$

and, therefore,

$$(p/\rho=1/3):\ G(x;x') = -\frac{\varepsilon(\xi)}{4\pi}\{a(\eta)a(\eta')\}^{-1}\left\{\frac{\delta(|\xi|-r)}{r}\right.$$

$$\left. -\theta(|\xi|-r)(m/z)J_1(mz)\right\}, \tag{4·7}$$

where $k_0\equiv(k^2+m^2)^{1/2}$, $\xi\equiv\eta-\eta'$, $z\equiv(\xi^2-r^2)^{1/2}$ and J_1 is the Bessel function of the first order. (The above $G(x;x')$ is very similar to that in the Minkowski space-time, except for the appearance of the time-dependent factor $\{a(\eta)a(\eta')\}^{-1}$ and the situation that the light cone is represented by $|\eta-\eta'|=r$, but not $|t-t'|=r$.) The second-step solution is easily obtained, so its expression is omitted. It would not be useless to point out that, when $\lambda=0$, the counterpart of Eq. (4·7) in the matter dominated stage has two terms (with opposite signs) surviving in the time-like region specified by $|\eta-\eta'|>r$.

§ 5. The bi-scalar function $D(x;x')$ in the spherical universe

The counterparts of Eqs. (4·1) and (4·2) in the universe with $\varepsilon=1$ are of the form

$$(p/\rho=1/3):\ \{d^2/d\eta^2+n^2+(ma_0)^2\sin^2\eta\}\begin{pmatrix}a_n\\ b_n\end{pmatrix}=0 \qquad (5\cdot1)$$

and

$$(p/\rho=0):\ \left\{d^2/d\eta^2+n^2-\frac{(1-\lambda)}{(1-\cos\eta)}+(ma_1)^2(1-\cos\eta)^2\right\}\begin{pmatrix}a_n\\ b_n\end{pmatrix}=0\,, \qquad (5\cdot2)$$

where a_n and b_n should satisfy the initial conditions (2·13). Of the two equations, the simpler one (5·1) is the Mathieu equation, but its solutions cannot be periodic because of its particular form. If we adopt the approximation mentioned in § 4, we can derive the counterparts of Eqs. (4·6) and (4·7) from Eqs. (5·1), (5·2) and (3·1·2).

However, our main concern in this section is to examine how the spherical topology of the universe with $\varepsilon=1$ has influence upon the structure of various Green's functions. Accordingly we shall confine ourselves to the derivation of $D(x;x')$ for a massless field. In this case, we can solve Eqs. (5·1) and (5·2) without any approximation. Namely we have

$$(p/\rho=1/3):\ a_n(\eta)b_n(\eta')-a_n(\eta')b_n(\eta)=-\sin\{n(\eta-\eta')\}/n \qquad (5\cdot3)$$

and

$$(p/\rho=0):\ a_n(\eta)b_n(\eta')-a_n(\eta')b_n(\eta)=-\frac{1}{n(4n^2-1)}[\{4n^2+\cot(\eta/2)\cot(\eta'/2)\}\sin\{n(\eta-\eta')\}+2n\{\cot(\eta/2)-\cot(\eta'/2)\}\times\cos\{n(\eta-\eta')\}]\quad(\text{if }\lambda=0). \qquad (5\cdot4)$$

On inserting these expressions into Eq. (3·1·2) for $D(x;x')$ in place of $G(x;x')$ and making use of the formulae

$$\begin{cases}\sum_n\cos\{n(\xi-u)\}=\pi\delta(\xi-u)-1/2\,,\\ \sum_n\cos\{n(\xi-u)\}/(4n^2-1)=1/2-(\pi/4)\,\varepsilon(\xi-u)\sin\{(\xi-u)/2\},\end{cases}$$

we arrive at

$$(p/\rho=1/3):\ D(x;x')=-\frac{\varepsilon(\xi)}{4\pi}\{a(\eta)a(\eta')\}^{-1}\frac{\delta(|\xi|-u)}{\sin u} \qquad (5\cdot5)$$

and

$$(p/\rho=0):\ D(x;x')=-\frac{\varepsilon(\xi)}{4\pi}\{a(\eta)a(\eta')\}^{-1}\left[\frac{\delta(|\xi|-u)}{\sin u}+(1-\lambda)\,\theta(|\xi|-u)\left\{\frac{\sec(u/2)}{4\sin(\eta/2)\sin(\eta'/2)}\right\}\right], \qquad (5\cdot6)$$

where $\xi \equiv \eta - \eta'$, and $|\xi| = u$ stands for a light cone.

As is easily seen, the above expressions for $D(x;x')$ are reduced to the ones given by Eqs. (4·3) and (4·4) in the limit of $(\eta, \eta', u) \sim 0$, but the difference between them becomes large for other values of η, η' and u. Since $u(\boldsymbol{x}, \boldsymbol{x}')$ defined by $\cos u = \cos\chi \cos\chi' + \sin\chi \sin\chi' \cos\Theta$ is invariant under the transformations $\chi \to \chi + 2\pi$ and/or $\chi' \to \chi' + 2\pi$, the above bi-scalar functions are also the case.

Remark The $D(x;x')$-function in the open universe with $\varepsilon = -1$ is obtained from Eq. (5·6) by replacing $\sin u$ and $\sec(u/2)$ with $\mathrm{sh}\, u$ and $\mathrm{sech}(u/2)$, on the prescription that $\mathrm{ch}\, u = \mathrm{ch}\,\chi\, \mathrm{ch}\,\chi' - \mathrm{sh}\,\chi\, \mathrm{sh}\,\chi' \cos\Theta$.

§ 6. A massless scalar field generated from two kinds of sources

As a typical problem in which various Green's functions obtained in §§ 3~5 play a positive role, we shall consider a massless scalar field (e.g., photons or gravitons whose wave character is specified by $(\Box - R/6)$ or $\Box$) to be generated from two kinds of sources.

(1) The case where the source distribution is spatially uniform:

Such a source distribution may be realized in an early stage of the universe. For simplicity, let us assume that the early stage was dominated by radiation, while there is also an opposite view.[10)]

At the radiation-dominated stage ($p/\rho = 1/3$ and $R = 0$), the scalar field in question will be described by the following inhomogeneous wave equation:

$$\Box\phi = -a^{-2}(\eta) S(\eta) \theta(\eta - \eta_i) \theta(\eta_f - \eta) + b^{-2}(\eta)\phi\,, \tag{6·1}$$

where $S(\eta)$ and $b(\eta)$ stand for the uniform source-distribution (whose duration is specified by $\eta_i \leq \eta \leq \eta_f$) and an effective attenuation length of the generated field, respectively. If the attenuation length is not very large compared with $a(\eta)$, the above equation is formally transformed into the one for a massive field with mass $m(\eta) \equiv b^{-1}(\eta)$, i.e.,

$$\{\Box - m^2(\eta)\}\phi = -a^2(\eta) S(\eta) \theta(\eta - \eta_i) \theta(\eta_f - \eta)\,. \tag{6·1'}$$

For definiteness, let us further assume that $m(\eta) = \text{const} \equiv m$ or $m(\eta) = \mu a^{-1}(\eta)$ (η is a dimensionless constant). This means that, in the derivation of $G_{\text{ret}}(x;x') = -\theta(\eta - \eta') G(x;x')$ which, together with Eqs. (3·1), permits us to obtain the retarded solution of Eq. (6·1'), we can make use of either Eqs. (4·1) and (5·1) (when $m(\eta) = \text{const} \equiv m$) or their modified ones (when $m(\eta) = \mu a^{-1}(\eta)$) in such a way as $m^2 a^2(\eta) \to \mu^2$. On this premise, the retarded solution of Eq. (6·1') is given by

$$\phi_{\text{ret}}(\eta, \boldsymbol{x}) = -a^{-1}(\eta) \int_{\eta_i}^{\eta} \theta(\eta_f - \eta') a(\eta') S(\eta') \{a_c(\eta) b_c(\eta') - a_c(\eta') b_c(\eta)\} d\eta', \tag{6·2}$$

where $c = 0$ (if $\varepsilon = 0$) or $c = 1$ (if $\varepsilon = 1$). The above expression shows that only the lowest wave-number modes ($k=0$ and $n=1$) contribute to the solution $\phi_{\rm ret}(\eta, \boldsymbol{x})$ which, by the way, depends only on η.

(i) The universe with $\varepsilon=0$:

In this universe, the expression for $\{a_0(\eta)b_0(\eta')-a_0(\eta')b_0(\eta)\}$ when $m(\eta)=m$ is given by Eq. (4·5). On the other hand, its counterpart when $m(\eta)=\mu a^{-1}(\eta)$ is obtained from Eq. (4·6) by putting $k=0$ and replacing k_0 with μ, i.e.,

$$a_0(\eta)b_0(\eta')-a_0(\eta')b_0(\eta)=-\sin\{\mu(\eta-\eta')\}/\mu\,. \tag{6·3}$$

A substitution of Eq. (6·3) into Eq. (6·2) gives

$$\phi_{\rm ret}(\eta, \boldsymbol{x})=\eta^{-1}\int_{\eta_i}^{\eta}\theta(\eta_f-\eta')\eta' S(\eta')\frac{\sin\{\mu(\eta-\eta')\}}{\mu}d\eta', \tag{6·4}$$

where $a(\eta)=a_0\eta$ has been used.

(ii) The universe with $\varepsilon=1$:

The counterpart of Eq. (6·3) in the spherical universe is obtained by solving Eq. (5·1) with $n=1$ and μ^2 in place of $(ma_0)^2\sin^2\eta$, i.e.,

$$a_1(\eta)b_1(\eta')-a_1(\eta')b_1(\eta)=-\sin\{(\eta-\eta')\sqrt{\mu^2+1}\}/\sqrt{\eta^2+1}\,. \tag{6·5}$$

On inserting Eq. (6·5) and $a(\eta)=a_0\sin\eta$ into Eq. (6·2), we obtain

$$\phi_{\rm ret}(\eta, \boldsymbol{x})=(\sin\eta)^{-1}\int_{\eta_i}^{\eta}\theta(\eta_f-\eta')(\sin\eta')S(\eta')\frac{\sin\{\hat{\mu}(\eta-\eta')\}}{\hat{\mu}}d\eta', \tag{6·6}$$

where $\hat{\mu}\equiv(\mu^2+1)^{1/2}$.

(2) The case of a point source:

Let us consider a massless scalar field whose source is situated at $\boldsymbol{x}_0=(x_0, y_0, z_0)$ or $(\chi_0, \theta_0, \varphi_0)$ according as $\varepsilon=0$ or 1. The scalar field should be described by the following inhomogeneous wave equation:

$$(\Box-\lambda R/6)\phi=-a^{-2}(\eta)\{4\pi A^2(\eta)\}^{1/2}\cdot\begin{Bmatrix}\delta(\boldsymbol{x}-\boldsymbol{x}_0), & (\varepsilon=0)\\ \{\bar{\gamma}(\boldsymbol{x}, \boldsymbol{x}_0)\}^{-1/2}\delta(\boldsymbol{x}, \boldsymbol{x}_0), & (\varepsilon=1)\end{Bmatrix} \tag{6·7}$$

where $A(\eta)$ is a given quantity whose physical meaning is shown in the next section. The retarded Green's function $D_{\rm ret}(x;x')=-\theta(\eta-\eta')D(x;x')$ necessary to solve Eq. (6·7) is already given, by virtue of Eqs. (4·3), (4·4), (5·5) and (5·6). However, such a point-source problem is mainly realized in a later and matter-dominated stage of the universe, so that we shall confine ourselves to the stage $p/\rho=0$. Then we obtain the retarded solution in the following form:

$$(\varepsilon=0):\ \phi_{\rm ret}(\eta, \boldsymbol{x})=(1/4\pi)^{1/2}\left\{\frac{a(\eta-r)}{a(\eta)r}\right\}\Big[A(\eta-r)$$

$$+(1-\lambda)(r/\eta)\int_0^{\eta-r}A(\eta')\left\{\frac{a(\eta')}{a(\eta-r)}\right\}(d\eta'/\eta')\Big] \tag{6·8}$$

and

$$(\varepsilon=1):\ \phi_{\rm ret}(\eta,\boldsymbol{x})=(1/4\pi)^{1/2}\left\{\frac{a(\eta-u)}{a(\eta)\sin u}\right\}\Big[A(\eta-u)$$

$$+(1-\lambda)\left\{\frac{\sin(u/2)}{\sin(\eta/2)}\right\}\int_0^{\eta-u}A(\eta')\left\{\frac{a(\eta')}{a(\eta-u)}\right\}\frac{(d\eta'/2)}{\sin(\eta'/2)}\Big], \tag{6·9}$$

where $r\equiv|\boldsymbol{x}-\boldsymbol{x}_0|$ (for $\varepsilon=0$) and $\cos u=\cos\chi\cos\chi_0+\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)\}\sin\chi\sin\chi_0$ (for $\varepsilon=1$).

Now let us perform the spectral decomposition of $A(\eta)$ as follows :

$$A(\eta)=\begin{cases}(1/2\pi)\displaystyle\int_{-\infty}^{\infty}d\omega\hat{A}(\omega)e^{i\omega\eta},\ \hat{A}^*(\omega)=\hat{A}(-\omega)\ \text{ if }\ \varepsilon=0\,,\\ (1/2\pi)\displaystyle\sum_{-\infty}^{\infty}{}_n\hat{A}(n)e^{in\eta},\ \hat{A}^*(n)=\hat{A}(-n)\ \text{ if }\ \varepsilon=1\,,\end{cases} \tag{6·10}$$

where $\hat{A}^*$ is the complex conjugate of $\hat{A}$. Then the respective factors $[A(\eta-r)+\cdots]$ and $[A(\eta-u)+\cdots]$ in Eqs. (6·8) and (6·9) are transformed into

$$(\varepsilon=0):\ (1/2\pi)\int_{-\infty}^{\infty}d\omega\hat{A}(\omega)e^{i\omega(\eta-r)}\{1+(1-\lambda)(r/\eta)B_0(i\omega,\eta-r)\} \tag{6·11}$$

and

$$(\varepsilon=1):\ (1/2\pi)\sum_{-\infty}^{\infty}{}_n\hat{A}(n)e^{in(\eta-u)}\left[1+(1-\lambda)\left\{\frac{\sin(u/2)}{\sin(\eta/2)}\right\}B_1(in,\eta-u)\right], \tag{6·12}$$

where

$$\begin{cases}B_0(i\omega,\eta)\equiv(i\omega\eta)^{-1}-(1-e^{-i\omega\eta})(i\omega\eta)^{-2},\\ B_1(in,\eta)\equiv\left(\dfrac{4n^2}{4n^2-1}\right)[\{2in\sin(\eta/2)\}^{-1}-\{\cos(\eta/2)-e^{-in\eta}\}\\ \qquad\qquad\times\{2in\sin(\eta/2)\}^{-2}].\end{cases} \tag{6·13}$$

§ 7. Energy-momentum tensor of the scalar field obtained in § 6

In this section, we shall examine the energy-momentum tensor of the scalar field obtained in § 6.

(1) The case where the source distribution is spatially uniform:

As shown by Eq. (6·2), the scalar field $\phi_{\rm ret}(\eta,\boldsymbol{x})$ in this case is independent of $\boldsymbol{x}$, so we shall henceforth abbreviate it as $\phi(\eta)$. If we apply the expression for T_i^j or Eq. (2·8) to this field (by replacing m with $m(\eta)=$const or $\mu a^{-1}(\eta)$), it is easily seen that there are only two surviving components T_0^0 and $T_\alpha^\beta=(T_\lambda^\lambda/3)\delta_\alpha^\beta$. In other words, we can regard this scalar field as a perfect fluid whose density and pressure are of the form

$$\begin{cases} -T_0{}^0 \equiv \rho_{(\phi)} = \frac{1}{2}\{a^{-2}(\eta)\,(d\phi/d\eta)^2 + m^2(\eta)\,\phi^2\}, \\ T_\lambda{}^\lambda/3 \equiv p_{(\phi)} = \frac{1}{2}\{a^{-2}(\eta)\,(d\phi/d\eta)^2 - m^2(\eta)\,\phi^2\}. \end{cases} \quad (7\cdot1)$$

When $m(\eta) = \mu a^{-1}(\eta)$, it follows from Eqs. (6·4), (6·6) and (7·1) that

$$\rho_{(\phi)} = \tfrac{1}{2}a^{-6}(\eta)\,\{T^2(\eta) + V^2(\eta)\}\,,\ p_{(\phi)} = \tfrac{1}{2}a^{-6}(\eta)\,\{T^2(\eta) - V^2(\eta)\} \quad (7\cdot2)$$

with

$$\begin{cases} T(\eta) \equiv \displaystyle\int_{\eta_i}^{\eta} \theta(\eta_f - \eta')\,a(\eta')\,S(\eta')\left[a(\eta)\cos\{\nu(\eta-\eta')\} - a'(\eta)\frac{\sin\{\nu(\eta-\eta')\}}{\nu}\right]d\eta', \\ V(\eta) \equiv (1-\varepsilon/\nu^2)^{1/2}a(\eta)\displaystyle\int_{\eta_i}^{\eta}\theta(\eta_f-\eta')\,a(\eta')\,S(\eta')\sin\{\nu(\eta-\eta')\}\,d\eta', \end{cases} \quad (7\cdot3)$$

where $a(\eta) = a_0\eta$ or $a_0 \sin\eta$ according as $\varepsilon = 0$ or 1, and $\nu \equiv (\mu^2+\varepsilon)^{1/2}$. If $\mu = 0$ (corresponding to the case where the attenuation length $b(\eta) = \mu^{-1}a(\eta)$ is infinitely large) in particular, we have $V(\eta) = 0$ and, therefore,

$$p_{(\phi)} = \rho_{(\phi)} = \tfrac{1}{2}\,a^{-6}(\eta)\left\{\int_{\eta_i}^{\eta}\theta(\eta_f-\eta')\,a^2(\eta')\,S(\eta')\,d\eta'\right\}^2. \quad (7\cdot4)$$

It is noticeable that the relation $p_{(\phi)} = \rho_{(\phi)}$ is formally identical with the one envisaged by Zel'dovich in a different context.[11]

Moreover, it follows from Eq. (7·4) and its counterpart for the background substratum, i.e., $3p = \rho = \rho_*(a_*/a)^4$, that

$$\Gamma(\eta) \equiv \{\rho_{(\phi)}/\rho\}^{1/2} = Ka^{-1}(\eta)\int_{\eta_i}^{\eta}\theta(\eta_f-\eta')\,a^2(\eta')\,S(\eta')\,d\eta', \quad (\text{if } \mu=0) \quad (7\cdot5)$$

where $K \equiv (2\rho_* a_*{}^4)^{-1/2}$ is an observable quantity. The η-dependence of $a(\eta)$, i.e., $a(\eta) = a_0\eta$ or $a_0\sin\eta$ according as $\varepsilon = 0$ or 1, in Eq. (7·5), has been derived without recourse to the existence of the perfect fluid specified by Eq. (7·4). In order that such a premise may be assured, we must have $\Gamma(\eta) \ll 1$. On the contrary, if there is some process which leads to $\Gamma(\eta) \sim 1$ in the relevant stage, the reaction effect of the fluid on the background universe must be taken into consideration.

(2) The case of a point source:

Let us impose the following conditions for the spectral function $\hat{A}(\omega)$ or $\hat{A}(n)$ appearing in Eq. (6·10):

$$(\varepsilon=0):\ \hat{A}(\omega) \neq 0 \text{ only for } |i\omega| \gg r^{-1},\ \eta^{-1}, |n-r|^{-1} \quad (7\cdot6\cdot1)$$

and

$$(\varepsilon=1):\ \hat{A}(n) \neq 0 \text{ only for } |in| \gg 1,\ \cot u, \cot\eta, |\cot(\eta-u)|. \quad (7\cdot6\cdot2)$$

Then, from Eqs. (6·8), (6·9) and (6·11) ~ (6·13), we can approximately derive the following expressions:

$$(\varepsilon=0):\ \begin{pmatrix}\phi_{,0}\\ \phi_{,\alpha}\end{pmatrix}=(1/4\pi)^{1/2}\left\{\frac{a(\eta-r)}{a(\eta)r}\right\}(1/2\pi)\int_{-\infty}^{\infty}d\omega\,(i\omega)\begin{pmatrix}1\\ -r_{,\alpha}\end{pmatrix}\hat{A}(\omega)\,e^{i\omega(\eta-r)} \tag{7·7·1}$$

and

$$(\varepsilon=1):\ \begin{pmatrix}\phi_{,0}\\ \phi_{,\alpha}\end{pmatrix}=(1/4\pi)^{1/2}\left\{\frac{a(\eta-u)}{a(\eta)\sin u}\right\}(1/2\pi)\sum_{-\infty}^{\infty}{}_n\,(in)\begin{pmatrix}1\\ -u_{,\alpha}\end{pmatrix}\hat{A}(n)\,e^{in(\eta-u)}, \tag{7·7·2}$$

where $\phi(\eta,\boldsymbol{x})$ is an abbreviation for $\phi_{\rm ret}(\eta,\boldsymbol{x})$. The above expressions for $\phi_{,i}$ satisfy the relation

$$g^{ij}\phi_{,i}\phi_{,j}=0\,,\ \ (\text{for both } \varepsilon=0 \text{ and } 1) \tag{7·8}$$

because $\delta^{\alpha\beta}r_{,\alpha}r_{,\beta}=\gamma^{\alpha\beta}u_{,\alpha}u_{,\beta}=1$. This means that the above $\phi_{,i}$ is a null-field. A substitution of Eq. (7·8) in Eq. (2·8) with $m=0$ gives

$$T_{ij}=\phi_{,i}\phi_{,j}\,.\qquad (\phi_{,i}\phi^{,i}=0) \tag{7·9}$$

Let us introduce here two unit vectors in the η- and r- (or u-) directions at an observation event $(\eta,\boldsymbol{x})$, i.e.,

$$\overset{(0)}{e^i}=a^{-1}(\eta)\,(1,0),\ \ \overset{(r)}{e^i}=a^{-1}(\eta)\begin{pmatrix}(0,r^{,\alpha}), & (\varepsilon=0)\\ (0,u^{,\alpha}). & (\varepsilon=1)\end{pmatrix} \tag{7·10}$$

Then we can define two physical components of T_{ij} given by Eq. (7·9) as follows:

$$\begin{cases} U\equiv\overset{(0)}{e^i}\overset{(0)}{e^j}T_{ij}=a^{-2}(\eta)\,(\phi_{,0})^2,\\ F\equiv-\overset{(0)}{e^i}\overset{(r)}{e^j}T_{ij}=-a^{-2}(\eta)\,(\phi_{,0})\begin{pmatrix}\phi_{,\alpha}r^{,\alpha}, & (\varepsilon=0)\\ \phi_{,\alpha}u^{,\alpha}, & (\varepsilon=1)\end{pmatrix}\end{cases} \tag{7·11}$$

which specify the energy density and the energy flow, respectively. Let us further introduce the following quantity:

$$L^{1/2}(\eta)\equiv\begin{Bmatrix}(1/2\pi)\int_{-\infty}^{\infty}(i\omega)\,\hat{A}(\omega)\,e^{i\omega\eta}d\omega\,, & (\varepsilon=0)\\ (1/2\pi)\sum_{-\infty}^{\infty}{}_n\,(in)\,\hat{A}(n)\,e^{in\eta}. & (\varepsilon=1)\end{Bmatrix} \tag{7·12}$$

On inserting Eqs. (7·7) into Eq. (7·11) and making use of Eq. (7·12), we obtain

$$U=F=\begin{Bmatrix}\dfrac{L(\eta-r)}{4\pi a^2(\eta)\,r^2(1+z)^2}, & (\varepsilon=0)\\ \dfrac{L(\eta-u)}{4\pi a^2(\eta)\,(\sin u)^2(1+z)^2}, & (\varepsilon=1)\end{Bmatrix} \tag{7·13}$$

where $z\equiv a(\eta)/a(\eta-r)-1$ or $a(\eta)/a(\eta-u)-1$ is the redshift of the radiation measured at $(\eta,\boldsymbol{x})$. It is of course that, if we replace $\sin u$ by $\operatorname{sh} u$ in the second of Eq. (7·13), there arises the expression applicable in the open universe with

$\varepsilon=-1$.

If the observer is situated at the origin of coordinates $\boldsymbol{x}=0$, we can put $r=u=\chi_0$ in Eq. (7·13). Then we arrive at Robertson's well-known formula[12] for the apparent luminosity $l(=F)$ of a receding galaxy:

$$l=\frac{L(\eta-\chi_0)}{4\pi a^2(\eta)\sigma^2(\chi_0)(1+z)^2}, \tag{7·14}$$

where $\sigma(\chi_0)=\sin\chi_0$, χ_0 or $\mathrm{sh}\,\chi_0$ according as $\varepsilon=1, 0$ or -1, and $1+z=a(\eta)/a(\eta-\chi_0)$. In the above formula, the quantity $L(\eta-\chi_0)$ stands for the absolute luminosity of the galaxy at the emission epoch $(\eta-\chi_0)$.

Solving Eq. (7·12) with respect to $\widehat{A}(\omega)$ or $\widehat{A}(n)$ and inserting the solution into Eq. (6·10), we obtain

$$A(\eta)=\int_0^\eta L^{1/2}(\eta)\,d\eta, \quad \text{(irrespective of } \varepsilon=1,\ 0 \text{ or } -1) \tag{7·15}$$

which, together with Eq. (7·14), gives the physical meaning of $A(\eta)$ in question.

§ 8. The behavior of photons or gravitons generated from a turbulent medium in an early universe

Similarly to case (1) in § 6, let us assume that an early universe was dominated by radiation; we shall consider only the universe with $\varepsilon=0$ for mathematical simplicity in a later discussion. If the universe was in a turbulent state, a massless scalar field (e.g., photons or gravitons) generated from the turbulent medium should obey a stochastic version of Eq. (6·1). For this purpose, we shall at first replace the source term by $-a^{-2}(\eta)B(\eta,\boldsymbol{x})$, where $B(\eta,\boldsymbol{x})$ is a stochastic quantity which consists in general of several fundamental stochastic variables (whose mean-values vanish) and their derivatives. The attenuation length $b(\eta)$ should also be replaced by its stochastic version, but we shall adopt here $b(\eta)=\mu^{-1}a(\eta)$ used in § 6 for simplicity. Then we have

$$\{\Box-\mu^2a^{-2}(\eta)\}\phi=-a^{-2}(\eta)B(\eta,\boldsymbol{x}), \tag{8·1}$$

where $a(\eta)=a_0\eta$ corresponding to $p/\rho=1/3$ and $\varepsilon=0$.

The retarded Green's function $G_{\text{ret}}(x;x')=-\theta(\eta-\eta')G(x;x')$ necessary to solve formally the above stochastic equation is obtained from Eq. (4·7) by replacing m with μ. Then the formal solution of Eq. (8·1) is of the form

$$\phi(\eta,\boldsymbol{x})=\frac{1}{4\pi a(\eta)}\int\frac{d\boldsymbol{r}}{r}\Big[a(\eta-r)B(\eta-r,\boldsymbol{x}+\boldsymbol{r})-\mu r\int_r^\eta a(\eta-t)B(\eta-t,\boldsymbol{x}+\boldsymbol{r})$$
$$\times\{z^{-1}J_1(\mu z)\}\,dt\Big], \tag{8·2}$$

where $z\equiv(t^2-r^2)^{1/2}$, and $\phi(\eta,\boldsymbol{x})$ is an abbreviation for $\phi_{\text{ret}}(\eta,\boldsymbol{x})$. At this stage, let us perform the following approximation to the second integration term:

$$\mu r\{a(\eta-t)B(\eta-t,\boldsymbol{x}+\boldsymbol{r})\}_{t=r}\cdot\{z^{-1}J_1(\mu z)\}_{t=\eta}\cdot\int_r^\eta dt\,. \tag{8·3}$$

Accordingly we can reduce Eq. (8·2) to

$$\phi(\eta,\boldsymbol{x})=(1/4\pi)\int\frac{d\boldsymbol{r}}{r}\,h(\eta,r;\mu)B(\eta-r,\boldsymbol{x}+\boldsymbol{r}), \tag{8·4}$$

where

$$h(\eta,r;\mu)\equiv(1-r/\eta)\left\{1-\mu r\left(\frac{\eta-r}{\eta+r}\right)^{1/2}J_1(\mu\sqrt{\eta^2-r^2})\right\}. \tag{8·5}$$

Now let us assume that the turbulent field is homogeneous and isotropic similarly to the background universe itself, so that correlations of, say, $B(\eta,\boldsymbol{x})B(\eta,\boldsymbol{x}+\boldsymbol{r})$ and $\partial_\eta B(\eta,\boldsymbol{x})\partial_\mu B(\eta,\boldsymbol{x}+\boldsymbol{r})$ are of the form

$$\begin{cases}\langle B(\eta,\boldsymbol{x})B(\eta,\boldsymbol{x}+\boldsymbol{r})\rangle=C(\eta,r),\\ \langle\partial_\eta B(\eta,\boldsymbol{x})\,\partial_\eta B(\eta,\boldsymbol{x}+\boldsymbol{r})\rangle=D(\eta,r),\end{cases} \tag{8·6}$$

which are even with respect to $r\equiv|\boldsymbol{r}|$. Once a stochastic dynamical equation for $B(\eta,\boldsymbol{x})$ is given, we can determine the respective functional forms of these mean-value quantities. By making use of Eq. (8·6), it follows from Eq. (8·4) that

$$\langle\phi(\eta,\boldsymbol{x})\phi(\eta,\boldsymbol{x}+\boldsymbol{r})\rangle=(1/4\pi)^2\int\frac{d\boldsymbol{r}'}{r'}\{h(\eta,r';\mu)\}^2\hat{C}(\eta-r',R) \tag{8·7}$$

with

$$\begin{aligned}\hat{C}(\eta-r',R)&\equiv\int\frac{d\boldsymbol{r}''}{r''}\,C(\eta-r',|\boldsymbol{r}-\boldsymbol{r}'+\boldsymbol{r}''|)\qquad(R\equiv|\boldsymbol{r}-\boldsymbol{r}'|)\\ &=(2\pi/R)\int_0^\infty dz\int_{|R-z|}^{R+z}C(\eta-r',x)\,xdx\\ &=(4\pi/R)\int_0^\infty\{\theta(R-x)x+\theta(x-R)R\}C(\eta-r',x)\,xdx\,.\end{aligned} \tag{8·8}$$

In the derivation of Eq. (8·7), we have assumed that the stochastic character of $h(\eta,r;\mu)B(\eta-r,\boldsymbol{x}+\boldsymbol{r})$ in Eq. (8·4) refers to its $(\boldsymbol{x}+\boldsymbol{r})$-dependence rather than its $(\eta-r)$-dependence. The last form in Eq. (8·8) is a consequence of dz-integration after exchanging the order of two successive integrations.

On inserting Eq. (8·8) into Eq. (8·7) and performing a lengthy calculation, we obtain

$$\begin{aligned}\langle\phi(\eta,\boldsymbol{x})\phi(\eta,\boldsymbol{x}+\boldsymbol{r})\rangle=&\int_0^\infty x^2dx\Bigg[\theta(r-x)\left\{\frac{1}{r}\int_0^{r-x}ydy+\int_{r-x}^\infty dy\right.\\ &\left.-\int_{r-x}^{r+x}\frac{(r+x-y)^2}{4rx}dy\right\}+\theta(x-r)\left\{\frac{1}{x}\int_0^{x-r}ydy+\int_{x-r}^\infty dy\right.\\ &\left.-\int_{x-r}^{x+r}\frac{(r+x-y)^2}{4rx}dy\right\}\Bigg]\{h(\eta,y;\mu)\}^2C(\eta-y,x)\,.\end{aligned} \tag{8·9}$$

Similarly, it follows from Eqs. (8·4) and (8·6) that

$$\langle \partial_\eta \phi(\eta, \boldsymbol{x}) \partial_\eta \phi(\eta, \boldsymbol{x}+\boldsymbol{r}) \rangle = \int_0^\infty x^2 dx [\theta(r-x)\{\cdots\} + \theta(x-r)\{\cdots\}]$$
$$\times \left[\{h(\eta, y; \mu)\}^2 D(\eta-y, x) + \{\partial_\eta h(\eta, y; \mu)\} \frac{\partial}{\partial \eta} \{h(\eta, y; \mu) C(\eta-y, x)\} \right], \tag{8·10}$$

where $\{\cdots\}$'s are an abbreviation for the terms appearing in Eq. (8·9). If $r=0$ in particular, the above expressions are reduced to

$$\langle \phi(\eta, \boldsymbol{x}) \phi(\eta, \boldsymbol{x}) \rangle = \int_0^\infty x dx \int_0^\infty dy \{\theta(x-y) y + \theta(y-x) x\} \{h(\eta, y; \mu)\}^2 C(\eta-y, x) \tag{8·11}$$

and

$$\langle \partial_\eta \phi(\eta, \boldsymbol{x}) \partial_\eta \phi(\eta, \boldsymbol{x}) \rangle = \int_0^\infty x dx \int_0^\infty dy \{\theta(x-y) y + \theta(y-x) x\}$$
$$\times \left[\{h(\eta, y; \mu)\}^2 D(\eta-y, x) + \{\partial_\eta h(\eta, y; \mu)\} \frac{\partial}{\partial \eta} \{h(\eta, y; \mu) C(\eta-y, x)\} \right]. \tag{8·12}$$

The last two quantities may be regarded as the counterparts of $\phi^2(\eta)$ and $(d\phi/d\eta)^2$, respectively, appearing in Eq. (7·1) with $m(\eta) = \mu a^{-1}(\eta)$, when the source distribution is turbulent.

Moreover, it would not be useless to point out that the above procedure for deriving Eq. (8·9) may also be applied to refine our formula[13] for the intensity of sound generated from primordial cosmic turbulence.

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