Research Article

Properties and Bounds of Jensen-Type Functionals via Harmonic Convex Functions

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Dragomir introduced the Jensen-type inequality for harmonic convex functions (HCF) and Baloch et al. studied its different variants, such as Jensen-type inequality for harmonic h-convex functions. In this paper, we aim to establish the functional form of inequalities presented by Baloch et al. and prove the superadditivity and monotonicity properties of these functionals. Furthermore, we derive the bound for these functionals under certain conditions. Furthermore, we define more generalized functionals involving monotonic nondecreasing concave function as well as evince superadditivity and monotonicity properties of these generalized functionals.

1. Introduction

Convexity is natural and simple notion which has found applications in business, industry, and medicine. During the study of convexity, many researchers have been fascinated by generalization of this class and have tried to find out those classes of functions which have close relation with this class (but not convex in general). Harmonic convex functions (HCFs) [1], harmonic (α, m)-convex functions [2], harmonic (s, m)-convex functions [3, 4], and harmonic $(p, (s, m))$-convex functions [5] are among these classes. For a quick glance on importance of these classes and applications, see [6–9] and references therein. The class of harmonic convex functions (HCFs) and its different variants are very important classes that gained prominence in the theory of inequalities and applications as well as in other branches of mathematics. Many researchers have been working on the class of harmonic convex functions (HCFs) due to its significance and have been trying to explore about it more and

more. During this study, recently different generalizations of the class of harmonic convex functions (HCFs) have been found, for example, see [10–13] and references therein.

The importance of the class of HCFs continuously encourages us and many other researchers to explore more about it, and the following paper is a link to it. For the better understanding of the results of present paper, we first recall some basic definitions.

Definition 1. Consider $I \subset \mathbb{R} \backslash \{0\}$. A function $f: I \longrightarrow \mathbb{R}$ is HCF on I if

$$f\left(\frac{w_1 w_2}{t w_1 + (1-t)w_2}\right) \leq t f(w_2) + (1-t)f(w_1), \qquad (1)$$

holds, for all $w_1, w_2 \in I$ and $t \in [0, 1]$. If we reverse the above inequality, the function f becomes harmonic concave.

Remark 1 (see [14]).

(i) The function $f(w) = \ln w$ is a HCF on the interval $(0, \infty)$, but it is not a convex function.

(ii) The function

$$f(w) = \begin{cases} \dfrac{1-w}{w}, & 0 < w \leq 1, \\[2mm] 0, & 1 < w \leq 2, \\[2mm] \dfrac{w-2}{w}, & w > 2, \end{cases} \quad (2)$$

is another example of HCF, which is neither convex nor concave.

Baloch et al., in [7], observed some remarkable facts for the class of HCFs.

Proposition 1. *For $I \subseteq \mathbb{R} \setminus \{0\}$, a function $f: I \longrightarrow \mathbb{R}$, we have the following facts:*

(1) If $I \subset (0, \infty)$ and f is nondecreasing and convex function, then f is HCF

(2) If $I \subset (0, \infty)$ and f is nonincreasing function and harmonically convex, then f is convex function

(3) If $I \subset (-\infty, 0)$ and f is nondecreasing function and harmonically convex, then f is convex function

(4) If $I \subset (-\infty, 0)$ and f is nonincreasing function and convex, then f is HCF

Varošanec, in [15], proposed the concept of h-convexity (also, see [16–18]) to unify numerous generalized aspects of convex functions. In a similar fashion, harmonic h-convexity unifies the various types of harmonic convexities.

Definition 2 (see [19]). Consider a nonnegative function $h: [0, 1] \longrightarrow \mathbb{R}^+$. Then, the function $f: I \subseteq \mathbb{R} \setminus \{0\} \longrightarrow (0, \infty)$ is said to be harmonic h-convex on I if the inequality

$$f\left(\frac{w_1 w_2}{t w_1 + (1-t) w_2}\right) \leq h(t) f(w_2) + h(1-t) f(w_1), \quad (3)$$

holds, for all $w_1, w_2 \in I$ and $t \in [0, 1]$. Furthermore, if we reverse inequality (3), then f becomes harmonic h-concave.

Remark 2. We provide few examples of harmonic h-convex (concave) functions as follows:

(i) Obviously, with $h(t) = t$, the class of nonnegative harmonic convex (concave) functions on I become a particular case of the class of harmonic h-convex (concave) functions on I.

(ii) Let $t \in (0, 1)$ and $h(t) = t^2$. Consider a function $f: [-1, 0) \cup t(0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = 1$, which is neither nonincreasing nor nondecreasing

h-convex function. Therefore, f is a harmonic h-convex function by Proposition 1 given in [7].

(iii) Let $t \in (0, 1)$ and $h: (0, 1) \longrightarrow (0, \infty)$ be a real-valued function such that $h(t) \geq t$ on $(0, 1)$. Then, the following four functions, $h_1(t) = t$, $h_2(t) = t^s$ ($s \in (0, 1)$), $h_3(t) = 1/t$, and $h_4(t) = 1$, satisfy the conditions of the function h mentioned above. Therefore, f is a harmonic h_α-convex function for $\alpha = 1, 2, 3, 4$ if $f: I \subseteq (0, \infty) \longrightarrow (0, \infty)$ is a nondecreasing convex function, or harmonic s-convex function, or harmonic Godunova–Levin function or harmonic P-function.

(iv) Let $f: (0, \infty) \longrightarrow (0, \infty)$ be a nondecreasing continuous function and $h: [0, 1] \longrightarrow (0, \infty)$ be a continuous self-concave function such that $f(t w_1 + (1-t) w_2) \leq h(t) f(w_1) + (1-t) f(w_2)$, for some $t \in (0, 1)$ and all $w_1, w_2 \in (0, \infty)$. Then, f is a h-convex function by Lemma 1 of [20], and hence, f is a harmonic h-convex function by Proposition 1 of [7].

Definition 3. A function $h: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a submultiplicative function if

$$h(w_1 w_2) \leq h(w_1) h(w_2), \quad (4)$$

for all $w_1, w_2 \in I$. If inequality (4) is reversed, then h is said to be supermultiplicative function. If just equality holds in relation (4), then h is said to be multiplicative function.

Definition 4. A function $h: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a subadditive function if

$$h(w_1 + w_2) \leq h(w_1) + h(w_2), \quad (5)$$

for all $w_1, w_2 \in I$. If inequality (5) is reversed, then h is said to be superadditive function. If just the equality holds in relation (5), then h is said to be additive function.

Jensen-type inequality for HCFs is proposed by Dragomir [21].

Theorem 1. *Let $I \subseteq (0, \infty)$ be an interval. If $f: I \longrightarrow \mathbb{R}$ is HCF, then*

$$f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha / w_\alpha}\right) \leq \left(\frac{1}{B_n}\right) \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha), \quad (6)$$

Holds, for all $w_1, \ldots, w_n \in I$ and $b_1, \ldots, b_n \geq 0$ with $\sum_{\alpha=1}^{n} b_\alpha = B_n$.

In [22], Baloch et al. derived the following results:

Theorem 2. *Let $I \subseteq \mathbb{R} \setminus \{0\}$. If $f: I \longrightarrow \mathbb{R}$ is HCF, then, for any finite positive sequence $(w_\alpha)_{\alpha=1}^{n} \in I$ and $b_1, \ldots, b_n \geq 0$ with $\sum_{\alpha=1}^{n} b_\alpha = B_n$, we have*

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha / w_\alpha}\right) \leq f(w_1) + f(w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha). \quad (7)$$

Theorem 3. *Let $b_1, \ldots, b_n$ be positive real numbers $(n \geq 2)$ and $B_n = \sum_{\alpha=1}^{n} b_\alpha$. If $h: I \supseteq (0,1) \longrightarrow \mathbb{R}$ is a nonnegative supermultiplicative function and if f is harmonic h-convex function, $(w_\alpha)_{\alpha=1}^{n} \in I$, then*

$$f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) \leq \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha). \qquad (8)$$

Theorem 4. *Let $h: I \supseteq (0,1) \longrightarrow \mathbb{R}$ be a nonnegative supermutiplicative function on I. Let $b_1, \ldots, b_n$ be positive real numbers $(n \geq 2)$ such that $B_n = \sum_{\alpha=1}^{n} b_\alpha$ and $\sum_{\alpha=1}^{n} h(b_\alpha/B_n) \leq 1$. If f is harmonic h-convex on $I \subseteq \mathbb{R} \backslash \{0\}$, then, for any finite positive increasing sequence $(w_\alpha)_{k=1}^{n} \in I$, we have*

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) \leq f(w_1) + f(w_n) - \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha). \qquad (9)$$

If h is a submultiplicative function, $\sum_{\alpha=1}^{n} h(b_\alpha/B_n) \geq 1$ and f is harmonic h-concave then inequality (9) is reversed.

Remark 3. Importance of the class of HCFs can be guessed by the following applications in the field of mathematics:

   (i) Harmonic convexity provides a useful analytic tool to calculate several known definite integrals such as $\int_a^b (e^w/x^n) \mathrm{d}w$, $\int_a^b e^{w^2} \mathrm{d}w$, $\int_a^b (\sin w/w^n) \mathrm{d}w$, and $\int_a^b (\cos w/w^n) \mathrm{d}x \, \forall n \in \mathbb{N}$, where $a, b \in (0, \infty)$, see [7]

   (ii) Inequality (6) provides a very short proof of the discrete form of Hölder's inequality (see [22])

   (iii) By inequalities (6) and (7), we can easily prove weighted HGA inequality (see [14])

Many researchers considered the functionals related to Jensen's inequality and tried to find properties and bound for these functionals (for example, see [23–30]). In the sequel, the set of all nonnegative n-tuples $\mathbf{b} = (b_1, \ldots, b_n)$, such that $B_n := \sum_{\alpha=1}^{n} b_\alpha > 0$, will be denoted with $\mathscr{B}_n^0$.

The difference between the right-hand and the left-hand side of inequalities (6)–(9) defines the following functionals:

$$\mathscr{M}_1(f, \mathbf{w}, \mathbf{b}) := \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) - B_n f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \qquad (10)$$

$$\mathscr{M}_2(f, \mathbf{w}, \mathbf{b}) := B_n[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) - B_n f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \qquad (11)$$

$$\mathscr{M}_3(f, \mathbf{w}, \mathbf{b}) := \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha) - f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right), \qquad (12)$$

$$\mathscr{M}_4(f, \mathbf{w}, \mathbf{b}) := f(w_1) + f(w_n) - \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha) - f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right). \qquad (13)$$

For a fixed function f and n-tuple $\mathbf{w}$, $\mathscr{M}_1(f, \mathbf{w}, .)$, $\mathscr{M}_2(f, \mathbf{w}, .)$, $\mathscr{M}_3(f, \mathbf{w}, .)$, and $\mathscr{M}_4(f, \mathbf{w}, .)$ can be considered as functions on $\mathscr{B}_n^0$, which is a convex subset in $\mathbb{R}^n$. Furthermore, because of inequalities (6)–(9), we have $\mathscr{M}_1(f, \mathbf{w}, \mathbf{b}) \geq 0$, $\mathscr{M}_2(f, \mathbf{w}, \mathbf{b}) \geq 0$, $\mathscr{M}_3(f, \mathbf{w}, \mathbf{b}) \geq 0$, and $\mathscr{M}_4(f, \mathbf{w}, \mathbf{b}) \geq 0$, for all $\mathbf{b} \in \mathscr{B}_n^0$.

2. Main Results

In this section, we establish some properties of functionals related to Jensen-type inequalities for HCFs.

Theorem 5. *Let $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$ be two n-tuples from $\mathscr{B}_n^0$. Let $I \subseteq (0, \infty)$ be an interval. If $f: I \longrightarrow \mathbb{R}$ is a HCF, $h: I \supseteq (0,1) \longrightarrow \mathbb{R}$ is a nonnegative multiplicative and additive function on J, and if $\mathbf{w} = (w_1, \ldots, w_n) \in I^n$, $h(\alpha) + h(1 - \alpha) \geq 2$, then $\mathscr{M}_i(f, \mathbf{w}, .)$, for $i = 1, 2, 3, 4$, defined by (10)–(12) are superadditive on $\mathscr{B}_n^0$, i.e.,*

$$\mathscr{M}_i(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq \mathscr{M}_i(f, \mathbf{w}, \mathbf{b}) + \mathscr{M}_i(f, \mathbf{w}, \mathbf{c}) \geq 0, \qquad (14)$$

for $i = 1, 2, 3, 4$.

Proof. Take $i = 1$ in (28) and starting from definition, we have

$$\mathcal{M}_1(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) = \sum_{\alpha=1}^{n} (b_\alpha + c_\alpha) f(w_\alpha) - (B_n + C_n) f\left(\frac{1}{(1/B_n + C_n) \sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right)$$

$$= \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) + \sum_{\alpha=1}^{n} c_\alpha f(w_\alpha) - (B_n + C_n) f\left(\frac{1}{(1/B_n + C_n) \sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right),$$

$$(15)$$

while, after arranging and harmonic convexity of f, yields

$$f\left(\frac{1}{(1/B_n + C_n) \sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right) = f\left(\frac{1}{(B_n/B_n + C_n) \cdot (1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha + (C_n/B_n + C_n) \cdot (1/C_n) \sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right)$$

$$\leq \frac{B_n}{B_n + C_n} f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) + \frac{C_n}{B_n + C_n} f\left(\frac{1}{(1/C_n) \sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right).$$

$$(16)$$

Finally, combining relation (15) and inequality (16), we obtain

$$\mathcal{M}_1(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) + \sum_{\alpha=1}^{n} c_\alpha f(w_\alpha) - B_n f\left(\frac{1}{(1/B_n) \sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right)$$

$$- C_n f\left(\frac{1}{(1/C_n) \sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right) = \mathcal{M}_1(f, \mathbf{w}, \mathbf{b}) + \mathcal{M}_1(f, \mathbf{w}, \mathbf{c}).$$

$$(17)$$

Now, taking $i = 2$ in (28) and starting from the definition, we have

$$\mathcal{M}_2(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) = (B_n + C_n)[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} (b_\alpha + c_\alpha) f(w_\alpha)$$

$$- (B_n + C_n) f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n + C_n) \sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right)$$

$$= B_n[f(w_1) + f(w_n)] + C_n[f(w_1) + f(w_n)]$$

$$- \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha) - \sum_{\alpha=1}^{n} c_\alpha f(w_\alpha) - (B_n + C_n)$$

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n + C_n) \sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right),$$

$$(18)$$

while, after arranging and harmonic convexity of f, yields

$$f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n + C_n)\sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right) = f\left(\frac{1}{(1/B_n + C_n)\sum_{\alpha=1}^{n}(b_\alpha + c_\alpha)((1/w_1) + (1/w_n) - (1/w_\alpha))}\right)$$

$$= f\left(\frac{1}{(B_n/B_n + C_n)\sum_{\alpha=1}^{n}(b_\alpha/B_n)((1/w_1) + (1/w_n) - (1/w_\alpha)) + (C_n/B_n + C_n)\sum_{\alpha=1}^{n}(c_\alpha/C_n)((1/w_1) + (1/w_n) - (1/w_\alpha))}\right)$$

$$\leq \frac{B_n}{B_n + C_n} f\left(\frac{1}{\sum_{\alpha=1}^{n}(b_\alpha/B_n)((1/w_1) + (1/w_n) - (1/w_\alpha))}\right) + \frac{C_n}{B_n + C_n} f\left(\frac{1}{\sum_{\alpha=1}^{n}(c_\alpha/C_n)((1/w_1) + (1/w_n) - (1/w_\alpha))}\right)$$

$$= \frac{B_n}{B_n + C_n} f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right) + \frac{C_n}{B_n + C_n} f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/C_n)\sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right).$$

$$(19)$$

Finally, combining relation (18) and inequality (19), we obtain

$$\mathcal{M}_2(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq B_n[f(w_1) + f(w_n)] + C_n[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} b_\alpha f(w_\alpha)$$

$$- \sum_{\alpha=1}^{n} c_\alpha f(w_\alpha) - (B_n + C_n).\frac{B_n}{B_n + C_n} f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/B_n)\sum_{\alpha=1}^{n} b_\alpha/w_\alpha}\right)$$

$$(20)$$

$$- (B_n + C_n).\frac{C_n}{B_n + C_n} f\left(\frac{1}{(1/w_1) + (1/w_n) - (1/C_n)\sum_{\alpha=1}^{n} c_\alpha/w_\alpha}\right) = \mathcal{M}_2(f, \mathbf{w}, \mathbf{b}) + \mathcal{M}_2(f, \mathbf{w}, \mathbf{c}).$$

Taking $i = 3$ in (28) and starting from the definition, we have

$$\mathcal{M}_3(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) = \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha + c_\alpha}{B_n + C_n}\right) f(w_\alpha) - f\left(\frac{1}{(1/B_n + C_n)\sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right)$$

$$= \sum_{\alpha=1}^{n} h\left(\frac{b_\alpha}{B_n + C_n}\right) f(w_\alpha) + \sum_{\alpha=1}^{n} h\left(\frac{c_\alpha}{B_n + C_n}\right) f(w_\alpha)$$

$$- f\left(\frac{1}{(1/B_n + C_n)\sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right) \qquad (21)$$

$$= \sum_{\alpha=1}^{n} h\left(\frac{B_n}{B_n + C_n}\right) h\left(\frac{b_\alpha}{B_n}\right) f(w_\alpha) + \sum_{\alpha=1}^{n} h\left(\frac{C_n}{B_n + C_n}\right) h\left(\frac{c_\alpha}{C_n}\right) f(w_\alpha)$$

$$- f\left(\frac{1}{(1/B_n + C_n)\sum_{\alpha=1}^{n} b_\alpha + c_\alpha/w_\alpha}\right),$$

while, after arranging and harmonic h-convexity of f, yields

$$f\left(\frac{1}{(1/B_n + C_n)\sum_{\alpha=1}^{n}(p_i + c_\alpha/w_\alpha)}\right) = f\left(\frac{1}{(B_n/B_n + C_n).(1/B_n)\sum_{\alpha=1}^{n}(b_\alpha/w_\alpha) + (C_n/B_n + C_n).(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)$$

$$\leq h\left(\frac{B_n}{B_n + C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right) + h\left(\frac{C_n}{B_n + C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right) \tag{22}$$

$$= h\left(\frac{B_n}{B_n + C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right) + h\left(\frac{C_n}{B_n + C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right).$$

Finally, combining relation (21) and inequality (22), we obtain

$$\mathcal{M}_3(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq h\left(\frac{B_n}{B_n + C_n}\right)\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha) + h\left(\frac{C_n}{B_n + C_n}\right)\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha)$$

$$- h\left(\frac{B_n}{B_n + C_n}\right)f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right) - h\left(\frac{C_n}{B_n + C_n}\right)f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)$$

$$= h\left(\frac{B_n}{B_n + C_n}\right)\left[\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha) - f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}b_\alpha/w_\alpha}\right)\right] \tag{23}$$

$$+ h\left(\frac{C_n}{B_n + C_n}\right)\left[\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha) - f\left(\frac{1}{(1/C_n)\sum_{i=1}^{n}c_\alpha/w_\alpha}\right)\right]$$

$$\geq \left[\sum_{\alpha=1}^{n}h\left(\frac{b_\alpha}{B_n}\right)f(w_\alpha) - f\left(\frac{1}{(1/B_n)\sum_{\alpha=1}^{n}(b_\alpha/w_\alpha)}\right)\right] + \left[\sum_{\alpha=1}^{n}h\left(\frac{c_\alpha}{C_n}\right)f(w_\alpha) - f\left(\frac{1}{(1/C_n)\sum_{\alpha=1}^{n}c_\alpha/w_\alpha}\right)\right]$$

$$= \mathcal{M}_3(f, \mathbf{w}, \mathbf{b}) + \mathcal{M}_3(f, \mathbf{w}, \mathbf{c}).$$

Similarly, it can be proved that

$$\mathcal{M}_4(f, \mathbf{w}, \mathbf{b} + \mathbf{c}) \geq \mathcal{M}_4(f, \mathbf{w}, \mathbf{b}) + \mathcal{M}_4(f, \mathbf{w}, \mathbf{c}). \tag{24}$$
□

Theorem 6. *Let* $\mathbf{b} = (b_1, \ldots, b_n)$ *and* $\mathbf{c} = (c_1, \ldots, c_n)$ *be two n-tuples from* $\mathscr{B}_n^0$ *such that* $\mathbf{b} \geq \mathbf{c}$, *(i.e.,* $b_\alpha \geq c_\alpha, \alpha = 1, \ldots, n$). *Let* $I \subseteq (0, \infty)$ *be an interval. If* $f: I \longrightarrow \mathbb{R}$ *is a HCF and if* $\mathbf{w} = (w_1, \ldots, w_n) \in I^n$, *then* $\mathcal{M}_i(f, \mathbf{w}, .)$, *for* $i = 1, 2, 3, 4$, *defined by (10)–(12) satisfy the following inequality:*

$$\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{c}), \tag{25}$$

on $\mathscr{B}_n^0$.

Proof. The monotonicity property follows directly from superadditivity. Since $\mathbf{b} \geq \mathbf{c}$, $\mathbf{b}$ can be represented as the sum of two n-tuples: $\mathbf{b} - \mathbf{c}$ and $\mathbf{c}$. Applying (28), we have

$$\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) = \mathcal{M}_i(f, \mathbf{w}, (\mathbf{b} - \mathbf{c}) + \mathbf{c}) \geq \mathcal{M}_i(f, \mathbf{w}, (\mathbf{b} - \mathbf{c})) + \mathcal{M}_i(f, \mathbf{w}, \mathbf{c}). \tag{26}$$

Finally, $\mathcal{M}_i(f, \mathbf{w}, (\mathbf{b} - \mathbf{c})) \geq 0$ by (10)–(12). So, we have that $\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{c})$, which proves the theorem. □

Theorem 7. *Let* $\mathbf{b} = (b_1, \ldots, b_n)$ *and* $\mathbf{c} = (c_1, \ldots, c_n)$ *be two n-tuples from* $\mathscr{B}_n^0$. *Let* m *and* M *be real constants such that*

$$m \geq 0, \quad b_\alpha - mc_\alpha \geq 0,$$
$$Mc_\alpha - b_\alpha \geq 0, \quad \alpha = 1, \ldots n. \tag{27}$$

Let $I \subseteq (0, \infty)$ be an interval. If $f: I \longrightarrow \mathbb{R}$ is a HCF and if $\mathbf{w} = (w_1, \ldots, w_n) \in I^n$ be any n-tuple, then

$$M\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq m\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}), \tag{28}$$

for $i = 1, 2, 3, 4$.

Proof. Since $m \geq 0$, $b_\alpha - mc_\alpha \geq 0$ and $Mc_\alpha - b_\alpha \geq 0$, $\alpha = 1, \ldots n$, this implies that $\mathbf{b} - m\mathbf{c}$ and $M\mathbf{c} - \mathbf{b}$ are in $\mathscr{B}_n^0$. Then, by Theorem 5, we obtain

$$\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq \mathcal{M}_i(f, \mathbf{w}, (\mathbf{b} - m\mathbf{c})) + \mathcal{M}_i(f, \mathbf{w}, m\mathbf{c})$$
$$\geq m\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}). \tag{29}$$

Similarly, we obtain

$$\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \leq M\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}), \tag{30}$$

that is,

$$M\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq m\mathcal{M}_i(f, \mathbf{w}, \mathbf{c}). \tag{31}$$
$\square$

Corollary 1. *Let* $\mathbf{b}$, $\mathbf{w}$, f, *and functional* $\mathcal{M}_i$ *be as in Theorem 5. Then,*

$$\max_{1 \leq k \leq n} \{b_\alpha\} \mathcal{M}_i^{\mathcal{N}}(f, \mathbf{w}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq \min_{1 \leq k \leq n} \{b_\alpha\} \mathcal{M}_i^{\mathcal{N}}(f, \mathbf{w}) \quad \text{(for } i = 1, 2\text{)}, \tag{32}$$

where

$$\mathcal{M}_1^{\mathcal{N}}(f, \mathbf{w}) = \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right),$$

$$\mathcal{M}_2^{\mathcal{N}}(f, \mathbf{w}) = n[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/w_1) + (1/w_n) - (1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right). \tag{33}$$

Proof. Let $\mathbf{b}_{\min} \in \mathcal{B}_n^0$ be a constant n-tuple, i.e.,

$$\mathbf{b}_{\min} = \left(\min_{1 \leq k \leq n} \{b_\alpha\}, \ldots, \min_{1 \leq k \leq n} \{b_\alpha\}\right). \tag{34}$$

Then, for any $\mathbf{b} \in \mathcal{B}_n^0$, we have $\mathbf{b} \geq \mathbf{b}_{\min}$. So, by applying Theorem 6, we have

$$\mathcal{M}_i(f, \mathbf{w}, \mathbf{b}) \geq \mathcal{M}_i(f, \mathbf{w}, \mathbf{b}_{\min}). \tag{35}$$

On the contrary,

$$\mathcal{M}_1(f, \mathbf{w}, \mathbf{b}_{\min}) = \min_{1 \leq k \leq n} \{b_\alpha\} \left\{ \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right) \right\},$$

$$\mathcal{M}_1(f, \mathbf{w}, \mathbf{b}_{\min}) = \min_{1 \leq k \leq n} \{b_\alpha\} \left\{ n[f(w_1) + f(w_n)] - \sum_{\alpha=1}^{n} f(w_\alpha) - nf\left(\frac{1}{(1/w_1) + (1/w_n) - (1/n)\sum_{\alpha=1}^{n} 1/w_\alpha}\right) \right\}, \tag{36}$$

i.e., $\mathcal{M}_1(f, \mathbf{w}, \mathbf{b}_{\min}) = \min_{1 \leq k \leq n} \{b_\alpha\} \mathcal{M}_i^{\mathcal{N}}(f, \mathbf{w})$. So, it proves the right-hand side of inequality (35). The left-hand inequality is obtained similarly by exchanging the role of min and max.

To present our next results, we need to introduce the following notations:

$$J(\mathbb{R}) := \{\mathbf{b} = \{b_\alpha\}_{k \in \mathbb{N}} : b_\alpha \in \mathbb{R} \quad \text{are} \quad \text{such} \quad \text{that} \\ B_K := \sum_{\alpha \in I} b_\alpha \neq 0, \text{ for all } K \in P_f(\mathbb{N})\}$$

$F^+(C, \mathbb{R}) := \{f \in F(C, \mathbb{R}) : f(x) > 0, \text{ for all } x \in C\}$

$J^+(\mathbb{R}) := \{\mathbf{b} \in J(\mathbb{R}) : b_\alpha \geq 0, \text{ for all } k \in \mathbb{N}\}$

$J_*(\mathbb{R}) := \{\mathbf{w} = \{w_\alpha\}_{\alpha \in \mathbb{N}} : w_\alpha \in C, \text{ for all } k \in \mathbb{N}\}$

$P_f(\mathbb{N}) := \{K \subset \mathbb{N} : K \text{ is finite}\}$

$H\text{Conv}(C, \mathbb{R}) :=$ the cone of all HCFs on C

$F(C, \mathbb{R}) :=$ the linear space of all real functions on C

Now, we consider more general functionals:

$$\mathcal{D}_1(f, K, \mathbf{b}, \mathbf{w}; \Psi) \coloneqq B_K \Psi\left[\frac{1}{B_K}\sum_{\alpha \in K} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha}\right)\right],$$

$$\mathcal{D}_2(f, K, \mathbf{b}, \mathbf{w}; \Psi) \coloneqq B_K \Psi\left[f(W_1) + f(X_2)\frac{1}{B_K}\sum_{\alpha \in K} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/w_1) + (1/x_2) - (1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha}\right)\right],$$

$$\mathcal{D}_3(f, K, \mathbf{b}, \mathbf{w}; \Psi) \coloneqq \Psi\left[\sum_{\alpha \in K} h\left(\frac{b_\alpha}{B_K}\right) f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha}\right)\right],$$

$$\mathcal{D}_4(f, K, \mathbf{b}, \mathbf{w}; \Psi) \coloneqq \Psi\left[f(W_1) + f(X_2)\frac{1}{B_K}\sum_{\alpha \in K} h\left(\frac{b_\alpha}{B_K}\right) f(w_\alpha) - f\left(\frac{1}{(1/w_1) + (1/x_2) - (1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha}\right)\right],$$

$$(37)$$

where $f \in HConv(C, \mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{b} \in J^+(\mathbb{R})$, $\mathbf{w} \in J_*(C)$, and $\Psi: (0, \infty) \longrightarrow \mathbb{R}$ is a convex function whose properties will determine the behavior of functional $\mathcal{D}_i, i = 1, 2, 3, 4$, as follows. Obviously, for $\Psi(t) = t$, we recapture, from functional $\mathcal{D}_i$, the functional $\mathcal{M}_i$ considered in Theorem 5.

First of all, we observe that, by Jensen-type inequality, the functional $\mathcal{D}_i$ is well defined and positive homogenous in the third variable, that is,

$$\mathcal{D}_i(f, K, m\mathbf{b}, \mathbf{w}; \Psi) = m\mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi), \quad (38)$$

for any $m > 0$ and $\mathbf{b} \in J^+(\mathbb{R})$.

The following result concerning the superadditivity and the monotonicity of the functional $\mathcal{D}_i, i = 1, 2$, as function of weights holds. □

Theorem 8. *Let* $f \in HConv(C, \mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{w} \in J_*(C)$, *and* $\Psi: (0, \infty) \longrightarrow \mathbb{R}$ *be monotonic nondecreasing and concave function.*

(i) If $\mathbf{b}, \mathbf{c} \in J^+(\mathbb{R})$, *then*

$$\mathcal{D}_i(f, K, \mathbf{b} + \mathbf{c}, \mathbf{w}; \Psi) \geq \mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) + \mathcal{D}_i(f, K, \mathbf{c}, \mathbf{w}; \Psi).$$

$$(39)$$

That is, $\mathcal{D}_i$ *is superadditive as a function of weights.*

(ii) If $\mathbf{b}, \mathbf{c} \in J^+(\mathbb{R})$, *with* $b \geq c$, *meaning that* $b_i \geq c_i$, *for each* $i \in \mathbb{N}$ *and* $\Psi: (0, \infty) \longrightarrow (0, \infty)$, *then*

$$\mathcal{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) \geq \mathcal{D}_i(f, K, \mathbf{c}, \mathbf{w}; \Psi) \geq 0. \quad (40)$$

That is, $\mathcal{D}_i$ *is monotonic nondecreasing as function of weights.*

Proof

(i) Let $\mathbf{b}, \mathbf{c} \in J^+(\mathbb{R})$; then, by the harmonic convexity of f on C,

$$\frac{1}{B_K + C_K}\sum_{\alpha \in K}(b_\alpha + c_\alpha)f(w_\alpha) - f\left(\frac{1}{(1/B_K + B_K)\sum_{\alpha \in K}(b_\alpha + c_\alpha)/w_\alpha}\right)$$

$$= \frac{B_K\left((1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha)\right) + C_K\left((1/C_K)\sum_{\alpha \in K} c_\alpha f(w_\alpha)\right)}{B_K + C_K}$$

$$- f\left(\frac{1}{(B_K/B_K + C_K).(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha + (C_K/B_K + C_K).(1/C_K)\sum_{\alpha \in K} c_\alpha/w_\alpha}\right)$$

$$\geq \frac{B_K\left((1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha)\right) + C_K\left((1/C_K)\sum_{\alpha \in K} c_\alpha f(w_\alpha)\right)}{B_K + C_K}$$

$$-\frac{B_K f\left(1/\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha/w_\alpha\right)+C_K f\left(1/\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha/w_\alpha\right)}{B_K+C_K}$$

$$=\frac{B_K\left[\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha f\left(w_\alpha\right)-f\left(1/\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha/w_\alpha\right)\right]}{B_K+C_K} \tag{41}$$

$$+\frac{C_K\left[\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha f\left(w_\alpha\right)-f\left(1/\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha/w_\alpha\right)\right]}{B_K+C_K}.$$

Since Ψ is monotonically nondecreasing and concave, then, by (40),

$$\Psi\left[\frac{1}{B_K+C_K}\sum_{\alpha\in K}\left(b_\alpha+c_\alpha\right)f\left(w_\alpha\right)-f\left(\frac{1}{\left(1/B_K+B_K\right)\sum_{\alpha\in K}\left(b_\alpha+c_\alpha\right)/w_\alpha}\right)\right]$$

$$\geq\frac{B_K\Psi\left[\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha f\left(w_\alpha\right)-f\left(1/\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha/w_\alpha\right)\right]}{B_K+C_K} \tag{42}$$

$$+\frac{C_K\Psi\left[\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha f\left(w_\alpha\right)-f\left(1/\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha/w_\alpha\right)\right]}{B_K+C_K},$$

which, by multiplication with $B_K+C_K>0$, produces the desired result (39) for $i=1$.

Similarly, we can easily verify result (39), for $i=2$.

(ii) If $\mathbf{b}\geq\mathbf{c}$, then by (i),

$$\mathcal{D}_i\left(f,K,\mathbf{b},\mathbf{w};\Psi\right)=\mathcal{D}_i\left(f,K,\left(\mathbf{b}-\mathbf{c}\right)+\mathbf{c},\mathbf{w};\Psi\right)$$

$$\geq\mathcal{D}_i\left(f,K,\left(\mathbf{b}-\mathbf{c}\right),\mathbf{w};\Psi\right)+\mathcal{D}_i\left(f,K,\mathbf{c},\mathbf{w};\Psi\right)\geq\mathcal{D}_i\left(f,K,\mathbf{c},\mathbf{w};\Psi\right), \tag{43}$$

since $\mathcal{D}_i\left(f,K,\left(\mathbf{b}-\mathbf{c}\right),\mathbf{w};\Psi\right)\geq0$. □

Corollary 2. *Let $f\in HConv(C,\mathbb{R})$, $K\in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\Psi:(0,\infty)\longrightarrow\mathbb{R}$ be a monotonically nondecreasing and concave function.*

If there exist the numbers m and M with $M\geq m\geq0$ such that $M\mathbf{c}\geq\mathbf{b}\geq m\mathbf{c}$, then

$$MC_K\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f\left(w_\alpha\right)-f\left(\frac{1}{\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right]$$

$$\geq B_K\Psi\left[\frac{1}{B_K}\sum_{\alpha\in K}b_\alpha f\left(w_\alpha\right)-f\left(\frac{1}{\left(1/B_K\right)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right] \tag{44}$$

$$\geq mC_K\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f\left(w_\alpha\right)-f\left(\frac{1}{\left(1/C_K\right)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right].$$

In particular,

$$\frac{M}{m}\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f(w_\alpha) - f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right]$$

$$\geq \Psi\left[\frac{1}{B_K}\sum_{\alpha\in K}b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/B_K)\sum_{\alpha\in K}b_\alpha/w_\alpha}\right)\right] \qquad (45)$$

$$\geq \frac{m}{M}\Psi\left[\frac{1}{C_K}\sum_{\alpha\in K}c_\alpha f(w_\alpha) - f\left(\frac{1}{(1/C_K)\sum_{\alpha\in K}c_\alpha/w_\alpha}\right)\right].$$

For a function $\Phi\colon (0,\infty) \longrightarrow (0,\infty)$, we now consider the functionals

$$\mathfrak{D}_1(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi) := \sum_{\alpha\in K}\Phi(b_\alpha)\Psi\left[\frac{1}{\sum_{\alpha\in K}\Phi(b_\alpha)}\sum_{\alpha\in K}\Phi(b_\alpha)f(w_\alpha) - f\left(\frac{1}{(1/\sum_{\alpha\in K}\Phi(b_\alpha))\sum_{\alpha\in K}\Phi(b_\alpha)/w_\alpha}\right)\right],$$

$$\mathfrak{D}_2(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi) := \sum_{\alpha\in K}\Phi(b_\alpha)\Psi\left[f(w_1) + f(w_n) - \left(1/\sum_{\alpha\in K}\Phi(b_\alpha)\right)\sum_{\alpha\in K}\Phi(b_\alpha)f(w_\alpha)\right. \qquad (46)$$

$$\left. - f\left(\frac{1}{(1/w_1)+(1/w_n)-(1/\sum_{\alpha\in K}\Phi(b_\alpha))\sum_{\alpha\in K}\Phi(b_\alpha)/w_\alpha}\right)\right],$$

where $f \in H\mathrm{Conv}(C,\mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\mathbf{b}\in J^+(\mathbb{R})$. Now, if we denote by $\Phi(\mathbf{b})$ the sequence $\{\Phi(b_\alpha)\}_{k\in\mathbb{N}}$, then we observe that, for $i=1,2$,

$$\mathfrak{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi) = \mathscr{D}_i(f,K,\Phi(\mathbf{b}),\mathbf{w};\Psi). \qquad (47)$$

The following result may be stated.

Corollary 3. *Let $f \in H\mathrm{Conv}(C,\mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\Psi\colon (0,\infty) \longrightarrow (0,\infty)$ be monotonically nondecreasing and concave function. If $\Phi\colon (0,\infty) \longrightarrow (0,\infty)$ is concave, then $\mathfrak{D}_i(f,K,.,\mathbf{w};\Psi,\Phi)$ is also concave on $J^+(\mathbb{R})$, for $i=1,2$.*

Proof. Utilizing the properties of monotonicity, superadditivity, and positive homogeneity of functional $\mathscr{D}_i(f,K,.,\mathbf{w};\Psi)$, we have

$$\mathfrak{D}_i(f,K,t\mathbf{b}+(1-t)\mathbf{c},\mathbf{w};\Psi,\Phi) = \mathscr{D}_i(f,K,\Phi(t\mathbf{b}+(1-t)\mathbf{c}),\mathbf{w};\Psi)$$

$$\geq \mathscr{D}_i(f,K,t\Phi(\mathbf{b})+(1-t)\Phi(\mathbf{c}),\mathbf{w};\Psi)$$

$$\geq \mathscr{D}_i(f,K,t\Phi(\mathbf{b}),\mathbf{w};\Psi) + \mathscr{D}_i(f,K,(1-t)\Phi(\mathbf{c}),\mathbf{w};\Psi) \qquad (48)$$

$$= t\mathscr{D}_i(f,K,\Phi(\mathbf{b}),\mathbf{w};\Psi) + (1-t)\mathscr{D}_i(f,K,\Phi(\mathbf{c}),\mathbf{w};\Psi)$$

$$= t\mathfrak{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi,\Phi) + (1-t)\mathfrak{D}_i(f,K,\mathbf{c},\mathbf{w};\Psi,\Phi),$$

for any $\mathbf{b},\mathbf{c}\in J^+(\mathbb{R})$ and $t\in[0,1]$, which proves the statement.

The following result concerning the superadditivity and monotonicity of the functional $\mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi)$, for $i=1,2$, as an index set function holds.   □

Theorem 9. *Let $f \in H\mathrm{Conv}(C,\mathbb{R})$, $K \in P_f(\mathbb{N})$, $\mathbf{w}\in J_*(C)$, and $\mathbf{b}\in J^+(\mathbb{R})$. Assume that $\Psi\colon (0,\infty) \longrightarrow (0,\infty)$ is a monotonically nondecreasing and concave function.*

(i) If $K,L \in P_f(\mathbb{N})$ with $K\cap L\neq\phi$,

$$\mathscr{D}_i(f,K\cup L,\mathbf{b},\mathbf{w};\Psi) \geq \mathscr{D}_i(f,K,\mathbf{b},\mathbf{w};\Psi) + \mathscr{D}_i(f,L,\mathbf{b},\mathbf{w};\Psi). \qquad (49)$$

That is, $\mathscr{D}_i(f,K,.,\mathbf{w};\Psi)$ is superadditive as an index set function on $P_f(\mathbb{N})$.

(ii) If $K,L \in P_f(\mathbb{N})$ with $L\subset K$ and $\Psi\colon (0,\infty) \longrightarrow (0,\infty)$, then

$$\mathscr{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) \geq \mathscr{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi)(\geq 0). \quad (50)$$

That is, $\mathscr{D}_i(f, K, ., \mathbf{w}; \Psi)$ is monotonically nondecreasing as an index set function on $P_f(\mathbb{N})$.

Proof

(i) Let $K, L \in P_f(\mathbb{N})$ with $K \cap L \neq \phi$. By the harmonic convexity of f on C, we have

$$\frac{1}{P_{K \cup L}} \sum_{j \in K \cup L} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/P_{K \cup L})\sum_{j \in K \cup L} b_\alpha/w_\alpha}\right)$$

$$= \frac{B_K\left((1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha)\right) + P_L\left((1/P_L)\sum_{l \in L} q_l f(x_l)\right)}{B_K + P_L}$$

$$- f\left(\frac{1}{(B_K/B_K + P_L).(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha + (P_L/B_K + P_L).(1/P_L)\sum_{l \in L} q_l/x_l}\right)$$

$$\geq \frac{B_K\left((1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha)\right) + P_L\left((1/P_L)\sum_{l \in L} q_l f(x_l)\right)}{B_K + P_L} \quad (51)$$

$$- \frac{B_K f\left(1/(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha\right) + P_L f\left(1/(1/P_L)\sum_{l \in L} q_l/x_l\right)}{B_K + P_L}$$

$$= \frac{B_K\left[(1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha) - f\left(1/(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha\right)\right]}{B_K + P_L}$$

$$+ \frac{P_L\left[(1/P_L)\sum_{l \in L} q_l f(x_l) - f\left(1/(1/P_L)\sum_{l \in L} q_l/x_l\right)\right]}{B_K + P_L}.$$

Since Ψ is monotonically nondecreasing and concave, then, by (50),

$$\Psi\left[\frac{1}{P_{K \cup L}} \sum_{j \in K \cup L} b_\alpha f(w_\alpha) - f\left(\frac{1}{(1/P_{K \cup L})\sum_{j \in K \cup L} b_\alpha/w_\alpha}\right)\right]$$

$$\geq \frac{B_K \Psi\left[(1/B_K)\sum_{\alpha \in K} b_\alpha f(w_\alpha) - f\left(1/(1/B_K)\sum_{\alpha \in K} b_\alpha/w_\alpha\right)\right]}{B_K + P_L} \quad (52)$$

$$+ \frac{P_L \Psi\left[(1/P_L)\sum_{l \in L} q_l f(x_l) - f\left(1/(1/P_L)\sum_{l \in L} q_l f(x_l)\right)\right]}{B_K + P_L},$$

which, by multiplication with $B_K + P_L > 0$, produces the desired result (48) for $i = 1$.

Similarly, we can easily verify result (48), for $i = 2$.

(ii) Let $K, L \in P_f(\mathbb{N})$ with $L \subset K$; then,

$$\mathscr{D}_i(f, K, \mathbf{b}, \mathbf{w}; \Psi) = \mathscr{D}_i(f, (K\backslash L) \cup L, \mathbf{b}, \mathbf{w}; \Psi) \geq \mathscr{D}_i(f, K\backslash L, \mathbf{b}, \mathbf{w}; \Psi) + \mathscr{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi)$$

$$\geq \mathscr{D}_i(f, L, \mathbf{b}, \mathbf{w}; \Psi)(\geq 0), \quad (53)$$

since $\mathscr{D}_i(f, K\backslash L, \mathbf{b}, \mathbf{w}; \Psi) \geq 0$. □

3. Conclusion

First of all, we have presented the refinement of Jensen-type inequality, and further, we have discussed several important aspect of functionals associated with Jensen-type inequalities for the HCFs. On the basis of ideas discussed in this paper along with the literature present on HCFs, we encourage the interested researcher to explore more interesting results for this class of functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally in this research article. All authors read and approved the final manuscript.

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