PROPER r-HARMONIC SUBMANIFOLDS INTO ELLIPSOIDS AND ROTATION HYPERSURFACES

S. MONTALDO AND A. RATTO

Abstract. The study of r-harmonic maps was proposed by Eells-Sampson in 1965 and by Eells-Lemaire in 1983. These maps are a natural generalization of harmonic maps and are defined as the critical points of the r-energy functional $E_r(\varphi) = (1/2) \int_M |(d^*+d)^r(\varphi)|^2 \, dv_M$, where $\varphi : M \to N$ denotes a smooth map between two Riemannian manifolds. If an r-harmonic map $\varphi : M \to N$ is an isometric immersion and it is not minimal, then we say that $\varphi(M)$ is a proper r-harmonic submanifold of N. In this paper we prove the existence of several new, proper r-harmonic submanifolds into ellipsoids and rotation hypersurfaces.

1. Introduction

Harmonic maps are the critical points of the *energy* functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, dv_M \ , \tag{1.1}$$

where $\varphi : M \to N$ is a smooth map between two Riemannian manifolds M and N. In particular, φ is harmonic if it is a solution of the Euler-Lagrange system of equations associated to (1.1), i.e.

$$-d^* d\varphi = \operatorname{trace} \nabla d\varphi = 0 \ . \tag{1.2}$$

The left member of (1.2) is a vector field along the map φ or, equivalently, a section of the pull-back bundle $\varphi^{-1}(TN)$: it is called *tension field* and denoted $\tau(\varphi)$. In addition, we recall that, if φ is an *isometric* immersion, then φ is a harmonic map if and only if $\varphi(M)$ is a minimal submanifold of N (see [7, 8] for background).

A related topic of growing interest deals with the study of *polyharmonic maps*, or *r-harmonic maps*: these maps, which provide a natural generalization of harmonic maps, are the critical points of the r-energy functional (as suggested in [8], [11])

$$E_r(\varphi) = \frac{1}{2} \int_M |(d^* + d)^r(\varphi)|^2 \, dv_M \ . \tag{1.3}$$

In the case that $r = 2$, the functional (1.3) is called bienergy and its critical points are the so-called biharmonic maps. There have been extensive studies on biharmonic maps (see [14, 19] for an introduction to this topic and [20, 22] for an approach which is related to this paper). In 1989 Wang [28] studied the first variational formula of the r-energy (1.3) while the expression for its second variation was derived in [17], where it was shown that a biharmonic map is not always r-harmonic ($r \geq 3$) and, more generally, that an s-harmonic

2000 *Mathematics Subject Classification.* Primary: 58E20; Secondary: 53C43.

Key words and phrases. harmonic maps, r-harmonic maps, r-harmonic immersions, Chen's conjecture.

Work supported by P.R.I.N. 2015 – Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis – Italy and by Fondazione di Sardegna and Regione Autonoma della Sardegna (Project GESTA).

map is not always r-harmonic $(2 \leq s < r)$. On the other hand, any harmonic map is trivially r-harmonic for all $r \geq 2$. Therefore, we say that an immersed submanifold of N is a *proper* r-harmonic submanifold if the immersion is an r-harmonic map which is *not* harmonic (or, equivalently, *not* minimal).

As a general fact, when the ambient has nonpositive sectional curvature there are several results which assert that, under suitable conditions, an r-harmonic submanifold is minimal (see [5], [15], [18] and [25], for instance), but the Chen conjecture that a biharmonic submanifold of $\mathbb{R}^n$ must be minimal is still open (see [6] for recent results in this direction). More generally, the Maeta conjecture (see [15]) that any r-harmonic submanifold of the Euclidean space is minimal is open. By contrast, in our recent paper [23] we produced several new proper r-harmonic submanifolds of the Euclidean unit sphere $\mathbb{S}^m$ $(r \geq 4$, extending the previous results of [16] for $r = 3)$. The aim of this paper is to continue the work started in [23] and describe some extensions to cases where the ambient manifold does not have constant sectional curvature. In particular, we shall construct new examples of r-harmonic submanifolds into Euclidean ellipsoids. The search for critical points in the presence of ellipsoidal deformation of an Euclidean sphere was started by R.T Smith in [27] and has a long history in the theory of harmonic maps (see [1, 2, 9, 12], for instance). In order to explain the type of results which we will obtain in this context, let $E^m(b) \subset \mathbb{R}^{m+1}$ $(b > 0)$ be the Euclidean ellipsoid defined by

$$x_1^2 + \ldots + x_m^2 + \frac{x_{m+1}^2}{b^2} = 1 \tag{1.4}$$

(of course, $E^m(1) = \mathbb{S}^m$). We shall study submanifolds of $E^m(b)$ obtained by intersection with a hyperplane of the type $x_{m+1} =$ constant and prove the following

Theorem 1.1. *Assume $r, m \geq 2$ and $b > 0$. Let us consider submanifolds of $E^m(b) \subset \mathbb{R}^{m+1}$ of the following type:*

$$\begin{array}{cccccc} \varphi_{\alpha^*} : & \mathbb{S}^{m-1} & \to & E^m(b) \subset & \mathbb{R}^m \times \mathbb{R} \\ & w & \mapsto & & (\sin\alpha^* w, b\cos\alpha^*) \ , \end{array}$$

where $0 < \alpha^* < \pi$ is a fixed value. Then $\varphi_{\alpha^*}(\mathbb{S}^{m-1}) = \mathbb{S}^{m-1}(\sin\alpha^*)$ is a proper r-harmonic submanifold of $E^m(b)$ if and only if

$$\sin\alpha^* = \frac{1}{\sqrt{\frac{b}{2}\left(\sqrt{b^2(r-2)^2 + 4(r-1)} + b(r-2)\right) + 1}} \tag{1.5}$$

Remark 1.2. We point out that the right member of equation (1.5) is always a real number in the interval $(0, 1)$. Therefore, for all $r, m \geq 2$ and $b > 0$, we have the existence of two proper r-harmonic parallel hyperspheres which lie in a symmetric position with respect to the equator plane $x_{m+1} = 0$. The case $r = 2$ of Theorem 1.1 was proved in [21] by different methods, while the spherical case $a = b, r \geq 3$ was first obtained in [16, 23].

In the context of rotation surfaces, we shall prove the following result:

Theorem 1.3. *Assume that $C > 0$. Let $S_{\text{par}} \subset \mathbb{R}^3$ be the paraboloid of revolution defined by*

$$z = C\left(x^2 + y^2\right). \tag{1.6}$$

Then a circle $C_{\alpha^*}=S_{\rm par}\cap\{z=\alpha^{*2}\}$ *is a proper* r*-harmonic submanifold of* $S_{\rm par}$ $(r\geq 3)$ *if and only if*

$$(\alpha^*)^2=\frac{1}{4C^2(r-2)}. \tag{1.7}$$

Remark 1.4. It is well-known that $S_{\rm par}$ admits no closed geodesic. In addition, it was proved in [24] that there exists no proper biharmonic curve in $S_{\rm par}$, a fact which, together with Theorem 1.3, suggests that the study of proper r-harmonic submanifolds may display phenomena which are substantially different from those observed in the biharmonic case. In this order of ideas, we point out that the original motivation of the Eells-Sampson work [10] was to investigate the existence of a harmonic map in a given homotopy class: in Example 4.2 below we shall provide an instance where a homotopy class does not admit any harmonic representative, but it does contain an r-harmonic map for each $r\geq 2$.

Our paper is organized as follows: in Section 2 we recall some basic facts and fundamental formulas concerning r-harmonic maps. In Section 3 we introduce two families of rotationally symmetric manifolds which are suitable to prepare the ground for our study of r-harmonic submanifolds of ellipsoids and rotation hypersurfaces. In this section we shall also compute all the relevant equations which we will need in our symmetric context. Next, in Section 4, we shall first provide the proof of Theorem 1.1 and 1.3. In the second part of Section 4 we shall prove the existence of certain proper, r-harmonic Clifford's torus type submanifolds and also study r-harmonic immersions into rotation hypersurfaces. In the latter context we shall find that the existence of proper r-harmonic parallel hyperspheres requires positive radial curvature of the ambient space.

2. Generalities on r-harmonic maps

In order to make this paper reasonably self-contained we recall here some basic facts about r-harmonic maps (details and proofs can be found in [8] and [15]). Let $\varphi:(M,g_M)\to(N,g_N)$ be a smooth map between two Riemannian manifolds M and N of dimension m and n respectively. Then $d\varphi$ is a 1-form with values in the vector bundle $\varphi^{-1}TN$ or, equivalently, a section of $T^*M\otimes\varphi^{-1}TN$. In local charts $d\varphi$ is described by

$$d\varphi=\frac{\partial\varphi^\gamma}{\partial x_i}\,dx^i\otimes\frac{\partial}{\partial y_\gamma}=\varphi_i^\gamma\;dx^i\otimes\frac{\partial}{\partial y_\gamma}\;, \tag{2.1}$$

where, here and below, the Einstein's sum convention over repeated indices is adopted. The second fundamental form $\nabla d\varphi$ is a covariant differentiation of the 1-form $d\varphi$, i.e., a section of $\odot^2T^*M\otimes\varphi^{-1}TN$. The local coordinates expression for the second fundamental form is

$$\nabla d\varphi=(\nabla d\varphi)_{ij}^\gamma\;dx^idx^j\otimes\frac{\partial}{\partial y_\gamma}\;, \tag{2.2}$$

where

$$\begin{aligned}(\nabla d\varphi)_{ij}^\gamma\;&=\;\left[\nabla_{\partial/\partial x_i}\left(\varphi_\ell^\beta\;dx^\ell\otimes\frac{\partial}{\partial y_\beta}\right)\frac{\partial}{\partial x_j}\right]^\gamma\\ &=\;\varphi_{ij}^\gamma-{}^M\Gamma_{ij}^k\varphi_k^\gamma+{}^N\Gamma_{\beta\delta}^\gamma\varphi_i^\beta\varphi_j^\delta\;.\end{aligned} \tag{2.3}$$

Now, since $\tau(\varphi) = -d^* d\varphi$ is the trace of the second fundamental form, its description in local coordinates is

$$(\tau(\varphi))^\gamma = (-d^* d\varphi)^\gamma = g_M^{ij} \; (\nabla d\varphi)_{ij}^\gamma \; . \tag{2.4}$$

For our purposes, it is useful to recall the definitions of the rough laplacian and of the sectional curvature operator. We shall denote by ∇^M, ∇^N and ∇^φ the induced connections on the bundles TM, TN and $\varphi^{-1}TN$ respectively. The *rough Laplacian* on sections of $\varphi^{-1}(TN)$, denoted by $\overline{\Delta}$, is defined by

$$\overline{\Delta} = d^* d = -\sum_{i=1}^{m} \{ \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla^M_{e_i} e_i} \} \; , \tag{2.5}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame on M. The curvature operator on N is the $(1,3)$-tensor defined by:

$$R^N(X,Y)W = \nabla^N_X \nabla^N_Y W - \nabla^N_Y \nabla^N_X W - \nabla^N_{[X,Y]} W \; . \tag{2.6}$$

We now proceed to a general description of the r-energy (1.3) when $r \geq 2$. If $r = 2s$, $s \geq 1$:

$$\begin{aligned} E_{2s}(\varphi) &= \frac{1}{2} \int_M \langle \underbrace{(d^* d) \ldots (d^* d)}_{s \text{ times}} \varphi, \underbrace{(d^* d) \ldots (d^* d)}_{s \text{ times}} \varphi \rangle_{{}_N} \; dv_M \\ &= \frac{1}{2} \int_M \langle \overline{\Delta}^{s-1} \tau(\varphi), \overline{\Delta}^{s-1} \tau(\varphi) \rangle_{{}_N} \; dv_M \end{aligned} \tag{2.7}$$

Now, the map φ is $2s$-harmonic if, for all variations φ_t,

$$\frac{d}{dt} E_{2s}(\varphi_t) \Big|_{t=0} = 0 \; .$$

Setting

$$V = \frac{\partial \varphi_t}{\partial t} \Big|_{t=0} \in \Gamma(\varphi^{-1} TN) \; ,$$

$$\frac{d}{dt} E_{2s}(\varphi_t) \Big|_{t=0} = - \int_M \langle \tau_{2s}(\varphi), V \rangle_{{}_N} \; dv_M \; , \tag{2.8}$$

where the explicit formula for the $2s$-tension field $\tau_{2s}(\varphi)$ is:

$$\begin{aligned} \tau_{2s}(\varphi) &= \overline{\Delta}^{2s-1} \tau(\varphi) - R^N \left(\overline{\Delta}^{2s-2} \tau(\varphi), d\varphi(e_j) \right) d\varphi(e_j) \\ & - \sum_{\ell=1}^{s-1} \Big\{ R^N \left(\nabla^\varphi_{e_j} \overline{\Delta}^{s+\ell-2} \tau(\varphi), \overline{\Delta}^{s-\ell-1} \tau(\varphi) \right) d\varphi(e_j) \\ & \qquad - R^N \left(\overline{\Delta}^{s+\ell-2} \tau(\varphi), \nabla^\varphi_{e_j} \overline{\Delta}^{s-\ell-1} \tau(\varphi) \right) d\varphi(e_j) \Big\} \; , \end{aligned} \tag{2.9}$$

where $\overline{\Delta}^{-1} = 0$ and $\{e_j\}_{j=1}^m$ is a local orthonormal frame on M (here and below, the sum over j is not written but understood). Of course, φ is $2s$-harmonic if $\tau_{2s}(\varphi)$ vanishes identically.

In the case that $r=2s+1$, the relevant modifications are:

$$
\begin{aligned}
E_{2s+1}(\varphi) &= \frac{1}{2}\int_M \langle\, d\,\underbrace{(d^*d)\ldots(d^*d)}_{s\ \text{times}}\varphi,\; d\,\underbrace{(d^*d)\ldots(d^*d)}_{s\ \text{times}}\varphi\,\rangle_{_N}\; dv_M \\
&= \frac{1}{2}\int_M \langle\, \nabla^{\varphi}_{e_j}\,\overline{\Delta}^{s-1}\tau(\varphi),\; \nabla^{\varphi}_{e_j}\,\overline{\Delta}^{s-1}\tau(\varphi)\,\rangle_{_N}\; dv_M\ ;
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
\tau_{2s+1}(\varphi) = {} & \overline{\Delta}^{2s}\tau(\varphi) - R^N\left(\overline{\Delta}^{2s-1}\tau(\varphi), d\varphi(e_j)\right) d\varphi(e_j) \\
& - \sum_{\ell=1}^{s-1}\Big\{R^N\left(\nabla^{\varphi}_{e_j}\,\overline{\Delta}^{s+\ell-1}\tau(\varphi), \overline{\Delta}^{s-\ell-1}\tau(\varphi)\right) d\varphi(e_j) \\
& \qquad - R^N\left(\overline{\Delta}^{s+\ell-1}\tau(\varphi), \nabla^{\varphi}_{e_j}\,\overline{\Delta}^{s-\ell-1}\tau(\varphi)\right) d\varphi(e_j)\Big\} \\
& - R^N\left(\nabla^{\varphi}_{e_j}\,\overline{\Delta}^{s-1}\tau(\varphi), \overline{\Delta}^{s-1}\tau(\varphi)\right) d\varphi(e_j)\ .
\end{aligned} \tag{2.11}
$$

3. Rotationally symmetric manifolds and the relevant equations

First, let us introduce a family of manifolds which will be suitable for our purposes. We set

$$
(M, g_M) = \left(\mathbb{S}^{m-1}\times I, f^2(\alpha)g_{\mathbb{S}^{m-1}} + k^2(\alpha)d\alpha^2\right), \tag{3.1}
$$

where $I\subset\mathbb{R}$ is an open interval and $f(\alpha), k(\alpha)$ are smooth, positive functions on I. In some instances, one or both the endpoints of I will be admitted: by way of example, if $I=[0,+\infty)$, $k(\alpha)\equiv 1$ and

$$
\begin{cases} f(0)=0\,, \quad f'(0)=1; \\ f^{(2\ell)}(0)=0 \quad \text{for all } \ell\geq 1\ , \end{cases} \tag{3.2}
$$

then our manifold (3.1) is a *model* in the sense of Greene and Wu (see [13]). In particular, if $f(\alpha)=\alpha$ (respectively, $f(\alpha)=\sinh\alpha$) we have the Euclidean space $\mathbb{R}^m$ (respectively, the hyperbolic space $\mathbb{H}^m$). In a similar spirit, if $I=[0,\pi]$, $f(\alpha)=\sin\alpha$ and $k^2(\alpha)=b^2\sin^2\alpha+\cos^2\alpha$ ($b>0$), then the manifold (3.1) is isometric to the Euclidean ellipsoid $E^m(b)\subset\mathbb{R}^{m+1}$ defined in (1.4). By way of summary, we shall refer to a manifold as in (3.1) as to a *rotationally symmetric manifold* and, to shorten notation, we shall write $M_{f,k}$ to denote it.

Remark 3.1. Performing a suitable change of coordinates, it is always possible to reduce the study of manifolds of the type (3.1) to the case that $k\equiv 1$. However, in order to describe more directly some applications to ellipsoids and rotation surfaces (see Section 4 below), it is convenient to consider here the general case of metrics as in (3.1).

We work with coordinates w_j, α on $M_{f,k}$, where $w_1, \ldots, w_{m-1}$ is a set of local coordinates on $\mathbb{S}^{m-1}$. A straightforward computation, based on the well-known formula

$$
\Gamma^k_{ij} = \frac{1}{2}\,g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial y_i}+\frac{\partial g_{\ell i}}{\partial y_j}-\frac{\partial g_{ij}}{\partial y_\ell}\right)\ , \tag{3.3}
$$

leads us to establish the following lemma:

Lemma 3.2. *Let $w_1, \ldots, w_{m-1}, \alpha$ be local coordinates as above on $M_{f,k}$. Then their associated Christoffel's symbols Γ_{ij}^k are described by the following table:*

$$
\begin{array}{lll}
\text{(i)} & \text{If } 1 \leq i,j,k \leq m-1: & \Gamma_{ij}^k = {}^{\mathbb{S}}\Gamma_{ij}^k \\
\text{(ii)} & \text{If } 1 \leq i,j \leq m-1: & \Gamma_{ij}^m = -\frac{f(\alpha)\,f'(\alpha)}{k^2(\alpha)}\,(g_{\mathbb{S}})_{ij} \\
\text{(iii)} & \text{If } 1 \leq i,j \leq m-1: & \Gamma_{im}^j = \frac{f'(\alpha)}{f(\alpha)}\,\delta_i^j \\
\text{(iv)} & \text{If } 1 \leq j \leq m-1: & \Gamma_{mm}^j = 0 = \Gamma_{jm}^m\ , \\
\text{(v)} & \text{If } i=j=k=m: & \Gamma_{mm}^m = \frac{k'(\alpha)}{k(\alpha)}\ .
\end{array} \tag{3.4}
$$

where ${}^{\mathbb{S}}\Gamma_{ij}^k$ and $g_{\mathbb{S}}$ denote respectively the Christoffel symbols and the metric tensor of $\mathbb{S}^{m-1}$ with respect to the coordinates $w_1, \ldots, w_{m-1}$.

Now we are in the right position to start our process of computing the quantities and equations which are relevant to the study of r-harmonicity in the context of rotationally symmetric manifolds (3.1). Our first goal is to derive the condition of r-harmonicity for a parallel hypersphere in $M_{f,k}$. More precisely, let us consider maps of the following type:

$$
\begin{array}{rccc}
\varphi_{\alpha^*}: & \mathbb{S}^{m-1} & \to & M_{f,k} \\
 & w & \mapsto & (w, \alpha^*)\ ,
\end{array} \tag{3.5}
$$

where α^* is a fixed constant value in the interior of the interval I. We focus on finding under which conditions a map φ_{α^*} as in (3.5) is proper r-harmonic. As a first step we shall prove the following

Proposition 3.3. *Assume that $m, r \geq 2$. Let $\varphi_{\alpha^*}: \mathbb{S}^{m-1} \to M_{f,k}$ be a map of the type (3.5). Then its r-energy is*

$$E_r(\varphi_{\alpha^*}) = \frac{1}{2}\,\mathrm{Vol}(\mathbb{S}^{m-1})\,\varepsilon_r(\alpha^*), \tag{3.6}$$

where

$$\varepsilon_r(\alpha) = (m-1)^r\, f^2(\alpha) \left[\frac{f'^2(\alpha)}{k^2(\alpha)}\right]^{(r-1)}. \tag{3.7}$$

The proof of Proposition 3.3 is based on a series of lemmata in which we compute, using Lemma 3.2, the relevant covariant derivatives. To this purpose, it is convenient to begin with a specific preliminary work. First, we observe that $d\varphi_{\alpha^*}(\partial/\partial w_j) = \partial/\partial w_j$ for all $1 \leq j \leq m-1$ and so, if it is clear from the context, we will not state explicitly whether a vector field $\partial/\partial w_i$ is to be considered on the domain or on the codomain. The next lemma is elementary:

Lemma 3.4. *Let φ_{α^*} be a map as in (3.5). Then*

$$\tau(\varphi_{\alpha^*}) = F(\alpha^*)\frac{\partial}{\partial\alpha} \tag{3.8}$$

where we have set

$$F(\alpha^*) = -\frac{1}{2}\,\frac{\varepsilon_1'(\alpha^*)}{k^2(\alpha^*)} = -\,(m-1)\,\frac{f(\alpha^*)\,f'(\alpha^*)}{k^2(\alpha^*)}. \tag{3.9}$$

Now we prove three lemmata which will play a key role:

Lemma 3.5. *Let φ_{α^*} be a map as in* (3.5)*. Then*

$$d\tau(\varphi_{\alpha^*}) = G(\alpha^*) \sum_{i=1}^{m-1} dw^i \otimes \frac{\partial}{\partial w_i} \ , \tag{3.10}$$

where we have set

$$G(\alpha^*) = F(\alpha^*) \, \frac{f'(\alpha^*)}{f(\alpha^*)}. \tag{3.11}$$

Proof. $d\tau(\varphi_{\alpha^*})$ is a section of $T^*M \otimes \varphi^{-1}TN$. Equation (3.10) tells us that we just have to verify that

$$(d\tau(\varphi_{\alpha^*}))^i_i = \ F(\alpha^*) \, \frac{f'(\alpha^*)}{f(\alpha^*)} \qquad \text{if } 1 \leq i \leq m-1 \tag{3.12}$$

and

$$(d\tau(\varphi_{\alpha^*}))^i_j = 0 \qquad \text{whenever } i \neq j \ . \tag{3.13}$$

Now, (3.12) and (3.13) are a simple consequence of the following:

$$\begin{aligned} d\tau(\varphi_{\alpha^*}) \left(\frac{\partial}{\partial w_i} \right) &= \nabla^{\varphi}_{\partial/\partial w_i} \, \tau(\varphi_{\alpha^*}) \\ &= \nabla^{\varphi}_{\partial/\partial w_i} \, F(\alpha^*) \frac{\partial}{\partial \alpha} = F(\alpha^*) \, \nabla^{\varphi}_{\partial/\partial w_i} \, \frac{\partial}{\partial \alpha} \\ &= F(\alpha^*) \, \left[\Gamma^k_{im} \, \frac{\partial}{\partial w_k} + \Gamma^m_{im} \, \frac{\partial}{\partial \alpha} \right] \\ &= F(\alpha^*) \, \frac{f'(\alpha^*)}{f(\alpha^*)} \, \delta^k_i \, \frac{\partial}{\partial w_k} = F(\alpha^*) \, \frac{f'(\alpha^*)}{f(\alpha^*)} \, \frac{\partial}{\partial w_i} \ , \end{aligned} \tag{3.14}$$

where, in (3.14), we have used the explicit expression of the Christoffel symbols given in Lemma 3.2. □

Lemma 3.6. *Let φ_{α^*} be a map as in* (3.5)*. Then*

$$d^*d\,(\tau(\varphi_{\alpha^*})) = H(\alpha^*) \, \frac{\partial}{\partial \alpha} \ , \tag{3.15}$$

where

$$H(\alpha^*) = (m-1) \, G(\alpha^*) \, \frac{f(\alpha^*) \, f'(\alpha^*)}{k^2(\alpha^*)}. \tag{3.16}$$

Proof. We compute (see [8]):

$$\begin{aligned} \sum_{\ell=1}^{m-1} \left(\nabla_{\partial/\partial w_i} \left(dw^\ell \otimes \frac{\partial}{\partial w_\ell} \right) \right) &= -\, {}^{\mathbb{S}}\Gamma^{\gamma}_{ik} \, dw^k \otimes \frac{\partial}{\partial w_\gamma} \\ &\quad + \Gamma^{\beta}_{i\gamma} \, dw^\gamma \otimes \frac{\partial}{\partial w_\beta} + \Gamma^m_{i\gamma} \, dw^\gamma \otimes \frac{\partial}{\partial \alpha} \\ &= \Gamma^m_{i\gamma} \, dw^\gamma \otimes \frac{\partial}{\partial \alpha} = -\frac{f(\alpha^*) \, f'(\alpha^*)}{k^2(\alpha^*)} \, (g_{\mathbb{S}})_{i\gamma} \, dw^\gamma \otimes \frac{\partial}{\partial \alpha} \ . \end{aligned} \tag{3.17}$$

Next, using Lemma 3.5 and (3.17), we obtain

$$\begin{aligned} d^*d\left(\tau(\varphi_{\alpha^*})\right) &= -\,g_{\mathbb{S}}^{ij}\sum_{i,\ell=1}^{m-1}\left(G(\alpha^*)\,\nabla_{\partial/\partial w_i}\left(dw^\ell\otimes\frac{\partial}{\partial w_\ell}\right)\frac{\partial}{\partial w_j}\right)\\ &= (m-1)\,G(\alpha^*)\,\frac{f(\alpha^*)\,f'(\alpha^*)}{k^2(\alpha^*)}\,\frac{\partial}{\partial\alpha}\ , \end{aligned} \tag{3.18}$$

so ending the proof of this lemma. □

We are now in the right position to complete the proof of Proposition 3.3:

Proof. The proof of Proposition 3.3 reduces to an iteration of the calculations which we have performed in Lemmata 3.4, 3.5, 3.6. We have:

$$\begin{aligned} E_2(\varphi_{\alpha^*}) &= \frac{1}{2}\int_{\mathbb{S}^{m-1}}|\tau(\varphi_{\alpha^*})|^2\,dv_{\mathbb{S}^{m-1}} = \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,F^2(\alpha^*)\,k^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,\varepsilon_2(\alpha^*). \end{aligned} \tag{3.19}$$

$$\begin{aligned} E_3(\varphi_{\alpha^*}) &= \frac{1}{2}\int_{\mathbb{S}^{m-1}}|d\tau(\varphi_{\alpha^*})|^2\,dv_{\mathbb{S}^{n-1}}\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)\,f^2(\alpha^*)\,G^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)\,f'^2(\alpha^*)\,F^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)\,\frac{f'^2(\alpha^*)}{k^2(\alpha^*)}\,\varepsilon_2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,\varepsilon_3(\alpha^*). \end{aligned} \tag{3.20}$$

$$\begin{aligned} E_4(\varphi_{\alpha^*}) &= \frac{1}{2}\int_{\mathbb{S}^{m-1}}\left|d^*d\left(\tau(\varphi_{\alpha^*})\right)\right|^2 dv_{\mathbb{S}^{m-1}} = \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,H^2(\alpha^*)\,k^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)^2\,\frac{f^2(\alpha^*)\,f'^2(\alpha^*)}{k^2(\alpha^*)}\,G^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)^2\,\frac{f'^4(\alpha^*)}{k^2(\alpha^*)}\,F^2(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,(m-1)\,\frac{f'^2(\alpha^*)}{k^2(\alpha^*)}\,\varepsilon_3(\alpha^*)\\ &= \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{m-1}\right)\,\varepsilon_4(\alpha^*). \end{aligned} \tag{3.21}$$

Now the iterative procedure can be made explicit and we recognize the pattern

$$E_{r+1}(\varphi_{\alpha^*}) = (m-1)\,\frac{f'^2(\alpha^*)}{k^2(\alpha^*)}\,E_r(\varphi_{\alpha^*}) \tag{3.22}$$

from which the conclusion follows by induction. □

Remark 3.7. The conclusion of Proposition 3.3 is also true for $r = 1$, but this is not of interest for our purposes.

There are some geometrically significant instances where, instead of (3.1), it is convenient to consider the following family of manifolds:

$$(M, g_M) = \left(\mathbb{S}^{p_1} \times \mathbb{S}^{p_2} \times I, f_1^2(\alpha) g_{\mathbb{S}^{p_1}} + f_2^2(\alpha) g_{\mathbb{S}^{p_2}} + k^2(\alpha) d\alpha^2\right), \tag{3.23}$$

where p_1, p_2 are positive integers, $I \subset \mathbb{R}$ is an interval and $f_1(\alpha), f_2(\alpha), k(\alpha)$ are smooth, positive functions in the interior of I. In particular, if we choose $I = [0, (\pi/2)]$, $f_1(\alpha) = a \sin\alpha$, $f_2(\alpha) = b \cos\alpha$ and $k^2(\alpha) = a^2 \cos^2\alpha + b^2 \sin^2\alpha$ $(a, b > 0)$, then the manifold (3.23) is isometric to the Euclidean ellipsoid $E^{p_1+p_2+1}(a, b) \subset \mathbb{R}^{p_1+p_2+2}$ defined by

$$\frac{x_1^2 + \ldots + x_{p_1+1}^2}{a^2} + \frac{y_1^2 + \ldots + y_{p_2+1}^2}{b^2} = 1 \tag{3.24}$$

(of course, $E^{p_1+p_2+1}(1, 1) = \mathbb{S}^{p_1+p_2+1}$). We shall refer to a manifold as in (3.23) as to a *generalized rotationally symmetric manifold* and, to shorten notation, we shall write $M_{f_1,f_2,k}$ to denote it. Let $\mathbb{S}^n(R)$ denote the Euclidean, n-dimensional sphere of radius R. Next, instead of (3.5), we study:

$$\begin{array}{rrcl} \varphi_{\alpha^*}^C : & \mathbb{S}^{p_1}(R_1) \times \mathbb{S}^{p_2}(R_2) & \to & M_{f_1,f_2,k} \\ & (R_1\, w, R_2\, z) & \mapsto & (w, z, \alpha^*)\ , \end{array} \tag{3.25}$$

where $R_1, R_2 > 0$, w, z denote the generic coordinates of a point of $\mathbb{S}^{p_1}$ and $\mathbb{S}^{p_2}$ respectively, and α^* is a fixed constant value in the interior of the interval I. Now, as above, we want to determine under which conditions a map $\varphi_{\alpha^*}^C$ as in (3.25) is proper r-harmonic: a straightforward adaptation (whose details are omitted) of the proof of Proposition 3.3 leads us to the following result:

Proposition 3.8. *Assume that $p_1, p_2 \geq 1, r \geq 2$ and $R_1, R_2 > 0$. Let $\varphi_{\alpha^*}^C$ be a map of the type* (3.25)*. Then its r-energy is*

$$E_r(\varphi_{\alpha^*}^C) = \frac{1}{2} \operatorname{Vol}(\mathbb{S}^{p_1}(R_1)) \operatorname{Vol}(\mathbb{S}^{p_2}(R_2)) \ \ \varepsilon_r^C(\alpha^*), \tag{3.26}$$

where

$$\varepsilon_r^C(\alpha) = \frac{\left(\frac{p_1}{R_1^2} f_1(\alpha) f_1'(\alpha) + \frac{p_2}{R_2^2} f_2(\alpha) f_2'(\alpha)\right)^2}{k^2(\alpha)} \left[\frac{\frac{p_1}{R_1^2} f_1'^2(\alpha) + \frac{p_2}{R_2^2} f_2'^2(\alpha)}{k^2(\alpha)}\right]^{(r-2)} . \tag{3.27}$$

4. New examples of r-harmonic immersions

Our first general result is the following:

Theorem 4.1. *Assume that $m, r \geq 2$. Let $\varphi_{\alpha^*} : \mathbb{S}^{m-1} \to M_{f,k}$ be a map of the type* (3.5)*. Then its image $\varphi_{\alpha^*}(\mathbb{S}^{m-1})$ is a proper r-harmonic submanifold of $\mathbb{S}^m$ if and only if $f'(\alpha^*) \neq 0$ and $\varepsilon_r'(\alpha^*) = 0$, where $\varepsilon_r(\alpha)$ is the function defined in* (3.7)*.*

Proof. If φ_{α^*} is a map of the type (3.5), then the induced metric on $\mathbb{S}^{m-1}$ is simply given by $f^2(\alpha^*)\, g_{\mathbb{S}^{m-1}}$. Therefore, since r-harmonicity is preserved by multiplication of the Riemannian metric of the domain manifold by a positive constant, we conclude that if φ_{α^*} is a proper r-harmonic map, then its image $\varphi_{\alpha^*}(\mathbb{S}^{m-1})$ is a proper r-harmonic parallel hypersphere, of radius $R = f(\alpha^*)$, into $M_{f,k}$. Therefore, we just have to show that a map as in (3.5) is proper r-harmonic if and only if $f'(\alpha^*) \neq 0$ and $\varepsilon_r'(\alpha^*) = 0$. According to Proposition 3.3 $\varepsilon_r(\alpha^*)$ is, up to a constant, the r-energy of φ_{α^*}. Now, a direct computation as in Proposition 3.1 of [23] (see this reference for details) shows that, in the spirit of [26],

$$\tau_r(\varphi_{\alpha^*}) = \overline{T_r}(\alpha^*)\ \frac{\partial}{\partial\alpha} \quad \text{for all} \quad r \geq 2 \tag{4.1}$$

for some functions $\overline{T_r}(\alpha)$. Next, we can obtain the explicit expression of the r-tension field $\tau_r(\varphi_{\alpha^*})$ by means of the following argument. Let us consider the variation

$$\varphi_{\alpha^*,t} = (w, \alpha^* + t). \tag{4.2}$$

Clearly,

$$\left.\frac{\partial\varphi_{\alpha^*,t}}{\partial t}\right|_{t=0} = \frac{\partial}{\partial\alpha}. \tag{4.3}$$

Then, because of (2.8) (which also holds when r is odd), (4.1) yields:

$$\begin{aligned} \left.\frac{d}{dt}\, E_r(\varphi_{\alpha^*,t})\right|_{t=0} &= -\int_{\mathbb{S}^{m-1}} \langle\, \tau_r(\varphi_{\alpha^*}), \frac{\partial}{\partial\alpha}\, \rangle_{M_{f,k}}\ dv_{\mathbb{S}^{m-1}} \\ &= -\operatorname{Vol}(\mathbb{S}^{m-1})\, \overline{T_r}(\alpha^*) k^2(\alpha^*). \end{aligned} \tag{4.4}$$

On the other hand, direct calculation using (3.6) gives:

$$\begin{aligned} \left.\frac{d}{dt}\, E_r(\varphi_{\alpha^*,t})\right|_{t=0} &= \frac{1}{2}\int_{\mathbb{S}^{m-1}} \left.\frac{d\, \varepsilon_r(\alpha^*+t)}{dt}\right|_{t=0} dv_{\mathbb{S}^{m-1}} \\ &= \frac{1}{2}\operatorname{Vol}(\mathbb{S}^{m-1})\, \varepsilon_r'\,(\alpha^*). \end{aligned} \tag{4.5}$$

Comparing (4.4) and (4.5) we conclude immediately that

$$\tau_r(\varphi_{\alpha^*}) = -\,\frac{1}{2}\,\frac{\varepsilon_r'\,(\alpha^*)}{k^2(\alpha^*)}\,\frac{\partial}{\partial\alpha} \quad \text{for all} \quad r \geq 2. \tag{4.6}$$

Finally, observing that the harmonicity of $\varphi(\alpha^*)$ is equivalent to the condition $f'(\alpha^*) = 0$, the conclusion of the proof follows immediately from (4.6). ∎

As a first application of Theorem 4.1, we can now proceed to the proof of the Theorem 1.1 stated in the introduction:

Proof of Theorem 1.1. The proof is a consequence of Theorem 4.1 with $f^2(\alpha) = \sin^2\alpha$ and $k^2(\alpha) = b^2\sin^2\alpha + \cos^2\alpha$. Indeed, in this case the condition $\varepsilon_r'(\alpha) = 0$ takes the form

$$-2\sin\alpha\cos\alpha\left(\frac{\cos^2\alpha}{b^2\sin^2\alpha+\cos^2\alpha}\right)^r\left[b^2(r-1)\tan^4\alpha + b^2(r-2)\tan^2\alpha - 1\right] = 0.$$

Now, $f'(\alpha^*) = \cos\alpha^*$ does not vanish if (1.5) holds, and it follows that α^* must satisfy

$$b^2(r-1)\tan^4\alpha^* + b^2(r-2)\tan^2\alpha^* - 1 = 0$$

from which (1.5) follows by a routine computation. □

Now we give the proof of Theorem 1.3:

Proof of Theorem 1.3. It suffices to observe that $S_{\rm par}/\{O\}$ is isometric to a 2-dimensional rotationally symmetric manifold as in (3.1), with

$$I=(0,+\infty), \quad f(\alpha)=\alpha \quad \text{and} \quad k^2(\alpha)=1+4C^2\alpha^2.$$

Then we apply Theorem 4.1 and deduce that the submanifold

$$C_{\alpha^*}=(\alpha^*\sin\vartheta,\alpha^*\cos\vartheta, C\,\alpha^{*2}) \quad (0\leq\vartheta\leq 2\pi)$$

is proper r-harmonic in $S_{\rm par}$ if and only if $\varepsilon_r'(\alpha^*)=0$, where

$$\varepsilon_r(\alpha)=\alpha^2\left[\frac{1}{1+4C^2\alpha^2}\right]^{(r-1)}.$$

Finally, computing $\varepsilon_r'(\alpha)$, we easily obtain (1.7). □

Example 4.2. Let us consider the following *complete* surface of revolution S parameterized by

$$S\ :\qquad (\alpha\sin\vartheta,\alpha\cos\vartheta,\tan(\alpha-\pi/2))\,, \tag{4.7}$$

where $0\leq\vartheta\leq 2\pi$ and $0<\alpha<\pi$. As the distance from the rotation axis is an increasing function of α, we know from Clairaut's relation that S does not admit any closed geodesic. Now, for a value $\overline{\alpha}\in(0,\pi)$, we define a map $\varphi_{\overline{\alpha}}:\mathbb{S}^1\to S$ by

$$\varphi_{\overline{\alpha}}(\vartheta)=(\overline{\alpha}\sin\vartheta,\overline{\alpha}\cos\vartheta,\tan(\overline{\alpha}-\pi/2))\,.$$

Because of the previous observation $\varphi_{\overline{\alpha}}$ is not homotopic to any harmonic map. By contrast, using Theorem 4.1 with $f(\alpha)=\alpha$ and $k^2(\alpha)=1+(1/\sin^4\alpha)$, it is easy to deduce that $\varphi_{\alpha^*}=(\alpha^*\sin\vartheta,\alpha^*\cos\vartheta,\tan(\alpha^*-\pi/2))$ is r-harmonic if and only if α^* verifies

$$\sin^4\alpha+2(r-1)\alpha\cot\alpha+1=0.$$

In particular, a simple analysis of this equation enables us to conclude that, for each $r\geq 2$, $\varphi_{\overline{\alpha}}$ is homotopic to an r-harmonic map.

Next, we consider a revolution torus T^2 in $\mathbb{R}^3$ parameterized by

$$T^2\ :\qquad (\sin\vartheta(C+\cos\alpha),\cos\vartheta(C+\cos\alpha),\sin\alpha)\,, \tag{4.8}$$

where $0\leq\vartheta,\alpha\leq 2\pi$, while $C>1$ is a fixed constant. We have

Corollary 4.3. *Assume $r\geq 2$ and $C>1$. Let us consider submanifolds of $T^2\subset\mathbb{R}^3$ of the type*

$$(\sin\vartheta(C+\cos\alpha^*),\cos\vartheta(C+\cos\alpha^*),\sin\alpha^*)\,, \tag{4.9}$$

where $0\leq\alpha^*\leq 2\pi$ is a fixed value. Then (4.9) is a proper r-harmonic submanifold if and only if

$$\cos\alpha^*=\frac{\sqrt{C^2(r-1)^2+4r}-C(r-1)}{2r}. \tag{4.10}$$

Proof. The proof is an application of Theorem 4.1 with $f^2(\alpha) = [C + \cos\alpha]^2$ and $k^2(\alpha) \equiv 1$. Indeed, in this case the condition $\varepsilon_r'(\alpha) = 0$ takes the form

$$\sin^{2r-3}(\alpha)(\cos\alpha + C)\left[r\cos(2\alpha) + 2C(r-1)\cos\alpha + r - 2\right] = 0.$$

Now, $\sin\alpha^* \neq 0$ otherwise the submanifold is minimal. It follows that α^* must satisfy

$$r\cos(2\alpha^*) + 2C(r-1)\cos\alpha^* + r - 2 = 0. \tag{4.11}$$

Finally, a simple analysis of (4.11) shows that the only admissible solution is (4.10). □

Remark 4.4. The special case $r = 2$ in Corollary 4.3 was first proved in [4] by different methods.

More generally, we can consider a hypersurface of revolution $S_{\text{rev}} \subset \mathbb{R}^{m+1}$ given by

$$S_{\text{rev}} \quad : \qquad (y(\alpha)w,\, z(\alpha)) \in \mathbb{R}^m \times \mathbb{R}, \tag{4.12}$$

where $w \in \mathbb{S}^{m-1}$ and the rotating profile curve is such that $y(\alpha) > 0$ and $y'^2(\alpha) + z'^2(\alpha) \equiv 1$. Then we have the following more general version of Corollary 4.3 (the proof is similar and so we omit it):

Corollary 4.5. *Assume $r, m \geq 2$. Let us consider a parallel hypersphere $\mathbb{S}^{m-1}(y(\alpha^*))$ of $S_{\text{rev}} \subset \mathbb{R}^{m+1}$ of the type*

$$(y(\alpha^*)w,\, z(\alpha^*))\,, \tag{4.13}$$

where α^* is a fixed value. Then this hypersphere is a proper r-harmonic submanifold of S_{rev} if and only if $y'(\alpha^*) \neq 0$ and

$$(r-1)\frac{y''(\alpha^*)}{y(\alpha^*)} + \frac{{y'}^2(\alpha^*)}{y^2(\alpha^*)} = 0. \tag{4.14}$$

Remark 4.6. Condition (4.14) can be reformulated as follows:

$$-(r-1)K_{\text{rad}}(\alpha^*) + H^2(\alpha^*) = 0, \tag{4.15}$$

where $K_{\text{rad}}(\alpha^*)$ is the sectional curvature of S_{rev} with respect to any section parallel to $\partial/\partial\alpha$ and $H(\alpha^*)$ is the mean curvature of $\mathbb{S}^{m-1}(y(\alpha^*))$ in S_{rev}. Therefore, on any proper, parallel r-harmonic hyperphere K_{rad} must be a *strictly positive* constant. We also observe that, in the case that S_{rev} is a surface of revolution in $\mathbb{R}^3$, (4.15) becomes

$$-(r-1)K(\alpha^*) + \kappa_g^2(\alpha^*) = 0,$$

where K is the Gaussian curvature of the rotation surface and κ_g is the geodesic curvature of the parallel. This condition was found by Maeta in the case of r-harmonic curves with constant geodesic curvature in 2-dimensional space forms (see [17] and [4] for the case $r = 2$).

Remark 4.7. By integrating the ordinary differential equation

$$(r-1)y''(\alpha)y(\alpha) + {y'}^2(\alpha) = 0 \tag{4.16}$$

associated to condition (4.14) it is possible to construct a rotation hypersurface with the property that *all* its parallel hyperspheres are proper r-harmonic submanifolds. All the explicit solutions of (4.16) are given by:

$$y(\alpha) = C\alpha^{\frac{r-1}{r}}, \quad z(\alpha) = \int_{\alpha_0}^{\alpha} \sqrt{1 - C^2\,\frac{(r-1)^2}{r^2}\,u^{-\frac{2}{r}}}\,du,$$

where $C > 0$ is a fixed constant and the rotating curve is defined for α in $(\alpha_0, +\infty)$, where $\alpha_0 = \sqrt[r]{C(r-1)/r}$, so that it produces a *not complete* rotation hypersurface. This example was found in [4] in the case $r = 2$.

Now we state the version of Theorem 4.1 in the context of generalized rotational manifolds (3.23). The proof is a simple modification of the arguments of Theorem 4.1 and so we omit it.

Theorem 4.8. *Let $\varphi^C_{\alpha^*}$ be a map as in* (3.25). *Then $\varphi^C_{\alpha^*}$ is a proper r-harmonic map ($r \geq 2$) if and only if*

$$\frac{p_1}{R_1^2}\, f_1(\alpha^*)\, f_1'(\alpha^*) \;+\; \frac{p_2}{R_2^2}\, f_2(\alpha^*)\, f_2'(\alpha^*) \;\neq\; 0 \tag{4.17}$$

and α^* is a critical point of the function ε_r^C defined in (3.27).

Next, we want to study the existence of proper, r-harmonic generalized Clifford's tori in the family of ellipsoids $E^{p_1+p_2+1}(a,b)$ defined in (3.24). As a first application of Theorem 4.8 we have the following proposition:

Proposition 4.9. *Assume $r \geq 2$, $p_1, p_2 \geq 1$ and $a, b, R_1, R_2 > 0$. Let us consider:*

$$\begin{array}{rcccl} \varphi_{\alpha^*} : & \mathbb{S}^{p_1}(R_1) \times \mathbb{S}^{p_2}(R_2) & \to & E^{p_1+p_2+1}(a,b) \subset & \mathbb{R}^{p_1+1} \times \mathbb{R}^{p_2+1} \\ & (R_1 w, R_2 z) & \mapsto & & (a \sin\alpha^* w, b \cos\alpha^* z)\ , \end{array} \tag{4.18}$$

where w, z denote the generic coordinates of a point of $\mathbb{S}^{p_1}$, $\mathbb{S}^{p_2}$ respectively and α^* is a fixed value in $(0, (\pi/2))$. Then φ_{α^*} is proper r-harmonic if and only if

$$\left[a^2 \frac{p_1}{R_1^2} \;-\; b^2 \frac{p_2}{R_2^2} \right] \neq 0 \tag{4.19}$$

and α^* is a critical point of the function

$$\varepsilon_r(\alpha) = \frac{\cos^2\alpha \sin^2\alpha}{k^2(\alpha)} \left[\frac{a^2 \frac{p_1}{R_1^2} \cos^2\alpha \;+\; b^2 \frac{p_2}{R_2^2} \sin^2\alpha}{k^2(\alpha)} \right]^{(r-2)}, \tag{4.20}$$

i.e., $0 < t = \sin^2\alpha^* < 1$ is a root of the third order polynomial

$$\begin{aligned} Q_r(t) \;=\; & a^4 \frac{p_1}{R_1^2} - a^2 \left((r-1) \left(a^2 \frac{p_1}{R_1^2} - b^2 \frac{p_2}{R_2^2} \right) - (a^2 - b^2) r \frac{p_1}{R_1^2} + 2(2a^2 - b^2) \frac{p_1}{R_1^2} \right) t \\ & + a^2 \left(3(a^2 - b^2) \frac{p_1}{R_1^2} + b^2 r \left(\frac{p_1}{R_1^2} - \frac{p_2}{R_2^2} \right) \right) t^2 \\ & - (a^2 - b^2) \left(a^2 \frac{p_1}{R_1^2} - b^2 \frac{p_2}{R_2^2} \right) t^3. \end{aligned} \tag{4.21}$$

Proof. The proof follows by Theorem 4.8 with $I = (0, (\pi/2))$, $f_1(\alpha) = a \sin\alpha$, $f_2(\alpha) = b \cos\alpha$ and $k^2(\alpha) = a^2 \cos^2\alpha + b^2 \sin^2\alpha$. In particular, here (4.17) becomes (4.19) and the condition $\varepsilon_r'(\alpha^*) = 0$ is equivalent to the requirement that $t = \sin^2\alpha^*$ is a root of the polynomial $Q_r(t)$ defined in (4.21). □

We can now state our main result in this context:

Theorem 4.10. *Assume $r \geq 2$, $p_1, p_2 \geq 1$ and $a, b > 0$. Let us consider a Clifford's torus type submanifold $\mathbb{S}^{p_1}(a \sin \alpha^*) \times \mathbb{S}^{p_2}(b \cos \alpha^*)$ of $E^{p_1+p_2+1}(a,b) \subset \mathbb{R}^{p_1+1} \times \mathbb{R}^{p_2+1}$ defined by*

$$(a \sin \alpha^* w, b \cos \alpha^* z) \in E^{p_1+p_2+1}(a,b) \subset \mathbb{R}^{p_1+1} \times \mathbb{R}^{p_2+1}, \tag{4.22}$$

where w, z denote the generic coordinates of a point of $\mathbb{S}^{p_1}$, $\mathbb{S}^{p_2}$ respectively and α^* is a fixed value in $(0, (\pi/2))$. Set $t = \sin^2 \alpha^*$ for convenience. Then the submanifold (4.22) is proper r-harmonic if and only if

$$t \neq \frac{p_1}{p_1 + p_2} \tag{4.23}$$

and $0 < t < 1$ is a root of the following fourth order polynomial:

$$\begin{aligned} P_r(t) \; = \; & a^2 p_1 - \left(4a^2 + (r-2)b^2\right) p_1 t \\ & + \left((6a^2 + (2r-5)b^2)p_1 + a^2(r-1)p_2\right) t^2 \\ & - \left((4a^2 + (r-4)b^2)p_1 + a^2 r p_2\right) t^3 \\ & + (a^2 - b^2)(p_1 + p_2) t^4 . \end{aligned} \tag{4.24}$$

Proof. We want to use Proposition 4.9. In the case of maps as in (4.18) the induced pull-back metric identifies the domain with $\mathbb{S}^{p_1}(a \sin \alpha^*) \times \mathbb{S}^{p_2}(b \cos \alpha^*)$. Therefore, in order to ensure that an r-harmonic map of type (4.18) is an *isometric* immersion, it is enough to consider those roots of the polynomial in (4.21) which satisfy the additional condition $R_1^2 = a^2 \sin^2 \alpha^*$ and $R_2^2 = b^2 \cos^2 \alpha^*$. Using this remark, it is not difficult to check that the condition that a Clifford's torus type submanifold of the type (4.22) is not minimal is equivalent to (4.23), while the polynomial (4.21) becomes (4.24). □

Remark 4.11. We point out that

$$P_r(0) = a^2 p_1 > 0 \quad \text{and} \quad P_r(1) = -b^2 p_2 < 0 .$$

Therefore, $P_r(t)$ always admits at least one root t in the interval $(0,1)$. It is not difficult to verify that this root satisfies (4.23) (i.e., the associated submanifold is not minimal) provided that

$$\frac{a^2}{b^2} \neq \frac{p_1(p_1 + p_2(r-2))}{p_2(p_2 + p_1(r-2))}. \tag{4.25}$$

A direct numerical analysis also shows that, for suitable large values of r, there are cases where we have three distinct roots in $(0,1)$ which all give rise to proper r-harmonic submanifolds. By way of example, let $a = 2, b = 1, p_1 = 1, p_2 = 1$. If $r < 60$ then there is only one admissible root, but if $r \geq 60$ there are three distinct admissible solutions.

Remark 4.12. The case $r = 2$ of Corollary 4.10 was proved in [21] by different methods, while the spherical case $a = b, r \geq 3$ was first obtained in [16, 23].

References

[1] P. Baird. Harmonic maps with symmetry, harmonic morphisms and deformations of metrics. *Research Notes in Mathematics* 87, Pitman (Advanced Publishing Program), Boston, 1983.

[2] A. Baldes. Stability and uniqueness properties of the equator map from a ball into an ellipsoid. *Math. Z.* 185 (1984), 505–516.

[3] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds in spheres. *Israel J. Math.* 130 (2002), 109–123.

[4] R. Caddeo, S. Montaldo, P. Piu. Biharmonic curves on a surface. *Rend. Mat. Appl.* 21 (2001), 143–157.
[5] B.-Y. Chen, *Total mean curvature and submanifolds of finite type.* Second edition. Series in Pure Mathematics, 27. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2015).
[6] B.-Y. Chen, Some open problems and conjecture on the submanifolds of finite type: recent development. *Tamkang J. Math.* 45 (2014), 87–108.
[7] J. Eells, L. Lemaire. Another report on harmonic maps. *Bull. London Math. Soc.* 20 (1988), 385–524.
[8] J. Eells, L. Lemaire. *Selected topics in harmonic maps.* CBMS Regional Conference Series in Mathematics, 50. American Mathematical Society, Providence, RI, 1983.
[9] J. Eells, A. Ratto. Harmonic maps between spheres and ellipsoids. *Int. J. Math.* 1 (1993), 1–27.
[10] J. Eells, J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* 86 (1964), 109–160.
[11] J. Eells, J.H. Sampson. Variational theory in fibre bundles. *Proc. U.S.-Japan Seminar in Differential Geometry*, Kyoto (1965), 22–33.
[12] A. Fardoun. Stability for the p-energy of the equator map of the ball into an ellipsoid. *Diff. Geom. and Appl.* 8 (1998), 171–176.
[13] R.E. Greene, H. Wu. Function theory on manifolds which possess a pole. *Lecture Notes in Mathematics* 699. Springer, Berlin, 1979.
[14] G.Y. Jiang. 2-harmonic maps and their first and second variation formulas. *Chinese Ann. Math. Ser. A 7*, 7 (1986), 130–144.
[15] S. Maeta. k-harmonic maps into a Riemannian manifold with constant sectional curvature. *Proc. Amer. Math. Soc.*, 140 (2012), 1835–1847.
[16] S. Maeta. Construction of triharmonic maps. *Houston J. Math.* 41 (2015), 433–444.
[17] S. Maeta. The second variational formula of the k-energy and k-harmonic curves. *Osaka J. Math.* 49 (2012), 1035–1063.
[18] S. Maeta, N. Nakauchi, H. Urakawa. Triharmonic isometric immersions into a manifold of non-positively constant curvature. *Monatsh. Math.* 177 (2015), 551–567.
[19] S. Montaldo, C. Oniciuc. A short survey on biharmonic maps between riemannian manifolds. *Rev. Un. Mat. Argentina*, 47 (2006), 1–22.
[20] S. Montaldo, C. Oniciuc, A. Ratto. Rotationally symmetric maps between models. *J. Math. Anal. and Appl.* 431 (2015), 494–508.
[21] S. Montaldo, A. Ratto. Biharmonic submanifolds into ellipsoids. *Monatsh. Math.* 176 (2015), 589–601.
[22] S. Montaldo, A. Ratto. A general approach to equivariant biharmonic maps. *Med. J. Math.* 10 (2013), 1127–1139.
[23] S. Montaldo, A. Ratto. New examples of r-harmonic immersions into the sphere. *J. Math. Anal. and Appl.* 458 (2018), 849–859.
[24] S. Montaldo, A. Ratto. Biharmonic curves into quadrics. *Glasgow Math. J.* 57 (2015), 131–141.
[25] N. Nakauchi, H. Urakawa. Polyharmonic maps into the Euclidean space. *arXiv:1307.5089.*
[26] R.S. Palais. The principle of symmetric criticality. *Comm. Math. Phys.* 69 (1979), 19–30.
[27] R.T. Smith. Harmonic mappings of spheres. *Am. J. Math.* 97 (1975), 229–236.
[28] S.B. Wang. The first variation formula for k-harmonic mappings. *Journal of Nanchang University* 13, N.1 (1989).

Università degli Studi di Cagliari, Dipartimento di Matematica e Informatica, Via Ospedale 72, 09124 Cagliari, Italia
E-mail address: montaldo@unica.it

Università degli Studi di Cagliari, Dipartimento di Matematica e Informatica, Viale Merello 93, 09123 Cagliari, Italia
E-mail address: rattoa@unica.it