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PROPER HOLOMORPHIC IMAGES OF STRICTLY PSEUDOCONVEX DOMAINS.

Klas Diederich and John E. Fornæss
Universität Wuppertal, Mathematik
Inst. of Math., Univ. of Oslo

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Proper holomorphic images of strictly pseudoconvex domains.

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0. Introduction.

H. Poincaré showed for the first time that the ball in $\mathbb{C}^2$ and the bidisc are not biholomorphically equivalent. Later R. Remmert and K. Stein [11], G.M. Henkin [8] and A.T. Huckleberry [9] generalized this result more and more by considering larger classes of domains and also proper holomorphic mappings. In all their results the existence of local complex analytic foliations of parts of the boundary for one domain and some strict pseudoconvexity of the other boundary play an essential role.

On the other hand, there are well-known examples of proper holomorphic mappings f with non-empty branching locus from certain bounded, C^∞-smooth pseudoconvex domains Ω_1 onto strictly pseudoconvex domains Ω_2. But it has been conjectured that any proper holomorphic mapping $f: \Omega_1 \to \Omega_2$ is necessarily unbranched if Ω_1 is strictly pseudoconvex and Ω_2 is weakly speudoconvex and C^∞-smooth.

For both Ω_1 and Ω_2 being strictly pseudoconvex the conjecture has first been fully verified by S. Pinçuk [10] building on work of H. Alexander [1](see also D. Burns and St. Shnider [5] and W. Rudin [12]). If Ω_1 and Ω_2 as in the conjecture are in addition known to be complete Reinhardt domains, St. Bell [3] has confirmed the claim. In the case of real-analytic boundaries the result is contained in Bell [4].

In this paper we prove now

<u>Theorem 1.</u> Let $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be domains with C^∞-boundaries and Ω_1 strictly pseudoconvex. Then any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$ is unbranched and, therefore, extends to a C^∞-covering $\hat{f} : \bar{\Omega}_1 \to \bar{\Omega}_2$. In particular, f extends to an unbranched C^∞-covering $\hat{f} : \bar{\Omega}_1 \to \bar{\Omega}_2$ (because of [7]) and Ω_2 is also strictly pseudoconvex.

In section 1 we explain the notations and the relevant results of St. Bell [2] which are basic for our proof. In section 2 we find a generic branching point z_o of f on $b\Omega_1$ where the branching locus of f hits $b\Omega_1$ at z_o as a transverse manifold, and we show that f extends in a C^∞ way to $b\Omega_1$ near z_o. For section 3 we use this to show that the branching locus has to be empty. In section 4 we mention some more general results than the theorem above that can be derived with the same methods.

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1. Notations and tools.

For a C^∞-smooth domain $\Omega \subset \mathbb{C}^n$ we denote by $A^\infty(\Omega)$ the algebra of functions in $C^\infty(\bar{\Omega})$ holomorphic on Ω and we always will write u for the Jacobian determinant of the given mapping f. The following statement is a special case of theorem 2 (and it's proof) of St. Bell [2] and will be the basic tool in our proof of the theorem:

Proposition. In the situation of the theorem the function $u\cdot(h\circ f)\in A^{\infty}(\Omega_1)$ if $h\in A^{\infty}(\Omega_2)$. In particular, $u\in A^{\infty}(\bar{\Omega}_1)$.

2. C^{∞}-extension at generic branching points on $b\Omega_1$.

We assume that the branching locus $\hat{X}:=\{z\in\Omega_1 : u(z)=0\}$ of the mapping f is non-empty.

2.1. At first, we want to find sufficiently generic points on $b\Omega_1$ where, in particular, $\hat{X}$ hits $b\Omega_1$ transversally.

Along each connected component of the regular locus of $\hat{X}$ the function u has a well-defined constant order of vanishing. Let X_1 be one such component on which this order is minimal, say k. The set $X:=\bar{X}_1\cap\Omega_1$ is an irreducibel branch of $\hat{X}$.

There is a multiindex α, $|\alpha| = k-1$, such that the function

$$v := \frac{\partial^{\alpha} u}{\partial z^{\alpha}} \in A^{\infty}(\Omega_1)$$

vanishes along some non-empty relatively open set $U_1\subset X_1$ with order 1 and we can find an index β, $1\leq\beta\leq n$ such that

$$\frac{\partial v}{\partial z_\beta}(p) \neq 0$$

for some $p\in U_1$. Notice that $v|\hat{X}\equiv 0$. We put

$$S: = \bar{X}\cap b\Omega.$$

Because of the maximum principle applied to X there is a $q\in S$ with

$$\frac{\partial v}{\partial z_\beta}(q) \neq 0 \tag{1}$$

and we can extend v to a C^{∞}-function $\tilde{v}$ on an open neighborhood U of q such that $\bar{\partial}\tilde{v}$ vanishes to infinite order along $b\Omega_1 \cap U$. Because of (1) the set

$$Y: = \{z \in U : \tilde{v}(z) = 0\}$$

is a smooth C^{∞}-submanifold of U, if U was chosen small anough. Notice that $\hat{X} \cap U \subset Y$. Furthermore, after shrinking U again, we can write Y as a graph over its tangent space T_qY at q, which is complex, in the follwing way:
After a linear coordinate change we may assume that

$$T_qY = \{t = (t_1, \ldots, t_n) = (t', t_n) \in \mathbb{C}^n : t_n = 0\} = \mathbb{C}^{n-1}.$$

Let $\pi : \mathbb{C}^n \to T_qY$, $\pi((t', t_n)) = t'$, be the projection and $U' : = \pi(U)$. Then there is a C^{∞}-function $g : U' \to \mathbb{C}$ which is holomorphic on $\pi(Y \cap \Omega_1)$ and whose differential $\bar{\partial}g$ vanishes to infinite order along $\pi(Y \cap b\Omega_1)$ such that

$$Y = \{(z', g(z')) : z' \in U'\}.$$

Let now ρ be a strictly plurisubharmonic defining function of Ω_1 defined in a neighborhood of $\bar{\Omega}_1$ and put

$$\sigma(z'): = \rho((z', g(z'))) \quad \text{for} \quad z' \in U'.$$

Then, after shrinking U again, σ becomes a strictly plurisubharmonic function on U'.
We call $S' : = \pi(S \cap U)$ such that $\sigma | S' = 0$.

<u>Claim</u>: $d\sigma | S' \neq 0$. (2)

Suppose $d\sigma | S' = 0$. The Taylorexpansion of σ around $q' : = \pi(q)$ in real coordinates $z' = x' + iy'$ after a suitable linear change of coordinates has the form

$$\sigma = \sum_{j=1}^{n-1} (x_j^2 + \alpha_j y_j^2) + \text{higher order terms}$$

with $\alpha_j > -1$. Therefore, the set

$$\Sigma = \{z' : \frac{\partial\sigma}{\partial x_1}(z') = 0\}$$

is a real hypersurface in U' which can be supposed to divide U' into exactly two connected components. We choose a component intersecting $\pi(X \cap U)$ and call it Σ^-. Notice that $X \cap U \subset Y \cap \Omega_1$ is a closed subvariety and $\pi|Y$ is proper. Therefore, $\pi(X \cap U)$ is a closed subvariety of $\pi(Y \cap \Omega_1)$ of full dimension. The boundary of $\pi(X \cap U)$ in U' is S' and $S' \subset \Sigma$. Hence $\pi(X \cap U) \supset \Sigma^-$. This shows that $\sigma|\Sigma^- < 0$ such that the Hopf-lemma applied to σ at q' gives

$$d\sigma(q') \neq 0$$

contradicting the assumption $d\sigma|S' = 0$.
As a consequence of (2) we can now move q on S such that Y intersects $b\Omega_1$ at q transversally. This implies in particular because of the choice of v that

$$\hat{X} \cap U = Y \cap \Omega_1 \quad \text{(shrink } U \text{ if necessary)} \qquad (3)$$

2.2. Next we want to show that the mapping f can be extended in a C^∞ way to $b\Omega_1$ near q. Because of the proposition of Bell from section 1 applied to the coordinate functions $w_j \in A^\infty(\Omega_2)$ it is enough for this purpose to prove that the functions $u \circ f_j \in A^\infty(\Omega_1)$ can be divided near q by u without destroying the differentiability.

For this we choose suitable coordinates near q in the following way: we may assume that

$$\frac{\partial \tilde{v}}{\partial z_1}(q) \neq 0$$

such that

$$z_1^* = \tilde{v}(z)$$
$$z_j^* = z_j - z_j(q), \quad j = 2,\dots,n$$

is a C^∞-coordinate change holomorphic on $\Omega_1 \cap U$. It, therefore, does not destroy the strict pseudoconvexity of $\Omega_1 \cap U$ at $b\Omega_1$. We call the new coordinates again z and now have

$$Y = \{z \in U : z_1 = 0\} \tag{4}$$

Let now $g \in A^\infty(\bar{\Omega}_1)$ with $g|\hat{X} = 0$ be arbitrary and let $\tilde{g}$ be any C^∞-extension of g to U. We want to normalize this extension along Y in a suitable way by showing

Lemma 1. For any given integer $l > 0$ the extension $\tilde{g}$ can always be chosen in such a way that it's Taylor expansion at the points of Y in z_1, $\bar{z}_1$ has the form

$$\tilde{g} = \sum_{i=1}^{l} g_i z_1^i + R_l$$

where g_j are C^∞-functions on Y and

$$R_l = O(|z_1|^{l+1})$$

Proof. We choose a C^∞-retraction

$$\pi : U \to Y$$

$$\text{with } \pi(U \cap \bar{\Omega}_1) = \bar{\hat{X}} \cap U.$$

Let $\tilde{g}$ be an arbitrary C^∞-extension of g to U and define

$$\tilde{g}_1 := \tilde{g} - \tilde{g}\circ\pi .$$

If $\tilde{g}_r$ has been defined and

$$\tilde{g}_r = \sum_{i+j\leq r} g_{ij}^{(r)} z_1^i \bar{z}_1^j + R_r \tag{4}$$

is its Taylor series along Y with

$$R_r = O(|z_1|^{r+1}),$$

we put inductively:

$$\tilde{g}_{r+1} := \tilde{g}_r - \sum_{\substack{i+j\leq r\\ j\geq 1}} (g_{ij}^{(r)}\circ\pi) z_1^i \bar{z}_1^j .$$

Then the Taylor expansion of $\tilde{g}_{r+1}$ along Y has the shape f (4) for $r+1$ and $\tilde{g}_{l+1}$ satisfies the requirements of the lemma. □

A simple consequence of this is

<u>Lemma 2</u>. Let $g \in A^{\infty}(\Omega_1)$ be a function with $g|\hat{X} = 0$. Then the function $\hat{g} := g/z_1$ extends from $\Omega_1 \cap U$ to $b\Omega_1 \cap U$ in a C^{∞}-way.

<u>Proof</u>. Let α be any multiindex and $l > |\alpha|$ a positive integer. Choose an extension $\tilde{g}$ of g according to lemma 1 with this l. Then one obviously has near Y for $z_1 \neq 0$

$$\frac{\partial^{\alpha}(\tilde{g}/z_1)}{\partial z^{\alpha}} = O(|z_1|^{l-|\alpha|}) .$$

This proves the lemma. □

We now can easily prove

Lemma 3. If $q \in \bar{\hat{X}}$ has been chosen as in the beginning of this section, the mapping f extends in a C^∞-way to $b\Omega_1$ near q.

Proof. 1) Since u vanishes along $Y \cap \bar{\Omega}_1$ to the order k exactly, lemma 2 gives that

$$u|\Omega_1 \cap U = z_1^k \tilde{u} \tag{5}$$

with a holomorphic function $\tilde{u}$ which extends to $b\Omega_1 \cap U$ in a C^∞-way and such that

$$\tilde{u}(q) \neq 0. \tag{6}$$

2) The proposition of Bell from section 1 says

$$u \cdot f_j \in A^\infty(\Omega_1) \quad \text{for} \quad j = 1, \ldots, n.$$

(5) and (6) therefore imply that

$$g_j := z_1^k f_j$$

extends to $b\Omega_1 \cap U$ in a C^∞-way. Because of lemma 2 this must therefore also be true for f_j. □

3. Elimination of the branching.

In section 2 we worked under the assumption that the branching locus $\hat{X}$ of f is non-empty and we found the point q of lemma 3 in $\bar{\hat{X}} \cap b\Omega_1$. Hence we will have obtained a contradiction and, therefore, proved the theorem if we will have shown:

Lemma 4. Let $f : U_1 \to U_2$, $U_i \subset \mathbb{C}^n$ open, be a proper holomorphic mapping. Suppose, there are relatively open sets $M_i \subset bU_i$ where bU_i is a C^∞-smooth pseudo-convex hypersurface and let M_1 even be strictly pseudoconvex. Furthermore, suppose that f extends in a C^∞-way to $U_1 \cup M_1$ and that $f(M_1) \subset M_2$. Then f is unbranched near M_1 (and M_2 is strictly pseudoconvex at $f(M_1)$).

<u>Remark</u>. The statement is purely local at points of M_1. We, therefore, will shrink the U_i during the proof suitably without mentioning it explicitly.

<u>Proof</u>. We will use the transformation formula for a complex Monge-Ampere-equation in a way which is due to N. Kerzman, J.J. Kohn and L. Nirenberg. We call $p := f(q) \in M_2$. According to [6] we can choose a (local) C^∞-defining function ρ_2 of M_2 on U_2 such that

$$\psi_2 := -(-\rho_2)^{2/3}$$

is (strictly) plurisubharmonic on U_2. Define

$$\rho_1 := \rho_2 \circ f \in C^\infty(U_1 \cup M_1).$$

Then $\psi_1 := -(-\rho_1)^{2/3} = \psi_2 \circ f$ is negative and plurisubharmonic on U_1 and

$$\lim_{z \to M_1} \psi_1(z) = 0.$$

Therefore, by the Hopf lemma there is a constant $C > 0$ such that

$$\psi_1(z) \leq -C \operatorname{dist}(z, M_1).$$

This means that

$$\rho_1(z) \leq -C^{3/2} \operatorname{dist}^{3/2}(z, M_1) \quad \text{for} \quad z \in U_1$$

such that $d\rho_1(z) \neq 0$ for $z \in M_1$. Hence, ρ_1 is a defining function of U_1 along M_1. Because M_1 is strictly pseudoconvex we can find a constant $L > 0$ such that

$$\varphi_1 := \rho_1 e^{-L\rho_1}$$

is even a strongly plurisubharmonic defining function of U_1 along M_1. Notice that $\varphi_1 = \varphi_2 \circ f$ with $\varphi_2 = \rho_2 e^{-L\rho_2}$ being a defining function of U_2 along M_2. Since $\bar{\partial} f$ vanishes to infinite order at M_1 we have

$$0 \lneqq \det\left(\frac{\partial^2 \varphi_1}{\partial z_i \partial \bar{z}_j}\right)(z) = |u(z)|^2 \det\left(\frac{\partial^2 \varphi_2}{\partial w_i \partial \bar{w}_j}\right)(f(z))$$

for all $z \in M_1$. Therefore, $u(z) \neq 0$ for all $z \in M_1$ and M_2 is also strictly pseudoconvex at all points in $f(M_1)$. □

4. Remarks.

1) Our proof shows, in fact, that the following statement holds:

Theorem 2. Let $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be pseudoconvex domains with C^∞-smooth boundaries. Suppose Ω_1 satisfies condition R (in the sense of Bell [2]) for the Bergman projection operator on Ω_1. Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Then f does not have any branching points near the strictly pseudoconvex boundary points of Ω_1.

In order to reduce this to what has been done in the proof of theorem 1 it is enough to show: if there is a strictly pseudoconvex boundary point p of $b\Omega$ with $p \in \bar{\hat{X}}$, then q as in section 2 can be chosen arbitrarily close to p. For this, we define X as an irreducible branch of $\hat{X}$ on which u vanishes to minimal order among all branches of $\hat{X}$ clustering on $b\Omega$ in a given neighborhood U of p. We may assume that $p \in \bar{X}$ and define v and S with respect to X as in section 2. We claim: There is a $q \in S \cap U$ with

$$\frac{\partial v}{\partial z_\beta}(q) \neq 0.$$

Suppose $v' = \partial v | \partial z_\beta$ vanishes identically on $S \cap U$. Notice that there is a function $f \in A^\infty(\Omega_1)$ with

$$f(p) = 1 \quad \text{and} \quad |f| \Big| \bar{\Omega} \setminus \{p\} < 1 .$$

Therefore, it is easy to find an $\varepsilon > 0$, a $z_o \in X \cap U$ and an $N \in \mathbb{N}$ such that

$$|(f+\varepsilon)^N v'|(z_o) > 1 \quad \text{and} \quad |(f+\varepsilon)^N v'| \Big| b\Omega \setminus U < 1$$

Because $v'|S \cap U = 0$ this contradicts the maximum principle for v' on X. - We now can apply the proof in section 2 to the situation near q.

2) Theorem 2 shows that in the situation as given there the branching locus of f hits $b\Omega_1$ only at weakly pseudoconvex points. One might, therefore, ask whether this excludes all branching of f if the set of weakly pseudoconvex points on $b\Omega_1$ is small enough. This is, indeed, the case. More precisely we have

<u>Theorem 3</u>. Let $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be pseudoconvex domains with C^∞-smooth boundaries. Suppose that Ω_1 satisfies condition R and that the set E of weakly pseudoconvex boundary points of Ω_1 has Hausdorff-measure

$$\Lambda_{2n-3}(E) = 0.$$

Then any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$ is unbranched, and, therefore, extends to a covering map

$$\hat{f} : \bar{\Omega}_1 \to \bar{\Omega}_2 .$$

Proof. Because of theorem 2 we only have to show that the branching locus of f has to hit $b\Omega_1$ at a strictly pseudoconvex point if it is non-empty. For this using the notations of section 2 we have to observe that a point $q \in b\Omega_1$ where X intersects $b\Omega_1$ as a transversal C^∞ real manifold of real codimension 2 (in $\mathbb{C}^n$) can be found without using strict pseudoconvexity of $b\Omega_1$ at q. Namely, to achieve this one replaces the strict plurisubharmonic defining function ρ by a local defining function ρ of Ω_1 near $q \in b\Omega_1$ as chosen in (1) such that

$$\varphi: = -(-\rho)^{2/3}$$

is strictly plurisubharmonic on Ω_1 near q, thereby getting a C^∞-function σ on U' with

$$\psi: = -(-\sigma)^{2/3}$$

being plurisubharmonic on $\pi(Y \cap \Omega_1)$ and $\sigma(z') = 0$ for $z' \in \pi(Y \cap b\Omega_1)$. Now we choose a point $z'_o \in \pi(Y \cap \Omega_1)$ very close to q' and let $B' \subset U' \cap \pi(Y \cap \Omega_1)$ be the largest ball around z'_o. Then there is a point $\tilde{q}' \in bB' \cap S'$. Applying Hopf lemma to $\psi|B'$ at $\tilde{q}'$ gives

$$d\sigma(\tilde{q}') \neq 0$$

such that at $\tilde{q}: = (\tilde{q}', g(\tilde{q}')) \in S$

$$d(\rho|Y)(\tilde{q}) \neq 0.$$

This shows that Y intersects $b\Omega_1$ at $\tilde{q}$ transversally. Therefore, S has to be near $\tilde{q}$ a real manifold of real codimension 3 (in $\mathbb{C}^n$) and cannot be contained in E since $\Lambda_{2n-3}(E) = 0$. □

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Universität Wuppertal, Mathematik, Gauss-str. 20, D-5600 Wuppertal 1, W. GERMANY.

Matematisk Institutt, Universitetet i Oslo, Blindern, Oslo 3, Norge.