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Properties and conjectures for the flux of TASEP with site disorder

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Abstract. We consider the TASEP with site disorder and show that several representations of the flux are equivalent. We then describe heuristically the phase transition induced by site disorder and relate it to the problem of homogenization of conservation laws.

1 Introduction

Disorder can have a dramatic impact on the properties of particle systems. It has been shown in several models of equilibrium statistical physics (Aizenman and Wehr (1990)) that the disorder can suppress phase transitions. In non-equilibrium statistical mechanics, the disorder has also a variety of effects which differ depending on the models. Even in the reduced class of disordered driven system, one has to distinguish the behaviors of site and particle disorder TASEP. The latter case has been extensively studied in Andjel et al. (2000), Bahadoran et al. (2014), Benjamini, Ferrari and Landim (1996), Ferrari and Sisko (2007), Georgiou, Kumar and Seppäläinen (2010), Krug and Seppäläinen (1999), Lin and Seppäläinen (2012) and an important mathematical feature of this model is that the invariant measures can be computed explicitly. When the jump rates are site dependent, all the knowledge on the invariant measures is lost and an analytic analysis turns out to be very challenging. In this paper, we will focus on the case of site disorder, for which mathematical results in the literature are quite rare (Seppäläinen (1999), Bahadoran et al. (2014), Chayes and Liggett (2007), Schütz (1993, 2014), Szavits-Nossan (2013)) compared to particle disorder.

The flux is a key macroscopic parameter in non-equilibrium systems. In particular, it governs the hydrodynamic behavior of such systems via the scalar conservation law

$$\partial_t \rho(t,x) + \partial_x f\big[\rho(t,x)\big] = 0, \qquad (1.1)$$

where $\rho(t,x)$ denotes the local density of particles after hyperbolic space–time rescaling. It is well-known that in the particular case of homogeneous TASEP, the

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flux function is given by

$$f(\rho) := \rho(1-\rho). \tag{1.2}$$

In the case of site disorder TASEP, the flux function cannot be computed explicitly. It has been shown in Tripathy and Barma (1998), Harris and Stinchcombe (2004), Krug (2000) that the flux carries the signature of a phase transition. Contrary to the homogenous TASEP for which the flux is strictly concave, the flux of the site disordered TASEP is constant for densities in an interval $[\rho_c, 1-\rho_c]$ (Tripathy and Barma (1998)). There are several ways to define the macroscopic flux: some are suited for simulations, others are easier to handle mathematically. Our goal in this paper is twofold. First, we are going to prove that different definitions of the flux are equivalent and in this way, the numerical results for the flux on large periodic domains (Tripathy and Barma (1998), Harris and Stinchcombe (2004)) are compatible with the mathematical results obtained on the flux of the disordered TASEP in $\mathbb{Z}$ (Bahadoran and Bodineau (2015a)). Second, we recall some heuristics on the phase transition and explain how the existence of a plateau in the flux can be understood in terms of homogenization of a multiscale system (Bahadoran and Bodineau (2015a)).

2 Definitions of the flux

In this section, we introduce three different microscopic representations of the flux and we show that they all lead to the same macroscopic quantity.

2.1 TASEP with site disorder

From a numerical point of view (Tripathy and Barma (1998)), it is convenient to define the TASEP dynamics on a ring of size N and to extrapolate the macroscopic properties of the system by considering the large N limit. Let $\mathbb{T}_N := \{0, \dots, N-1\}$ with periodic boundary conditions. The particle configuration in $\mathbb{T}_N$ is denoted by $\eta = (\eta(x) : x \in \mathbb{T}_N)$, where $\eta(x) \in \{0, 1\}$ is the occupation number at site x. The jump rates are site dependent $\alpha = (\alpha(x) : x \in \mathbb{T}_N)$ where the $\alpha(x)$ take values in $(0, 1]$. Typically, the $\alpha(x)$ are independent and identically distributed random variables. A particle at site x will jump at site $x+1$ with rate $\alpha(x)$ if the site $x+1$ is empty. The TASEP is a Markov process on $\{0, 1\}^{\mathbb{T}_N}$ with generator given by

$$L_N^{\alpha} f(\eta) = \sum_{x \in \mathbb{T}_N} \alpha(x)\eta(x)[1-\eta(x+1)][f(\eta^{x,x+1}) - f(\eta)], \tag{2.1}$$

where $\eta^{x,x+1} = \eta - \delta_x + \delta_{x+1}$ denotes the new configuration after a particle has jumped from x to $x+1$ and N is identified with 0 so that a particle at $N-1$ jumps to 0. This defines a Markov jump process with finite state space. Due to the conservation of particle number, each set

$$\mathbf{X}_{N,k} := \left\{ \eta : \sum_{x \in \mathbb{T}_N} \eta(x) = k \right\} \tag{2.2}$$

is invariant. The restriction of the process to $\mathbf{X}_{N,k}$ is irreducible. Thus, it has a unique invariant measure $\nu^{\alpha}_{N,k}$.

Given α, we denote by $J^{\alpha}_x(t,\eta_0)$ the number of jumps from x to $x+1$ up to time t, in the TASEP starting from initial state η_0, and evolving in environment α. By the ergodic theorem, this measures the current flowing in the ring $\mathbb{T}_N$ for a given number k of particles

$$f_N(k) := \lim_{t\to\infty} \frac{1}{t} J^{\alpha}_x(t,\eta_0) = \int j^{\alpha}_x(\eta)\, d\nu^{\alpha}_{N,k}(\eta), \tag{2.3}$$

where the microscopic flux is defined as

$$j^{\alpha}_x(\eta) = \alpha(x)\eta(x)\big[1-\eta(x+1)\big]. \tag{2.4}$$

Since there is a finite number k of particles,

$$\forall x, y \in \mathbb{T}_N \qquad \big|J^{\alpha}_x(t,\eta_0) - J^{\alpha}_y(t,\eta_0)\big| \le k,$$

which implies that the asymptotic quantity (2.3) does not depend on x. A natural definition of the macroscopic current at density ρ is therefore

$$f(\rho) := \lim_{N\to\infty, k/N\to\rho} \int j^{\alpha}_x(\eta)\, d\nu^{\alpha}_{N,k}(\eta). \tag{2.5}$$

The expectation in (2.3) is well defined for all $N \in \mathbb{N}^*$ and $k \in \mathbb{N}$, but the existence of the thermodynamic limit (2.5) and the independence on α is a non-trivial issue which will be addressed in Theorem 2.1.

Alternatively, one can consider the TASEP on $\mathbb{Z}$ with site disorder $\alpha = (\alpha(x) : x \in \mathbb{Z}) \in \mathbf{A} := (0,1]^{\mathbb{Z}}$, where α is a stationary ergodic sequence of positive bounded random variables. A particle configuration on $\mathbb{Z}$ is of the form $\eta = (\eta(x) : x \in \mathbb{Z})$ and the state space is denoted by $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$. As in the periodic case, the jump rate from site x is $\alpha(x)$ and the generator of the process is given

$$L^{\alpha} f(\eta) = \sum_{x\in\mathbb{Z}} \alpha(x)\eta(x)\big[1-\eta(x+1)\big]\big[f(\eta^{x,x+1}) - f(\eta)\big]. \tag{2.6}$$

A second definition of the flux $f(\rho)$ is given as the limit of the microscopic current starting from macroscopically homogeneous states at density ρ. For $\rho \in [0,1]$, let η^{ρ} be an initial particle configuration with uniform density profile ρ in the following sense

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=0}^{n} \eta^{\rho}(x) = \rho = \lim_{n\to\infty} \frac{1}{n} \sum_{x=-n}^{0} \eta^{\rho}(x). \tag{2.7}$$

As before, $J^{\alpha}_x(t,\eta^{\rho})$ stands for the rightward current across site x up to time t starting from η^{ρ}. We then set

$$f(\rho) := \lim_{t\to\infty} \frac{1}{t} J^{\alpha}_x(t,\eta^{\rho}), \tag{2.8}$$

where the limit is understood in probability with respect to the law of the quenched process. We will show in Theorem 2.1 that this limit exists and is independent of $x \in \mathbb{Z}$, of the disorder realization α, and of the choice of initial data η^ρ with average density ρ in the sense (2.7). Definition (2.8) is the one for which a hydrodynamic limit of the form (1.1) is established in Seppäläinen (1999) for the infinite disordered TASEP with generator (2.6). It is also the definition that arises in Bahadoran and Bodineau (2015a).

A third definition of the flux $f(\rho)$ is to consider the expectation of the microscopic flux function j_x^α defined in (2.4) with respect to the stationary measure at density ρ on $\mathbb{Z}$. This definition is the usual one in hydrodynamic limit theory, but it arises from the study of classical models (such as the homogeneous TASEP) for which invariant measures on the line are explicitly known, and most often of product form. It happens to be more problematic for asymmetric models which do not have this property, in these cases almost nothing is known about invariant measures of the infinite system. Contrary to the finite size system, a stationary measure does not always exist for all density values ρ, as phase transition may occur for some values of the density. Thus further notation is required before stating this third definition.

Let $\mathcal{R}$ be the set of densities $\rho \in [0, 1]$ with the following property: there exists a subset $\mathbf{B} \subset \mathbf{A}$ of the jump rate configurations, and a family of probability measures $(\nu_\rho^\alpha)_{\alpha \in \mathbf{B}}$, such that:

(o) $\tau_1 \mathbf{B} = \mathbf{B}$ a.s., and $\mathbf{B}$ has probability 1 with respect to the law of the environment.

(i) For every $\alpha \in \mathbf{B}$, ν_ρ^α is invariant for L^α.

(ii) ν_ρ^α has density ρ in the sense that

$$\lim_{l \to \infty} \frac{1}{2l+1} \sum_{x \in \mathbb{Z}:\, |x| \le l} \eta(x) = \rho, \qquad \nu_\rho^\alpha\text{-a.s.} \tag{2.9}$$

(iii) For a.e. $\alpha \in \mathbf{B}$ and $x \in \mathbb{Z}$, $\nu_\rho^{\tau_x \alpha} = \tau_x \nu_\rho^\alpha$.

Note that $\mathcal{R}$ contains 0 and 1, since the deterministic configurations with either 0's or 1's everywhere are invariant for any environment α. The following is shown in Bahadoran et al. (2014):

(1) $\mathcal{R}$ is a closed subset of [0, 1];

(2) the quantity $f(\rho)$ defined by

$$f(\rho) := \begin{cases} \displaystyle\int j_x^\alpha(\eta)\, d\nu_\rho^\alpha(\eta), & \text{if } \rho \in \mathcal{R}, \\ \displaystyle\frac{\rho^+ - \rho}{\rho^+ - \rho^-} f(\rho^-) + \frac{\rho - \rho^-}{\rho^+ - \rho^-} f(\rho^+), & \text{if } \rho \notin \mathcal{R}, \end{cases} \tag{2.10}$$

where

$$\rho^- = \sup \mathcal{R} \cap [0, \rho), \qquad \rho^- = \inf \mathcal{R} \cap (\rho, 1] \tag{2.11}$$

does not depend on the disorder realization α;

(3) the system has a quenched hydrodynamic limit given by a scalar conservation law with flux function $f(\rho)$, see Proposition 2.1 below for a precise statement.

Remark 2.1. Little is known about the above set $\mathcal{R}$. In particular, whether it is a strict subset of [0, 1], and whether it contains values other than 0 and 1, are open questions.

An easy observation (see Seppäläinen (2001)) is that $\mathcal{R}$ and f are symmetric if the distribution of α is invariant by symmetry $x \mapsto -x$.

The main result of this section is the equivalence of the three definitions for the flux.

Theorem 2.1. *For $\mathcal{P}$-a.e. $\alpha \in \mathbf{A}$, the flux $f(\rho)$ defined by the limits* (2.5) *and* (2.8) *exists for all $x \in \mathbb{Z}$ and $\rho \in [0,1]$. These limits do not depend on x or α, and coincide with the quantity defined by* (2.10) *and* (2.11).

For notational convenience, we have written Theorem 2.1 in the context of site-disordered TASEP. However, the proof uses fairly general arguments and would apply more generally to the setting of Bahadoran et al. (2014), where L^α in (2.6) is a family of attractive generators indexed by an abstract ergodic environment α, such that $\tau_x L_\alpha = L_{\tau_x \alpha}$, and τ_x denotes the space shift. This includes in particular translation invariant systems with unknown invariant measures, such as the K-exclusion process.

The rest of this section is devoted to the proof of Theorem 2.1. The main ingredients for this proof are a localization argument, stated in Section 2.2 below, that derives the hydrodynamic limit for site-disordered TASEP on the torus from the result on the line, and a result on the asymptotic flux in (1.1) on the torus, stated in Section 2.3 below.

It is shown in Seppäläinen (2001) that f is a concave function. Apart from concavity, little can be said about f. This is due to the fact that invariant measures for site-disordered TASEP are not explicit (in contrast with particle-disordered TASEP or equivalently site-disordered zero-range process, which has explicit product invariant measures). One of the key issues is whether f is strictly concave or not. For i.i.d. disorder, numerical works (see, e.g., Tripathy and Barma (1998)) confirm the occurrence of a plateau for the flux function f, that is an interval $[\rho_c, 1-\rho_c]$ (with $0 \le \rho_c < 1/2$) on which f is constant. The occurrence of such a plateau is a signature of a phase transition and it will be discussed in Section 3.

2.2 Hydrodynamic limit of site-disordered TASEP

Hydrodynamic limit was established in Seppäläinen (2001) for TASEP with i.i.d. site disorder, and more generally in Bahadoran et al. (2014), for attractive systems in ergodic random environment. Both approaches are designed intrinsically for the infinite system on $\mathbb{Z}$ and are not naturally localizable. In this section, we state a general implication from hydrodynamic limit on the line to hydrodynamic limit on the torus.

In order to compare the periodic and the infinite volume dynamics, it is convenient to introduce an alternative view of the dynamics defined by (2.1) as a Markov process on the subset

$$\mathbf{X}_N^{\text{per}} := \{\eta \in \mathbf{X} \colon \forall x \in \mathbb{Z}, \eta(x+N) = \eta(x)\}$$

of periodic particle configurations on $\mathbb{Z}$ with generator

$$L_N^\alpha f(\eta) = \sum_{x=0}^{N-1} \alpha(x)\eta(x)\big[1-\eta(x+1)\big]\big[f\big(\eta^{x,x+1,\text{per}}\big) - f(\eta)\big], \tag{2.12}$$

where $\eta^{x,x+1,\text{per}}$ is the periodic configuration obtained by letting particles jump from $x+kN$ to $x+1+kN$ for all $k \in \mathbb{Z}$. The process defined by (2.12) is equivalent to (2.1) in the sense that the restriction to $\{0, \dots, N-1\}$ is a Markov process with generator (2.1). Conversely, the periodic extension of a process generated by (2.1) is a process generated by (2.12).

For the purpose of forthcoming statements, we now recall some standard definitions from hydrodynamic limit theory. Let $\mathcal{M}(\mathbb{R})$ denote the set of Radon measures on $\mathbb{R}$ equipped with the topology of vague convergence. Define

$$\pi^N(\eta)(dx) := \frac{1}{N}\sum_{y\in\mathbb{Z}} \eta(y)\delta_{y/N}(dx) \in \mathcal{M}(\mathbb{R}) \tag{2.13}$$

as the empirical density field viewed at scale N under the particle configuration η. Let $(\eta^N)_{N\in\mathbb{N}^*}$ be a sequence of $\mathbf{X}$-valued random variables, and $\rho(\cdot)$ be an $L^\infty(\mathbb{R};[0,1])$-valued random field, viewed as the random measure $\pi(dx) = \rho(\cdot)\,dx$ in $\mathcal{M}(\mathbb{R})$. We write $\eta^N \sim \rho(\cdot)$, iff the sequence of random measures $\pi^N(\eta^N)(dx)$ converges in distribution to $\rho(\cdot)\,dx$. In the periodic setting, when working on $\mathbf{X}_N^{\text{per}}$ with generator (2.12), $\rho(\cdot)$ is a.s. a 1-periodic function taking values in $[0,1]$.

Suppose $(\eta_0^N)_{N\geq 1}$ is a sequence of (random or deterministic) particle configurations on $\mathbb{Z}$. We denote by $(\eta_t^N)_{t\geq 0}$ the process with initial state η_0^N and generator L^α defined by (2.6). Similarly, if η_0^N are configurations on $\mathbb{T}^N$ (viewed as periodic configurations on $\mathbb{Z}$), we denote by $(\eta_t^N)_{t\geq 0}$ the process with initial state η_0^N and generator L_N^α defined by (2.12).

We say that the generator (according to the context) L^α or L_N^α has hydrodynamic limit (1.1) if the following holds: for any $\rho_0(\cdot) \in L^\infty(\mathbb{R},[0,1])$ and any initial sequence $\eta_0^N \sim \rho_0(\cdot)$, the $\mathcal{M}(\mathbb{R})$-valued process $(\pi_{Nt}^N)_{t\geq 0}$ converges in probability in the Skorokhod space $\mathcal{D}([0,\infty),\mathcal{M}(\mathbb{R}))$ to the deterministic path $\pi_t(dx) = \rho(t,\cdot)\,dx$, where $\rho(\cdot,\cdot)$ denotes the entropy solution to (1.1) with

Cauchy datum $\rho_0(\cdot)$. When L_N^α is considered, we restrict to 1-periodic data $\rho_0(\cdot)$ on $\mathbb{R}$.

The following result is proved in Section 2.5.

Proposition 2.1. *Assume that for some $\alpha(\cdot) \in \mathbf{A}$, the generator L^α defined by* (2.6) *has hydrodynamic limit* (1.1) *on $\mathbb{R}$. Then the generator L_N^α defined by* (2.12) *has hydrodynamic limit* (1.1) *on $\mathbb{T}$.*

For notational simplicity, we stated and will prove this proposition in the context of disordered TASEP, but it is valid as such for any conservative dynamics with local interactions and bounded number of particles per site. Using the results of Seppäläinen (1999), Bahadoran et al. (2014), we immediately deduce the following corollary.

Corollary 2.1. *For $\mathcal{P}$-a.e. environment $\alpha \in \mathbf{A}$, the generators L^α and L_N^α have the hydrodynamic limit* (1.1), *respectively, on $\mathbb{R}$ and $\mathbb{T}$, where f is given by* (2.8) *or* (2.10) *and* (2.11).

We next state two consequences of the hydrodynamic limit that will be useful for us. The proofs of these statements are postponed to Section 2.6. The first of these consequences is the extension to random initial conditions.

Corollary 2.2. *Let $(\eta_t^N)_{t\geq 0}$ denote the process with generator* (2.6) *or* (2.12) *starting from η_0^N, where $\eta_0^N \sim \rho_0(\cdot)$ for some random $\rho_0(\cdot)$. Then the $\mathcal{M}(\mathbb{R})$-valued process $(\pi_{Nt}^N)_{t\geq 0}$ converges in law in the Skorokhod space $\mathcal{D}([0,\infty), \mathcal{M}(\mathbb{R}))$ to the random path $\pi_t(dx) = \rho(t,\cdot)\,dx$, where $\rho(\cdot,\cdot)$ denotes the entropy solution to* (1.1) *with Cauchy datum $\rho_0(\cdot)$.*

The next corollary implies a scaling limit for the current $J_x^\alpha(t,\eta)$ across site x.

Corollary 2.3. *Under the assumptions of Corollary* 2.2, *for $\mathcal{P}$-a.e. $\alpha \in \mathbf{A}$, the following holds: for all $t \geq 0$ and $x \in \mathbb{T}$,*

$$\lim_{N\to\infty} \frac{1}{N} J_{\lfloor Nx \rfloor}^\alpha (Nt, \eta_0^N) = \int_0^t f\big[\rho(s,x)\big]\,ds \tag{2.14}$$

in probability with respect to the law of the quenched process.

2.3 Asymptotic current for periodic solutions of (1.1)

In this subsection, we state and prove the following result.

Proposition 2.2. *Let $\rho_0(\cdot) \in L^\infty(\mathbb{R})$ be 1-periodic and $[0,1]$-valued, with*

$$\bar{\rho} = \int_0^1 \rho_0(x)\,dx. \tag{2.15}$$

Let $\rho(t,x)$ denote the entropy solution to (1.1) *with Cauchy datum $\rho_0(\cdot)$. Then, for every $x \in \mathbb{R}$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(\rho(s,x)\big)\, ds = f(\bar\rho). \tag{2.16}$$

Proof. Let $h_0(x) := \int_0^x \rho_0(y)\, dy$ and $\tilde h_0(x) = \bar\rho x$. Since ρ_0 is 1-periodic and satisfies (2.15), h_0 and $\tilde h_0$ coincide on the integers, hence

$$\sup_{x\in\mathbb{R}} \big|h_0(x) - \tilde h_0(x)\big| < +\infty. \tag{2.17}$$

Let $h_t(x)$ and $\tilde h_t(x)$ denote the viscosity solutions of the Hamilton–Jacobi equation

$$\partial_t h + f(\partial_x h) = 0 \tag{2.18}$$

with respective initial data h_0 and $\tilde h_0$. It is well-known (Barles (1994)) that:

(i) $\partial_x h_t =: \rho_t$ is the entropy solution to (1.1) with initial datum $\partial_x h_0 = \rho_0$.
(ii) $\tilde h_t$ is given by

$$\tilde h_t(x) = \bar\rho x - t f(\bar\rho). \tag{2.19}$$

(iii) h_t and $\tilde h_t$ satisfy the maximum principle

$$\sup_{x\in\mathbb{R}} \big|h_t(x) - \tilde h_t(x)\big| \le \sup_{x\in\mathbb{R}} \big|h_0(x) - \tilde h_0(x)\big|. \tag{2.20}$$

It follows from (2.19), (2.20) and (2.17) that

$$\lim_{t\to\infty} \frac{1}{t}\big[h_t(x) - h_0(x)\big] = \lim_{t\to\infty} \frac{1}{t} h_t(x) = -f(\bar\rho).$$

But by (2.18)

$$h_t(x) - h_0(x) = -\int_0^t ds\, f\big(\partial_x h_s(x)\big).$$

Since $\rho_s = \partial_x h_s$, the LHS of the above equality is equal to the LHS of (2.16). □

2.4 Proof of Theorem 2.1

We proceed in two steps. First, we prove the following proposition.

Proposition 2.3. *The limit in* (2.8) *exists, coincides with the quantity defined by* (2.10) *and* (2.11), *and is independent of x and α.*

Proof. We recalled above from Bahadoran et al. (2014) that the flux function f defined by (2.10) and (2.11) was independent of x and α. By Theorem 2.1, the hydrodynamic limit of L^α is given by (1.1) with this f. Assumption (2.7) implies that the sequence of initial configurations defined by $\eta_0^N = \eta^\rho$ has uniform density profile $\rho_0(\cdot) \equiv \rho$. The corresponding entropy solution is uniform and constant, that is, $\rho(t,x) \equiv \rho$. Corollary 2.3 implies that the limit (2.8) exists and equals $f(\rho)$. $\square$

Next, we have to show that the thermodynamic limit in (2.5) is well defined and coincides with the function f of Proposition 2.3.

Proposition 2.4. *Assume $\alpha \in \mathbf{A}$ is such that L_N^α in (2.12) has hydrodynamic limit (1.1). Then the thermodynamic limit (2.5) exists and is equal to $f(\rho)$.*

Proof. Let $(k_N)_{N\in\mathbb{N}^*}$ be a sequence of integers such that

$$\lim_{N\to\infty} \frac{k_N}{N} = \rho. \tag{2.21}$$

For notational simplicity, we write ν_N^α for ν_{N,k_N}^α. By stationarity of the latter,

$$\int_{\mathbf{X}_N^{\mathrm{per}}} j_0(\eta)\, d\nu_N^\alpha(\eta) = \mathbb{E}_N^\alpha\left[\frac{1}{Nt}\int_0^{Nt} j_0^\alpha(\eta_s^\alpha)\, ds\right] = \mathbb{E}_N^\alpha\left[\frac{1}{Nt} J_0^\alpha(Nt,\eta)\right], \tag{2.22}$$

where $\mathbb{E}_N^\alpha$ denotes the expectation w.r.t. the process with generator (2.12) and initial distribution ν_N^α. The law of this process will be denoted below by $\mathbb{P}_N^\alpha$. Let $Q^N = \nu_N^\alpha \circ (\pi^N)^{-1}$ denote the distribution of $\pi^N(\eta)$ when η has distribution ν_N^α. Since $\mathcal{M}(\mathbb{R})$ is compact for the topology of vague convergence, $(Q^N)_{N\in\mathbb{N}^*}$ is tight. Let Q be one of its limit points. In the following, $\lim_{N\to\infty}$ implicitly denotes convergence along the associated subsequence. Since $\eta(x) \in \{0,1\}$ for all x, Q is supported on Radon measures π such that $\pi(I) \le |I|$ for every interval I of $\mathbb{R}$. Hence, we may view Q as a distribution on the random 1-periodic element $\rho_0(\cdot) \in L^\infty(\mathbb{R})$ with values in $[0,1]$. Besides, (2.21) implies that

$$\int_0^1 \rho_0(x)\, dx = \rho \tag{2.23}$$

with probability 1. By Corollary 2.3, for a.e. disorder realization α, the limit

$$\frac{1}{Nt} J_{\lfloor Nx\rfloor}^\alpha(Nt,\eta) \stackrel{N\to\infty}{\longrightarrow} \Gamma_t := \frac{1}{t}\int_0^t f[\rho(s,x)]\, ds \tag{2.24}$$

holds in $\mathbb{P}_N^\alpha$-probability with respect to η, where $\rho(\cdot,\cdot)$ is the entropy solution to (1.1) with random Cauchy datum $\rho_0(\cdot)$ (note that the randomness of Γ_t comes only from the randomness of $\rho_0(\cdot)$). Since $J_0^\alpha(t,\eta)$ is bounded above in distribution by a Poisson random variable with parameter t, the LHS of (2.24) is a uniformly integrable family of random variables, hence

$$\lim_{N\to\infty} \mathbb{E}_N^\alpha\left[\frac{J_0^\alpha(Nt,\eta)}{Nt}\right] = \mathbb{E}(\Gamma_t).$$

By Proposition 2.2, 1-periodicity of $\rho_0(\cdot)$ and (2.23), $\Gamma_t \to f(\rho)$ in probability as $t \to \infty$. Since f is uniformly bounded, Γ_t is uniformly bounded, thus

$$\lim_{t\to\infty} \mathbb{E}(\Gamma_t) = f(\rho).$$

Finally, since we may choose t arbitrarily large in (2.22), the result follows. □

2.5 Proof of Proposition 2.1

For this proof, we recall some standard material.

Entropy inequalities (Kružkov (1970)). The entropy solution $\rho(t,x)$ to (1.1) with Cauchy datum $\rho_0(\cdot)$ is uniquely characterized (see, e.g., Kružkov (1970)) by the fact that, for every $c \in [0,1]$, and every $\varphi \in C_K^\infty([0,+\infty)\times\mathbb{R})$,

$$\begin{aligned}\int\int_{[0,+\infty)\times\mathbb{R}} & h_c[\rho(t,x)]\partial_t\varphi(t,x)\,dx\,dt \\ & + \int\int_{[0,+\infty)\times\mathbb{R}} g_c[\rho(t,x)]\partial_x\varphi(t,x)\,dx\,dt + \int_{\mathbb{R}} h_c[\rho_0(x)]\,dx \geq 0,\end{aligned} \tag{2.25}$$

where (h_c, g_c) is the Kružkov's entropy-flux pair defined by

$$h_c(\rho) = |\rho - c|, \qquad g_c(\rho) = \operatorname{sgn}(\rho - c)\big[f(\rho) - f(c)\big]. \tag{2.26}$$

It is equivalent to require (2.25) for c in a dense subset (independent of φ) of [0, 1].

Graphical (*or Harris*) *construction* (Harris (1972)). The process with generator (2.6) can be constructed as follows on the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, where Ω_1 is the set of space–time point measures ω_1 of the form

$$\omega_1(dt,dx) = \sum_{n\in\mathbb{N}} \delta_{(t_n,x_n)}$$

equipped with the σ-field generated by the mappings $\omega \mapsto \omega((0,t]\times\{x\})$, where $t > 0$, $x \in \mathbb{Z}$, and the probability measure $\mathbb{P}_1$ that makes ω a random Poisson point process with intensity $dt \otimes (\sum_{x\in\mathbb{Z}} \alpha(x)\delta_x)$. One can show that for a.e. realization of ω, for any given initial state η_0, there exists a unique cadlag path $(\eta_t = \eta_t(\eta_0, \omega_1))_{t\geq 0}$ such that:

(i) for every atom (t,x) of ω_1, $\eta_t = \eta_{t-}^{x,x+1}$ if $\eta_{t-}(x) = 1$ and $\eta_{t-}(x+1) = 0$,

(ii) for every $x \in \mathbb{Z}$ and $0 < s < t$, $\eta_.(x)$ is constant on (s,t) if $\omega((s,t)\times\{x\}) = 0$.

The above construction yields a natural way of coupling different TASEP's by using the same Poisson clocks. On the other hand, we may construct the process with generator (2.12) by using only Poisson events (t,x) such that $x \in \{0,\dots,N-1\}$. Whenever such an event occurs, we simultaneously attempt all jumps from $x+k$ to $x+k+1$, where $k \in \mathbb{Z}$. We use this construction to couple processes with generators (2.6) and (2.12).

Tightness and finite propagation. The following tightness result can be found in Rezakhanlou and Tarver (2000), Section 4, and immediately adapted to our setting.

Lemma 2.1. *Let $\alpha \in \mathbf{A}$, and $(\eta^N_\cdot)_{N\in\mathbb{N}^*}$ be a sequence of processes with generator* (2.6). *Let $\Pi^N_t := \pi^N(\eta^N_{Nt})(dx)$. Then the sequence $(\Pi^N_\cdot)_{N\in\mathbb{N}}$ of $\mathcal{M}(\mathbb{R})$-valued processes is tight with respect to the Skorokhod topology on $\mathcal{D}([0,+\infty),\mathcal{M}(\mathbb{R}))$, and any subsequential limit in distribution is supported on $\mathcal{C}([0,+\infty),\mathcal{M}(\mathbb{R}))$.*

The following classical finite propagation result states that discrepancies in two coupled TASEP's propagate at finite speed.

Lemma 2.2. *There exists constants $V > 0$ and $C > 0$ with the following property. Let $(\eta_\cdot, \xi_\cdot)$ be a coupled process, where each component has generator either* (2.6) *or* (2.12), *including the possibility of two different generators. Let $a, b \in \mathbb{R}$ such that $0 < b - a < 1$. Suppose that initially η_0 and ξ_0 coincide on $[Na, Nb]$, then*

$$\mathbb{P}\big(\eta_s(x) = \xi_s(x), \forall s \in \big[0, N(b-a)/(2V)\big], \forall x \in \big[N(a+Vs), N(b-Vs)\big]\big) \\ \le C^{-1}e^{-CN}. \tag{2.27}$$

We are now ready for the following proof.

Proof of Proposition 2.1. Let $\pi_t(dx) = \rho(t,x)\,dx$ be a limit in law of the rescaled empirical measure process $\Pi^N_\cdot = \pi^N(\eta^N_{N\cdot})$. We have to verify that, with probability 1, the random function $\rho(\cdot,\cdot)$ satisfies (2.25) for all $\varphi \in C^\infty_K([0,+\infty)\times\mathbb{R})$. Let D be a countable dense subset of $[0,+\infty)\times\mathbb{R}$. Using partitions of unity, it is enough to show the following: for every $(t_0, x_0) \in D$, there exists a neighborhood $V(t_0,x_0)$ of (t_0,x_0) in $[0,+\infty)\times\mathbb{R}$, such that (2.25) holds for every $\varphi \in C^\infty_K(V(t_0,x_0))$. Considering a countable dense subset C of values of c, we only have to show that for every $(t_0,x_0,c) \in D\times C$, and every $\varphi \in C^\infty_K(V(t_0,x_0))$, (2.25) holds with probability 1.

Let $t_1 := \max[0, t_0 - \delta/(3V)]$. Choose $\delta < 1/2$ with V as in Lemma 2.2. We couple the process $\eta^N_\cdot$ with a process $\xi^N_\cdot$ defined at times $t \ge Nt_1$ with generator (2.6), and such that at time Nt_1 we have $\eta^N_{Nt_1}(x) = \xi^N_{Nt_1}(x)$ for every $x \in \{0,\dots,N-1\}$. Lemma 2.2, with $a = x_0 - \delta$, $b = x_0 + \delta$, implies that for $t \in (t_1, t_0 + \delta/(3V))$, $\pi^N(\eta^N_{Nt})$ and $\pi^N(\xi^N_{Nt})$ coincide on $(x_0 - \delta/2, x_0 + \delta/2)$. We then set $V(t_0,x_0) = (t_1, t_0+\delta/(3V)) \times (x_0-\delta/2, x_0+\delta/2)$. Let us consider a subsequence along which $(\pi^N(\xi^N_{Nt_1}))_N$ converges in distribution to some random measure $\tilde\pi_0 = \tilde\rho_0(\cdot)\,dx$ on $\mathbb{R}$. Any weak limit in law of the joint process $(\pi^N(\eta^N_{Nt}), \pi^N(\xi^N_{Nt}))_{t\in[t_1,t_0+\delta/(3V)]}$ can be viewed as a random function $(\rho(t,x), \tilde\rho(t,x))$ such that $\rho(t,x) = \tilde\rho(t,x)$ a.s. on $V(t_0,x_0)$. Along this subsequence we know by step one that the process $(\pi^N(\xi^N_{N(t_1+\cdot)}))$ converges in distribution to the random path $\tilde\pi_\cdot$ defined by $\tilde\pi_t(dx) = \tilde\rho(t,\cdot)\,dx$, where $\tilde\rho(\cdot,\cdot)$ is the entropy solution to (1.1) with random Cauchy datum $\tilde\rho_0(\cdot)$. Thus the random function $\tilde\rho(\cdot,\cdot)$, and therefore also $\rho(\cdot,\cdot)$, a.s. satisfies (2.25) on $V(t_0,x_0)$. If $t_1 > 0$, there is no initial term in (2.25). If $t_1 = 0$, then $\tilde\rho_0 = \rho_0$ on $(x_0-\delta/2, x_0+\delta/2)$, and thus the initial term obtained with $\tilde\rho_0$ is the same as with ρ_0. □

2.6 Proofs of Corollaries 2.2 and 2.3

Proof of Corollary 2.2. By Skorokhod's embedding theorem, there exists on some probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ a sequence of $\mathcal{M}(\mathbb{R})$-valued random variables $(\tilde{\Pi}_0^N)_{N\in\mathbb{N}^*}$ and a $\mathcal{M}(\mathbb{R})$-valued random variable $\tilde{\Pi}_0$ such that:

(i) $\tilde{\Pi}_0^N$ has the same distribution as Π_0^N;
(ii) $\tilde{\Pi}_0^N \to \tilde{\Pi}_0$ a.s. when $N \to \infty$.

Note that $\tilde{\Pi}_0$ is a.s. of the form $\tilde{\rho}_0(\omega_0)(x)\,dx$ for some random $\tilde{\rho}_0(\omega_0) \in L^\infty(\mathbb{R}; [0,1])$. Since π^N is a one-to-one mapping from $\mathbf{X}$ to $\mathcal{M}(\mathbb{R})$, we have $\Pi^N = (\pi^N)(\tilde{\eta}_0^N)$, where $\tilde{\eta}_0^N$ is a $\mathbf{X}$-valued random variable with the same distribution as η_0^N. We set

$$\tilde{\Pi}_t^N(\omega_0, \omega_1) := \pi_{Nt}^N\big(\eta_t(\tilde{\eta}_0^N, \omega_1)\big)$$

with ω_1 as in the graphical construction. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ denote the product of the probability spaces $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ and $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. Under $\tilde{\mathbb{P}}$, the process $\omega = (\omega_0, \omega_1) \mapsto \eta_.(\tilde{\eta}_0^N(\omega_0), \omega_1)$ has the same distribution as the original process with initial state η_0^N. Let $(S_t)_{t\geq 0}$ denote the evolution semigroup of (1.1) on $L^\infty(\mathbb{R}; [0,1])$, that is, for every $u_0 \in L^\infty(\mathbb{R}; [0,1])$, $(t,x) \mapsto u_t(x) := S_t u_0(x)$ is the entropy solution to (1.1) with Cauchy datum $u_0(\cdot)$. For fixed ω_0, with randomness coming only from ω_1, we can view $\tilde{\rho}_0(\omega_0)$ as a deterministic Cauchy datum for (1.1). By (ii) above and Corollary 2.1, for $\mathbb{P}_0$-a.e. $\omega_0 \in \Omega_0$, $\Pi_.^N(\omega_0, \omega_1)$ converges in $\mathbb{P}_1$-probability to $\Pi_.(\omega_0)$ defined by $\Pi_t(\omega_0) = S_t[\tilde{\rho}_0(\omega_0)](\cdot)\,dx$. Thus $\tilde{\Pi}_.^N(\omega_0, \omega_1)$, which has the same distribution as $\Pi_.^N$, converges in $\mathbb{P}_0 \otimes \mathbb{P}_1$-probability to $\Pi_.(\omega_0)$, which has the same distribution as $S_.[\rho_0(\cdot)]\,dx$. □

Before proving Corollary 2.3, we recall the standard finite propagation property (see, e.g., Kružkov (1970)) for (1.1), that is analogous to Lemma 2.2.

Lemma 2.3. *Let W be a Lipschitz constant for f. Then, for any two functions $\rho_0^1(\cdot), \rho_0^2(\cdot) \in L^\infty(\mathbb{R}, [0,1])$ that coincide on some interval $[a,b] \subset \mathbb{R}$, it holds that for every $t \in (0, (b-a)/(2W))$, $\rho^1(t,\cdot)$ and $\rho^2(t,\cdot)$ coincide on $[a+Wt, b-Wt]$, where for $i \in \{1,2\}$, $\rho^i(\cdot,\cdot)$ denotes the entropy solution to (1.1) with Cauchy datum ρ_0^i.*

Proof of Corollary 2.3. It is enough to consider the case of deterministic $\rho_0(\cdot)$, from which the case of random $\rho_0(\cdot)$ can be deduced by means of Skorokhod embedding in the spirit of Corollary 2.2. Let $j_N(t) := N^{-1} J^\alpha_{\lfloor Nx\rfloor}(Nt, \eta_0^N)$. For any $0 \leq s \leq t$,

$$\big|j_N(t) - j_N(s)\big| \leq N_t - N_s,$$

where $(N_t)_{t\geq 0}$ is the rate 1 Poisson process of events at site $\lfloor Nx \rfloor$ in the Harris construction of the process. It follows that the family of $\mathbb{R}$-valued processes $\{j_N(\cdot) : N \geq 1\}$ is tight in $\mathcal{D}([0,+\infty),\mathbb{R})$, and that its weak limits are supported on $C([0,+\infty),\mathbb{R})$. Let $j(\cdot)$ be a subsequential limit of this sequence. It is clear that $j(0)=0$ a.s. It is enough to show that a.s. with respect to the law of $j(\cdot)$, for every $t_0>0$, there exists $\varepsilon \in (0,t_0)$ such that

$$j(v) - j(u) = \int_u^v f\big[\rho(t,x)\big]\,dt \tag{2.28}$$

for all $u, v \in (t_0-\varepsilon, t_0+\varepsilon)$, for this and $j(0)=0$ imply $j(t)=\int_0^t f[\rho(s,x)\,dx]$ for every $t>0$. To show (2.28), we have to show that, for every $t_0>0$, there exists $\varepsilon \in (0,t_0)$ such that

$$\lim_{N\to+\infty} j_N(v) - j_N(u) = \int_u^v f\big[\rho(t,x)\big]\,dt \tag{2.29}$$

in probability with respect to the law of the quenched process for all $u, v \in (t_0-\varepsilon, t_0+\varepsilon)$. To prove (2.29), we couple the process $(\eta_t^N)_{t\geq 0}$ on the time interval $[Nu, Nv]$ with a process $(\zeta_t^N)_{t\geq 0}$ defined as follows:

(i) at time Nu, ζ_{Nu}^N coincides with η_{Nu}^N at sites $y\in\mathbb{Z}$ such that $|y-\lfloor Nx\rfloor| \leq \delta N$, and has no particles at other sites, where δ is chosen such that $0<\delta<\min(x,1-x)/3$;

(ii) $(\zeta_t^N)_{t\geq Nu}$ is a Markov process with generator (2.6), that is, has the dynamics of disordered TASEP on $\mathbb{Z}$.

The coupling is performed via the Harris construction as explained in Section 2.5 above. We also couple these processes with the "empty" process $(\xi_t^N)_{t\geq Nu}$ starting from $\xi_{Nu}^N \equiv 0$, with generator (2.6).

Set $\varepsilon := \delta/2\max(V,W)$, where V is the constant in Lemma 2.2. Let $\tilde{j}_N(t) := J_{\lfloor Nx\rfloor}(Nt, \zeta_0^N)$ be the current up to time t across x in $\zeta_{\cdot}^N$. On the one hand, by Lemma 2.2, for $t\in(u,v)$, η_{Nt}^N and ζ_{Nt}^N coincide with high probability in some neighborhood of $\lfloor Nx\rfloor$. On the event of their coincidence, we have $\tilde{j}_N(t) - \tilde{j}_N(u) = j_N(t) - j_N(u)$, for the instantaneous variation of the current in each system depends only on the number of particles at $\lfloor Nx\rfloor$ and $1+\lfloor Nx\rfloor$. On the other hand, since $\zeta_{\cdot}^N$ is a finite system on $\mathbb{Z}$, we also have that

$$\tilde{j}_N(v) - \tilde{j}_N(u) = \sum_{y>\lfloor Nx\rfloor} \zeta_{Nv}^N(y) - \sum_{y>\lfloor Nx\rfloor} \zeta_{Nu}^N(y). \tag{2.30}$$

Finally, applying Lemma 2.2 to $\zeta_{\cdot}^N$ and $\xi_{\cdot}^N$, we find that with high probability, over the time interval $[Nu, Nv]$, $\zeta_{\cdot}^N$ has no particle outside the space interval $[\lfloor Nx\rfloor - 2N\delta, \lfloor Nx\rfloor + 2N\delta]\cap\mathbb{Z}$. This implies that with negligible error in the limit, we can replace (2.30) by

$$\tilde{j}_N(v) - \tilde{j}_N(u) \simeq \sum_{\lfloor Nx\rfloor+3N\delta>y>\lfloor Nx\rfloor} \zeta_{Nv}^N(y) - \sum_{\lfloor Nx\rfloor+3N\delta>y>\lfloor Nx\rfloor} \zeta_{Nu}^N(y). \tag{2.31}$$

By Corollary 2.1, the above RHS converges in probability (with respect to the law of the quenched process) to

$$\begin{aligned}
&\int_x^{x+3\delta} \tilde{\rho}(v,y)\,dy - \int_x^{x+3\delta} \tilde{\rho}(u,y)\,dy \\
&\quad = \int_x^{+\infty} \tilde{\rho}(v,y)\,dy - \int_x^{+\infty} \tilde{\rho}(u,y)\,dy \qquad (2.32)\\
&\quad = \int_u^v f[\tilde{\rho}(s,x)]\,ds,
\end{aligned}$$

where $\tilde{\rho}(\cdot,\cdot)$ is the entropy solution to (1.1) on the time interval $[u,+\infty)$ with initial datum $\rho(u,\cdot)\mathbf{1}_{(x-\delta,x+\delta)}(\cdot)$ at time u. The first equality in (2.32) follows from Lemma 2.3 applied to the entropy solutions $\rho(\cdot,\cdot)$ and 0, which implies that $\rho(s,y)=0$ for $s\in(u,v)$ and $y>x+3\delta$. The second equality follows from (1.1). Finally, applying Lemma 2.3 to $\rho(\cdot,\cdot)$ and $\tilde{\rho}(\cdot,\cdot)$ shows that these solutions coincide in a neighborhood of x over the time interval $[u,v]$. Thus in the rightmost integral in (2.32), we may replace $\tilde{\rho}(s,x)$ by $\rho(s,x)$. □

3 Phase transition

Driven disordered systems have been extensively studied in the physics literature and physicists established that a phase transition occurs in site disorder driven TASEP. In this section, we recall the physical mechanisms behind this phase transition which were understood in Tripathy and Barma (1998), Harris and Stinchcombe (2004), Krug (2000) based on numerical results. We then develop these heuristics in the langage of the homogenization of driven systems. Using the homogenization framework, we conclude this section by reporting on the recent work (Bahadoran and Bodineau (2015a)) where the occurrence of a plateau in the flux has been proven for a wide range of disorder distributions.

3.1 Results from physics

The simplest disordered dynamics to define is the TASEP with fast and slow randomly distributed jump rates. At each site, the jump rates $(\alpha(x))_{x\in\mathbb{Z}}$ are independently distributed according to

$$Q_\varepsilon = (1-\varepsilon)\delta_1 + \varepsilon\delta_r, \qquad (3.1)$$

where $r<1$ stands for the slow rate and $\varepsilon\le 1/2$ represents the density of defects.

The randomness of the jump rates triggers a phase transition for some values of the density. A signature of this phase transition is the occurrence of a plateau in the flux which becomes constant for densities in the range $[\rho_c, 1-\rho_c]$ where ρ_c is a critical density. This plateau is interpreted in Tripathy and Barma (1998), Harris and Stinchcombe (2004), Krug (2000) as a phase separation mechanism which we describe below. Suppose that ε is a small parameter so that the slow rates r are very rare. Typically, there are long stretches of sites with only jump rates equal to 1 where the flux should depend on the local density ρ and be of the form $\rho(1-\rho)$. However a single slow site cannot carry a flux larger than r. In a stationary regime, the flux is constant through the system and this imposes a constraint on the maximum flux (for $r < 1/4$)

$$\rho(1-\rho) < r \quad \Rightarrow \quad \rho \notin \left[\frac{1}{2}(1-\sqrt{1-4r}), \frac{1}{2}(\sqrt{1-4r}+1)\right]. \tag{3.2}$$

These heuristics explain why in a steady state regime, some densities cannot be found in a fast stretch. The true mechanism is more complex as the limiting current is due to an atypical accumulation of slow sites which forces the current to be less than $r/4$. Furthermore the typical fast regions contain a certain proportion of defects which alter the flux. But it is believed that qualitatively the previous blockage heuristics describe correctly the phase transition regime and that there is a critical value ρ_c such that densities in the range $[\rho_c, 1-\rho_c]$ cannot be reached by an invariant measure. For an initial data at constant density ρ in $[\rho_c, 1-\rho_c]$, a coarsening phenomenon will take place and blocks of density ρ_c or $1-\rho_c$ will appear. On larger time scales, the typical length of these blocks grows: some blocks disappear and others merge. Thus a phase separation takes place leading ultimately to a system locally at density ρ_c or $1-\rho_c$. Interesting conjectures on the coarsening time scale are presented in Krug (2000). In Section 3.2, we will detail these mechanisms in a simpler mesoscopic framework which can be studied by means of homogenization.

3.2 Homogenization

In this section, we give a homogenization-based point of view to explain the existence of a plateau, the phase coexistence, and the underlying dynamical picture of shock coalescence. This point of view will be useful to explain, in Section 3.3, the results obtained in Bahadoran and Bodineau (2015a). In fact, we will see in Section 3.4 that some simplified versions of our problem can be directly mapped to one of the homogenization problems described below.

Homogenization and plateau. If we think of the homogeneous Burgers' equation as the hydrodynamic limit of homogenous TASEP, a natural problem in relation to disordered TASEP is the large-scale homogenization problem for

$$\partial_t \rho(t,x) + \partial_x\big[\alpha(x) f\big(\rho(t,x)\big)\big] = 0 \tag{3.3}$$

with f a bell-shaped Lipschitz function on $[0, 1]$ such that $f(0) = f(1) = 0$, that is, that there exists $\rho^* \in (0, 1)$ such that f is increasing on $[0, \rho^*]$ and decreasing on $[\rho^*, 1]$. We denote by $f^* := f(\rho^*)$ the maximum of f. The variation of the jump rates is now modeled by a smooth function $\alpha(\cdot)$ which is $(0, 1]$-valued, 1-periodic on $\mathbb{R}$. The minimum value of $\alpha(\cdot)$ is denoted by $\alpha_0 > 0$. Consider the entropy solution $\rho^\varepsilon(t,x)$ whose Cauchy datum $\rho_0^\varepsilon(x)$ is such that $\rho_0^\varepsilon(\varepsilon^{-1}x)$ converges to some $\rho_0(x)$ as $\varepsilon \to 0$. One would like to say that the rescaled solution $\tilde{\rho}^\varepsilon(t,x) := \rho^\varepsilon(\varepsilon^{-1}t, \varepsilon^{-1}x)$ converges to some limit $\bar{\rho}(t,x)$ that is the entropy solution of a homogeneous conservation law

$$\partial_t \bar{\rho}(t,x) + \partial_x[\bar{f}(\bar{\rho}(t,x))] = 0 \tag{3.4}$$

with some effective flux $\bar{f}(\rho)$. Here (3.3) is our "microscopic" model, and $\tilde{\rho}^\varepsilon(\cdot,\cdot)$ is its hyperbolic scaling, where $\varepsilon \to 0$ plays the role of the scaling parameter in the hydrodynamic limit. The smoothness requirement on α is not fundamental. Piecewise smoothness can be handled too, but the definition of entropy solutions in this case is more difficult, since the classical entropy conditions of Krug (2000) no longer make sense (see, e.g., Audusse and Perthame (2005), Bachmann and Vovelle (2006) for possible approaches).

In fact, homogenization of scalar conservation laws is a difficult problem which, in space dimension bigger than one, does not in general yield a scalar conservation law (Dalibard (2009)). On the other hand, homogenization of Hamilton–Jacobi equations, which is equivalent in one dimension, does yield Hamilton–Jacobi equations in any dimension (Lions, Papanicolaou and Varadhan (1988), Rezakhanlou and Tarver (2000)). We may understand the construction of $\bar{f}$ in a "hydrodynamic limit" spirit as follows. The effective flux should be given by the flux in the "local equilibrium" state. Since (3.4) is locally a large-scale version of (3.3), we expect ρ^ε to look locally like a stationary solution of (3.3). A 1-periodic stationary solution $\rho(\cdot)$ of (3.3) is constructed by solving

$$\alpha(x) f[\rho(x)] = \lambda \qquad \forall x \in [0,1), \tag{3.5}$$

where $\lambda \in [0, \alpha_0 f^*]$. For $\lambda \in [0, \alpha_0 f^*]$, we may thus construct two one-parameter families $\{\rho_\lambda^\pm, \lambda \in [0, \alpha_0 f^*]\}$ of stationary solutions to (3.3), by defining $\rho_\lambda^-(x)$, respectively, $\rho_\lambda^+(x)$, as the unique solution on (3.5) on $[0, \rho^*)$, respectively, $[\rho^*, 1)$. The two families are related by the symmetry of the flux f, that we denote by $\sigma(\cdot)$, that is the unique decreasing function $\sigma : [0,1] \to [0,1]$ such that $f \circ \sigma = f$. Then we have $\rho_\lambda^+ = \sigma \circ \rho_\lambda^-$.

The mean density associated to the stationary solution $\rho_\lambda^\pm$ for a flux value λ in $[0, \alpha_0 f^*]$ is given by

$$R^\pm(\lambda) := \int_0^1 \rho_\lambda^\pm(x)\, dx. \tag{3.6}$$

By definition of $\rho_\lambda^\pm(\cdot)$, R^- is an increasing continuous bijection from $[0, \alpha_0 f^*]$ to $[0, r^-]$, while R^+ is a decreasing continuous bijection from $[0, \alpha_0 f^*]$ to $[r^+, 1]$.

For $\lambda = \alpha_0 f^*$, we may construct additional periodic stationary entropy solutions by connecting ρ_λ^- and ρ_λ^+ by an upper shock at a point $y \in [0,1]$ such that $\alpha(y) > \alpha_0$, and using a minimum $x > y$ of $\alpha(\cdot)$, where $\alpha(x) = \alpha_0$, to make a

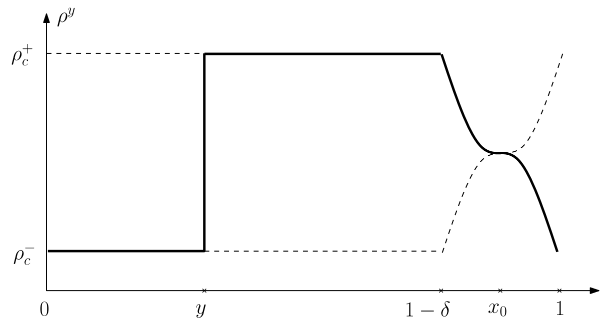

Figure 1 *The critical density profiles $\rho^{\pm}_{\alpha_0 f^*}$ are represented in dashed lines and they coincide at x_0 where α reaches its minimum. The solution ρ^y is in plain line and jumps from $\rho^-_{\alpha_0 f^*}$ to $\rho^+_{\alpha_0 f^*}$ at y. When δ vanishes an antishock is formed at* 1.

smooth junction from ρ_λ^+ back to ρ_λ^-, since at such a point x we have $\rho_\lambda^{\pm}(x) = \rho^*$. We may construct solutions with as many such shocks in [0, 1) as there are minima for $\alpha(\cdot)$, and arbitrarily choose the location of the shock before each minimum of $\alpha(\cdot)$. The point is that since the flux is bell-shaped, a zero-speed shock is entropic if and only if it is increasing (see Figure 1).

One can show that the set of stationary entropy solutions to (3.3) is the set $\mathcal{S}$ of periodic functions $\rho(\cdot)$ satisfying (3.5) for some $\lambda \in [0, \alpha_0 f^*]$ (that is the stationary current carried by this solution), such that $\rho(\cdot)$ has left and right limits at all points, and $\rho(x-) \le \rho(x+)$ for all $x \in \mathbb{R}$. The above construction describes all these solutions. Let

$$\mathcal{D} := \left\{ \bar\rho := \int_0^1 \rho(x)\, dx : \rho(\cdot) \in \mathcal{S}, \alpha(x) f[\rho(x)] \equiv \alpha_0 f^* \right\}$$

denote the range of the mean densities for the profiles in $\mathcal{S}$ with maximum current $\alpha_0 f^*$. Let us define $r^{\pm} := R^{\pm}[\alpha_0 f^*]$. Since any $\rho \in \mathcal{S}$ satisfies $\rho(x) \in \{\rho_\lambda^+(x), \rho_\lambda^-(x)\}$ for $\lambda = \alpha_0 f^*$, it is clear that $\bar\rho \in [r^-, r^+]$. Conversely, by tuning the shock locations between successive minima of $\alpha(\cdot)$, one can achieve any intermediate density in this interval. Hence, $\mathcal{D} = [r^-, r^+]$.

The effective flux $\bar f$ is given by

$$\forall \rho \in [0,1] \qquad \bar f(\rho) := \begin{cases} \alpha_0 f^*, & \text{if } \rho \in [r^-, r^+], \\ (R^+)^{-1}(\rho), & \text{if } \rho \in [0, r^-], \\ (R^-)^{-1}(\rho), & \text{if } \rho \in [r^+, 1]. \end{cases} \tag{3.7}$$

Coexistence. The homogenization result (3.7) is quite general, but to be more explicit, we shall now somewhat specialize our model. We are considering a mostly homogeneous conservation law with sufficiently sparse “defects,” that are regions where the current is slowed down. To this end, given a small dilution parameter $\delta > 0$, we divide our periodic cell [0, 1) into $[0, 1-\delta)$ and $[1-\delta, 1)$, and assume that $\alpha(\cdot) = 1$ on $[0, 1-\delta]$, while $\min_{[1-\delta,1]} \alpha(\cdot) = \alpha_0 \in (0, 1)$, and $\alpha(\cdot)$ has a single minimum at $x_0 \in (1-\delta, 1)$. We may interpret $[1-\delta, 1]$ as the "defect subcell" and δ as the "defect density." Let f^- and f^+, respectively, denote restrictions of f to $[0, \rho^*]$ and $[\rho^*, 1]$, so that f^- is an increasing continuous bijection from $[0, \rho^*]$ to $[0, f^*]$, while f^+ is a decreasing continuous bijection from $[\rho^*, 1]$ to $[0, f^*]$. Now the restrictions of $\rho_\lambda^\pm(\cdot)$ to $[0, 1-\delta]$ are constant functions with value $(f^\pm)^{-1}(\lambda)$. Let us define "critical densities" $\rho_c^\pm := (f^\pm)^{-1}(\alpha_0 f^*)$. The family of additional shock stationary solutions can now be described by means of one parameter, that is the position of the single shock inside the cell. Precisely, for every $y \in [0, 1-\delta)$, let $\rho^y(\cdot)$ be the 1-periodic function whose restriction to $[0, 1]$ is defined by (see Figure 1)

$$\begin{aligned} \rho^y(x) &:= \rho_c^- \mathbf{1}_{[0,y)}(x) + \rho_c^+ \mathbf{1}_{[y,1-\delta]}(x) + \rho^+_{\alpha_0 f^*}(x)\mathbf{1}_{(1-\delta,x_0)}(x) \\ &\quad + \rho^-_{\alpha_0 f^*}(x)\mathbf{1}_{(x_0,1)}(x). \end{aligned} \tag{3.8}$$

We observe that the defect sub-cell makes a (here smooth) transition between the high density ρ_c^+ and the low density ρ_c^-. One can show that the set of 1-periodic stationary entropy solutions to (3.3) consists exactly of functions $\rho_\lambda^\pm(\cdot)$ for $\lambda \in [0, \alpha_0 f^*]$, and $\rho^y(\cdot)$ for $y \in [0, 1-\delta)$. In this case, we have

$$r^- = (1-\delta)\rho_c^- + \delta\bar{\rho} \quad \text{and} \quad r^+ = (1-\delta)\rho_c^+ + \delta\bar{\rho},$$

where the quantity

$$\bar{\rho} := \frac{1}{\delta}\int_{1-\delta}^1 \rho^y(x)\,dx$$

does not depend on y, since the restriction of $\rho^y(\cdot)$ to $[1-\delta, 1]$ is independent of y. Notice that the mean cell density for the stationary solution (3.8) is given by

$$R^y := y\rho_c^- + (1-\delta-y)\rho_c^+ + \delta\bar{\rho}.$$

The particular stationary solution (3.8) is expected to represent the "microscopic" structure of ρ^ε around some point X where $\bar{\rho}(X) = r \in (r^-, r^+)$, with value y tuned in a unique way so that $R^y = r$. This can be viewed as a naive deterministic picture of phase transition with the occurrence of phase coexistence. However, the coarsening dynamics due to coalescence of shocks is not captured by this picture as randomness is missing. A natural example corresponds to the *fully segregated model* (see Tripathy and Barma (1998) and Section 3.4), where

$$\alpha(x) = \mathbf{1}_{(0,1-\delta)}(x) + \alpha_0 \mathbf{1}_{(1-\delta,1)}(x)$$

in which case

$$\begin{aligned} R^\pm(\lambda) &= (1-\delta)(f^\pm)^{-1}(\lambda) + \delta(f^\pm)^{-1}\left(\frac{\lambda}{\alpha_0}\right) \qquad \text{for } \lambda < \alpha_0 f^*, \\ \bar{\rho} &= \rho^*. \end{aligned} \tag{3.9}$$

In the case of TASEP, where f is given by (1.2), one has $f^* = 1/4$, and the explicit expressions (see also (3.2))

$$(f^\pm)^{-1}(\lambda) = \frac{1 \pm \sqrt{1-4\lambda}}{2} \tag{3.10}$$

from which (3.7) can be computed explicitly. In particular, the density range $[r^-, r^+]$ for the plateau (with maximum current $\alpha_0/4$) is given by

$$r^\pm = \frac{1 \pm (1-\delta)\sqrt{1-\alpha_0}}{2}. \tag{3.11}$$

In order to introduce a simple minimalistic model to capture the coarsening, we will yet slightly simplify the treatment of inhomogeneity. We consider to this end the dilute limit $\delta \to 0$. In this limit, we have a localized point defect at the interface of two cells, and $R^\pm(\lambda) = (f^\pm)^{-1}(\lambda)$ for $\lambda \in [0, \alpha_0 f^*)$, so that the homogenized flux function is simply the cutoff function

$$\bar{f}(\rho) = \bar{f}_{f,\alpha_0} := \max\{f(\rho), \alpha_0 f^*\}. \tag{3.12}$$

One can interpret (3.12) as follows. For fixed values of δ, the effective flux is a complex mixture of the original flux f and the slow flux in the defect cell, with maximum value given by the slow flux, and some plateau at this flux value. As $\delta \to 0$, the proportion of the slow flux in the mixture vanishes, thus the effective flux becomes the original flux f, except that the (only) memory it retains from the defect cell is the restriction on the maximum value of the current.

The above stationary solutions $\rho_\lambda^\pm(\cdot)$ converge weakly, as $\delta \to 0$, to the family of uniform profiles taking values in $[0, \rho_c^-] \cup [\rho_c^+, 1]$, and the family of shock profiles converges to the periodic profiles (that we still denote by $\rho^y(\cdot)$ by an abuse of notation) whose restrictions to $[0, 1]$ are given by

$$\rho^y(x) := \rho_c^- \mathbf{1}_{[0,y)}(x) + \rho_c^+ \mathbf{1}_{[y,1)}(x). \tag{3.13}$$

Now the transition from high to low density around the defect is sharp and gives rise to an antishock which no longer satisfies the entropy solution. Note that so far we have considered first the homogenization limit $\varepsilon \to 0$ and then studied the dilute limit $\delta \to 0$ of the homogenized problem. It is interesting to observe that these limits actually commute. To see this, one has to study the evolution problem that arises as the $\delta \to 0$ limit of (3.3), that is the evolution problem for which the non-entropic shock profiles (3.13) are stationary solutions.

Burgers equation with localized defects. It turns out that such a notion of Burgers equation with a "localized" defect arises when studying the hydrodynamic behavior of TASEP with a slow bond (Seppäläinen (2001)), or more generally driven particle systems with a local perturbation of the dynamics (Bahadoran (2004)).

Let us start with the case of a single defect at $x = 0$. One has to define a notion of entropy solution for the conservation law

$$\partial_t \rho(t, x) + \partial_x\big[f\big(\rho(t, x)\big)\big] = 0 \tag{3.14}$$

with a defect at $x = 0$. The strength of the defect is described by a parameter $\alpha_0 \in [0, 1]$ which specifies the maximum current $\alpha_0 f^*$ across it.

The relevant notion of solution captures the following dynamic behavior. If the current across the defect does not exceed this threshold value, the solution is the usual entropy solution. As soon as the current exceeds the threshold, a traffic jam is generated around the defect, with high density ρ_c^+ to the left and low density ρ_c^- to the right, such that $f(\rho_c^\pm) = \alpha_0 f^*$. This creates an antishock which violates the usual entropy condition, at $x = 0$, while away from $x = 0$ the solution evolves still like the usual entropy solution (locally in space–time). From that point, the solution will globally differ from the standard one, but the evolution may again coincide with the usual entropic evolution on later time intervals where the current across the defect comes back to an admissible value, which causes the antishock to disappear. Formally, the above $\rho(\cdot, \cdot)$ can be understood as "the entropy solution of" (3.3) where α would have the degenerate form

$$\alpha(x) = \alpha_0 \mathbf{1}_{\{0\}}(x) + \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x). \tag{3.15}$$

The problem with this interpretation is that, from the point of view of weak solutions, (3.15) is really undistinguishable from $\alpha(\cdot) \equiv 1$. In fact, these solutions are indeed weak solutions of the homogeneous conservation law (3.14), but they generally differ from *the* entropy solution. However, they do satisfy an existence and uniqueness result for given initial data. The unique solution is characterized by entropy conditions that are modified at $x = 0$, or equivalently some interface conditions.

More generally, it is possible to define the entropy solution to (3.3) with a locally finite sequence of singular defects of the form

$$\alpha(x) = \sum_{i\in\mathbb{Z}} \alpha_i \mathbf{1}_{\{x_i\}}(x) + \mathbf{1}_{\mathbb{R}\setminus\mathcal{X}}(x), \tag{3.16}$$

where $\mathcal{X} := \{x_i, i \in \mathbb{Z}\}$. This really means that we consider (3.14) with a defect of parameter α_i placed at every $x_i \in \mathbb{Z}$. Such solutions are define by localizing the entropy conditions around each defect. It is shown in Bahadoran (2004) that the entropy solution of (3.3)–(3.16) is indeed the limit as $\delta \to 0$ of the entropy solution of (3.3) with α replaced by a smooth approximation α_δ that has constant value 1 except in a δ-neighborhood of $\mathcal{X}$, and minimum value α_i in a δ-neighborhood of x_i. In the periodic setting

$$x_i = i, \qquad \alpha_i = \alpha_0, \tag{3.17}$$

the homogenization of (3.3)–(3.17) leads to (3.4) with effective flux (3.12). This, as announced above, corresponds to taking the dilute limit $\delta \to 0$ before the homogenization limit $\varepsilon \to 0$.

Coalescence and coarsening. We now introduce a toy-model for site disordered TASEP, that we claim captures essential mechanisms of shock coalescence and coarsening. We consider the conservation law (3.3) with (3.16) and $x_i = i$. It can be shown that for a given sequence $(\alpha_n)_{n\in\mathbb{Z}}$, under the conditions

$$\liminf_{n\to+\infty} \alpha_n = \liminf_{n\to-\infty} \alpha_n = r, \tag{3.18}$$

the hyperbolic scaling limit leads again to the homogenized equation (3.4) with effective flux $\bar{f} = \bar{f}_{f,r}$ defined in (3.12). Thus, a phase separation occurs for densities in $[\rho_c^-, \rho_c^+]$.

Assume that $(\alpha_i)_{i\in\mathbb{Z}}$ is a family of i.i.d. random variables taking values in $(r, 1)$, where $r \in (0, 1)$ is the infimum of the support of the distribution of α_i. Suppose that the initial density is constant equal to $\bar{\rho} \in [\rho_c^-, \rho_c^+]$, then the defects i such that the instantaneous flux $f(\bar{\rho})$ is less than the maximum admissible value $\alpha_i f^*$ will not modify the density, but those for which $\alpha_i f^* < f(\bar{\rho})$ will induce a phase separation. Indeed these defects generate an antishock wave with left density value $\rho_c^{i,+}$ and right density value $\rho_c^{i,-}$ given by

$$\rho_c^{i,\pm} := \left(f^\pm\right)^{-1}(\alpha_i f^*)$$

and carrying the current $\alpha_i f^*$. We call this wave a i-wave. This i-wave starts propagating: an upward shock with densities $(\bar{\rho}, \rho_c^{i,+})$ moves to the left of the defect and another shock $(\rho_c^{i,-}, \bar{\rho})$ moves to the right. Since these shocks have not reached another defect, their velocity is given by

$$v(\rho_1, \rho_2) := \frac{f(\rho_1) - f(\rho_2)}{\rho_1 - \rho_2} \tag{3.19}$$

the Rankine–Hugoniot speed of a shock connecting densities ρ_1 and ρ_2. In fact, a wave will cross all the defects with larger current threshold, but it will start interacting with the other waves. When an i-wave propagating a flux $\alpha_i f^*$ meets a j-wave with a lower flux $\alpha_j f^*$, the i-wave starts disappearing and the j-wave will keep progressing but at a lower velocity (3.19) which depends on the wave densities and the difference in their fluxes.

The coalescence of the different waves leads to a complex phase separation mechanism in which the waves with low fluxes dominate. Thus the asymptotic behavior of the dynamics will be determined by the rare defects, far from the origin, which generate the waves with the lowest fluxes. However, as time goes, the remaining waves carry fluxes which are approaching the minimum flux rf^* and therefore the coarsening mechanism slows down as the velocity of the shocks (3.19) tends to 0. This complex mechanism leads to a coarsening with ultimately only the densities $\rho_c^\pm$. The coarsening law for the domain size $\zeta(t)$ at time t has been predicted for the disordered TASEP (3.1) by Krug (Krug (2000)) and scales like

$$\zeta(t) = \frac{t/t_0}{\log(t/t_0)}.$$

The statistics of the coarsening in the toy model with random defects will be investigated in a future work (Bahadoran and Bodineau (2015b)).

3.3 Existence of a plateau in the flux

In this section, we turn back to the disordered TASEP and review a recent result on the existence of a plateau obtained in Bahadoran and Bodineau (2015a). Using the formalism of homogenization, we will explain the renormalization procedure on which the proof of Bahadoran and Bodineau (2015a) is based.

3.3.1 *Notation and result.* We consider a more general form of the disorder distribution than (3.1)

$$Q_\varepsilon = (1-\varepsilon)\delta_1 + \varepsilon Q, \tag{3.20}$$

where Q is a probability measure on $[r, R]$ with $0 < r < R < 1$ and such that r is the infimum of the support of Q. The law of $\alpha = (\alpha(x), x \in \mathbb{Z})$ is the product measure with marginal Q_ε at each site

$$\mathcal{P}_\varepsilon(d\alpha) := \bigotimes_{x\in\mathbb{Z}} Q_\varepsilon\big[d\alpha(x)\big].$$

Expectation with respect to $\mathcal{P}_\varepsilon$ is denoted by $\mathcal{E}_\varepsilon$. We can interpret this by saying that each site is chosen independently at random to be, with probability $1-\varepsilon$, a "fast" site with normal rate 1, or with probability ε to be a microscopic "defect" with rate distribution Q supported away from 1 and 0. Thus, ε is the mean density of defects. Let us denote by f_ε the flux function (2.8) for this disorder distribution.

The first result is the occurrence of a plateau in the flux for sufficiently dilute disorder.

Theorem 3.1. *Assume Q satisfies the following lower tail assumption for some $\kappa > 1$*

$$Q\big((r, r+u)\big) = O\big(u^\kappa\big) \qquad \text{as } u \to 0^+. \tag{3.21}$$

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the flux f_ε has a plateau with value $r/4$:

$$\exists \rho_c \in]0, 1/2[, \forall \rho \in [\rho_c, 1-\rho_c], \qquad f_\varepsilon(\rho) = r/4.$$

It is easy to see that the flux will always be less or equal than $r/4$. Indeed when the system size diverges, one can find arbitrarily large stretches where all the jump rates are very close to the minimum value r. Thus in these long stretches, the flux cannot be larger than the maximal flux in a TASEP where all the jump rates are equal to r. The latter system can be explicitly handled and its maximal flux is $r/4$. The difficult part in the proof is to derive a lower bound on the flux.

Theorem 3.1 is a consequence of a more general result based on Assumption (H) below, on the flux in a finite size system which we explain below. Consider the TASEP in $[1, N]$ with disordered rates and the following boundary dynamics: a particle enters at site 1 with rate 1 if this site is empty, and leaves from site N with rate $\alpha(N)$ if this site is occupied. The generator of this process is given by

$$\begin{aligned}\mathcal{L}_N^\alpha f(\eta) &= \sum_{x=1}^{N-1} \alpha(x)\eta(x)[1-\eta(x+1)][f(\eta^{x,x+1}) - f(\eta)] \\ &\quad + [1-\eta(1)][f(\eta+\delta_1) - f(\eta)] + \alpha(N)\eta(N)[f(\eta-\delta_N) - f(\eta)],\end{aligned} \tag{3.22}$$

where $\eta \pm \delta_x$ denotes creation/annihilation of a particle at x. Thanks to these boundary conditions, the current flowing in the finite system is maximal and is denoted by

$$\forall x \in [1, N] \qquad j_{\infty,N}(\alpha_N) := \int \alpha(x)\eta(x)[1-\eta(x+1)]\, d\nu_{[1,N]}^\alpha(\eta), \tag{3.23}$$

where $\nu_{[1,N]}^\alpha$ is the unique invariant measure for the process on $[1, N]$ and $\alpha_N = (\alpha(x); x \in [1, N])$ denotes the disorder in the interval $[1, N]$.

It is well known (Liggett (1975), Derrida et al. (1993), Schütz and Domany (1993)) that in the homogeneous case, that is, when $\alpha(x) = r$ for all x in $[1, N]$ (with r a positive constant), then $j_{\infty,N}$ is no longer a random variable and it decreases to the value $\frac{r}{4}$ when N diverges. We now introduce an assumption on the disorder distribution, which quantifies the finite-size fluctuation of the maximal current as a function of disorder.

Assumption (H). There exists $b \in (0, 2)$, $a > 0$, $c > 0$ and $\beta > 0$ such that, for ε small enough, the following holds for any N

$$\mathcal{P}_\varepsilon\left(j_{\infty,N}(\alpha_N) \le \frac{r}{4} + \frac{a}{N^{b/2}}\right) \le \frac{c}{N^\beta}. \tag{3.24}$$

The main result of Bahadoran and Bodineau (2015a) is the following theorem.

Theorem 3.2. *Assume Q_ε satisfies Assumption (H). Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the flux f_ε has a plateau with value $r/4$.*

Remark 3.1. We were not able to check Assumption (H) for the TASEP with binary disorder (3.1). In fact, it is possible to check that this distribution satisfies (3.24) with $b = 2$ but dealing with this case is more challenging.

Theorem 3.1 is in fact a simple consequence of Theorem 3.2, we only have to check that the tail assumption (3.21) implies (H). This follows from tail estimates for $\alpha^* := \min[\alpha(1), \dots, \alpha(n)]$, and coupling with the homogeneous rate α^* TASEP to get a lower bound on the current.

3.3.2 *Renormalization of driven systems.* We now describe the main ideas of the renormalization scheme introduced in Bahadoran and Bodineau (2015a) for the proof of Theorem 3.2. We define successive scales by considering block sizes $K_n(\varepsilon)$ such that $\lim_{\varepsilon\to 0} K_1(\varepsilon) = +\infty$ and $K_{n+1}(\varepsilon) = l_n(\varepsilon)K_n(\varepsilon)$ as follows. Let $n \in \mathbb{N} \setminus \{0\}$ be the renormalization level and $K_n = K_n(\varepsilon)$ the size of a renormalized block of order n (by block we mean a finite sub-interval of $\mathbb{Z}$). For $n = 1$, we initialize $K_1 = K_1(\varepsilon)$, and define a block B of order 1 to be good if it contains no defect, that is, $\alpha(x) = 1$ for every $x \in B$. Otherwise, the block is said to be *bad*.

For $n \geq 1$, we set $l_n = [K_n^\gamma]$ with $\gamma \in (0,1)$. For $n \geq 1$, a block B_{n+1} of order $n+1$ has size K_{n+1} and is partitioned into l_n disjoint blocks of order n. A block of order $n+1$ is *good* if the two conditions below are satisfied

$$\begin{aligned}&\text{the block contains at most one } \textit{bad} \text{ block of order } n, \text{ and}\\ &j_{\infty,B_{n+1}}(\alpha_{B_{n+1}}) \geq j_{n+1} \text{ with } j_{n+1} := \frac{r}{4} + \frac{a}{K_{n+1}^{b/2}},\end{aligned} \tag{3.25}$$

where the constants a, b were defined in Assumption (H) and $j_{\infty,B_{n+1}}(\alpha_{B_{n+1}})$ is the flux restricted to B_{n+1} as in (3.23). Otherwise the block B_{n+1} is said to be *bad*. We stress the fact that the status (*good* or *bad*) of B_{n+1} depends only on the disorder variables $\alpha_{B_{n+1}}$ in B_{n+1}.

The renormalization is built such that the large blocks are good with high probability. Let $q_n(\varepsilon)$ denote the probability under $\mathcal{P}_\varepsilon$ that $[0, K_n - 1] \cap \mathbb{Z}$ is a *bad* block. Then by tuning appropriately the parameters γ and $K_1(\varepsilon)$, one can prove (Bahadoran and Bodineau (2015a)) that there is $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \leq \varepsilon_0 \qquad \lim_{n\to\infty} q_n(\varepsilon) = 0.$$

It turns out that to obtain the existence of the plateau, it is enough to bound from below the flux function f_ε with a reference flux function which itself has a plateau with maximum flux value $r/4$. This comes from the following facts: first, we know a priori that f_ε takes maximum value $r/4$; next, from Seppäläinen (2001), we also know a priori that f_ε is a concave function, and the smallest possible concave flux function with a plateau of value $J = r/4$ between densities $\rho_0 < 1/2$ and $1 - \rho_0$ is

$$f^{\rho_0,J}(\rho) := J \max\left[1, \min\left(\frac{\rho}{\rho_0}, \frac{1-\rho}{1-\rho_0}\right)\right]. \tag{3.26}$$

Therefore our goal is to show that $f_\varepsilon \geq f^{\rho_0,r/4}$ for some $\rho_0 \in [0, 1/2)$. Assume now that in a box B of size K_n, we can approximately describe the behavior of our disordered TASEP by a "finite-scale hydrodynamic approximation" of Burgers' type

$$\partial_t \rho(t,x) + \partial_x \bar{f}_n^B[\rho(t,x)] = 0, \tag{3.27}$$

where $\bar{f}_n^B$ is (up to some fluctuations) the "homogenized" flux in B. A priori this "mesoscopic flux" $\bar{f}_n^B$ depends on the box. The idea is to use the decomposition of a *good* $(n+1)$-block B into n-blocks to recover a kind of homogenization picture, from which we can approximately describe the behavior at scale $(n+1)$ by a conservation law of the type (3.27) with an effective flux function $\bar{f}^B_{n+1}$ at scale $n+1$. The key point is to ensure that the plateau survives at each scale change, by maintaining *in any good box B at any scale n* a lower bound of the form

$$\bar{f}^B_n \geq f^{\rho_n, J_n} \tag{3.28}$$

(thus depending only on the scale and no longer on the box itself) for some sequences $(\rho_n)_{n\geq 1}$ and $(J_n)_{n\geq 1}$ depending on ε. We do this by deriving recursion relations and showing that

$$\limsup_{\varepsilon\to 0}\limsup_{n\to+\infty}\rho_n(\varepsilon) < 1/2, \qquad \liminf_{n\to+\infty} J_n(\varepsilon) = r/4. \tag{3.29}$$

Due to fluctuations, the maximum current J_n will be slightly worse (that is smaller) than j_n defined in (3.25), and Assumption (H) is tailored exactly such that despite this degradation we always have $J_n > r/4$, so that the flux f_n is always bounded from below by a plateau at $r/4$.

Let us explain informally how to propagate the bound (3.28). The *good* blocks of size K_n introduced in (3.25) are those in which we know that the mesoscopic description (3.27) satisfies the bound (3.28) at scale n. In a *good* block of level $n+1$, this is the case for all the n-blocks but maybe one of the subboxes. This *bad* subbox acts as a defect with an unknown flux function. Patching the scale n-blocks together, we obtain an $(n+1)$-block with $l_n - 1$ *good* flux functions satisfying (3.28), and one unknown flux satisfying the *default* bound (3.30) below. However, gluing together two conservation laws gives no apriori knowledge on the interface conditions between them. As a matter of fact (Andreianov, Karlsen and Risebro (2011)), there happen to be an infinite choice of such interface conditions. For instance, when gluing two conservation laws with exclusion-like flux, the “canonical” interface condition corresponds to the usual entropy solution, while other choices correspond to a defect of variable strength. At the microscopic level, the approximation (3.27) does not encode information about the neighborhood of the boundary at a smaller scale than K_n, therefore the gluing of two boxes depends on additional boundary information not contained in (3.27). Fortunately, Assumption (H) tells us about the maximum current through the box of size K_{n+1}, and the maximum current across each of the interfaces cannot be smaller, since the former results from the latter *plus* possible larger defects inside certain boxes. We can essentially lowerbound the “unknown” flux in the *bad* box B by

$$\bar{f}^B_n \geq f^{1/2, j_{n+1}}. \tag{3.30}$$

The value $1/2$ for ρ_0 means that since the box is bad, we have no control on the length or even the existence of the plateau in this box; remark that $\rho_0 = 1/2$ in (3.26) corresponds to the absence of a plateau. Thus, if we perform the rescaling

$[0, K_{n+1}] \mapsto [0, 1]$, we find ourselves in a situation where the inhomogeneous flux inside the cell of size K_{n+1} has formally a lower bound of the type

$$f_{n+1}(x,\rho) := f^{\rho_n, J_n}(\rho)\mathbf{1}_{(0,1-l_n^{-1})}(x) + \sum_{l=1}^{l_n} f^{1/2, j_{n+1}}(\rho)\mathbf{1}_{\{l/l_n\}}(x) + \mathbf{1}_{(1-l_n^{-1},1)}(x) f^{1/2, j_{n+1}}(\rho), \tag{3.31}$$

which means that we have the *good* flux of order n on the first $n-1$ subcells, the “unknown” flux in the last subcell, and defects with maximum current j_{n+1} at each interface. Note that in (3.31), the formal “punctual” flux function at l/l_n is somewhat arbitrary (though notationally convenient): in fact, only its maximum value j_{n+1} happens to be relevant to the definition of the corresponding solution. To obtain $\bar{f}_{n+1}$, we must homogenize the flux function (3.31). To this end, as above, let us examine stationary solutions $x \mapsto \rho(x)$ defined by the equation

$$f_{n+1}\big(x, \rho(x)\big) = \lambda. \tag{3.32}$$

The length of the homogenized plateau at scale $n+1$ is given by the possible range of mean densities for the stationary solutions with flux $\lambda = j_{n+1}$. On the one hand, we have stationary solutions that are uniform in $(0, 1-l_n^{-1})$ and do not feel the defect threshold j_{n+1}. On the other hand, we have solutions with saturated defects, in which a piece of antishock is placed around each defect, and then connected with shocks between two consecutive defects. The upper and lower density of each antishock are solutions of $f_n(\rho) = j_{n+1}$, namely $\rho'_{n+1} := \rho_n j_{n+1}/j_n$ for the lower density, and $1-\rho'_{n+1}$ for the higher density. Tuning the position of shocks between defects yields the range of possible densities. The lowest possible density corresponds to having the uniform low density ρ'_{n+1} on $(0, 1-l_n^{-1})$, while the highest value corresponds to a uniform high density $1-\rho'_{n+1}$. Correspondingly, we have a uniform density $1/2$ in the last cell, that is the unique solution of $f^{1/2, j_{n+1}}(\rho) = j_{n+1}$. Thus, the lower and upper extremity of the homogenized plateau are ρ''_{n+1} and $1-\rho''_{n+1}$, where

$$\rho''_{n+1} := \left(1-\frac{1}{l_n}\right)\rho'_{n+1} + \frac{1}{l_n}\frac{1}{2} \leq \left(1-\frac{1}{l_n}\right)\rho_n + \frac{1}{l_n} =: \rho_{n+1}$$

while the maximum current in the homogenized flux should in principle be $J_{n+1} = j_{n+1}$. However, we have so far neglected fluctuations of the current. If we take these into account, we obtain slightly perturbed recursions for the plateau half-length and the maximum current

$$\rho_{n+1} := \left(1-\frac{1}{l_n}\right)\rho_n + \frac{1}{l_n} + \delta_n(\varepsilon), \tag{3.33}$$

$$J_{n+1} := \min(J_n, j_{n+1}) - \delta_n(\varepsilon), \tag{3.34}$$

where δ_n can be shown to be of at most Gaussian order $K_n^{-1/2}$ up to logarithmic corrections. Recall that by definition, *good* boxes at level 1 contain no defect. Thus the recursion can be initialized with the values ρ_1 and J_1 at scale $K_1(\varepsilon)$ given by those of a homogeneous rate 1 TASEP. Since we only want a lower bound for the flux, using the subadditivity property, this bound is exact for given ε. By choosing ε small enough, the perturbation $\delta_n(\varepsilon)$ remains small even for $n = 1$. Iterating the recursion (3.33) and (3.34) one can prove (3.29) and therefore the existence of a plateau.

The initialization step is tailored so that we *start* with an interval of non-negligible length above $r/4$. This is why we need to start with a low enough density of defects so that as ε decays and $K_1(\varepsilon)$ grows, the level 1 boxes contain essentially no defect. Without this assumption, the effective flux at scale 1 could have a maximum value already close to $r/4$, so that the initial interval above $r/4$, which gives the initial length of the plateau for the lower bound flux (3.26), would get small. This means that the recursion (3.33) would start with a value ρ_1 arbitrarily close to $1/2$ and we would fail to obtain (3.29).

3.4 Related explicitly solvable models

We discuss now disordered versions of TASEP on the torus for which $\bar{f}$ can be computed explicitly and turns out to be exactly the homogenized effective flux described in Section 3.2. These models are more tractable than TASEP with i.i.d. disorder in the sense that the computation of the flux (2.5) is directly amenable to a hydrodynamic limit problem followed by homogenization. Indeed, in these models there are long enough (growing) stretches of homogeneity on which TASEP can be replaced by Burgers' equation, so one can decouple a first hydrodynamic limit step from a second homogenization step for the hydrodynamic equation.

Assume that the generators (2.1)–(2.6) are now of the form

$$L^{\alpha}_{N,\mathbb{T}_N} f(\eta) = \sum_{x\in\mathbb{T}_N} \alpha^N(x)\eta(x)[1-\eta(x+1)][f(\eta^{x,x+1}) - f(\eta)], \quad (3.35)$$

$$L^{\alpha}_{N,\mathbb{Z}} f(\eta) = \sum_{x\in\mathbb{Z}} \alpha^N(x)\eta(x)[1-\eta(x+1)][f(\eta^{x,x+1}) - f(\eta)] \quad (3.36)$$

with $\alpha^N(x) = \alpha(x/N)$, where $\alpha(\cdot)$ is a 1-periodic function on $\mathbb{R}$ (we have used slightly different notation than in (2.1)–(2.6) because now both generators depend on a scaling parameter). We consider the following examples, where the restriction of $\alpha(\cdot)$ to $\mathbb{T} = [0, 1)$ is defined as follows:

(1) The fully segregated model, where

$$\alpha(x) := \mathbf{1}_{(0,1-\delta)}(x) + r\mathbf{1}_{(1-\delta,1)}(x)$$

for some $\delta \in (0, 1)$.

(2) The slow-bond model, where (in the sense of localized defects, cf. Section 3.2)

$$\alpha(x) := r\mathbf{1}_{\{0\}}(x) + \mathbf{1}_{(0,1)}(x).$$

The known hydrodynamic result on $\mathbb{Z}$ is established in Georgiou, Kumar and Seppäläinen (2010) in case (1), and in Seppäläinen (2001) in case (2). Thus, the system on $\mathbb{Z}$ with generator (3.36) has a hydrodynamic limit given as $N \to +\infty$ by entropy solutions on $\mathbb{R}$ of (1.1) and (1.2).

In this context, we can obtain an extension of Proposition 2.4. Indeed, on the one hand, the very same localization argument as in the proof of Proposition 2.1 implies that this hydrodynamic limit is still true for (3.35). On the other hand, the following extension of Proposition 2.2 holds when starting from a non-homogeneous conservation law on the torus

Proposition 3.1. *Let $\rho_0(\cdot) \in L^\infty(\mathbb{R})$ be 1-periodic and $[0, 1]$-valued, with*

$$\bar{\rho} = \int_0^1 \rho_0(x)\,dx. \tag{3.37}$$

Let $\rho(t, x)$ denote the entropy solution to (3.3) with Cauchy datum $\rho_0(\cdot)$. Then, for every $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha(x) f\big(\rho(s, x)\big)\,ds = \bar{f}(\bar{\rho}), \tag{3.38}$$

where $\bar{f}$ is the homogenized flux for (3.3), cf. Section 3.2.

The proof of this extension is carried out by plugging the relevant homogenization theorem into the proof of Proposition 2.2. Next, using Proposition 3.1, the proof of Proposition 2.4 can be repeated to yield the following extension.

Proposition 3.2. *Assume $\alpha(\cdot)$ is such that $L^\alpha_{N,\mathbb{T}_N}$ in (3.35) has hydrodynamic limit (3.3). Then the thermodynamic limit (2.5) exists and is equal to the homogenized flux $\bar{f}(\rho)$ given by (3.7).*

Therefore, for both models the thermodynamic flux (2.5) is the homogenized flux of the corresponding problem (3.3). In case (1), we find (3.7) obtained from (3.9), (3.10), (3.11) and $\alpha_0 = r$. In case (2),

$$\bar{f}(\rho) = \max\big\{\rho(1 - \rho), j_{\max}(r)\big\},$$

where $j_{\max}(r)$ is the maximal current for TASEP with a slow bond of rate r. In this case, $\bar{f}$ has a plateau if and only if $j_{\max}(r) < 1/4$, that is, $r < r_c$, where the critical rate r_c is conjectured to be 1 (see Janowski and Lebowitz (1992)), a proof of which has been recently announced in Basu, Sidoravicius and Sly (2014). Let us mention that for a slow-bond TASEP on a ring with parallel update, the existence of the plateau in the thermodynamic limit was established in Schütz (2014) by means of Bethe ansatz.

A full extension of Theorem 2.1 to this setting is less clear. On the one hand, hydrodynamic limits and Proposition 3.1 again yield the following extension of (2.8)

$$\bar{f}(\rho) := \lim_{t\to+\infty} \lim_{N\to+\infty} \frac{1}{Nt} J_x^N(Nt, \eta^\rho), \tag{3.39}$$

where the superscript N indicates that the current is computed for the process with generator (3.36). On the other hand, a stronger version of (2.8) would be

$$\bar{f}(\rho) := \lim_{N\to+\infty} \lim_{t\to+\infty} \frac{1}{t} J_x^N(t, \eta^\rho), \tag{3.40}$$

which is a more difficult problem. Unlike (3.39), (3.40) is related to the invariant measures of (3.36), and thus to a possible version of (2.10) in this context; however such a version is unclear, as for fixed N and $\alpha^N(\cdot)$, a suitable characterization of the extremal invariant measures is missing for (3.36).

In the slow-bond case, since J_x^N is independent of N, the limit (3.40) is the same as (3.39), and both of them reduce to the initial form (2.8). Let us comment in this setting on a connection between Proposition 3.2 and (2.8), and a recent result of Costin et al. (2013). The latter proved the equivalence between the maximum current of the slow-bond TASEP whether defined from a thermodynamic limit on the torus, or directly on the line. Theorem 2.1 can be viewed as a broad generalization of this result in the sense that it addresses the equivalence of flux definitions for the whole range of densities (not only its maximal value), and also general models and local perturbations as studied in Bahadoran (2004).

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