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PROPER EFFICIENCY AND DUALITY FOR VECTOR VALUED OPTIMIZATION PROBLEMS

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Abstract

The duality results of Wolfe for scalar convex programming problems and some of the more recent duality results for scalar nonconvex programming problems are extended to vector valued programs. Weak duality is established using a 'Pareto' type relation between the primal and dual objective functions.

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1. Introduction

The scalar optimization problem may be expressed as

(P) minimize $f(x)$ subject to $g(x) \leqslant 0$

where f: $\mathbb{R}^n \to \mathbb{R}$ and g: $\mathbb{R}^n \to \mathbb{R}^m$. If f and g are differentiable functions the duality results of Wolfe [23] are well known and widely quoted. Wolfe showed that the problem

(D) maximize $f(u) + y^t g(u)$
subject to $\nabla f(u) + \nabla y^t g(u) = 0, \; y \geqslant 0$

is a dual to (P) if f and g are convex.

Various other approaches to duality for the scalar convex program (P) have also been widely studied; these include Lagrangian convex duality, see Geoffrion [8]; the use of the conjugate convex function concept, see Fenchel [6], Whinston

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[20], Rockafellar [16]; and the use of minimax theory, see Stoer [17], Mangasarian and Ponstein [13]. Some of these methods have been used as well in dealing with nonconvex programs see Crouziex [5] and Luenberger [10].

Many of the duality results for the scalar optimization problem have been extended to the vector optimization problem which may be expressed as

(PV) minimize $f(x)$ subject to $g(x) \leqslant 0$

where $f\colon \mathbb{R}^n \to \mathbb{R}^k$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are both real vector valued functions. This is the problem of finding the set of efficient or Pareto [15] optimal points for (PV): x_0 is said to be efficient if it is feasible for (PV) and there exists no other feasible point x such that $f(x) \leqslant f(x_0)$ and $f(x) \neq f(x_0)$.

The concept of proper efficiency is a slightly restricted definition of efficiency which eliminates efficient points of certain anomalous type: x_0 is said to be properly efficient if it is efficient for (PV) and if there exists a scalar $M > 0$ such that, for each i,

$$\frac{f_i(x_0) - f_i(x)}{f_j(x) - f_j(x_0)} \leqslant M$$

for some j such that $f_j(x) > f_j(x_0)$ whenever x is feasible for (PV) and $f_i(x) < f_i(x_0)$; thus unbounded tradeoffs between the various objectives $\{f_i\}$ are not allowed. An efficient point that is not properly efficient is said to be improperly efficient.

By studying a natural generalization of the scalar Lagrangian,

$$f(x) + y^t g(x) e \quad \left(e = (1, 1, \ldots, 1)^t \in \mathbb{R}^k\right),$$

Tanino and Sawaragi [18] (for properly efficient points) and White [22] have developed a Lagrangian duality theory for convex (PV), generalizing that of the scalar case. In addition, Tanino and Sawaragi [19] have extended the scalar Fenchel duality theory to the vector valued optimization problem. Using Lagrangians incorporating matrix Lagrange multipliers, Bitran [2], Corely [3], Craven [4] and Ivanov and Nehse [9] have also given duality results for (PV).

In this paper, using the Lagrangian $f(x) + y^t g(x)e$, some of the duality results of Wolfe [23], Bector *et al.* [1], Mahajan and Vartak [11] and Mond and Weir [14] are extended to the vector optimization problem. The relation for weak duality comparing the objective functions of the primal and dual may be loosely described as 'Pareto' and is similar to that given in [18].

2. Duality for convex programs

Before proceeding to duality theorems we give a definition of duality for vector valued optimization problems and some preliminary results.

DEFINITION. Given a vector valued optimization problem

(P^0) minimize $\psi(x)$ subject to $x \in F$

where $\psi: \mathbb{R}^n \to \mathbb{R}^k$ and $F \subseteq \mathbb{R}^n$ we define the problem

(D^0) maximize $\phi(x)$ subject to $x \in G$

where $\phi: \mathbb{R}^n \to \mathbb{R}^k$ and $G \subseteq \mathbb{R}^n$ to be a dual for (P^0) if

(i) (Weak Duality). Whenever x is feasible for (P^0) and u is feasible for (D^0)

$$\psi(x) \geqslant_P \phi(u)$$

where $\geqslant_P$ means there is no x feasible for (P^0) and u feasible for (D^0) such that for some $i \in \{1, 2, \ldots, k\}$

$$\psi_i(x) < \phi_i(u) \quad \text{and} \quad \psi_j(x) \leqslant \phi_j(u) \quad \text{for all } j \neq i.$$

(ii) (Strong Duality). If (P^0) has a properly efficient point x_0 then (D^0) has a properly efficient point u_0 and $\psi(x_0) = \phi(u_0)$.

In Section 3 we will allow a slight weakening of this definition so that if (i) holds and (ii) holds with efficient u_0 rather than properly efficient u_0, then (D^0) will be called a dual for (P^0). Note that the definition of weak duality accords with definition of efficiency.

In relation to (PV) consider the scalar minimization problem

$(PV\lambda)$ minimize $\lambda^t f(x)$

(1) subject to $g(x) \leqslant 0$

where $\lambda \geqslant 0$, $\lambda^t e = 1$, $e = (1, 2, \ldots, 1)^t \in \mathbb{R}^k$. Geoffrion [7] has established the following results:

LEMMA 2.1. (i) *Let* $\lambda_i > 0$ $(i = 1, 2, \ldots, k)$ *be fixed. If* x_0 *is optimal in* $(PV\lambda)$, *then* x_0 *is properly efficient in* (PV).

(ii) *If* f *and* g *are convex, then* x_0 *is properly efficient in* (PV) *if and only if* x_0 *is optimal in* $(PV\lambda)$ *for some* $\lambda > 0$.

REMARK 2.2. Note that in part (i) of Lemma 2.1 no convexity hypothesis is made. Similar results may be established for the problems

maximize $f(x)$ subject to $g(x) \leqslant 0$,

maximize $\lambda^t f(x)$ subject to $g(x) \leqslant 0$,

where $\lambda > 0$, $\lambda^t e = 1$, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^k$.

In relation to $(PV\lambda)$ consider the scalar maximization problem

$(DV\lambda)$ maximize $\lambda^t f(xu) + y^t g(u) \equiv \lambda^t(f(u) + y^t g(u) e)$

subject to $\nabla \lambda^t f(u) + \nabla y^t g(u) = 0$, $y \geqslant 0$

where $\lambda > 0$, $\lambda^t e = 1$, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^k$.

The results of Wolfe [23] establish the following theorem showing that $(DV\lambda)$ is a dual to $(PV\lambda)$ (for fixed λ).

THEOREM 2.3. *Let $\lambda > 0$ be fixed. If x is feasible for $(PV\lambda)$ and (u, y) feasible for $(DV\lambda)$ and if f and g are convex for all feasible (x, u, y) then*

$$\lambda^t f(x) \geqslant \lambda^t f(u) + y^t g(u).$$

Further, if x_0 is a local or global optimal solution for $(PV\lambda)$ and if a constraint qualification [12] *is satisfied at x_0, then there exists $y \geqslant 0$ such that $y^t g(x_0) = 0$ and (x_0, y) are global optimal solutions for $(PV\lambda)$ and $(DV\lambda)$ respectively.*

Now in relation to (PV) consider the vector valued optimization problem

(DV) maximize $f(u) + y^t g(u)e$ subject to

(2) $\nabla\lambda^t f(u) + \nabla y^t g(u) = 0$

(3) $y \geqslant 0,\ \lambda > 0,\ \lambda^t e = 1.$

THEOREM 2.4. (Weak Duality). *Let x be feasible for (PV) and (u, λ, y) feasible for (DV). If f and g are convex for all feasible (x, u, λ, y) then*

$$f(x) \geqslant_P f(u) + y^t g(u)e.$$

PROOF.

$$\begin{aligned}\lambda^t\{f(x) - (f(u) + y^t g(u)e)\} &= \lambda^t(f(x) - f(u)) - y^t g(u) \quad \text{(from (3))}\\ &\geqslant (x-u)^t \nabla\lambda^t f(u) - y^t g(u) \quad \text{(by convexity of } f)\\ &\geqslant (x-u)^t\{\nabla\lambda^t f(u) + \nabla y^t g(u)\} - y^t g(x) \quad \text{(by convexity of } g)\\ &= -y^t g(x) \quad \text{(by (2))}\\ &\geqslant 0 \quad \text{(by (1) and (3))}.\end{aligned}$$

Thus $f(x) \geqslant_P f(u) + y^t g(u)e$.

THEOREM 2.5. (Strong Duality). *Let f and g be convex and let x_0 be a properly efficient solution for (PV) at which a constraint qualification is satisfied. Then there exist (λ, y) such that (x_0, λ, y) is a properly efficient solution of (DV) and the objective values of (PV) and (DV) are equal.*

PROOF. Since f and g are convex and x_0 is a properly efficient solution of (PV) then, by Lemma 2.1 part (ii), x_0 is optimal for $(PV\lambda)$ for some $\lambda > 0$, $\lambda^t e = 1$. Since, also, a constraint qualification is satisfied at x_0 then, by Theorem 2.3, there is a $y \geqslant 0$ such that (x_0, λ, y) is optimal for $(DV\lambda)$. Since (x_0, λ, y) is

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optimal for $(DV\lambda)$, by Lemma 2.1 part (i) and Remark 2.2, (x_0, λ, y) is properly efficient for (DV). Clearly the optimal values of (PV) and (DV) are equal since $y^t g(x_0) = 0$.

REMARK 2.6. (i) If $k = 1$, then (PV) and (DV) becomes the scalar minimization problem (P) and its Wolfe dual (D), with duality holding if f and g are convex.

(ii) White [21] establishes the result that if f and g are convex and the Kuhn-Tucker constraint qualification is satisfied on the feasible set then every efficient point of (PV) is properly efficient and thus Theorems 2.4 and 2.5 could be strengthened. Unfortunately this result appears to be in error as the following simple counterexample shows:

$$\text{minimize } f(x) = (f_1(x), f_2(x))^t = (x, x^2)^t \text{ subject to } g(x) = x \leqslant 0;$$

clearly $x = 0$ is efficient; however the loss to gain ratio of f_1 to f_2 is $-1/x$ which, for feasible x of sufficiently small magnitude, can be made arbitrarily large.

The next result and corollary show that under certain conditions the dual objective function is (component-wise) unbounded from above. These are simple extensions of the corresponding results for the scalar minimization problem given by Mangasarian [12] and Wolfe [23].

THEOREM 2.7. *If there exists a dual feasible point (x, λ, y) such that the system*

$$g(x) + \nabla g(x) z \leqslant 0$$

has no solution $z \in \mathbb{R}^n$, then every component of the dual objective function is unbounded (from above) on the set of dual feasible points.

COROLLARY 2.8. *If the problem (DV) has a feasible point (x, λ, y) and if (PV) has no feasible point, then, if g is concave at x or linear, the dual problem (DV) has a (component-wise) unbounded objective function (from above) on the set of dual feasible points.*

Finally, for problems with linear functions, we give a theorem which tells when the primal problem (PV) has no efficient solutions.

THEOREM 2.9. *Let f and g be linear functions on $\mathbb{R}^n$, and let the feasible set of (PV) be nonempty. If the dual problem (DV) has no feasible point, (PV) has no solution.*

PROOF. A proof similar to that of Mangasarian [11] shows that, for $\lambda > 0$ and $\lambda^t e = 1$,

$$g(x+z) \leqslant 0 \quad \text{and} \quad \lambda^t f(x+z) < \lambda^t f(x)$$

for $\|z\|$ sufficiently small and x feasible for (PV). Hence there is no solution to $(PV\lambda)$ for each $\lambda > 0$, $\lambda^t e = 1$. Thus, by Lemma 2.1 (ii), there are no properly efficient solutions for (PV). For problems with linear functions every efficient point is properly efficient; hence there is no solution to (PV).

3. Duality for nonconvex programs

In this section some of the duality results for scalar minimization problems given by Bector *et al.* [1], Mahajan and Vartak [11] and Mond and Weir [14] are extended to the vector valued optimization problem. However, the results in this section are not as strong as that given in Section 2 because the dual optimal solution is not guaranteed to be properly efficient, but only efficient, whenever the solution to the primal is properly efficient.

Consider again (PV) and (DV).

THEOREM 3.1. (Weak Duality). *Let x be feasible for (PV) and (u, λ, y) feasible for (DV). If, for all feasible (x, u, λ, y), $f + y^t g e$ is pseudoconvex then*

$$f(x) \geqslant_P f(u) + y^t g(u) e.$$

PROOF. Suppose to the contrary that there is x feasible for (PV) and (u, λ, y) feasible for (DV) such that $f_i(x) < f_i(u) + y^t g(u)$ for some $i \in \{1, 2, \ldots, k\}$ and $f_j(x) \leqslant f_j(u) + y^t g(u)$ for all $j \neq i$. then

$$f_i(x) + y^t g(x) < f_i(u) + y^t g(u) \quad \text{(by (1) and (3))}$$

and

$$f_j(x) + y^t g(x) \leqslant f_j(u) + y^t g(u), \quad j \neq i \quad \text{(by (1) and (3))}.$$

Since $f + y^t g e$ is pseudoconvex then $(x-u)^t \nabla (f_i(u) + y^t g(u)) < 0$ and $(x-u)^t \nabla (f_j(u) + y^t g(u)) \leqslant 0$ for $j \neq i$. Thus

$$(x-u)^t \left(\nabla \lambda^t f(u) + \nabla y^t g(u) \right) < 0 \quad \text{(by (3))}$$

contradicting the constraint (2) of (DV). Thus $f(x) \geqslant_P f(u) + y^t g(u) e$.

THEOREM 3.2. (Strong Duality). *Let x_0 be a properly efficient solution for (PV) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist (λ, y) such that (x_0, λ, y) is feasible for (DV) and the objective values of (PV) and (DV) are equal. If, also, $f + y^t ge$ is pseudoconvex then (x_0, λ, y) is efficient for (DV).*

PROOF. Assuming that the Kuhn-Tucker constraint qualification [12] is satisfied at x_0, then by the Kuhn-Tucker necessary conditions [7], there exists $\lambda > 0$, $\lambda^t e = 1$, $y \geqslant 0$ such that

$$\nabla \lambda^t f(x_0) + \nabla y^t g(x_0) = 0, \qquad y^t g(x_0) = 0.$$

Thus (x_0, λ, y) is feasible for (DV) and clearly the objective values of (PV) and (DV) are equal.

If (x_0, λ, y) is not an efficient solution of (DV) then there exists feasible (u^*, λ^*, y^*) for (DV) such that

$$f_i(u^*) + y^{*t} g(u^*) > f_i(x_0) + y^{*t} g(x_0) \quad \text{for some } i \in \{1, 2, \dots, k\}$$

and

$$f_j(u^*) + y^{*t} g(u) \geqslant f_j(x_0) + y^{*t} g(x_0) \quad \text{for } j \neq i.$$

Since $f + y^t ge$ is pseudoconvex $(x_0 - u^*)^t \nabla (f_i(u^*) + y^{*t} g(u^*)) < 0$ and $(x_0 - u^*)^t \nabla (f_j(u^*) + y^{*t} g(u^*)) \leqslant 0$ for $j \neq i$. Thus

$$(x_0 - u^*)^t (\nabla \lambda^{*t}(u^*) + \nabla y^{*t} g(u^*)) < 0$$

contradicting the feasibility of (u^*, λ^*, y^*). Thus (x_0, λ, y) is an efficient solution of (DV).

REMARK 3.3. *Theorems* 3.1 *and* 3.2 *extend the results of Bector et al.* [1] *and Mahajan and Vartak* [11] *for the scalar minimization problem to the vector valued minimization problem.*

Mond and Weir [14] proposed a number of different duals to the scalar minimization problem (P). Here it is shown, as for the Wolfe dual, that there are analogous results for the vector valued optimization problem (PV).

In relation to (PV) consider the problem

$(DV1)$ maximize $f(u)$ *subject to*

(4) $\nabla \lambda^t f(u) + \nabla y^t g(u) = 0,$

(5) $y^t g(u) \geqslant 0$

(6) $y \geqslant 0,\ \lambda > 0,\ \lambda^t e = 1.$

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THEOREM 3.4. (Weak Duality). *Let x be feasible for (PV) and (u, λ, y) feasible for $(DV1)$. If for all feasible (x, u, λ, y) f is pseudoconvex and $y^t g$ is quasiconvex, then*

$$f(x) \geqslant_P f(u).$$

PROOF. Suppose to the contrary that there is x feasible for (PV) and (u, λ, y) feasible for $(DV1)$ such that $f_i(x) < f_i(u)$ for some $i \in \{1, 2, \ldots, k\}$ and $f_j(x) \leqslant f_j(u)$ for $j \neq i$. Since f is pseudoconvex $(x - u)^t \nabla f_i(u) < 0$ and $(x - u)^t \nabla f_j(u) \leqslant 0$ for $j \neq i$. Hence

$$(7) \qquad (x - u)^t \nabla \lambda^t f(u) < 0 \quad (\text{by } (6))$$

Also, from (1), (5), (6)

$$(8) \qquad y^t g(x) - y^t g(u) \leqslant 0$$

and since $y^t g$ is quasiconvex $(x - u)^t \nabla y^t g(u) \leqslant 0$. Combining (7) and (8) gives $(x - u)^t (\nabla f(u) + \nabla y^t g(u)) < 0$ contradicting the constraint (4) of $(DV1)$. Thus $(fx) \geqslant_P f(u)$.

THEOREM 3.5. (Strong Duality). *Let x_0 be a properly efficient solution for (PV) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists (x_0, λ, y) is feasible for $(DV1)$ and the objective values of (PV) and $(DV1)$ are equal. If also f is pseudoconvex and $y^t g$ is quasiconvex then (x_0, λ, y) is efficient for $(DV1)$.*

PROOF. Assuming that the Kuhn-Tucker constraint qualification [12] is satisfeid at x_0, then by the Kuhn-Tucker necessary conditions [7], there exists $\lambda > 0$, $\lambda^t e = 1$, $y \geqslant 0$ such that

$$\nabla \lambda^t f(x_0) + \nabla y^t g(x_0) = 0, \qquad y^t g(x_0) = 0.$$

Thus (x_0, λ, y) is feasible for $(DV1)$ and the objective values of (PV) and $(DV1)$ are equal since the objective functions are the same.

If (x_0, λ, y) is not an efficient solution for $(DV1)$ there exists feasible (u^*, λ^*, y^*) for $(DV1)$ such that $f_i(u^*) > f_i(x_0)$ for some $i \in \{1, 2, \ldots, k\}$ and $f_j(u^*) \geqslant f_j(x_0)$ for $j \neq i$. Since f is pseudoconvex $(x_0 - u^*)^t \nabla f_i(u^*) < 0$ and $(x_0 - u^*)^t \nabla f_j(u^*) \leqslant 0$. Thus

$$(9) \qquad (x_0 - u^*)^t \nabla \lambda^{*t} f(u^*) < 0 \quad (\text{by } (6)).$$

Also $y^{*t} g(x_0) - y^t g(u^*) \leqslant 0$ (by (1), (5), (6)) and since $y^{*t} g$ is quasiconvex

$$(10) \qquad (x_0 - u^*)^t \nabla y^{*t} g(u^*) \leqslant 0.$$

Combining (9) and (10) gives

$$(x_0 - u^*)^t(\nabla\lambda^{*t}f(u^*) + \nabla y^{*t}g(u^*)) < 0$$

contradicting the feasibility of (u^*, λ^*, y^*).

In a manner similar to that given in [14] we state a general dual for the vector value optimization problem. For completeness we shall consider the case where the primal problem has equality as well as inequality constraints.

Consider the problem:

(PEV) minimize $f(x)$ subject to

$$g(x) \leqslant 0, \; h(x) = 0 \tag{11}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^k$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$, $h\colon \mathbb{R}^n \to \mathbb{R}^l$ are all differentiable.

Let $M = \{1,2,\ldots,m\}$, $L = \{1,2,\ldots,l\}$, $I_\alpha \subseteq M$, $\alpha = 0,1,2,\ldots,r$, with $I_\alpha \cap I_\beta = \varnothing$, $\alpha \neq \beta$, and $\bigcup_{\alpha=0}^r I_\alpha = M$ and $J_\alpha \subseteq L$, $\alpha = 0,1,2,\ldots,r$, with $J_\alpha \cap J_\beta = \varnothing$, $\alpha \neq \beta$, and $\bigcup_{\alpha=0}^r J_\alpha = L$.

Note that any particular I_α or J_α may be empty. Thus if M has r_1 disjoint subsets and L has r_2 disjoint subsets, $r = \operatorname{Max}[r_1, r_2]$. So that if $r_1 > r_2$, then J_α, $\alpha > r_1$ is empty.

In relation to (PEV) consider the problem:

(DEV) Maximize $f(u) + \sum_{i\in I_0} y_i g_i(u)e + \sum_{j\in J_0} z_j h_j(u)e$ subject to

$$\nabla\lambda^t f(u) + \nabla y^t g(u) + \nabla z^t h(u) = 0 \tag{12}$$

$$\sum_{i\in I_\alpha} y_i g_i(u) + \sum_{j\in J_\alpha} z_j h_j(u) \geqslant 0, \qquad \alpha = 1,2,\ldots,r, \tag{13}$$

$$y \geqslant 0,\; \lambda > 0,\; \lambda^t e = 1. \tag{14}$$

THEOREM 3.6. (Weak Duality). *Let x be feasible for (PEV) and (u, λ, y, z) feasible for (DEV). If $f + \sum_{i\in I_\alpha} y_i g_i e + \sum_{j\in J_\alpha} z_j h_j e$ is pseudoconvex for all (x, u, λ, y, z) and if $\sum_{i\in I\alpha} y_i g_i + \sum_{j\in J_\alpha} z_j h_j$, $\alpha = 1,2,\ldots,r$, is quasiconvex for all feasible (x, u, λ, y, z), then*

$$f(x) \geqslant_P f(u) + \sum_{i\in I_0} y_i g_i(u)e + \sum_{j\in J_0} z_j h_j(u)e.$$

PROOF. Suppose to the contrary that there is x feasible for (PEV) and (u, λ, y, z) feasible for (DEV) such that

$$f_p(x) < f_p(u) + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j h_j(u) \quad \text{for some } p \in \{1,2,\ldots,k\}$$

and

$$f_q(x) \leqslant f_q(u) + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j h_j(u), \qquad q \neq p.$$

Since $f + \sum_{i \in I_0} y_i g_i e + \sum_{j \in J_0} z_j h_j e$ is pseudoconvex

$$(x-u)^t \nabla\Big(f_p(u) + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j h_j(u) \Big) < 0$$

and

$$(x-u)^t \nabla\Big(f_q(u) + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j h_j(u) \Big) \leqslant 0, \qquad q \neq p.$$

Thus, by (14),

$$(15) \qquad (x-u)^t \Big\{ \nabla \lambda^t f(u) + \nabla \sum_{i \in I_0} y_i g_i(u) + \nabla \sum_{j \in J_0} z_j h_j(u) \Big\} < 0.$$

Also, by (11), (13), (14),

$$\sum_{i \in I_\alpha} y_i g_i(x) + \sum_{j \in J_\alpha} z_j h_j(x) - \sum_{i \in I_\alpha} y_i g_i(u) - \sum_{j \in J_\alpha} z_j h_j(u) \leqslant 0, \qquad \alpha = 1, 2, \ldots, r.$$

Since $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j$ is quasiconvex, $\alpha = 1, 2, \ldots, r$, then

$$(x-u)^t \Big\{ \nabla \sum_{i \in I_\alpha} y_i g_i(u) + \nabla \sum_{j \in J_\alpha} z_j h_j(u) \Big\} \leqslant 0, \qquad \alpha = 1, 2, \ldots, r.$$

Thus

$$(16) \qquad (x-u)^t \Bigg\{ \sum_{i \in \bigcup_{\alpha=1}^{r} I_\alpha} \nabla y_i g_i(u) + \sum_{J \in \bigcup_{\alpha=1}^{r} J_\alpha} \nabla z_j h_j(u) \Bigg\} \leqslant 0$$

Combining (15) and (16) gives

$$(x-u)^t \{ \nabla \lambda^t f(u) + \nabla y^t g(u) + \nabla z^t h(u) \} < 0$$

which contradicts the constraint (12) of (DEV). Thus

$$f(x) \geqslant_P f(u) + \sum_{i \in I_0} y_i g_i(u) e + \sum_{j \in J_0} z_j h_j(u) e.$$

THEOREM 3.7. (Strong Duality). *Let x_0 be a properly efficient solution for (PEV) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists (λ, y, z) such that (x_0, λ, y, z) is feasible for (DEV) and the objective values of (PEV) and (DEV) are equal. If, also, $f + \sum_{i \in I_0} y_i g_i e + \sum_{j \in J_\alpha} z_j h_j e$ is pseudoconvex and if $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j$ is quasiconvex, $\alpha = 1, 2, \ldots, r$, then (x_0, λ, y, z) is efficient for (DEV).*

PROOF. Assuming that the Kuhn-Tucker constraint qualification [12] is satisfied at x_0, then by the Kuhn-Tucker necessary conditions [7], there exists $\lambda > 0$, $\lambda^t e = 1$, $y \geqslant 0$, z, such that

$$\nabla \lambda^t f(x_0) + \nabla y^t g(x_0) + \nabla z^t h(x_0) = 0, \qquad y^t g(x_0) = 0, \qquad z^t h(x_0) = 0.$$

Thus (x_0, λ, y, z) is feasible for (DEV) and the objective values of (PEV) and (DEV) are equal there.

If (x_0, λ, y, z) is not an efficient solution for (DEV) there exists feasible $(u^*, \lambda^*, y^*, z^*)$ for (DEV) such that

$$f_p(u^*) + \sum_{i \in I_0} y_i^* g_i(u^*) + \sum_{j \in J_0} z_j^* h_j(u^*) >$$
$$f_p(x_0) + \sum_{i \in I_0} y_i^* g_i(x_0) + \sum_{j \in J_0} z_j^* h_j(x_0) \quad \text{for some } p \in \{1, 2, \dots, k\}$$

and

$$f_q(u^*) + \sum_{i \in I_0} y_i^* g_i(u^*) + \sum_{j \in J_0} z_j^* h_j(u^*) \geqslant$$
$$f_q(x_0) + \sum_{i \in I_0} y_i^* g_i(x_0) + \sum_{j \in J_0} z_j^* h_j(x_0), \quad q \neq p.$$

The pseudoconvexity of $f + \sum_{i \in I_0} y_i g_i e + \sum_{j \in J_0} z_j h_j e$ implies that

$$(x_0 - u^*)^t \nabla \Big(f_p(u^*) + \sum_{i \in I_0} y_i^* g_i(u^*) + \sum_{j \in J_0} z_j^* h_j(u^*) \Big) < 0$$

and

$$(x_0 - u^*)^t \nabla \Big(f_q(u^*) + \sum_{i \in I_0} y_i^* g_i(u^*) + \sum_{j \in J_0} z_j^* h_j(u^*) \Big) < 0, \qquad q \neq p.$$

Thus

$$(17) \qquad (x_0 - u^*) \nabla \Big(\lambda^{*t} f(u^*) + \sum_{i \in I_0} y_i^* g_i(u^*) - \sum_{j \in J_0} z_j^* h_j(u^*) \Big) < 0.$$

Also, for $\alpha = 1, 2, \dots, r$,

$$\sum_{i \in I_\alpha} y_i^* g_i(x_0) + \sum_{j \in J_\alpha} z_j^* h_j(x_0) - \sum_{i \in I_\alpha} y_i^* g_i(u^*) - \sum_{j \in J_\alpha} z_j^* h_i(u^*) \leqslant 0.$$

Since $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j$ is quasiconvex, $\alpha = 1, 2, \dots, r$,

$$(x_0 - u^*)^t \nabla \Big(\sum_{i \in I_\alpha} y_i^* g_i(u^*) + \sum_{j \in J_\alpha} z_j^* h_j(u^*) \Big) \leqslant 0, \qquad \alpha = 1, 2, \dots, r,$$

and so

$$(18) \qquad (x_0 - u^*)^t \nabla \Bigg(\sum_{i \in \bigcup_{\alpha=1}^{r} I_\alpha} y_i^* g_i(u^*) + \sum_{j \in \bigcup_{\alpha=1}^{r} J_\alpha} z_j^* h_j(u^*) \Bigg) \leqslant 0.$$

Combining (17) and (18) gives

$$(x_0 - u^*)^t \big(\nabla \lambda^{*t} f(u^*) + \nabla y^{*t} g(u^*) + \nabla z^{*t} h(u^*) \big) < 0$$

contradicting the feasibility of $(u^*, \lambda^*, y^*, z^*)$ for (DEV).

Special cases of the pair (PEV) and (DEV) and their corresponding duality theorems follow in a similar fashion to those presented in [14] for scalar minimization problems.

4. Nonnegative variables

Consider now the vector valued problem with nonnegative variables

(PV') minimize $f(x)$
subject to $g(x) \leqslant 0,\ x \geqslant 0.$

Corresponding to the vector analog of the Wolfe dual given in Section 2 we have

(DV') maximize $f(u) + y^t g(u)e - u^t[\nabla \lambda^t f(u) + \nabla y^t g(u)]e$
subject to $\nabla \lambda^t f(u) + \nabla y^t g(u) \geqslant 0,\ y \geqslant 0,\ \lambda > 0,\ \lambda^t e = 1.$

Duality holds, relating properly efficient solutions of the primal and dual, if f and g are convex.

One can also show that if $f - y^t ge - s^t(\cdot)e$ for all $s \geqslant 0$, is pseudoconvex, then duality (in the sense used in Section 3) also holds. By applying some of the earlier results of Section 3 to (PV'), or directly, one can establish different dual problems to (PV') that will hold under certain convexity conditions. Two such duals are stated below.

$(DV1')$ maximize $f(u)$
subject to $\nabla \lambda^t f(u) + \nabla y^t g(u) \geqslant 0,\ y \geqslant 0,\ \lambda > 0,\ \lambda^t e = 1,$
$y^t g(u) - u^t[\nabla \lambda^t f(u) + \nabla y^t g(u)] \geqslant 0.$

$(DV2')$ maximize $f(u) + y^t g(u)e$
subject to $\nabla \lambda^t f(u) + \nabla y^t g(u) \geqslant 0,\ y \geqslant 0,\ \lambda > 0,\ \lambda^t e = 1,$
$u^t[\nabla \lambda^t f(u) + \nabla y^t g(u)] \leqslant 0.$

COROLLARY 4.1. (a) *If f is pseudoconvex and $y^t g + s^t(\cdot)$, $s \geqslant 0$, is quasiconvex for all feasible (x, u, λ, y) of (PV') and $(DV1')$, then*

$$f(x) \geqslant_P f(u).$$

(b) *If $f + y^t ge$ is pseudoconvex for all feasible (x, u, λ, y) of (PV') and $(DV2')$, then*

$$f(x) \geqslant_P f(u) + y^t g(u)e.$$

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The corollary, as well as the corresponding strong duality result, can be obtained directly, or by applying Theorems 3.6 and 3.7, respectively, to (PV') and then eliminating from the dual problems the multiplier corresponding to the constraints $x \geqslant 0$.

5. Examples

Consider again the example given in Section 2:
minimize $(x, x^2)^t$ subject to $x \leqslant 0$;
the objective and constraint function are convex and every $x < 0$ is properly efficient. The dual problem as given in Section 2 is

$$\begin{aligned}&\text{maximize } (x, x^2)^t + (yx, yx)^t\\&\text{subject to } \lambda_1 + 2\lambda_2 x + y = 0\\&\qquad \lambda_2 > 0,\ \lambda_1 > 0,\ \lambda_1 + \lambda_2 = 1,\ y \geqslant 0.\end{aligned}$$

The constraints give $x = -(y + \lambda_1)/2(1 - \lambda_1)$. Since $y \geqslant 0$ and $\lambda_1 \in (0, 1)$, $x < 0$. The properly efficient solutions of the objective are given by $x < \frac{1}{2}y$ for any $y \geqslant 0$. The global properly efficient maximal set then corresponds to $y = 0$; that is, $x < 0$. Furthermore, corresponding to every properly efficient $x_0 < 0$ for the primal is a properly efficient $(x_0, \lambda_1 = 2x_0/(2x_0 - 1), \lambda_2 = 1 - \lambda_1, y = 0)$ for the dual problem and the objective functions are equal there.

As another example consider the nonconvex problem

$$\text{minimize } (-e^{-x^2}, x^2)^t \text{ subject to } 1 - x \leqslant 0.$$

the only properly efficient point is $x = 1$. The dual as given in Section 3 is the problem

$$\begin{aligned}&\text{maximize } (-e^{-x^2}, x^2)\\&\text{subject to } 2\lambda_1 x e^{-x^2} + 2\lambda_2 x = y\\&\qquad y(1 - x) \geqslant 0\\&\qquad \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1,\ y \geqslant 0.\end{aligned}$$

The point $(x = 1, 0 < \lambda_1 < 1, 1 - \lambda_1, 2\lambda_1 e^{-1} + 2(1 - \lambda_1))$ is efficient for the dual problem. Weak duality holds, since the objective is pseudoconvex and the constraint linear, and the optimal values of the primal and dual are equal.

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