Proper Congruence-Preserving Extensions of Lattices — Source link

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PROPER CONGRUENCE-PRESERVING EXTENSIONS OF LATTICES

G. GRÄTZER AND F. WEHRUNG

Abstract. We prove that every lattice with more than one element has a proper congruence-preserving extension.

1. Introduction

Let L be a lattice. A lattice K is a *congruence-preserving extension* of L, if K is an extension and every congruence of L has exactly one extension to K. (Of course, then, the congruence lattice of L is isomorphic to the congruence lattice of K.)

In [4], the first author and E. T. Schmidt raised the following question:

Is it true that every lattice L with more than one element has a proper congruence-preserving extension K?

Here *proper* means that K properly contains L, that is, $K - L \neq \varnothing$.

The first author and E. T. Schmidt pointed out in [4] that in the finite case this is obviously true, and they proved the following general result:

Theorem 1. *Let L be a lattice. If there exist a distributive interval with more than one element in L, then L has a proper congruence-preserving extension K.*

Generalizing this result, in this paper, we provide a positive answer to the above question:

Theorem 2. *Every lattice L with more than one element has a proper congruence-preserving extension K.*

2. Background

Let K and L be lattices. If L is a sublattice of K, then we call K an *extension* of L. If K is an extension of L and Θ is a congruence relation of K, then Θ_L, the restriction of Θ to L is a congruence of L. If the map $\Theta \mapsto \Theta_L$ is a bijection between the congruences of L and the congruences of K, then we call K a *congruence-preserving extension* of L. Observe that if K a congruence-preserving extension of L, then the congruence lattice of L is isomorphic to the congruence lattice of K in a natural way.

The proof of Theorem 1 is based on the following construction of E. T. Schmidt [9], summarized below as Theorem 3. (A number of papers utilize this construction;

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see, for instance, E. T. Schmidt [10], [11] and the recent paper G. Grätzer and E. T. Schmidt [5].)

Let L be a bounded distributive lattice with bounds 0 and 1, and let $M_3 = \{o, a, b, c, i\}$ be the five-element nondistributive modular lattice. Let $M_3[L]$ denote the poset of triples $\langle x, y, z\rangle \in L^3$ satisfying the condition

$$x \wedge y = y \wedge z = z \wedge x. \tag{S}$$

Theorem 3.
Let D be a bounded distributive lattice with bounds 0 and 1.

(i) *$M_3[D]$ is a modular lattice.*
(ii) *The subset*

$$\overline{M}_3 = \{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}$$

of $M_3[D]$ is a sublattice of $M_3[D]$ and it is isomorphic to M_3.
(iii) *The subposet $\overline{D} = \{\, \langle x,0,0\rangle \mid x \in D \,\}$ of $M_3[D]$ is a bounded distributive lattice and it is isomorphic to D; we identify D with $\overline{D}$.*
(iv) *$\overline{M}_3$ and D generate $M_3[D]$.*
(v) *Let Θ be a congruence relation of $D = \overline{D}$; then there is a* unique *congruence $\overline{\Theta}$ of $M_3[D]$ such that $\overline{\Theta}$ restricted to $\overline{D}$ is Θ; therefore, $M_3[D]$ is a congruence-preserving extension of D.*

Unfortunately, $M_3[L]$ fails, in general, to produce a lattice, if L is not distributive.

In this paper, we introduce a variant on the $M_3[L]$ construction, which we shall denote as $M_3\langle L\rangle$. This lattice $M_3\langle L\rangle$ is a proper congruence-preserving extension of L, for any lattice L with more than one element, verifying Theorem 2.

3. The construction

For a lattice L, let us call the triple $\langle x, y, z\rangle \in L^3$ *Boolean*, if

$$\begin{aligned} x &= (x \vee y) \wedge (x \vee z),\\ y &= (y \vee x) \wedge (y \vee z),\\ z &= (z \vee x) \wedge (z \vee y). \end{aligned} \tag{B}$$

We denote by $M_3\langle L\rangle \subseteq L^3$ the poset of Boolean triples of L.

Here are some of the basic properties of Boolean triples:

Lemma 1. *Let L be a lattice.*

(i) *Every Boolean triple of L satisfies* (S)*, so $M_3\langle L\rangle \subseteq M_3[L]$.*
(ii) *$\langle x, y, z\rangle \in L^3$ is Boolean iff there is a triple $\langle u, v, w\rangle \in L^3$ satisfying*

$$\begin{aligned} x &= u \wedge v,\\ y &= u \wedge w,\\ z &= v \wedge w. \end{aligned} \tag{R}$$

(iii) *For every triple $\langle x, y, z\rangle \in L^3$, there is a smallest Boolean triple $\overline{\langle x, y, z\rangle} \in L^3$ such that $\langle x, y, z\rangle \leq \overline{\langle x, y, z\rangle}$; in fact,*

$$\overline{\langle x, y, z\rangle} = \langle (x \vee y) \wedge (x \vee z), (y \vee x) \wedge (y \vee z), (z \vee x) \wedge (z \vee y)\rangle.$$

(iv) *$M_3\langle L\rangle$ is a lattice with the meet operation defined as*

$$\langle x_0, y_0, z_0\rangle \wedge \langle x_1, y_1, z_1\rangle = \langle x_0 \wedge x_1, y_0 \wedge y_1, z_0 \wedge z_1\rangle$$

and the join operation defined by

$$\langle x_0, y_0, z_0\rangle \vee \langle x_1, y_1, z_1\rangle = \overline{\langle x_0 \vee x_1, y_0 \vee y_1, z_0 \vee z_1\rangle}.$$

(v) *If L has 0, then the subposet $\{\, \langle x, 0, 0\rangle \mid x \in L \,\}$ is a sublattice and it is isomorphic to L.*

If L has 0 and 1, then $M_3\langle L\rangle$ has a spanning M_3, that is, a $\{0,1\}$-sublattice isomorphic to M_3, namely,

$$\{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}.$$

(vi) *If $\langle x, y, z\rangle$ is Boolean, then one of the following holds:*
  (a) *the components form a one-element set, so $\langle x, y, z\rangle = \langle a, a, a\rangle$, for some $a \in L$;*
  (b) *the components form a two-element set and $\langle x, y, z\rangle$ is of the form $\langle b, a, a\rangle$, or $\langle a, b, a\rangle$, or $\langle a, a, b\rangle$, for some a, $b \in L$, $a < b$.*
  (c) *the components form a three-element set and two components are comparable and L has two incomparable elements a and b such that $\langle x, y, z\rangle$ is of the form $\langle a, b, a \wedge b\rangle$, or $\langle a, a \wedge b, b\rangle$, or $\langle a \wedge b, a, b\rangle$.*
  (d) *the components form a three-element set and the components are pairwise incomparable and L has an eight-element Boolean sublattice B so that the components are the atoms of B.*

Proof.
(i) If $\langle x, y, z\rangle$ is Boolean, then

$$\begin{aligned} x \wedge y &= ((x \vee y) \wedge (x \vee z)) \wedge ((y \vee x) \wedge (y \vee z)) \\ &= (x \vee y) \wedge (y \vee z) \wedge (z \vee x), \end{aligned}$$

which is the upper median of x, y, and z. So (S) holds.

(ii) If $\langle x, y, z\rangle$ is Boolean, then $u = x \vee y$, $v = x \vee z$, and $w = y \vee z$ satisfy (R). Conversely, if there is a triple $\langle u, v, w\rangle \in L^3$ satisfying (R), then by Lemma I.5.9 of [1], the sublattice generated by x, y, and z is isomorphic to a quotient of $\mathfrak{C}_2^3$ (where $\mathfrak{C}_2$ is the two element chain) and x, y, and z are the images of the three atoms of $\mathfrak{C}_2^3$. Thus $(x \vee y) \wedge (x \vee z) = x$, the first part of (B). The other two parts are proved similarly.

(iii) For $\langle x, y, z\rangle \in L^3$, define $u = x \vee y$, $v = x \vee z$, $w = y \vee z$. Set $x_1 = u \wedge v$, $y_1 = u \wedge w$, $z_1 = v \wedge w$. Then $\langle x_1, y_1, z_1\rangle$ is Boolean by (ii) and $\langle x, y, z\rangle \leq \langle x_1, y_1, z_1\rangle$ in L^3. Now if $\langle x, y, z\rangle \leq \langle x_2, y_2, z_2\rangle$ in L^3 and $\langle x_2, y_2, z_2\rangle$ is Boolean, then

$$\begin{aligned} x_2 &= (x_2 \vee y_2) \wedge (x_2 \vee z_2) && \text{(by (B))} \\ &\geq (x \vee y) \wedge (x \vee z) && \text{(by } \langle x, y, z\rangle \leq \langle x_2, y_2, z_2\rangle\text{)} \\ &= u \wedge v = x_1, \end{aligned}$$

and similarly, $y_2 \geq y_1$, $z_2 \geq z_1$. Thus $\langle x_2, y_2, z_2\rangle \geq \langle x_1, y_1, z_1\rangle$, and so $\langle x_1, y_1, z_1\rangle$ is the smallest Boolean triple containing $\langle x, y, z\rangle$.

(iv) $M_3\langle L\rangle \neq \varnothing$; for instance, for all $x \in L$, the diagonal element $\langle x, x, x\rangle \in M_3\langle L\rangle$. It is obvious from (ii) that $M_3\langle L\rangle$ is meet closed. By (iii), $M_3\langle L\rangle$ is a closure system in L^3, from which the formulas of (iv) follow.

The proofs of (v) and (vi) are left to the reader. □

4. Proof of the theorem

Let L be a lattice with more than one element. We identify $x \in L$ with the diagonal element $\langle x, x, x\rangle \in M_3\langle L\rangle$, so we regard $M_3\langle L\rangle$ an extension of L. This is an embedding of L into $M_3\langle L\rangle$ different from the embedding in Lemma 1.(v). Moreover, the embedding in Lemma 1.(v) requires that L have a zero, while the embedding discussed here always works.

Note that $M_3\langle L\rangle$ is a proper extension; indeed, since L has more than one element, we can choose the elements $a < b$ in L. Then $\langle a, a, b\rangle \in M_3\langle L\rangle$ but $\langle a, a, b\rangle$ is not on the diagonal, so $\langle a, a, b\rangle \in M_3\langle L\rangle - L$. In fact, if $L = \mathfrak{C}_2$, the two-element chain, then this is the only type of nondiagonal element:

$$M_3\langle \mathfrak{C}_2\rangle = \{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}.$$

For a congruence Θ of L, let Θ^3 denote the congruence of L^3 defined componentwise. Let $M_3\langle\Theta\rangle$ be the restriction of Θ^3 to $M_3\langle L\rangle$.

Lemma 2. *$M_3\langle\Theta\rangle$ is a congruence relation of $M_3\langle L\rangle$.*

Proof. $M_3\langle\Theta\rangle$ is obviously an equivalence relation on $M_3\langle L\rangle$. Since $M_3\langle L\rangle$ is a meet subsemilattice of L^3, it is clear that $M_3\langle\Theta\rangle$ satisfies the Substitution Property for meets. To verify for $M_3\langle\Theta\rangle$ the Substitution Property for joins, let $\langle x_0, y_0, z_0\rangle$, $\langle x_1, y_1, z_1\rangle \in M_3\langle L\rangle$, let

$$\langle x_0, y_0, z_0\rangle \equiv \langle x_1, y_1, z_1\rangle \quad (M_3\langle\Theta\rangle),$$

(that is,

$$x_0 \equiv x_1 \quad (\Theta), \quad y_0 \equiv y_1 \quad (\Theta), \quad \text{and} \quad z_0 \equiv z_1 \quad (\Theta)$$

in L) and let $\langle u, v, w\rangle \in M_3\langle L\rangle$. Set

$$\langle x_i', y_i', z_i'\rangle = \langle x_i, y_i, z_i\rangle \vee \langle u, v, w\rangle$$

(the join formed in $M_3\langle L\rangle$), for $i = 0$, 1.

Then, using Lemma 1.(iii) and (iv) for $x_0 \vee u$, $y_0 \vee v$, and $z_0 \vee w$, we obtain that

$$\begin{aligned} x_0' &= (x_0 \vee u \vee y_0 \vee v) \wedge (x_0 \vee u \vee z_0 \vee w) \\ &\equiv (x_1 \vee u \vee y_1 \vee v) \wedge (x_1 \vee u \vee z_1 \vee w) = x_1' \ (M_3\langle\Theta\rangle), \end{aligned}$$

and similarly, $y_0' \equiv y_1' \ (M_3\langle\Theta\rangle)$, $z_0' \equiv z_1' \ (M_3\langle\Theta\rangle)$, hence

$$\langle x_0, y_0, z_0\rangle \vee \langle u, v, w\rangle \equiv \langle x_1, y_1, z_1\rangle \vee \langle u, v, w\rangle \quad (M_3\langle\Theta\rangle).$$

□

Since L was identified with the diagonal of $M_3\langle L\rangle$, it is obvious that $M_3\langle\Theta\rangle$ restricted to L is Θ. So to complete the proof of Theorem 2, it is sufficient to verify the following statement:

Lemma 3. *Every congruence of $M_3\langle L\rangle$ is of the form $M_3\langle\Theta\rangle$, for a suitable congruence Θ of L.*

Proof. Let Φ be a congruence of $M_3\langle L\rangle$, and let Θ denote the congruence of L obtained by restricting Φ to the diagonal of $M_3\langle L\rangle$, that is, $x \equiv y$ (Θ) in L iff $\langle x, x, x\rangle \equiv \langle y, y, y\rangle$ (Φ) in $M_3\langle L\rangle$. We prove that $\Phi = M_3\langle\Theta\rangle$.

To show that $\Phi \subseteq M_3\langle\Theta\rangle$, let

$$\langle x_0, y_0, z_0\rangle \equiv \langle x_1, y_1, z_1\rangle \quad (\Phi). \tag{1}$$

Define

$$o = x_0 \wedge x_1 \wedge y_0 \wedge y_1 \wedge z_0 \wedge z_1, \tag{2}$$

$$i = x_0 \vee x_1 \vee y_0 \vee y_1 \vee z_0 \vee z_1. \tag{3}$$

Meeting the congruence (1) with $\langle i, o, o \rangle$ yields

$$\langle x_0, o, o \rangle \equiv \langle x_1, o, o \rangle \quad (\Phi). \tag{4}$$

Since

$$\langle x_0, o, o \rangle \vee \langle o, o, i \rangle = \overline{\langle x_0, o, i \rangle} = \langle x_0, x_0, i \rangle,$$

joining the congruence (4) with $\langle o, o, i \rangle$ yields

$$\langle x_0, x_0, i \rangle \equiv \langle x_1, x_1, i \rangle \quad (\Phi). \tag{5}$$

Similarly,

$$\langle x_0, i, x_0 \rangle \equiv \langle x_1, i, x_1 \rangle \quad (\Phi). \tag{6}$$

Now we meet the congruences (5) and (6) to obtain

$$\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \quad (\Theta^3) \tag{7}$$

in L^3, proving that $\Phi \subseteq M_3\langle \Theta \rangle$.

To prove the converse, $M_3\langle \Theta \rangle \subseteq \Phi$, take

$$\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \quad (M_3\langle \Theta \rangle) \tag{8}$$

in $M_3\langle L \rangle$, that is,

$$\begin{aligned} x_0 &\equiv x_1 \quad (\Theta),\\ y_0 &\equiv y_1 \quad (\Theta),\\ z_0 &\equiv z_1 \quad (\Theta) \end{aligned}$$

in L. Equivalently,

$$\langle x_0, x_0, x_0 \rangle \equiv \langle x_1, x_1, x_1 \rangle \quad (\Phi), \tag{9}$$

$$\langle y_0, y_0, y_0 \rangle \equiv \langle y_1, y_1, y_1 \rangle \quad (\Phi), \tag{10}$$

$$\langle z_0, z_0, z_0 \rangle \equiv \langle z_1, z_1, z_1 \rangle \quad (\Phi) \tag{11}$$

in $M_3\langle L \rangle$.

Now, define o, i as in (2) and (3). Meeting the congruence (9) with $\langle i, o, o \rangle$, we obtain

$$\langle x_0, o, o \rangle \equiv \langle x_1, o, o \rangle \quad (\Phi). \tag{12}$$

Similarly, from (10) and (11), we obtain the congruences

$$\langle o, y_0, o \rangle \equiv \langle o, y_1, o \rangle \quad (\Phi), \tag{13}$$

$$\langle o, o, z_0 \rangle \equiv \langle o, o, z_1 \rangle \quad (\Phi). \tag{14}$$

Finally, joining the congruences (12)–(14), we get

$$\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \quad (\Phi), \tag{15}$$

that is, $M_3\langle \Theta \rangle \subseteq \Phi$. This completes the proof of this lemma and of Theorem 2. $\square$

5. Discussion

Special extensions. We can get a slightly stronger result by requiring that the extension preserve the zero and the unit, provided they exist. To state this result, we need the following concept.

An extension K of a lattice L is *extensive*, provided that the convex sublattice of K generated by L is K.

Note that if L has a zero, 0, then an extensive extension is a $\{0\}$-extension (and similarly for the unit, 1); if L has a zero, 0, and unit 1, then an extensive extension is a $\{0, 1\}$-extension.

Theorem 4. *Every lattice L with more than one element has a proper congruence-preserving extensive extension K.*

Proof. Indeed, every $\langle x, y, z\rangle \in M_3\langle L\rangle$ is in the convex sublattice generated by L since

$$\langle x \wedge y \wedge z, x \wedge y \wedge z, x \wedge y \wedge z\rangle \leq \langle x, y, z\rangle \leq \langle x \vee y \vee z, x \vee y \vee z, x \vee y \vee z\rangle. \qquad \square$$

In Theorem 3.(iii), we pointed out that $M_3[D]$ is a congruence-preserving extension of $D = \overline{D}$, where $\overline{D}$ is an ideal of $M_3[D]$. This raises the question whether Theorem 2 can be strengthened by requiring that L be an ideal in K. This is easy to do, if L has a zero, 0, since then we can identify $x \in L$ with $\langle x, 0, 0\rangle \in M_3\langle L\rangle$.

Theorem 5. *Every lattice L with more than one element has a proper congruence-preserving extension K with the property that L is an ideal in K.*

Proof. Take an element $a \in L$ such that $[a)$ (the dual ideal generated by a) has more than one element. Then by Lemma 1.(v), $A = M_3\langle [a)\rangle$ is a proper congruence-preserving extension of $[a)$ and $I = [a)$ is an ideal in A. Now form the lattice K by gluing L with the dual ideal $[a)$ to A with the ideal I. It is clear that K is a proper congruence-preserving extension of L. $\square$

Modularity and semimodularity. R. W. Quackenbush [8] proved that if L is a modular lattice, then $M_3[L]$ is a semimodular lattice. For our construction, the analogous result fails: $M_3\langle P\rangle$ is not semimodular, where P is a projective plane (a modular lattice). Indeed, let a, b, c be a triangle in P, with sides l, m, n, that is, let l, m, n be three distinct lines in the plane P, and define the points $a = n \wedge m$, $b = n \wedge l$, $c = m \wedge l$. Let p be a point in P not on any one of these lines. Then $\langle p, \varnothing, \varnothing\rangle$ is an atom in $M_3\langle P\rangle$, $\langle a, b, c\rangle \in M_3\langle P\rangle$ but

$$\langle \{p\}, \varnothing, \varnothing\rangle \vee \langle a, b, c\rangle = \overline{\langle p \vee a, b, c\rangle} = \langle P, l, l\rangle$$

and

$$\langle a, b, c\rangle < \langle n, b, l\rangle < \langle P, l, l\rangle,$$

showing that $M_3\langle P\rangle$ is not semimodular.

Now we characterize when $M_3\langle L\rangle$ is modular.

Theorem 6. *Let L be a lattice with more than one element. Then $M_3\langle L\rangle$ is modular iff L is distributive.*

Proof. If L is distributive, then $M_3\langle L\rangle = M_3[L]$, so $M_3\langle L\rangle$ is modular by Theorem 3.

Conversely, if $M_3\langle L\rangle$ is modular, then L is modular since it is a sublattice of $M_3\langle L\rangle$. Now if L is not distributive, then L contains an $M_3 = \{o, a, b, c, i\}$ as a sublattice. By Lemma 1.(vi), the elements

$$\langle o, o, a\rangle, \langle o, c, a\rangle, \langle c, c, i\rangle, \langle i, i, i\rangle, \langle b, o, a\rangle$$

belong to $M_3\langle L\rangle$. Obviously,

$$\langle o, o, a\rangle < \langle o, c, a\rangle < \langle c, c, i\rangle < \langle i, i, i\rangle$$

and

$$\langle o, o, a\rangle < \langle b, o, a\rangle < \langle i, i, i\rangle.$$

To prove that these five elements form an N_5, it is enough to prove that

$$\langle c, c, i\rangle \wedge \langle b, o, a\rangle = \langle o, o, a\rangle$$

and

$$\langle o, c, a\rangle \vee \langle b, o, a\rangle = \langle i, i, i\rangle.$$

The meet is obvious. Now the join:

$$\langle o, c, a\rangle \vee \langle b, o, a\rangle = \overline{\langle b, c, a\rangle} = \langle i, i, i\rangle.$$

So $M_3\langle L\rangle$ contains N_5 as a sublattice, contradicting the assumption that $M_3\langle L\rangle$ is modular. Therefore, L is distributive. $\square$

Further results. $M_3[L]$ is not a lattice for a general L. See, however, G. Grätzer and F. Wehrung [6], where a new concept of n-modularity is introduced, for any natural number n. Modularity is the same as 1-modularity.

By definition, n-modularity is an identity; for larger n, a weaker identity. For an n-modular lattice L, $M_3[L]$ is a lattice, a congruence-preserving extension of L.

For distributive lattices (in fact, for n-modular lattices), the construction $M_3[L]$ is a special case of the tensor product construction of two semilattices with zero, see, for instance, G. Grätzer, H. Lakser, and R. W. Quackenbush [2] and R. W. Quackenbush [8]. The $M_3\langle L\rangle$ construction is generalized in G. Grätzer and F. Wehrung [7] to two bounded lattices; the new construction is called *box product.* Some of the arguments of this paper carry over to box products.

Problems

Lattices. As usual, let us denote by $\mathbf{T}$, $\mathbf{D}$, $\mathbf{M}$, and $\mathbf{L}$ the variety of one-element, distributive, modular, and all lattices, respectively. A variety $\mathbf{V}$ is *nontrivial* if $\mathbf{V} \neq \mathbf{T}$.

Let us say that a variety $\mathbf{V}$ of lattices has the *Congruence Preserving Extension Property* (CPEP, for short), if every lattice in $\mathbf{V}$ with more than one element has a proper congruence-preserving extension in $\mathbf{V}$. It is easy to see that no finitely generated lattice variety has CPEP. (Indeed, by Jónsson's lemma, a nontrivial finitely generated lattice variety $\mathbf{V}$ has a finite maximal subdirectly irreducible member L; if K is a proper congruence-preserving extension of L, then K is also subdirectly irreducible and $|L| > |K|$, a contradiction.) In particular, $\mathbf{D}$ does not have CPEP.

Theorem 2 can be restated as follows: $\mathbf{L}$ has CPEP.

Problem 1. Find all lattice varieties $\mathbf{V}$ with CPEP. In particular, does $\mathbf{M}$ have CPEP?

Groups. Let us say that a variety $\mathbf{V}$ of groups has the *Normal Subgroup Preserving Extension Property* (NSPEP, for short), if every group G in $\mathbf{V}$ with more than one element has a proper supergroup $\overline{G}$ in $\mathbf{V}$ with the following property: every normal subgroup N in G can be uniquely represented in the form $\overline{N} \cap G$, where $\overline{N}$ is a normal subgroup of $\overline{G}$.

Not every group variety $\mathbf{V}$ has NSPEP, for instance, the variety $\mathbf{A}$ of Abelian groups does not have NSPEP.

Problem 2. Does the variety $\mathbf{G}$ of all groups have NSPEP? Find all group varieties having NSPEP?

Rings. For ring varieties, we can similarly introduce the *Ideal Preserving Extension Property* (IPEP, for short). The variety $\mathbf{R}$ of all (not necessarily commutative) rings has IPEP. Indeed, if R is a ring with more than one element, then embed R into $M_2(R)$ (the ring of 2×2 matrices over R) with the diagonal map. The two-sided ideals of $M_2(R)$ are of the form $M_2(I)$, where I is a two-sided ideal of R, and $I = M_2(I) \cap R$.

Problem 3. Find all ring varieties having IPEP? In particular, does the variety of all commutative rings have IPEP?

The second author found a positive answer for Dedekind domains: every Dedekind domain with more than one element has a proper ideal-preserving extension that is, in addition, a principal ideal domain.

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