U.P.B. Sci. Bull., Series A, Vol. 69, No. 2, 2007 ISSN 1223-7027

PROPER WEYL COLLINEATIONS IN KANTOWSKI-SACHS AND BIANCHI TYPE III SPACE-TIMES

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A study of proper Weyl collineations in Kantowski-Sachs and Bianchi type III space-times is given by using the rank of the 6×6 Weyl matrix and direct integration techniques. Studying proper Weyl collineations in each of the above space-times, it is shown that there exists no such possibility when the above space-times admit proper Weyl collineations.

1 Introduction

The aim of this paper is to find the existence of proper Weyl collineations (WCS) in Kantowski-Sachs and Bianchi type-III space-times. These WCS are vector fields, along which the Lie derivative of the Weyl tensor is zero. Different approaches [5,9-11] were adopted to study WCS. In this paper an approach, which is given in [4], is used to study proper WCS in Kantowski-Sachs and Bianchi type-III space-times by using the rank of the 6×6 Weyl metric and direct integration techniques. Through out M denotes a (4-dimensional Connected, Hausdorff) smooth space-time manifold with Lorentz metric g of signature (-,+,+,+). The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form where R_{abcd}, the Ricci tensor components are $R_{ab}=R^{c}{}_{acb}$, the Weyl tensor components are $C^{a}{}_{bcd}$, and the Ricci scalar is $R=g^{ab}R_{ab}$. Round and square brackets denote the usual symmetrization and skew-symmetrization.

Let X be a smooth vector field on M then in any coordinate system on M, one may decompose X in the form

$$X_{a;b}=\frac{1}{2}h_{ab}+F_{ab}, \tag{1}$$

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where $h_{ab} = L_X g_{ab}$ and $F_{ab}(= -F_{ba})$ are symmetric and skew symmetric tensor on M, respectively. If $h_{ab} = \alpha g_{ab}$ and $\alpha(\alpha : M \to R)$ is a real valued function on M then X is called a conformal vector field where F_{ab} is called the conformal bivector. The vector field X is called a proper conformal vector field if α is not constant on M. For a conformal bivector F_{ab} one can show that [1]

$$F_{ab;c} = R_{abcd} X^d - 2\alpha_{;[a} g_{b]c} \tag{2}$$

and

$$\alpha_{a;b} = -\frac{1}{2} L_{ab;c} X^c - \alpha L_{ab} + R_{c(a} F_{b)}{}^c \tag{3}$$

where $L_{ab} = R_{ab} - (1/6) R g_{ab}$. If X is a conformal vector field on M then by using (3) one can show that

$$L_X R_{ab} = -2\alpha_{a;b} - (\alpha^c{}_{;c}) g_{ab}.$$

Further, the conformal vector field X also satisfies [3]

$$L_X C^a{}_{bcd} = 0 \tag{4}$$

equivalently,

$$C^a{}_{bcd;f} X^f + C^a{}_{bcf} X^f{}_{;d} + C^a{}_{bfd} X^f{}_{;c} + C^a{}_{fcd} X^f{}_{;b} - C^f{}_{bcd} X^a{}_{;f} = 0.$$

The vector field X satisfying the above equation is called a Weyl collineation (WC). The vector field X is called a proper WC if it is not conformal [2]. The vector field X is called a homothetic vector field if α is constant and a proper homothetic vector field if α = constant $\neq 0$. If $\alpha = 0$ on M then vector field X is called a Killing vector field.

2 Main Results

It has been shown [2,4] that much information on the solutions of (4) can be obtained without integrating it directly. To see this let $p \in M$ and consider the following algebraic classification of the Weyl tensor as a linear map β from the vector space of bivectors to itself; $\beta : F_{ab} \to F_{cd} C^{cd}{}_{ab}$, for any bivector F_{ab} at p. The range of the Weyl tensor at p is then the range of β at p and its dimension is the Weyl rank at p. It follows from [4] that the rank of the 6×6 Weyl matrix is always even i.e. 6, 4, 2 or 0. If the rank of the 6×6 Weyl matrix is 6 or 4 then every Weyl symmetry is a conformal symmetry [2,4]. For finding proper WCS, we restrict attention to those cases of rank 2 or less.

2.1 Proper WCS in Bianchi type III and Kantowski-Sachs space-times

Consider the space-times in the usual coordinate system (t, r, θ, ϕ) with line element [6,8]

$$ds^2 = -dt^2 + A(t)dr^2 + B(t)\left[d\theta^2 + f^2(\theta)\, d\phi^2\right], \tag{5}$$

where A and B are no where zero functions of t only. For $f(\theta) = \sin\theta$ or $f(\theta) = \sinh\theta$ the above space-time (5) become Kantowski-Sachs or Bianchi type III space-times, respectively. The above space-time admits four independent Killing fields which are

$$\frac{\partial}{\partial r},\ \frac{\partial}{\partial \phi},\ \cos\phi\frac{\partial}{\partial \theta} - \frac{f'}{f}\sin\phi\frac{\partial}{\partial \phi},\ \sin\phi\frac{\partial}{\partial \theta} + \frac{f'}{f}\cos\phi\frac{\partial}{\partial \phi},$$

where prime denotes the derivative with respect to θ. The non-zero independent components of Weyl tensor are

$$\begin{aligned}
C_{0101} &= \frac{1}{12AB^2}K(t) \equiv F1, & C_{0202} &= -\frac{1}{24A^2B}K(t) \equiv F2,\\
C_{0303} &= f^2(\theta)\ F2 \equiv F3, & C_{1212} &= \frac{1}{24AB}K(t) \equiv F4,\\
C_{1313} &= f^2(\theta)F4 \equiv F5, & C_{2323} &= -\frac{f^2(\theta)}{12A^2}\ K(t) \equiv F6,
\end{aligned} \tag{6}$$

where $K(t) = (B^2(-2\ddot{A}A + \dot{A}^2) + AB\dot{A}\dot{B} + 2A^2(\ddot{B}B - \dot{B}^2) + 4A^2B)$ and dot denotes the derivative with respect to t. The Weyl tensor of M can be described by components C_{abcd} written in a well known way [7]

$$C_{abcd} = diag(F1, F2, F3, F4, F5, F6).$$

We restrict attention to those cases of rank 2 or less, since by theorem [4] no proper WCS can exist when the rank of the 6×6 Weyl matrix is $> 2.$ For the rank less or equal to two one may set four components of Weyl tensor in (6) to be zero. One gets A and B to be zero which gives contradiction to our assumption that A and B are no where zero functions on M this implies that there exists no such possibility when the rank of the 6×6 Weyl matrix is less or equal to zero. Hence no proper Weyl collineations exist in the above space-time (5).

3. Summary

In this paper a study of proper Weyl collineations in Kantowski-Sachs and Bianchi type III space-times is given by using the rank of the 6×6 Weyl matrix and direct integration techniques and the theorem given in [4]. Studying proper Weyl collineations in the Kantowski-Sachs and Bianchi type III space-times, it is shown that the above space-times do not admit proper Weyl collineations.

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