J. Korean Math. Soc. **40** (2003), No. 3, pp. 341–371

PROPER HOLOMORPHIC MAPPINGS, POSITIVITY CONDITIONS, AND ISOMETRIC IMBEDDING

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ABSTRACT. This article discusses in detail how the study of proper holomorphic rational mappings between balls in different dimensions relates to positivity conditions and to isometric imbedding of holomorphic bundles. The first chapter discusses rational proper mappings between balls; the second chapter discusses seven distinct positivity conditions for real-valued polynomials in several complex variables; the third chapter reveals how these issues relate to an isometric imbedding theorem for holomorphic vector bundles proved by the author and Catlin.

0. Introduction

The titles of the three lectures were "Proper holomorphic mappings", "Positivity conditions", and "An isometric imbedding theorem for holomorphic bundles". The lectures were intended to be accessible to graduate students. The first lecture posed some basic questions about the existence of proper holomorphic polynomial or rational mappings between balls in different dimensional complex Euclidean spaces with certain specified properties. The second lecture showed how to answer these questions, using some ideas that revolve around positivity conditions for real-valued polynomial functions on complex Euclidean space. The third lecture discussed joint work with David Catlin, where these ideas were generalized and applied in the setting of holomorphic vector bundles. The author wishes to acknowledge Catlin's important contributions to the mathematics in these lectures.

Received May 7, 2002.
2000 Mathematics Subject Classification: 32H99, 32F20, 32B05, 14C30, 15A23, 32J25.
Key words and phrases: proper holomorphic mappings, unit ball, positivity conditions, Hermitian forms, holomorphic line bundles, isometric imbedding, CR mappings.
Partially supported by NSF grant DMS 99-70024.

The organization of this paper parallels that of the lectures, but somewhat more information appears here than was in the lectures. Since the lectures were given, the author has written a Carus monograph [10] that includes an elementary treatment of much of the material preliminary to these lectures. The book [10] also includes small improvements on some of the results here.

1. Proper holomorphic mappings

1.1. Elementary considerations

A continuous map $f : X \to Y$ between topological spaces is called *proper* when K compact in Y implies $f^{-1}(K)$ compact in X. For us X and Y will be bounded domains in complex Euclidean spaces, perhaps of different dimensions. We have a simple characterization of proper holomorphic mappings in this case.

PROPOSITION 1. *Let $\Omega \subset \mathbf{C}^n$ and $\Omega' \subset \mathbf{C}^N$ be bounded domains. A holomorphic mapping $f : \Omega \to \Omega'$ is proper if and only if the following condition holds. If $\{z_\nu\}$ is a sequence of points in Ω tending to its boundary, then the image sequence $\{f(z_\nu)\}$ tends to the boundary of Ω'.*

Proof. It is easy to prove the contrapositive of each required statement. If the condition fails, then there is some sequence $\{z_\nu\}$ tending to the boundary whose image does not. Hence there is a subsequence whose image stays within a compact set in the target Ω'. Then the inverse image of this compact set is not compact in Ω, and f is not proper. On the other hand, if f is not proper, we can find a compact K whose inverse image is not compact. Then there is a sequence $\{z_\nu\}$ in $f^{-1}(K)$ that tends to the boundary, while its image stays within K. Thus the condition about sequences fails. Hence this condition is equivalent to f being proper. □

Suppose in the situation of Proposition 1 that the boundaries $b\Omega$ and $b\Omega'$ are smooth manifolds, and that f is known to have a continuous extension F to $b\Omega$. It follows that $F(b\Omega) \subset b\Omega'$ and hence its restriction to $b\Omega$ defines a CR mapping between CR manifolds. This is one of the main reasons why proper holomorphic mappings are interesting to us. For completeness we review some simple facts about CR geometry; the main thing we will need is the definition of strong pseudoconvexity.

Suppose first that M is a real hypersurface in $\mathbf{C}^n$; we let $TM \otimes \mathbf{C}$ denote its complexified tangent bundle. Let $T^{10}M$ be the subbundle of $TM \otimes \mathbf{C}$ whose local sections are combinations of the $\frac{\partial}{\partial z_j}$ for $j = 1, ..., n$, and let $T^{01}M = \overline{T^{10}M}$ be the complex conjugate bundle. Note that these bundles are integrable in the sense that the Lie bracket of local sections of $T^{10}M$ is also a local section of $T^{10}M$. Note that $T^{10}M \cap T^{01}M = 0$. Also the direct sum $T^{10}M \oplus T^{01}M$ is a subbundle of $TM \otimes \mathbf{C}$ whose fibers are of codimension one.

Suppose more generally that M is a smooth real manifold of odd dimension. We denote its complexified tangent bundle by $TM \otimes \mathbf{C}$.

DEFINITION 1. The odd-dimensional real manifold M is called a CR manifold of hypersurface type if there is a subbundle $T^{10}M$ of $TM \otimes \mathbf{C}$ satisfying the following properties:

1.1.1) $T^{10}M \cap \overline{T^{10}M} = 0$.

1.1.2) $T^{10}M$ is integrable (closed under the Lie bracket operation).

1.1.3) The fibres of $T^{10}M \oplus \overline{T^{10}M}$ have codimension one in $TM \otimes \mathbf{C}$.

The most crucial features of the geometry of a CR manifold M of hypersurface type are determined by its Levi form. We recall its definition. First let η denote a purely imaginary nonvanishing one-form that annihilates $T^{10}M \oplus T^{01}M$.

We define the Levi form λ to be the Hermitian form on $T^{10}M$ given by

$$\lambda(L, \overline{K}) = \langle \eta, [L, \overline{K}] \rangle = \langle d\eta, L \wedge \overline{K} \rangle. \tag{1}$$

The Levi form is determined only up to sign. In case M is a compact real hypersurface we always choose the sign so that this form is *positive* definite at the point on M farthest from the origin. A real hypersurface or more generally a CR manifold of hypersurface type is *strongly pseudoconvex* if its Levi form is definite. We call a domain in a complex manifold *strongly pseudoconvex* if its boundary is smooth and strongly pseudoconvex. For us the most important CR manifold will be the unit sphere in $\mathbf{C}^n$.

1.2. Proper holomorphic mappings between balls

Next we discuss proper holomorphic mappings between balls in perhaps different dimensional complex Euclidean spaces. We write $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ to denote the usual Hermitian inner product between vectors z and w in $\mathbf{C}^n$, and we write $||z||^2$ for the corresponding squared norm. We let B_n denote the unit ball in $\mathbf{C}^n$, so $B_n = \{z : ||z||^2 < 1\}$.

The first observation is that there are no proper holomorphic mappings from B_n to B_N when n exceeds N. The reason is simple; consider for example the nonempty subset $f^{-1}(f(0))$ of B_n. As the inverse image of a compact set it must be compact. As the solution set of N holomorphic equations in n variables it must be a complex analytic variety of dimension at least $n - N$. It is a standard elementary fact in complex analysis that there are no positive-dimensional *compact* complex analytic subvarieties of the unit ball. Hence, if there is a proper holomorphic mapping from B_n to B_N, then $n - N \leq 0$.

We next discuss the equidimensional case; here the result in the case $n = 1$ differs fundamentally from the case $n \geq 2$. It turns out that the one-dimensional case motivates many of the questions (in higher dimensions) we answer in this paper.

PROPOSITION 2. *The proper holomorphic mappings from B_1 to itself are precisely the finite Blaschke products. Thus, if $f : B_1 \to B_1$ is proper, then there is a point $e^{i\theta}$ on the circle, finitely many points a_j in the disk, and positive integer multiplicities m_j such that*

$$f(z) = e^{i\theta} \prod (\frac{a_j - z}{1 - \bar{a}_j z})^{m_j}. \tag{2}$$

Proof. (Sketch) Given a proper holomorphic mapping f, consider the divisor

$$f^{-1}(0) = \sum m_j [a_j], \tag{3}$$

where the sum is finite. Let $B(z)$ denote the Blaschke product with the same divisor as f.

$$B(z) = \prod (\frac{a_j - z}{1 - \bar{a}_j z})^{m_j}. \tag{4}$$

Then the function $\frac{f}{B}$ is holomorphic and nonzero in the unit disk. Using an epsilon-delta formulation of Proposition 1, we obtain the following inequalities. Given $\epsilon > 0$, we can find $\delta > 0$ so that

$$1 - \epsilon \leq |\frac{f}{B}(z)| \leq \frac{1}{1 - \epsilon} \tag{5}$$

for $|z| \geq 1 - \delta$. It then follows from the maximum principle that both inequalities in (5) hold on the unit disk. Since ϵ is arbitrary, $|\frac{f}{B}(z)| = 1$ on the unit disk. Since $\frac{f}{B}$ is holomorphic, it must be constant, and hence equal to $e^{i\theta}$ for some θ. □

We make two simple observations, which we can extend to more variables only by allowing arbitrarily large dimensions for the target.

1.2.1) Consider an arbitrary divisor $\sum_{j=1}^{M} m_j[a_j]$, where $|a_j| < 1$ for each j and m_j are positive integers. Then there is a proper holomorphic mapping from B_1 to itself whose zero set is precisely this divisor. Furthermore f is determined up to a unitary transformation of $\mathbf{C}$. The function $|f|^2$ is completely determined by the divisor $f^{-1}(0)$.

1.2.2) Let q be an arbitrary polynomial that does not vanish on the closed unit disk. Then there is a polynomial p such that p and q are relatively prime, and $\frac{p}{q}$ is a proper holomorphic mapping from B_1 to itself. We give the simple proof. If q is a constant c, then we can take $p(z) = cz$. If q has degree d, with $d \geq 1$, we put $p(z) = z^d\overline{q} = (1/z)$. Note then on the circle where $\overline{z} = 1/z$,

$$(6) \qquad |\frac{p}{q}(z)| = |\frac{z^d\overline{q}(1/z)}{q(z)}| = |\frac{\overline{q}(1/z)}{q(z)}| = |\frac{\overline{q(z))}}{q(z)}| = 1.$$

This calculation amounts to replacing the factor $1 - \overline{a}_j z$ in q with the factor $z - a_j$ in p.

The equidimensional case in higher dimensions is totally different.

THEOREM. (Alexander, Pinchuk) *For $n \geq 2$, the proper holomorphic mappings from B_n to itself are precisely the automorphisms of B_n. Thus if $f : B_n \to B_n$ is proper, then*

$$(7) \qquad f(z) = U\frac{a - L_a z}{1 - \langle z, a\rangle},$$

where U is a unitary transformation, and $||a|| < 1$, and L_a is a linear transformation. In fact $L_a z = c_1 z + c_2\langle z, a\rangle a$, for appropriate positive numbers c_1 and c_2 depending on $||a||$.

Proof. (Sketch) One first proves that the proper mapping extends smoothly to the boundary. Next one considers the branch locus, which is the complex analytic variety V defined by $V = \{z : \det(df)(z) = 0\}$. Suppose that V is nonempty; As a complex analytic variety of positive dimension in the ball it must be noncompact. By this and the extension of f to the boundary, the branch locus then includes points on the boundary. A calculation (or the Hopf lemma) shows that $||f(z)||^2$ must have nonvanishing differential on the sphere. Thus the function $||f||^2 - 1$ is a defining equation for the sphere, which is strongly pseudoconvex, and hence its Levi form must be positive definite. But the Levi form has $|\det(df)|^2$ as a factor, so we conclude that V must be empty. Thus f is an unbranched covering map of a simply connected domain, and hence must be injective. See [19] for more details, and more information on the automorphism group of the ball. □

We make the following observations to contrast with the one-dimensional case.

1.2.1') The only possible divisor is $[a]$. In other words, f is determined up to a unitary transformation by $f^{-1}(0)$. Thus $f^{-1}(0) = [a]$. Unlike the one-dimensional case, the divisor must be a single point a with multiplicity unity.

1.2.2') If q is a polynomial that does not vanish on the closed unit ball, then q does not arise as a denominator of a proper rational mapping reduced to lowest terms unless q is first degree!

The contrast between the results and the one-dimensional case motivated the author to study proper holomorphic mappings $f : B_n \to B_N$ for all $N \geq n$ at the same time. The main idea is this: we can find a proper rational mapping between balls with any reasonable properties, as long as we are allowed to choose the target dimension N sufficiently large. In particular we will find that, the analogue of 1.2.1) holds, and one can do much more! The analogue of 1.2.2) also holds. Given a polynomial $q : \mathbf{C}^n \to \mathbf{C}$ that does not vanish on the closed unit ball, there always exists an N and a polynomial mapping $p : \mathbf{C}^n \to \mathbf{C}^N$ such that $\frac{p}{q}$ is reduced to lowest terms and maps B_n properly to B_N. We want $\frac{p}{q}$ to be in lowest terms, or else we have the trivial example where $p(z) = q(z)(z_1, ..., z_n)$. The surprising thing is that, even when $n = 2$ and the degree of q is two, the minimal possible N can be arbitrarily large.

In order to prove this and related results, we need the following theorem proved by Catlin and the author. Later in this paper we will prove this theorem also.

THEOREM 1. *Suppose that $z \to r(z, \overline{z})$ is a real-valued polynomial on $\mathbf{C}^n$, and suppose that $r(z, \overline{z}) > 0$ for $||z|| = 1$. Then there is an integer N and a holomorphic polynomial mapping $h : \mathbf{C}^n \to \mathbf{C}^N$ such that*

$$r(z, \overline{z}) = ||h(z)||^2 = \sum_{j=1}^{N} |h_j(z)|^2 \tag{8}$$

for $||z|| = 1$.

For now we assume Theorem 1, and derive some applications to proper holomorphic mappings. See Section 2.3 for the proof of Theorem 1.

THEOREM 2. *Let $q : \mathbf{C}^n \to \mathbf{C}$ be a holomorphic polynomial, and suppose that q does not vanish on the closed unit ball. Then there is an integer N and a holomorphic polynomial mapping $p : \mathbf{C}^n \to \mathbf{C}^N$ such that*

1. *$\frac{p}{q}$ is a rational proper mapping between B_n and B_N,*
2. *$\frac{p}{q}$ is reduced to lowest terms.*

Proof. The result is trivial when q is a constant. We have already given the easy proof when $n = 1$. So we assume that $n \geq 2$, that q is not a constant, and that $q(z) \neq 0$ on the closed ball. Let g be an arbitrary nonconstant polynomial such that q and g have no common factor. Then there is a constant c so that

$$|q(z)|^2 - |cg(z)|^2 > 0 \tag{9}$$

for $||z||^2 = 1$. We set $p_1 = cg$.

By Theorem 1 a polynomial $f(z, \overline{z})$ that is positive on the unit sphere agrees with a squared norm of a holomorphic polynomial mapping on the sphere. Therefore there is an integer N and polynomials $p_2, ..., p_N$ such that

$$|q(z)|^2 - |p_1(z)|^2 = \sum_{j=2}^{N} |p_j(z)|^2 \tag{10}$$

on the sphere. It follows that $||\frac{p}{q}||^2 = 1$ on the sphere. Since $||\frac{p}{q}||^2$ is plurisubharmonic and nonconstant, the maximum principle guarantees that $\frac{p}{q}$ is a proper mapping from B_n to B_N. Since $\frac{p_1}{q}$ is reduced to lowest terms, so is $\frac{p}{q}$. □

Theorem 1 can be used also to show that one can choose various components f of a proper holomorphic rational mapping arbitrarily, assuming only that they satisfy the necessary condition $||f(z)||^2 < 1$ on the sphere.

THEOREM 3. *Suppose that $\frac{p}{q}$ is a (vector-valued) rational mapping on $\mathbf{C}^n$ and that $||\frac{p}{q}(z)||^2 < 1$ on the unit sphere. Then there is an integer K and a polynomial mapping $g : \mathbf{C}^n \to \mathbf{C}^K$ such that $\frac{p \oplus g}{q}$ is a proper holomorphic mapping between balls.*

Proof. Note that $|q|^2 - ||p(z)||^2$ is a polynomial that is positive on the sphere. By Theorem 1 we can find a holomorphic polynomial mapping g such that

$$|q(z)|^2 - ||p(z)||^2 = ||g(z)||^2 \tag{11}$$

on the sphere, and we may assume that g is not a constant multiple of q. Then $\frac{p}{q} \oplus \frac{g}{q}$ is a non-constant holomorphic rational mapping whose squared norm $\frac{||p||^2+||g||^2}{|q|^2}$ equals unity on the sphere. By the maximum principle it is the required proper mapping. □

COROLLARY 1. *Suppose that p is a vector-valued polynomial mapping on $\mathbf{C}^n$ and that $||p||^2 < 1$ on the unit sphere. Then there is a polynomial mapping g such that $p \oplus g$ defines a proper holomorphic mapping between balls.*

Theorems 2 and 3 allow us to create rational proper mappings between balls with the desired properties. Since the proofs rely on Theorem 1, we have no control on the minimum possible target dimension. It is easy to show directly that the conclusions of the theorems are not possible if we fix the target dimension a priori. In Corollary 1, for example, suppose that we are given $p(z) = cz_1z_2$. The hypotheses of the corollary are met when $|c|^2 < 4$. We want to find polynomials $g_1, ..., g_N$ such that $1 = |p|^2 + ||g||^2$ on the sphere. It is easy to see that the minimal possible N tends to infinity as $|c|^2$ tends to 4.

Theorems due to Lempert ([14], [15]) and to Løw [16] are related to Theorem 1, but they cannot be used here. These results state that positive functions on the boundaries of strongly pseudoconvex domains agree with squared norms of holomorphic mappings there. In Lempert's work, the boundary is real-analytic, the given positive function is real-analytic, and the resulting holomorphic mapping takes values in an infinite-dimensional space. In Løw's work, the domain has $\mathbf{C}^2$ boundary, the given positive function is continuous, and the resulting holomorphic mapping takes values in a finite-dimensional space. One can conclude only that it is holomorphic on the interior of the domain. We need the additional information that the resulting holomorphic mapping is polynomial.

We close this section by mentioning several important results about proper mappings between balls. First of all, many authors have contributed to the study of proper mappings between balls that do not extend smoothly to the boundary. It is now known, whenever $n \geq 1$ and $N > n$, that there are proper holomorphic mappings $f : B_n \to B_N$ that are continuous on the boundary, but that are not continuously differentiable. On the other hand, Forstneric [13] proved that, if $n \geq 2$, and f has $N - n + 1$ continuous derivatives at the boundary, then a proper mapping $f : B_n \to B_N$ must be rational. Cima-Suffridge [6] showed that such rational maps have no poles on the sphere, and hence are holomorphic in a neighborhood of the closed ball. In these lectures we are interested in only *rational* proper holomorphic mappings between balls.

1.3. Constructing proper maps via tensor products and undoing

We continue to use the result about finite Blaschke products to motivate our work. Suppose that f and g are proper mappings of a domain in the plane to the unit disk. Then their product fg also is. The extension of this idea to more variables requires tensor products, and is more interesting. The simplest generalization, which we make first, turns out to be inadequate. If f and g are proper mappings of a domain Ω in $\mathbf{C}^n$ to unit balls in dimensions N and K, then the tensor product $f \otimes g$ defines a proper holomorphic mapping from Ω to B_{NK}. Here of course $f \otimes g$ denotes the mapping whose components are all possible products $f_j g_k$ of the components of f and g, in some fixed but irrelevant order. Note that $||f \otimes g||^2 = ||f||^2 ||g||^2$, and hence this statement about properness to B_{NK} follows from Proposition 1.

We pause to make a simple remark. In many of our considerations the appropriate object to consider is $||h||^2$ rather than h itself. For example, given f and g as above, the tensor product $f \otimes g$ is determined by f and g only up to linear transformations, whereas its squared norm $||f \otimes g||^2$ is determined completely by them. It is useful to note [7] that if f and g are $\mathbf{C}^N$-valued holomorphic mappings, and $||f||^2 = ||g||^2$, then there is a unitary transformation U such that $f = Ug$. We will generally treat f and g as the same when $||f||^2 = ||g||^2$.

From the tensor product we immediately recover one piece of the finite Blaschke product result in one variable. Let ϕ_a denote the automorphism of B_n defined by

$$\phi_a(z) = \frac{a - L_a z}{1 - \langle z, a \rangle}. \tag{12}$$

Recall that L_a was defined in the Alexander-Pinchuk theorem. Given a finite set of points a_j in B_n, and positive integer multiplicities m_j, let $d = \sum m_j$. We can form a proper holomorphic mapping $f : B_n \to B_N$ with $f^{-1}(0) = \sum m_j [a_j]$ and $N = n^d$ by

$$f = \prod_{\otimes} \phi_{a_j}^{\otimes m_j}. \tag{13}$$

Mappings defined by (13) are direct analogues of finite Blaschke products; there are many polynomial and rational proper holomorphic mappings between balls that cannot be expressed in this fashion. We need to generalize the tensor product operation and discuss the "undoing" of this operation.

Suppose that A is a subspace of $\mathbf{C}^N$. We write $A^\perp$ for its orthogonal complement, so we have $\mathbf{C}^N = A \oplus A^\perp$. When f takes values in $\mathbf{C}^N$ we can write $f = f_A \oplus f_{A^\perp}$.

We next extend the notion of the tensor product by allowing tensoring on A.

Let Ω be a domain in $\mathbf{C}^n$. Suppose that $f : \Omega \to \mathbf{C}^N$ and $g : \Omega \to \mathbf{C}^K$ are holomorphic mappings. Let A be a subspace of $\mathbf{C}^N$ of dimension d. We define a holomorphic map $E(A,g)f$ from Ω to $\mathbf{C}^L$ by

$$E(A,g)f = (f_A \otimes g) \oplus f_{A^\perp}. \tag{14}$$

When $A = 0$, so $d = 0$, we see that $E(A,g)f = f$. When $A = \mathbf{C}^N$, so $d = N$, we see that $E(A,g)f = f \otimes g$. By identifying $A \otimes \mathbf{C}^K$ with $\mathbf{C}^{dK}$ we think of $E(A,g)f$ as taking values in $\mathbf{C}^L$, where $L = (N-d) + dK$. The point of this operation is the following simple fact.

PROPOSITION 3. *Suppose that Ω is a domain in $\mathbf{C}^n$, and that $f : \Omega \to B_N$ and $g : \Omega \to \mathbf{C}^K$ are proper holomorphic mappings. Then $E(A,g)f : \Omega \to B_L$ is a proper holomorphic mapping.*

Proof. Observe that

$$\begin{aligned} ||E(A,g)f||^2 &= ||f_{A^\perp}||^2 + ||f_A \otimes g||^2 \\ &= ||f_{A^\perp}||^2 + ||f_A||^2||g||^2 = ||f||^2 + ||f_A||^2(||g||^2 - 1). \end{aligned} \tag{15}$$

It follows from (15) that when $||g||^2$ and $||f||^2$ tend to unity, so does $||E(A,g)f||^2$, so the result follows from Proposition 1. □

This operation generates a new proper map from two given proper maps; we next introduce the *undoing* of this operation.

Suppose h is a holomorphic function taking values in $\mathbf{C}^L$. Suppose further that there are holomorphic mappings f and g and a subspace A of the target of f such that $h = E(A,g)f$. We then write $f = E(A,g)^{-1}h$. In general of course this undoing operation is not defined. We next give a simple example of a polynomial mapping that is not an iterated composition of tensor products; undoing is required for a composition product factorization.

EXAMPLE 1. Consider the map (first found by Faran) $f : \mathbf{C}^2 \to \mathbf{C}^3$ defined by

$$f(z_1, z_2) = (z_1^3, \sqrt{3}z_1z_2, z_2^3). \tag{16}$$

Then $f : B_2 \to B_3$ is a proper holomorphic mapping. We may express it via the following composition product factorization:

1.3.1) We begin with the identity.

$$f_1(z) = z.$$

1.3.2) We tensor with the identity map (on the full space) to obtain

$$f_2(z) = z \otimes z = f_1(z) \otimes z = (z_1^2, z_1z_2, z_2z_1, z_2^2).$$

1.3.3) We tensor (on the full space) again with the identity map, obtaining a mapping to 8-dimensional space.

(17)
$$f_3(z) = z \otimes z \otimes z = f_2(z) \otimes z = (z_1^3, z_1^2z_2, z_1^2z_2, z_1^2z_2, z_1z_2^2, z_1z_2^2, z_1z_2^2, z_2^3).$$

1.3.4) We compose (17) with a linear isometry and see that the image of the mapping fits into 4 dimensions. Projecting into 4 dimensions gives

$$f_4(z) = Lf_3(z) = (z_1^3, \sqrt{3}z_1^2z_2, \sqrt{3}z_1z_2^2, z_2^3). \tag{18}$$

1.3.5) We "undo" the middle two components of (18) to obtain

$$f_5(z) = E(A, I)^{-1}f_4(z) = (z_1^3, \sqrt{3}z_1z_2, z_2^3)$$

Observe that the components of the mapping f_4 in the above are precisely the homogeneous monomials of degree 3, multiplied by certain coefficients. It is natural to next make the following definition.

DEFINITION 2. Given positive integers n and m, we define a map $H_m : \mathbf{C}^n \to \mathbf{C}^N$ by

$$H_m(z) = (\cdots, \sqrt{\binom{m}{\alpha}} z^\alpha, \cdots). \tag{19}$$

Since $||H_m(z)||^2 = ||z||^{2m}$, we see immediately that H_m defines a proper holomorphic mapping from B_n to B_N, where N is the number of linearly independent monomials of degree m in n variables. We also see that $||H_m(z)||^2 = ||z \otimes z \cdots \otimes z||^2$.

Using this notation we see that the map f from Example 1 can be written $f = E(A, I)^{-1}H_3$. This gives insight into the following general result.

THEOREM 4. *Suppose that $f : \mathbf{C}^n \to \mathbf{C}^N$ is a polynomial of degree m that defines a proper mapping from B_n to B_N. We may then write*

$$f = \prod_{j=0}^{m} E(A_j, I)^{-1} H_m. \tag{20}$$

In other words, every proper polynomial mapping between balls of degree m is obtained by first taking the m-fold tensor product of the identity with itself and then *undoing* the tensor product operation on various subspaces.

The proof appears in [7], and is based upon the following simple observation. Suppose that we write a (vector-valued) polynomial f as $f = f_0 + f_1 + \cdots + f_m$ where each f_j is homogeneous of degree j. If f is a proper mapping between balls, then necessarily f_0 is orthogonal to f_m. More generally, the lowest order non-vanishing part is orthogonal to the highest order non-vanishing part. This determines a subspace A of the target. After tensoring on A, we obtain a new proper mapping *of the same degree* as f whose lowest order part is of higher degree. Thus, given f, there are subspaces on which to tensor so that, after at most m steps, we obtain a homogeneous mapping of degree m. Up to linear transformations, the only homogeneous proper mapping of degree m is H_m. See [7] for simple proofs of this.

1.4. Proper mappings and invariance under finite groups

It is also interesting to construct proper holomorphic mappings that are invariant under finite subgroups of the unitary group. It turns out to be impossible to find rational proper mappings invariant under most such groups. D'Angelo-Lichtblau [11] solved the CR Spherical Space Form problem by proving Theorem 5 below. This has an interesting relationship to squared norms of holomorphic mappings.

Suppose that G is a finite group, and that $\Gamma = \pi(G)$ is a representation of G as a subgroup of the unitary group $U(n)$. We call Γ a finite unitary group, and we say that it is fixed point free if the only element in Γ with an eigenvalue of unity is the identity. We say that a holomorphic mapping f is Γ-invariant if $f \circ \gamma = f$ for all $\gamma \in \Gamma$.

THEOREM 5. [11] *Suppose that Γ is a fixed-point free unitary representation of a finite group. Suppose that $f = \frac{g}{h}$ is a (holomorphic) rational function invariant under Γ, and furthermore suppose that there is an integer N such that $f : S^{2n-1} \to S^{2N-1}$. In order that f be non-constant, Γ must be cyclic, and represented in one of the following three ways.*

1.4.1) *Γ is the cyclic group of order m generated by ϵI. Here ϵ is a primitive m-th root of unity, and I is the identity matrix.*

1.4.2) *Γ is the cyclic group of order $2r+1$ generated by $\epsilon I_k \oplus \epsilon^2 I_{n-k}$. Here ϵ is a primitive $(2r+1)$ -st root of unity, and I_k denotes the k-th order identity matrix.*

1.4.3) *Γ is the cyclic group of order seven generated by $\epsilon I_k \oplus \epsilon^2 \oplus I_j \oplus \epsilon^4 I_{n-k-j}$. Here ϵ is a primitive seventh root of unity.*

There are explicit formulas for the minimal monomial example in each of these cases, see Example 2 below. Both the proof of Theorem 5 and the formulas for the maps that do arise use properties of the Γ-invariant real-valued polynomial Φ defined below in (21). There are also interesting combinatorial aspects. In the first case the multinomial coefficients arise as squared absolute values of coefficients of the monomials; in the second case a new triangle of integers arises in this way. See [8] and Chapter 5 in [7] for combinatorial aspects of invariant holomorphic mappings.

COROLLARY 2. *Let Γ be a fixed-point free unitary representation of a finite group. Define its invariant polynomial Φ by*

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, z \rangle). \tag{21}$$

Then Φ is a squared norm of a holomorphic mapping only when Γ is one of the three cases from Theorem 6. In each case the holomorphic mapping is a proper polynomial mapping between balls that is Γ-invariant.

EXAMPLE 2. We compute (21) in several cases.

1.4.1) Suppose that Γ is the cyclic group generated by ϵI, where ϵ is a primitive m-th root of unity. Expanding (21) yields

$$\Phi_\Gamma(z, \overline{z}) = ||z||^{2m} = ||H_m(z)||^2. \tag{22}$$

1.4.2) Suppose that Γ is as in 1.4.2. When $n = 2$ and $r = 1$, expanding (21) gives $|z_1|^6 + 3|z_1|^2|z_2|^2 + |z_2|^6$, which is the squared norm of the Faran mapping from Example 1. We obtain from (21) similar examples of proper polynomial mappings of degree $2r+1$ from B_2 to B_{r+2} for each positive integer r. For example, when $r = 4$ we obtain for the squared norm

(23)

$$|z_1|^{18} + 9|z_1|^{14}|z_2|^2 + 27|z_1|^{10}|z_2|^4 + 30 = |z_1|^6|z_2|^6 + 9|z_1|^2|z_2|^8 + |z_2|^{18}.$$

1.4.3) Expanding (21) when $n = 3$ yields (the squared norm of) a polynomial mapping of degree 7 from B_3 to B_{17}; see [7] for the formula.

In [8] the author expands (21) in cases where a squared norm does not result, thereby obtaining group invariant proper mappings to domains whose boundaries are quadrics.

We close this section with a conjecture for which the author has found both heuristic justification and computer evidence, but has not found a proof. Note first that, for each N with $N \geq 2$, the polynomial proper mappings in 1.4.2 are of degree $2N-3$. We conjecture that this is sharp.

Conjecture. Suppose that $f : B_2 \to B_N$ is a proper holomorphic rational mapping. Then the degree of f does not exceed $2N-3$.

2. Positivity conditions

2.1. Seven positivity conditions

In this section we introduce various positivity conditions for real-analytic real-valued functions of several complex variables. Suppose that $r : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ is a holomorphic function such that, for all z, $r(z, \overline{z})$ is real. We will be concerned with both $r(z, \overline{z})$ and with $r(z, \overline{w})$, so we use the phrase *on the diagonal* when we are thinking of the real-analytic function $z \to r(z, \overline{z})$.

We put the usual complex Euclidean norms on all finite-dimensional complex vector spaces. Thus when h is a holomorphic function taking values in $\mathbf{C}^N$, we write

$$||h||^2 = \sum_{k=1}^{N} |h^k|^2,$$

where h^k is the k-th component function. We use the same notation for the norm when h takes values in a Hilbert space, and we use $\langle , \rangle$ for the inner product.

We now introduce the seven positivity conditions. In all cases we assume that r is not identically zero.

2.1.1) On the diagonal r is nonnegative as a function. For all z,

$$r(z, \overline{z}) \geq 0.$$

2.1.2) On the diagonal r is the quotient of squared norms of holomorphic mappings. Thus there is a Hilbert space $\mathcal{H}$ and holomorphic mappings f and g from $\mathbf{C}^n$ to $\mathcal{H}$ such that

$$r(z, \overline{z}) = \frac{||f(z)||^2}{||g(z)||^2}.$$

2.1.3) On the diagonal r is the squared norm of a holomorphic mapping. Thus there is a Hilbert space $\mathcal{H}$ and a holomorphic mapping $f : \mathbf{C}^n \to \mathcal{H}$ such that

$$r(z,\overline{z}) = ||f(z)||^2.$$

2.1.4) The underlying matrix C of Taylor coefficients of r is of the form A^*A. Thus there is a Hilbert space $\mathcal{H}$ and elements $f_\alpha \in \mathcal{H}$ such that

$$r(z,\overline{z}) = \sum c_{\alpha\beta} z^\alpha \overline{z}^\beta$$

and the entries $c_{\alpha\beta}$ of the (necessarily) Hermitian matrix C satisfy

$$c_{\alpha\beta} = \langle f_\alpha, f_\beta \rangle.$$

2.1.5) r is positive at a single point, and there is a positive integer N such that, on the diagonal, r^N is a squared norm of a holomorphic mapping. Thus there is a Hilbert space $\mathcal{H}$ and a holomorphic mapping $f : \mathbf{C}^n \to \mathcal{H}$ such that

$$r(z,\overline{z})^N = ||f(z)||^2.$$

2.1.6) r is positive at a single point, and satisfies the global Cauchy-Schwarz inequality. For all z and w,

$$r(z,\overline{z})r(w,\overline{w}) \geq |r(z,\overline{w})|^2.$$

2.1.7) On the diagonal, r is plurisubharmonic. For all $z, a \in \mathbf{C}^n$, we have

$$\sum_{i,j=1}^{n} r_{z_i\overline{z}_j}(z,\overline{z}) a_i \overline{a}_j = \partial\overline{\partial} r(z,\overline{z})(a,\overline{a}) \geq 0.$$

We first mention all the obvious implications.
2.1.3) implies 2.1.2) and 2.1.2) implies 2.1.1).
2.1.3) implies 2.1.5)
The following implications are easy to show; see [9] for proofs.
2.1.5) implies 2.1.6) and 2.1.6) implies 2.1.7).
2.1.5) implies 2.1.1).
2.1.3) holds if and only if 2.1.4).

We continue with a beautiful example from [9] showing that most of the other implications fail even for bihomogeneous polynomials.

EXAMPLE 3. For a real number a, define r_a by

$$r_a(z,\overline{z}) = ||z||^{4N} - a \prod_{j=1}^{n} |z_j|^4. \tag{24}$$

We consider values of a for which the seven positivity conditions hold. For simplicity we assume that $n = 2$.

2.1.1) holds if and only if $a \leq 16$.
2.1.2) holds if and only if $a < 16$.
2.1.3) holds if and only if $a \leq 6$.
2.1.4) holds if and only if $a \leq 6$.
2.1.5) holds for $a \leq 7.8$, and fails for $a > 8$.
2.1.6) holds for $a \leq 7.8$, and fails for $a > 8$.
2.1.7) holds if and only if $a \leq 12$.

First we discuss 2.1.1) and 2.1.2). It is easy to see that 2.1.1) holds, and that r_a is positive away from the origin when $a < 16$. It then follows from Theorem 6 (a nontrivial result) that there is an integer d and a holomorphic polynomial mapping h such that $r_a(z, \bar{z}) = \frac{||h(z)||^2}{||z||^{2d}}$. Since $||z||^{2d} = ||H_d(z)||^2$, we see that 2.1.2) holds if $a < 16$. To see that it fails when $a = 16$, we use the *jet pullback property* from [5].

Suppose that 2.1.2) holds, so that $r(z, \bar{z}) = \frac{||f(z)||^2}{||g(z)||^2}$. Let $t \to z(t)$ be a holomorphic curve in $\mathbf{C}^n$. Then the pullback z^*r is far from arbitrary. In fact we have, where $\cdots$ denotes terms of higher degree,

$$z^*r(t, \bar{t}) = \frac{||f(z(t))||^2}{||g(z(t))||^2} = \frac{||at^m + \cdots||^2}{||bt^k + \cdots||^2} = |t|^{2(m-l)} + \cdots . \tag{25}$$

This shows that the lowest order part of the Taylor series for z^*r must be independent of the argument of t. The author named this necessary condition for being a quotient of squared norms the *jet pullback property*.

We show that this property fails for r_a when $a = 16$. Let $z(t) = (1 + t, 1)$. Then one can compute that

$$z^*r(t, \bar{t}) = 8(t^2 + 2|t|^2 + \bar{t}^2) + \cdots \tag{26}$$

so that the lowest order part does not satisfy the necessary condition. Thus r_a is not a quotient of squared norms when $a = 16$. One can also observe this by simply noting that its zero set is not a complex analytic variety. There are simple examples of polynomials whose zero sets are complex analytic varieties, but for which the jet pullback property fails. See page 17 of [5].

Next we discuss 2.1.5) and 2.1.6). As stated above, 2.1.5) implies 2.1.6). By hand one can easily see that r_a^2 is a squared norm if and only if $a \leq 7$. The calculation is facilitated by writing $x = |z_1|^2$ and $y = |z_2|^2$, and replacing r_a by $(x + y)^4 - ax^2y^2$. A power of r_a will be a squared norm if and only if all the coefficients of that power of $(x+y)^4 - ax^2y^2$ are nonnegative. By using Mathematica, one can see for example that r_a^{32}

is a squared norm when $a = 7.8$, but not when $a = 7.9$. The proof that r_a fails to satisfy 2.1.6) for $a > 8$ appears in [9]. The author suspects that 2.1.5) holds if and only if $a < 8$, and that 2.1.6) holds if and only if $a \leq 8$. For this example, it follows from Theorem 6 that 2.1.5) and a sharp form of 2.1.6) are equivalent.

We refer to [9] for the proofs of the other statements.

2.2. Positivity for bihomogeneous polynomials

Suppose that

$$p(z,\overline{z}) = \sum_{|\alpha|+|\beta|\leq 2m} c_{\alpha\beta} z^{\alpha} \overline{z}^{\beta} \tag{27}$$

is a polynomial function on complex Euclidean space $\mathbf{C}^n$. Observe that $p(z,\overline{z})$ will be real for all z if and only if the *underlying matrix of coefficients* $C = (c_{\alpha\beta})$ is Hermitian symmetric. It is evident that, if C is non-negative definite, then the polynomial will take on non-negative values, and if C positive definite, then the polynomial will be strictly positive away from the origin. The polynomial p can be considered as the restriction of the Hermitian form in N variables

$$\sum_{\alpha,\beta=1}^{N} c_{\alpha\beta} \zeta_{\alpha} \overline{\zeta}_{\beta} \tag{28}$$

to a Veronese variety given by parametric equations $\zeta_{\alpha}(z) = z^{\alpha}$.

Positivity conditions for the function are weaker than positivity conditions for the Hermitian form. Example 3 shows that 2.1.1) does not imply 2.1.4), even for bihomogeneous polynomials.

A real-valued polynomial p on $\mathbf{C}^n$ is called *bihomogeneous of degree* $2m$ if

$$p(z,\overline{z}) = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}. \tag{29}$$

We write V_m for the complex vector space of homogeneous holomorphic polynomials of degree m, and we can thus identify p with an Hermitian form on V_m via its underlying matrix of coefficients. In this section we use the term *squared norm* for a *finite* sum of squared absolute values of holomorphic functions. We have observed that a power of the squared Euclidean norm is itself a squared norm; $||z||^{2d} = ||H_d||^2$. Observe also that the components of H_d form a basis for V_d.

There is a decisive theorem (see [3, 17]) in the bihomogeneous case: such a polynomial is positive away from the origin precisely when it is a quotient of squared norms of holomorphic polynomial mappings, and both the numerator and denominator vanish only at the origin. This is statement 4) below.

THEOREM 6. (Catlin-D'Angelo, Quillen) *The bihomogeneous case. Let* $f(z,\overline{z}) = \sum\sum c_{\alpha\beta} z^\alpha \overline{z}^\beta$ *be a real-valued polynomial that is homogeneous of degree* m *in* z *and also of degree* m *in* $\overline{z}$. *The following are equivalent.*

1) f *achieves a positive minimum value on the sphere.*

2) *There is an integer* d *such that the underlying Hermitian matrix for* $||z||^{2d} f(z,\overline{z})$ *is positive definite. Thus*

$$||z||^{2d} f(z,\overline{z}) = \sum E_{\mu\nu} z^\mu \overline{z}^\nu \tag{30}$$

where $(E_{\mu\nu})$ *is positive definite.*

3) *There is an integer* d *such that the operator* R_{m+d} *defined by the kernel* $k_d(z,\zeta) = \langle z,\zeta\rangle^d f(z,\overline{\zeta})$ *is a positive operator from* $V_{m+d} \subset A^2(B_n)$ *to itself.*

4) *There is an integer* d *and a holomorphic homogeneous vector-valued polynomial* g *of degree* $m+d$ *such that* $\mathbf{V}(g) = \{0\}$ *and such that* $||z||^{2d} f(z,\overline{z}) = ||g(z)||^2$. *Thus* f *is a quotient of squared norms.*

Theorem 6 says that, for bihomogeneous polynomials, a sharp form of 2.1.1) implies 2.1.2). Even for bihomogeneous polynomials, 2.1.1) does not imply 2.1.2), as Example 3 shows (when $a = 16$). It is also worth noting that the minimum d required in 4) hold may be smaller than the minimum d required in 2). Also, if for some d the matrix $E_{\mu\nu}$ in (30) is positive definite, then the same holds for all larger d. For that reason the author called this result a stabilization theorem for Hermitian forms.

The main assertion that 1) and 2) are equivalent was proved in 1967 by Quillen. Unaware of that result, Catlin and the author, motivated by trying to prove Theorem 1, found a different proof, which we give in Section 3.4. Both proofs use hard analysis; Quillen uses Gaussian integrals on all of $\mathbf{C^n}$, whereas Catlin and the author use the Bergman kernel function on the unit ball B_n. Quillen uses a priori estimates, whereas Catlin and the author use facts about compact operators on $L^2(B_n)$. In [4] they generalized to operators on weakly pseudoconvex domains some of these facts about compact operators. In the statement of Proposition 1 of that paper, the word tangential should appear before pseudodifferential operator, but was incorrectly omitted.

In [4] they reinterpreted Theorem 6 in terms of holomorphic vector bundles and generalized the result considerably in [5]. We state these in Theorems 7 and 8.

2.3. Proof of Theorem 1 from Theorem 6

Assuming that Theorem 6 is known, we prove Theorem 1. We homogenize to reduce to the bihomogeneous case, and add a term to ensure positivity.

THEOREM 1. *Suppose that* $z \to r(z,\overline{z})$ *is a real-valued polynomial on* $\mathbf{C}^n$, *and suppose that* $r(z,\overline{z}) > 0$ *for* $||z|| = 1$. *Then there is an integer* N *and a holomorphic polynomial mapping* $h : \mathbf{C}^n \to \mathbf{C}^N$ *such that*

$$r(z,\overline{z}) = ||h(z)||^2 = \sum |h_j(z)|^2 \tag{31}$$

for $||z|| = 1$.

Proof. We write

$$r(z,\overline{z}) = \sum c_{\alpha\beta} z^\alpha \overline{z}^\beta \tag{32}$$

where α and β are multi-indices, and $0 \leq |\alpha| \leq m$ and $0 \leq |\beta| \leq m$. Since r is real-valued, we have $c_{\alpha\beta} = \overline{c_{\beta\alpha}}$. By multiplying r by $||z||^2$ if necessary, we may assume that m is even without changing the hypotheses.

Let t be a complex variable. For a suitable positive constant C we define a bihomogeneous polynomial F_C on $\mathbf{C}^n \times \mathbf{C}$ by

$$F_C(z,t,\overline{z},\overline{t}) = C(||z||^2 - |t|^2)^m + \sum c_{\alpha\beta} z^\alpha \overline{z}^\beta t^{m-|\alpha|} \overline{t}^{m-|\beta|}. \tag{33}$$

We claim that the summation in (33) is positive when $||z||^2 = |t|^2 \neq 0$. To see this, observe that the summation equals $|t|^{2m} r(\frac{z}{t}, \frac{\overline{z}}{\overline{t}})$ and hence is positive when $||\frac{z}{t}||^2 = 1$. On the other hand, this expression is continuous in both z and t. It is positive when $||z||^2 = |t|^2 = \frac{1}{2}$, and hence there is a $\delta > 0$ and a positive number k so that

$$|t|^{2m} r(\frac{z}{t}, \frac{\overline{z}}{\overline{t}}) \geq k > 0 \tag{34}$$

when $|(||z||^2 - |t|^2)| < \delta$ and $||z||^2 + |t|^2 = 1$. Therefore, on the unit sphere in $\mathbf{C}^{n+1}$ and for $|(||z||^2 - |t|^2)| < \delta$ we have $F_C(z,t,\overline{z},\overline{t}) \geq k$. On the other hand, if $|(||z||^2 - |t|^2)| \geq \delta$, we have that

$$F_C(z,t,\overline{z},\overline{t}) \geq C\delta^m + \sum c_{\alpha\beta} z^\alpha \overline{z}^\beta t^{m-|\alpha|} \overline{t}^{m-|\beta|}. \tag{35}$$

The summation in (35) is a polynomial, so it is continuous, and hence achieves a minimum value m on the unit sphere in $\mathbf{C}^{n+1}$. If we choose

C so that $C\delta^m + m$ is positive, then the bihomogeneous polynomial F_C will be positive on the unit sphere. By Theorem 6, there is an integer d and a holomorphic polynomial mapping $h(z,t)$ so that

$$(||z||^2 + |t|^2)^d F_C(z,t,\overline{z},\overline{t}) = ||h(z,t)||^2 \tag{36}$$

holds everywhere. Setting $t = 1$ and then $||z||^2 = 1$ shows that

$$2^d r(z,\overline{z}) = ||h(z,1)||^2 \tag{37}$$

on the unit sphere in $\mathbf{C}^n$, and completes the proof. □

3. An isometric imbedding theorem for holomorphic bundles

3.1. Reinterpretation of Theorem 6

The purpose of the third lecture is to reinterpret Theorem 6 in the language of holomorphic line bundles, and then to generalize it. Let f be a bihomogeneous polynomial that is positive away from the origin in $\mathbf{C}^{N+1}$. The link to bundles arises by first considering complex projective space $\mathbf{P_N}$, the collection of lines through the origin in $\mathbf{C}^{N+1}$. We have the usual open covering of $\mathbf{P_N}$ given by open sets U_j; here U_j consists of those lines containing a point z with $z_j \neq 0$. In U_j we define f_j by

$$f_j(z,\overline{z}) = \frac{f(z,\overline{z})}{|z_j|^{2m}}. \tag{38}$$

On the overlap $U_j \cap U_k$ these functions then transform via

$$f_k = |(\frac{z_j}{z_k})^m|^2 f_j. \tag{39}$$

Since $(\frac{z_j}{z_k})^m$ are the transition functions for the m-th power of the universal line bundle $\mathbf{U}^m$, the positive functions f_j determine an Hermitian metric on $\mathbf{U}^m$. In Section 3.2 we discuss the universal bundle in more detail.

Suppose that f is a bihomogeneous polynomial of degree $2m$. Then it defines via (38) a metric on $\mathbf{U}^m$ if and only if it is positive as a function away from the origin. This is the sharp form of 2.1.1). This metric is a holomorphic pullback of the Euclidean metric on $\mathbf{U}$ over $\mathbf{P}_K$ for some K if and only if 2.1.3) or 2.1.4) holds. Some tensor power of the bundle with itself is a holomorphic pullback of the Euclidean metric if and only if 2.1.5) holds. In Section 3.2 we discuss the relationships with the Cauchy-Schwarz inequality 2.1.6) and with plurisubharmonicity. Strong plurisubharmonicity is equivalent to the negativity of the curvature of the bundle, or to the strong pseudoconvexity of the unit ball in the total space of the bundle.

The previous paragraph applies in particular to the function r_a from Example 3. When $a < 16$, this bihomogeneous polynomial is strictly positive away from the origin, and hence defines a metric on $\mathbf{U}^4$ on $\mathbf{P_1}$. By varying the parameter a we see that the corresponding positivity properties of bundle metrics are also distinct.

We next restate Theorem 6.

THEOREM 7. *Let* $(\mathbf{U}^m, f)$ *denote the* m*-th power of the universal line bundle over* $\mathbf{P_n}$ *with special metric defined by* f*. Then there is an integer* d *so that* $(\mathbf{U}^{m+d}, ||z||^{2d} f(z,\overline{z}))$ *is a (holomorphic) pullback* $g^*(\mathbf{U}, ||\zeta||^2)$ *of the standard metric on the universal bundle over* $\mathbf{P_N}$*. The mapping* $g : \mathbf{P_n} \to \mathbf{P_N}$ *is a holomorphic (polynomial) embedding.*

$$(\mathbf{U}^m, f) \otimes (\mathbf{U}^d, ||z||^{2d}) = (\mathbf{U}^{m+d}, ||z||^{2d} f(z,\overline{z})) = (\mathbf{U}^{m+d}, ||g(z)||^2).$$

We have the bundles and metrics

$$\pi_1 : (\mathbf{U}^m, f) \to \mathbf{P_n},$$
$$\pi_2 : (\mathbf{U}^{m+d}, ||z||^{2d} f) \to \mathbf{P_n},$$
$$\pi_3 : (\mathbf{U}, ||\zeta||^2) \to \mathbf{P_N}.$$

Thus π_1 is not an isometric pullback of π_3, but, for sufficiently large d, π_2 *is* such a pullback.

Yum-Tong Siu suggested to the author a reformulation of Theorem 6 in this language. This important suggestion led Catlin and the author to what we state as Theorem 8 in these lectures.

3.2. Globalizable metrics

The reinterpretation using line bundles of Theorem 6 for bihomogeneous polynomials extends to matrices of bihomogeneous polynomials. Let $G(z,\overline{z})$ be a matrix of bihomogeneous polynomials of the same degree. Suppose that, for each $z \neq 0$, the matrix $G(z,\overline{z})$ is positive definite. By elementary Hermitian linear algebra a p-by-p matrix R is positive definite if and only if it can be written as $R = A^*A$ for A of rank p. Here A^* denotes the conjugate transpose of A. Thus, for each fixed $z \neq 0$, we can factorize $G(z,\overline{z})$ as A^*A. In general we cannot make A depend holomorphically on z, even when G is a one-by-one matrix. If a bihomogeneous polynomial G is positive at each point, then it need not be a squared norm of a holomorphic mapping, as we have seen in Section

2. See [18] for some classical results about holomorphic factorization of operator-valued functions of one complex variable.

Suppose again that G is a matrix of bihomogeneous polynomials and that G is positive definite away from the origin. Theorem 6 and its reinterpretation Theorem 7 suggest that we might be able to holomorphically factorize $||z||^{2d}G(z,\overline{z})$ for sufficiently large d. This is true, and is a very special case of the general theorem we will prove.

The general result begins with a compact complex manifold M and a vector bundle E over M, equipped with an Hermitian metric G. We want to write $G = A^*A$, where A depends holomorphically on its variables, but this is generally impossible. So we assume that there is also a line bundle L over M, equipped with an Hermitian metric R satisfying properties analogous to those of the Euclidean metric on the universal bundle over projective space. Then we hope that the metric R^dG on the bundle $L^d \otimes E$ is of the form A^*A, where A is holomorphic. We interpret this factorization by saying that R^dG is the isometric pullback of the Euclidean metric on the universal bundle over the Grassman manifold.

Such a conclusion requires certain hypotheses on the metrics; for example both G and R must be real-analytic. The purpose of this section is to discuss properties of metrics that are necessary for such a theorem to hold.

Let $\mathbf{G}_{p,N}$ denote the Grassman manifold of p planes in complex N-space. When $p = 1$ we have complex projective space, and we write as usual $\mathbf{P}_{N-1}$ for $\mathbf{G}_{1,N}$. Let $\mathbf{U}_{p,N}$ denote the universal bundle over $\mathbf{G}_{p,N}$. This bundle is sometimes known as the tautological bundle; a point in $\mathbf{U}_{p,N}$ is a pair (S,ζ) where S is a p-dimensional subspace of $\mathbf{C}^N$ and $\zeta \in S$. We let g_0 denote the Euclidean metric on $\mathbf{U}_{p,N}$. By definition we have

$$g_0((S,u),(S,v)) = v^*u = \langle u,v\rangle = \sum u_j\overline{v}_j. \tag{40}$$

In (40), the Euclidean inner product $\langle u,v\rangle$ makes sense because we consider u,v as elements of $\mathbf{C}^N$.

The Euclidean metric on $\mathbf{U}_{p,N}$ is special. It makes sense to evaluate it at pairs of points in the total space, even if they have different base points. To clarify this, let $g_0' : \mathbf{U}_{p,N} \times \mathbf{U}_{p,N} \to \mathbf{C}$ be defined by

$$g_0'((S_1,v_1),(S_2,v_2)) = \langle v_1,v_2\rangle. \tag{41}$$

Then g_0' is holomorphic in the first variable, anti-holomorphic in the second, satisfies $g_0'(\alpha,\beta) = \overline{g_0'(\beta,\alpha)}$, and extends the metric in the sense that $g_0'(\alpha,\beta) = g_0(\alpha,\beta)$ when $\pi(\alpha) = \pi(\beta)$. Henceforth we drop the prime from the notation, and write g_0 for this function.

The Euclidean metric g_0 on $\mathbf{U}_{p,N}$ will be our model for a *globalizable* metric. First we study how it behaves under pullbacks. Let M be a complex manifold, and let $h : M \to \mathbf{G}_{p,N}$ be a holomorphic mapping. The pullback $\pi' : h^*(\mathbf{U}_{p,N}) \to M$ is then a bundle over M; a point in $h^*(\mathbf{U}_{p,N})$ is a pair (z, u), where $z \in M$, $u \in \mathbf{U}_{p,N}$, and $h(z) = \pi(u)$. Write α and β for points in $h^*(\mathbf{U}_{p,N})$. Writing h_* as usual for the map satisfying $h\pi' = \pi h_*$, there is a natural metric $h^*(g_0)$ defined on $h^*(\mathbf{U}_{p,N})$ by

$$h^*(g_0)(\alpha, \beta) = g_0(h_*\alpha, h_*\beta). \tag{42}$$

We see again that $h^*(g_0)$ makes sense when α and β are based at different points; we can therefore extend the definition of the metric to a function $h^*(g_0) : h^*(\mathbf{U}_{p,N}) \times h^*(\mathbf{U}_{p,N})$ as above. The metric $h^*(g_0)$ will also be globalizable.

For a vector bundle E over M we write E^* for its dual bundle and we write $H(M, E^*)$ for the holomorphic sections of E^*. When M is compact, $H(M, E^*)$ is a finite-dimensional complex vector space.

Definition 3. Let $\pi' : E \to M$ be a holomorphic vector bundle over a complex manifold M. We suppose that G is an Hermitian metric on E. We say that G is *globalizable* if there is a mapping $G' : E \times E \to \mathbf{C}$ such that the following properties hold:
1. G' extends the metric: $G'(u, v) = G(u, v)$ whenever $\pi'(u) = \pi'(v)$.
2. G' is holomorphic in the first variable: $G'(\cdot, v) \in H(M, E^*)$.
3. G' is Hermitian: $G'(u, v) = \overline{G'(v, u)}$.

Henceforth we will write G instead of G'. Suppose that G is globalizable and $\phi_1, ..., \phi_q$ form a basis for $H(M, E^*)$, Then there is a matrix G_{kj} of constants so that (43) holds.

$$G(u, v) = \sum_{j,k=1}^{q} G_{kj}\phi_j(u)\overline{\phi_k(v)}. \tag{43}$$

Not all bundles E admit globalizable metrics. A necessary and sufficient condition is that, for each non-zero vector $v \in E$ there is a section ϕ of E^* with $\phi(v) \neq 0$. The collection of these sections determines a holomorphic map to some $\mathbf{G}_{p,N}$.

Next we note that globalizable metrics are preserved under the tensor product. Suppose for $j = 1, 2$ that G_j is a globalizable Hermitian metric on a holomorphic vector bundle E_j over a complex manifold M. Then the formula

$$G(u_1 \otimes u_2, v_1 \otimes v_2) = G_1(u_1, v_1)G_2(u_2, v_2) \tag{44}$$

determines a globalizable metric G on $E_1 \otimes E_2$. It is natural to write $G = G_1 G_2$.

Suppose that G is globalizable, and that (43) holds. By property 3 of Definition 3, the matrix of coefficients (G_{kj}) must be Hermitian, but analogously to Example 3 it can have eigenvalues of both signs. Thus even when G is a metric, the underlying matrix of coefficients in (43) need not be positive definite. As in Section 2, if this is positive definite, then G defines a metric; furthermore we can think of G as defining an Hermitian inner product on $H(M, E^*)$. We then obtain a holomorphic mapping from M to $\mathbf{G}_{p,q}$ inducing the metric. To see this concretely, suppose that (G_{kj}) is a positive definite matrix. Then there are q independent vectors ζ_k in $\mathbf{C}^q$ so that $G_{kj} = \langle \zeta_j, \zeta_k \rangle$, and hence, from (43),

$$(45) \qquad G(u,v) = \sum_{j,k=1}^{q} \langle \zeta_j, \zeta_k \rangle \phi_j(u) \overline{\phi_k(v)} = \langle \sum \zeta_j \phi_j(u), \sum \zeta_k \phi_k(v) \rangle.$$

We can use (45) to construct a holomorphic map $h : M \to \mathbf{G}_{p,q}$. If we write $\psi(u) = \sum \zeta_j \phi_j(u)$, then ψ defines a map from E to $\mathbf{C}^q$. For $z \in M$, write E_z for the fibre over z. Then, since ψ is injective on each fibre, $\psi(E_z)$ is a p-dimensional subspace of $\mathbf{C}^q$; we write $h(z) = \psi(E_z)$. We see from (45) that $G(u, v) = g_0(h_*(u), h_*(v))$. Thus, as u_z varies over the p-dimensional fibre E_z, the formula $h(z) = \sum \phi_j(z)(u_z)\zeta_j$ defines a p-dimensional subspace of $\mathbf{C}^q$.

We summarize our discussion so far in the next proposition.

PROPOSITION 4. *Let G be a globalizable Hermitian metric on a holomorphic vector bundle V over M. Suppose that the matrix G_{kj} defined by (43) is positive definite. Then $G = h^*(g_0)$ for some holomorphic mapping $h : M \to \mathbf{G}_{p,N}$. Conversely, the pullback of a holomorphic map $h : M \to \mathbf{G}_{p,N}$ defines a globalizable metric on $h^*(\mathbf{U}_{p,N})$ by (42). Finally, the tensor product defined by (44) of globalizable metrics is globalizable.*

We return to our favorite example. Let $M = \mathbf{P}_{n-1}$ and let $V = \mathbf{U}^m_{1,N} = \mathbf{U}^m$. The sections of the dual bundle are then the homogeneous monomials z^α, and the globalizable metric G is nothing more than a bihomogeneous polynomial that is positive away from the origin.

Theorem 8 will involve both a vector bundle E of arbitrary rank and a "good" line bundle L by which we tensor. The theorem must hold when $E = L$; this is analogous to 2.1.5) for bihomogeneous polynomials.

DEFINITION 4. Let L be a line bundle over M with globalizable Hermitian metric R.

1) Then R satisfies the global Cauchy-Schwarz inequality (GCS) if

$$(GCS) \qquad |R(u_1, u_2)|^2 \leq R(u_1, u_1)R(u_2, u_2)$$

and R satisfies the sharp global Cauchy-Schwarz inequality ($SGCS$) if

$$(SGCS) \qquad |R(u_1, u_2)|^2 < R(u_1, u_1)R(u_2, u_2)$$

holds whenever $u_1 \neq u_2$ and their vector parts are non-zero.

2) L is negative if the unit disk $B = \{z : R(z, z) < 1\}$ is a strongly pseudoconvex domain in L.

These properties are analogues of 2.1.6) and 2.1.7). We recall some standard definitions and use them to clarify when a holomorphic mapping to the Grassman manifold is an imbedding. Let g be a fibre metric on a Hermitian vector bundle E over a complex manifold M. We write $\mathrm{Ric}(g)$ for the Ricci curvature form $\overline{\partial}\partial(\log(\mathrm{Det}(g))$. Let g_0 be the Euclidean metric on the universal bundle; its Ricci curvature form is negative. Thus for nonzero tangent vectors v we have $\mathrm{Ric}(g_0)(v, v) < 0$.

PROPOSITION 5. *Suppose that M is a compact complex manifold, and that $h : M \to \mathbf{G}_{p,N}$ is a holomorphic mapping. Let $E = h^*(\mathbf{U}_{p,N})$ denote the pullback bundle, and assume we are given the pullback metric $h^*(g_0)$ on E. Then*

1. *h is an immersion if and only if the $(1, 1)$ form $\mathrm{Ric}(h^*(g_0))$ is negative.*

2. *h is injective if and only if for all distinct points z, w in M, there is $u \in E_z$ such that, for all nonzero $v \in E_w$, the ($SGCS$) inequality*

$$(46) \qquad |h^*(g_0)(u, v)|^2 < |h^*(g_0)(u, u)|^2 |h^*(g_0)(v, v)|^2$$

holds.

Proof. We refer to [5] for a proof. □

PROPOSITION 6. *Suppose that L is a negative line bundle over a compact complex manifold M, and that R is a globalizable Hermitian metric on L. Let $N + 1$ denote the dimension of $H(M, (L^*)^d)$. Suppose that $h_d : M \to \mathbf{P}^N$ is a holomorphic immersion, and $R^d = h_d^*(g_0)$. Then (GCS) must hold for R, and the curvature of R must be negative. If also h_d is an imbedding, then ($SGCS$) holds.*

Proof. First we show that $(SGCS)$ holds when h_d is an imbedding. Write l^d for the d-fold tensor product. Since h_d is assumed to be injective, Proposition 5 implies that

$$|R^d(l_1^d, l_2^d)|^2 < R^d(l_1^d, l_1^d)R^d(l_2^d, l_2^d)$$

and hence that

$$|R(l_1, l_2)|^{2d} < R(l_1, l_1)^d R(l_2, l_2)^d. \tag{47}$$

Taking d-th roots of (47) yields $(SGCS)$.

Next we show that R has negative curvature. Since h_d is an immersion, Proposition 5 implies that $\mathrm{Ric}(R^d)$ is negative. Since $\mathrm{Ric}(R^d) = d\mathrm{Ric}(R)$ we see that $\mathrm{Ric}(R)$ is also negative.

When $R^d = h^*(g_0)$, we see that B is defined by the equation

$$r(u) = g_0(h_*u, h_*u) = ||H(u)||^2 < 1. \tag{48}$$

The domain B will be strongly pseudoconvex if the complex Hessian of a defining function is positive definite on the boundary. Choosing $a \in \mathbf{C}^n$, we have

$$\partial\bar{\partial}r(a, a) = ||\partial H(a)||^2 \geq c||a||^2 \tag{49}$$

whenever ∂H has maximal rank. Formula (49) shows that B is strongly pseudoconvex whenever h is an immersion. □

3.3. Isometric imbedding

Theorem 8 from these lectures is the main result from [5]. In certain situations where Kodaira's famous imbedding theorem (See [20]) applies, this result shows that the imbedding can be taken to be an isometry. This isometric imbedding theorem for holomorphic bundles includes Theorem 6 as a concrete special case. In [2] Catlin applies ideas similar to those in the proof to improve a result of Tian. Theorem 8 also generalizes Calabi's famous result [1] on isometric imbeddings of the tangent bundle.

We state Theorem 8 in this section. We complete these lectures in the next section by sketching a proof of the more concrete Theorem 6. The proofs are similar, although things are of course much easier in the setting of bihomogeneous polynomials. For Theorem 6 we use the Bergman projection and kernel function for the unit ball B_n. For Theorem 8 we need analogous results for the Bergman projection for the unit ball in the total space of the line bundle L.

THEOREM 8. *Suppose that M is a compact complex manifold. Let E be a vector bundle of rank p over M with globalizable Hermitian metric*

G. Let L be a line bundle over M with globalizable Hermitian metric R, and suppose that L is negative and that R satisfies (SGCS). Then there is an integer d_0 so that, for all d with $d \geq d_0$, there is a holomorphic imbedding h_d with $h_d : M \to \mathbf{G}_{p,N}$ so that $E \otimes L^d = h_d^* \mathbf{U}_{p,N}$ and $GR^d = h_d^*(g_0)$.

REMARK 1. Theorem 7 is an immediate corollary. It is the special case where M is complex projective space $\mathbf{P}_{n-1}$, where E is a power $\mathbf{U}^m$ of the universal bundle, with metric determined by the bihomogeneous polynomial p, and L is the universal bundle $\mathbf{U}$ with the Euclidean metric.

3.4. Proof of Theorem 6

To complete these lectures we sketch the proof of Theorem 6.

Proof. (Sketch) Theorem 6 states the equivalence of four statements. It is obvious that 2) implies 4) and 4) implies 1). It is easy to show that 2) and 3) are equivalent. The strategy of the proof is thus to show that 1) implies 3).

We do this using Hilbert space ideas. As usual $L^2(B_n)$ denotes the Hilbert space of square integrable holomorphic functions on the unit ball. We let $A^2(B_n)$ denote the closed subspace of $L^2(B_n)$ consisting of holomorphic functions. Let $P : L^2(B_n) \to A^2(B_n)$ denote the orthogonal projection, called the Bergman projection. The Bergman kernel function is the integral kernel corresponding to P. For the unit ball, the Bergman kernel is well known. We have

$$B(z,\zeta) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z,\zeta \rangle)^{n+1}}. \tag{50}$$

Observe that the Bergman kernel is holomorphic in z and conjugate holomorphic in ζ when both variables lie in the ball.

The Bergman kernel for a bounded domain Ω is always equal to $\sum \phi_j(z)\overline{\phi_j(\zeta)}$, where the ϕ_j denote a complete orthonormal set for $A^2(\Omega)$. In the case of the ball, normalized monomials form such a complete orthonormal set, as is easily seen by integration in polar coordinates. For the ball we also have the nice formula

$$B(z,\zeta) = \sum c_d \langle z,\zeta \rangle^d \tag{51}$$

for appropriate positive constants c_d. In fact c_d equals $\frac{n!}{\pi^n}$ times an appropriate binomial coefficient.

Next recall that V_m denotes the space of holomorphic homogeneous polynomials of degree m. We see that V_m is orthogonal to V_k for $m \neq k$, and thus $A^2(B_n)$ is the orthogonal sum of the V_m.

Let $r(z,\overline{z})$ be a bihomogeneous polynomial of degree $2m$; we can use r as an integral kernel. We define an integral operator S_r on $L^2(B_n)$ by

$$S_r h(z) = \int r(z,\overline{\zeta})h(\zeta)dV(\zeta). \tag{52}$$

It follows easily from (52) and the orthogonality of monomials that S_r annihilates V_k for $k \neq m$, and that S_r maps V_m to itself.

Recall that an operator on a Hilbert space is compact if and only if it is the norm limit of operators with finite-dimensional range. We will show that a certain operator T on $L^2(B_n)$ satisfies $T = Q + K$, where the restriction of Q to $A^2(B_n)$ is positive, and where K is compact on $L^2(B_n)$. Furthermore, we will have $T = \sum T_d$, where each T_d is zero on each V_j except for V_{m+d}. The compactness of K then implies that there is an integer d_0, so that T_d is positive on V_{m+d} for $d \geq d_0$.

We return to the setting of the theorem, where f denotes the given bihomogeneous polynomial, and f is assumed to be positive away from the origin. It is easy to show that statements 2) and 3) from Theorem 6 are equivalent. In fact, for any d, the underlying Hermitian matrix for $||z||^{2d}f(z,\overline{z})$ is positive definite if and only if the operator on $L^2(B_n)$ defined by the kernel $\langle z,\zeta\rangle^d f(z,\overline{\zeta})$ is a positive operator from $V_{m+d} \subset A^2(B_n)$ to itself. Thus we will choose T_d to be a positive constant times this operator.

Let T_d be the integral operator on $L^2(B_n)$ whose integral kernel is

$$c_d\langle z,w\rangle^d f(z,\overline{\zeta}), \tag{53}$$

where each c_d is a positive number. Choose these positive numbers as in (51). Then we see that $T = \sum T_d = PS_f$, where P is the Bergman projection.

In other words, the integral kernel of T is

$$B(z,\zeta)\sum c_{\alpha\beta} z^\alpha \overline{\zeta}^\beta. \tag{54}$$

Let S denote the operator on $L^2(B_n)$ given by multiplication by $f(z,\overline{z})$. Then $PS = SP$ because P integrates with respect to ζ. Notice that $S_f - S$ is the operator whose kernel $K(z,\zeta)$ is given by

$$K(z,\zeta) = \sum c_{\alpha\beta} z^\alpha(\overline{\zeta}^\beta - \overline{z}^\beta). \tag{55}$$

We add and subtract in order to take advantage of the positivity of f. Thus we write

$$T = \sum T_d = PS_f = P(S_f - S) + PS = P(S_f - S) + SP. \tag{56}$$

Finally we add and subtract the operator PM, where M is multiplication by a nonnegative cut off function χ that is positive at the origin and has compact support. From (56) we obtain

$$T = P(S_f - S) + (SP + PM) - PM. \tag{57}$$

In (57) the operator PM is compact, because its kernel is smooth and compactly supported. We claim that the operator $SP + PM$ is positive on $A^2(B_n)$. This is easy to show. For $h \in A^2(B_n)$ we have $Ph = h$. Also P is self-adjoint. Therefore we have

$$\langle SPh + PMh, h\rangle_2 = \langle (S + M)h, h\rangle_2. \tag{58}$$

But $S + M$ is a positive operator, because $f + \chi$ is strictly positive on the ball.

Finally we claim that the operator $P(S_f - S)$ is also compact. Its kernel is

$$B(z,\zeta)(f(z,\overline{\zeta}) - f(z,\overline{z})) = B(z,\zeta)\sum c_{\alpha\beta}z^\alpha(\overline{\zeta}^\beta - \overline{z}^\beta). \tag{59}$$

The only singularity for $B(z, \zeta)$ occurs on the boundary diagonal, but the other term in (59) vanishes there, and compensates for the singularity. A precise proof of the compactness then follows from Young's inequality.

There is a second way to see the compactness. Observe from (59) that $P(S_f - S)$ is a finite sum of bounded multiplication operators $c_{\alpha\beta}z^\alpha$ times commutators $[P, \overline{w}^\beta]$. We write $\overline{w}^\beta$ to avoid confusion. In one term we multiply by $\overline{\zeta}^\beta$ and then apply P; in the other term we apply P first, which changes the variable from ζ to z, and then we multiply by $\overline{z}^\beta$. Catlin and the author [4] proved in a general situation (of which the strongly pseudoconvex case is the easy one) that commutators of P with bounded multiplication operators are compact. Since the compact operators form an ideal in the space of bounded operators, we see from this and (59) that $P(S_f - S)$ is compact.

We have thus shown that $\sum T_d = Q + K$ where Q is positive on $A^2(B_n)$ and K is compact on $L^2(B_n)$. Hence, as indicated above, the restriction of T_d to V_{m+d} is strictly positive on that space for sufficiently large d. This completes the sketch of the proof of Theorem 6. □

REMARK 2. We have already noted in Proposition 6 that the $(SGCS)$ is needed for the conclusion of Theorem 8. It is interesting to see how this arises in the proof. The proof of Theorem 8 uses the Bergman kernel function in a manner similar to its use in the above proof of Theorem 6. One replaces $(1 - \langle z, \zeta\rangle)^{-n-1}$ by $(1 - R(z, \zeta))^{-n-1}$, where R