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Properties and Inequalities of Generalized k-Gamma, Beta and Zeta Functions

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Abstract

Recently R.Diaz and E.Pariguan introduced [2] the k-generalized Gamma function $\Gamma_k(x)$, Beta function $B_k(x,y)$ and Zeta function $\zeta_k(x,s)$ and gave some identities which they satisfy. We give some more properties and inequalities for the above k-generalized functions.

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1 Introduction

In [2] the authors introduced the generalized k-Gamma function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{x/k-1}}{(x)_{n,k}}, \quad k>0, \quad x \in C - kZ^-, \tag{1.1}$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad x \in C, \quad k \in R, \quad n \in N^+. \tag{1.2}$$

It is obvious that $\Gamma_k(x) \to \Gamma(x)$, for $k \to 1$, where $\Gamma(x)$ is the known Gamma function. Also, for $Re(x) > 0$, it holds

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-t^k/k} dt \tag{1.3}$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.4}$$

In the same paper they introduced the k-Beta function $B_k(x,y)$ as

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad Re(x) > 0, \quad Re(y) > 0, \tag{1.5}$$

and k-Zeta function as

$$\zeta_k(x,s) = \sum_{\nu=0}^{\infty} \frac{1}{(x+\nu k)^s}, \quad k, x > 0, \quad s > 1. \tag{1.6}$$

The function $B_k(x,y)$ satisfies the equality

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \tag{1.7}$$

from which follows

$$B_k(x,y) = \frac{1}{k}B(\frac{x}{k},\frac{y}{k}). \tag{1.8}$$

We mention that $\lim_{k\to 1} B_k(x,y) \to B(x,y)$ and k-Zeta function is a generalization of Hurwitz Zeta function $\zeta(x,s) = \sum_{\nu=0}^{\infty} \frac{1}{(x+\nu)^s}$ which is a generalization of the Riemann Zeta function $\zeta(s) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^s}$.

The motivation to study properties of generalized k-Gamma and k-Beta functions is the fact that $(x)_{n,k}$ appears in the combinatorics of creation and annihilation operators [3 and refs there in].

Recently M.Mansour [4] determined the k-generalized Gamma function by a combination of some functional equations.

In this paper we use the definitions of the above generalized functions to prove a formula for $\Gamma_k(2x)$ which is a generalization of the Legendre duplication formula for $\Gamma(x)$ and to prove inequalities for the function $B_k(x,y)$, for $x,y,k > 0$ and $x+y \neq k$ and the product $\Gamma_k(x)\Gamma_k(1-x)$, for $0 < x, k < 1$. We also give monotonicity properties for $\psi_k(x) = \partial_x\psi(k,x)$ where $\psi(k,x) = \log\Gamma_k(x)$ and $\zeta_k(x,s)$ for $s \in N$ and $s \geq 2$.

We mention that using (1.4) the following equalities hold:

$$\Gamma_k(ak) = k^{a-1}\Gamma(a), \quad k > 0, \quad a \in R \tag{1.9}$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!, \quad k > 0, \quad n \in N \tag{1.10}$$

$$\Gamma_k(k) = 1 \quad k > 0, \tag{1.11}$$

and

$$\Gamma_k((2n+1)\frac{k}{2}) = k^{\frac{2n-1}{2}} \frac{(2n)!\sqrt{\pi}}{2^n n!}, \quad k > 0, \quad n \in N. \tag{1.12}$$

Also, using (1.5) and (1.8) the following equalities hold:

$$B_k(x+k, y) = \frac{x}{x+y} B_k(x, y), \quad B_k(x, y+k) = \frac{y}{x+y} B_k(x, y), \quad x, y, k > 0 \tag{1.13}$$

$$B_k(x, k) = \frac{1}{x}, \quad B_k(k, y) = \frac{1}{y} \quad x, y, k > 0 \tag{1.14}$$

$$B_k(ak, bk) = \frac{1}{k} B(a, b), \quad a, b, k > 0, \tag{1.15}$$

and

$$B_k(nk, nk) = \frac{[(n-1)!]^2}{k(2n-1)!}, \quad k > 0, \quad n \in N. \tag{1.16}$$

2 The function $\Gamma_k(x)$

Theorem 2.1 Let $x, k > 0$ and $\psi_k(x)$ be the logarithmic derivative of $\Gamma_k(x)$. Then the function $\psi'_k(x)$ is completely monotonic.

Proof From (1.4) we get

$$\log \Gamma_k(x) = (\frac{x}{k} - 1) \log k + \log \Gamma(\frac{x}{k})$$

or by setting $\psi(k, x) = \log \Gamma_k(x)$ we obtain

$$\psi(k, x) = (\frac{x}{k} - 1) \log k + \log \Gamma(\frac{x}{k}). \tag{2.1}$$

From (2.1) we get

$$\partial_x \psi(k, x) = \frac{1}{k} \log k + \psi(\frac{x}{k}) \tag{2.2}$$

We remind that $\psi(\frac{x}{k}) = \partial_x(\log \Gamma(\frac{x}{k}))$. From (2.2) taking the derivative with respect to x we have

$$\partial_x^2 \psi(k, x) = \frac{1}{k} \psi'(\frac{x}{k}) \tag{2.3}$$

$$\partial_x^3 \psi(k, x) = \frac{1}{k^2} \psi^{(2)}(\frac{x}{k})$$

and by induction we obtain

$$\partial_x^{n+1} \psi(k, x) = \frac{1}{k^n} \psi^{(n)}(\frac{x}{k})$$

or if we call $\psi_k(x) = \partial_x \psi(k,x)$, then the equation

$$\psi_k^{(n)}(x) = \frac{1}{k^n}\psi^{(n)}(\frac{x}{k}) \tag{2.4}$$

holds. It is known [1] that $\psi'(x)$ is completely monotonic for $x > 0$, so from (2.4) it follows the desired result.

Remark 2.1 (i) From (2.3) it follows that $\Gamma_k(x)$ is logarithmic convex on $(0,\infty)$ which is proved in [2].
(ii) Theorem 2.1 is a generalization of the known [1] result that the function $\psi'(x) = \frac{d}{dx}\frac{\Gamma'(x)}{\Gamma(x)}$ is completely monotonic.

Proposition 2.1 For $x > 0$ the function $\psi(k,x) = \log \Gamma_k(x)$ satisfies the partial differential equation:

$$-x^2 k \partial_x^2 \psi(k,x) + 2k^2 \partial_k \psi(k,x) + k^3 \partial_k \psi(k,x) = -x - k. \tag{2.5}$$

Proof From (2.1) taking the first and second derivatives of $\psi(k,x)$ with respect to k we obtain

$$\partial_k \psi(k,x) = -\frac{x}{k^2}\log k + \frac{x}{k^2} - \frac{1}{k} - \frac{x}{k}\psi(\frac{x}{k}) \tag{2.6}$$

and

$$\partial_k^2 \psi(k,x) = \frac{2x}{k^3}\log k - \frac{3x}{k^3} + \frac{1}{k^2} + \frac{x}{k^2}\psi(\frac{x}{k}) + \frac{x^2}{k^3}\psi'(\frac{x}{k}) \tag{2.7}$$

From (2.3), (2.6) and (2.7) we get (2.5).

Remark 2.2 Theorem 11 of [2, page 6] has a mistake in the right hand side of the same partial differential equation.

Theorem 2.2 The function $\Gamma_k(x)$ satisfies the equality

$$\Gamma_k(2x) = \sqrt{\frac{k}{\pi}}2^{2\frac{x}{k}-1}\Gamma_k(x)\Gamma_k(x+\frac{k}{2}) \tag{2.8}$$

for $x \in C$ with $Re(x) > 0$.

Proof From (1.7) it follows that

$$B_k(x,x) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{x}{k}-1}dt$$

or by setting $t = \frac{1+r}{2}$

$$B_k(x,x) = \frac{2}{k2^{2\frac{x}{k}-1}}\int_0^1 (1-r^2)^{\frac{x}{k}-1}dr$$

or by setting $r^2 = u$ we obtain

$$B_k(x,x) = \frac{1}{k2^{2\frac{x}{k}-1}} \int_0^1 u^{\frac{1}{2}-1}(1-u)^{\frac{x}{k}-1}du = \frac{1}{k2^{2\frac{x}{k}-1}} B(\frac{x}{k}, \frac{y}{k}) = \frac{1}{2^{2\frac{x}{k}-1}} B_k(x, \frac{k}{2})$$

or

$$B_k(x,x) = \frac{1}{2^{2\frac{x}{k}-1}} \frac{\Gamma_k(x)\Gamma_k(k/2)}{\Gamma_k(x+k/2)}. \tag{2.9}$$

From (1.9) for $a = 1/2$ we get $\Gamma_k(k/2) = \sqrt{\frac{\pi}{k}}$, since $\Gamma(1/2) = \sqrt{\pi}$, so from (2.9) and (1.5) we get the equality (2.8).

Remark 2.3 Theorem 2.2 is a generalization of the Legendre duplication formula of $\Gamma(x)$ [1].

3 The function $\zeta_k(x,s)$

Theorem 3.1 **(i)** Let $x, k > 0$ and $s > 1$. Then the positive function $\zeta_k(x,s)$ decreases with respect to x and also decreases with respect to k.
(ii) Let $x > 0$ and $s > 1$. Then the positive function $\zeta_k(x,s)$ decreases with respect to s for $x > 1$, and $k > 0$, $\nu \geq 0$ and increases with respect to s for $\nu > 0$, $0 < k < \frac{1}{\nu}$ and $0 < x < 1 - \nu k$.

Proof From (1.6) we obtain

$$\partial_x \zeta_k(x,s) = \sum_{\nu=0}^{\infty} \frac{-s}{(x+\nu k)^{s+1}}, \quad k, x > 0, \quad s > 1$$

or

$$\partial_x \zeta_k(x,s) = -s\zeta_k(x, s+1) \tag{3.1}$$

and

$$\partial_k \zeta_k(x,s) = \sum_{\nu=0}^{\infty} \frac{-\nu s}{(x+\nu k)^{s+1}} = -s \sum_{\nu=1}^{\infty} \frac{\nu}{(x+\nu k)^{s+1}}, \quad k, x > 0, \quad s > 1. \tag{3.2}$$

So (3.1)and (3.2) prove theorem 3.1 (i).

Also, the definition (1.6) gives

$$\partial_s \zeta_k(x,s) = -\sum_{\nu=0}^{\infty} \frac{\ln(x+\nu k)}{(x+\nu k)^s} \tag{3.3}$$

If $x > 1$ then $x > 1 - \nu k$, for $\nu, k > 0$ thus $\ln(x + \nu k) > 0$, so from (3.3) it follows that the function $\zeta_k(x,s)$ decreases with $s > 1$ and if $0 < k < \frac{1}{\nu}$ and

$0 < x < 1 - \nu k$ then $\ln(x + \nu k) < 0$ so from (3.3) it follows that the function $\zeta_k(x, s)$ increases with $s > 1$.

Proposition 3.1 Let $x > 0$, $k > 0$ and $s > 1$. Then the function $\zeta_k(x, s)$ satisfies the identities:

$$\partial_x^n \zeta_k(x, s) = (-1)^n (s)_{n,1} \zeta_k(x, s + n) \tag{3.4}$$

$$\zeta_k(x, n) = (-1)^n \frac{\partial_x^n \psi(k, x)}{(n-1)!}, \quad n \geq 2 \tag{3.5}$$

and

$$\zeta_k(x + k, s) = \zeta_k(x, s) - \frac{1}{x^s}. \tag{3.6}$$

Proof From (3.1) we obtain

$$\partial_x^2 \zeta_k(x, s) = -s \partial_x \zeta_k(x, s + 1) = (-1)^2 s(s + 1) \zeta_k(x, s + 2)$$

and repeating the same procedure we get (3.4) since $s(s+1)...(s+n-1) = (s)_{n,1}$.

In [2] it was proved that

$$\partial_x^2 \psi(k, x) = \sum_{\nu=0}^{\infty} \frac{1}{(x + \nu k)^2}, \tag{3.7}$$

so, from (1.6) for $s = 2$ and (3.7) we get

$$\partial_x^2 \psi(k, x) = \zeta_k(x, 2). \tag{3.8}$$

Differentiating (3.7) with respect to x and using (3.1) for $s = 2$ we get

$$\partial_x^3 \psi(k, x) = (-1) 2 \zeta_k(x, 3),$$

$$\partial_x^4 \psi(k, x) = (-1)^2 3! \zeta_k(x, 4)$$

and by induction we obtain (3.5).

The equation (3.6) follows from the definition (1.6) since

$$\zeta_k(x, s) = \frac{1}{x^s} + \sum_{k=0}^{\infty} \frac{1}{(x + k + \nu k)^s} = \frac{1}{x^s} + \zeta_k(x + k, s).$$

4 Inequalities for $B_k(x,y)$ and $\Gamma_k(x)\Gamma_k(1-x)$

Theorem 4.1 Let $x, y, k > 0$ and $x + y \neq k$. Then the function $B_k(x,y)$ satisfies the inequalities:

$$\frac{2^{2-\frac{x+y}{k}}}{x+y-k} < B_k(x,y) < \frac{1-2^{2-\frac{x+y}{k}}}{x+y-k}. \tag{4.1}$$

Lemma 4.1 The function $B(x,y)$ satisfies the inequalities

$$\frac{2^{2-(x+y)}}{x+y-1} < B(x,y) < \frac{1-2^{2-(x+y)}}{x+y-1}, \quad x, y > 0, \quad x+y \neq 1. \tag{4.2}$$

Proof of Lemma 4.1 The function $B(x,y)$ is defined [1] by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

which it can be written as

$$B(x,y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1}dt + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}dt \tag{4.3}$$

If $0 < t < 1/2$ then $t < 1-t$, so the following inequalities hold

$$\int_0^{1/2} t^{x+y-2}dt < \int_0^{1/2} t^{x-1}(1-t)^{y-1}dt < \int_0^{1/2} (1-t)^{x+y-2}dt \tag{4.4}$$

and if $1/2 < t < 1$ then $1-t < t$, so the following inequalities hold

$$\int_{1/2}^1 (1-t)^{x+y-2}dt < \int_{1/2}^1 t^{x-1}(1-t)^{y-1}dt < \int_{1/2}^1 t^{x+y-2}dt. \tag{4.5}$$

From (4.3), using the inequalities (4.4) and (4.5) and evaluating the integrals on the left and right side of the above inequalities we obtain the inequalities (4.2).

Proof of theorem 4.1 By setting $\frac{x}{k}$ and $\frac{y}{k}$, instead of x and y respectively in (4.2) and taking in account the relation (1.8) we get the inequalities (4.1).

Corollary 4.1 Let $x, y, k > 0$. Then the function $B_k(x,y)$ satisfies the inequalities:

$$\frac{2^{1-\frac{x+y}{k}}}{x} < B_k(x,y) < \frac{1-2^{1-\frac{x+y}{k}}}{x}. \tag{4.6}$$

or

$$\frac{2^{1-\frac{x+y}{k}}}{y} < B_k(x,y) < \frac{1-2^{1-\frac{x+y}{k}}}{y}. \tag{4.7}$$

Proof The above inequalities follow from (4.1) by setting $x+k$ (or $y+k$) instead of x (or y) and taking in account relations (1.13).

Corollary 4.2 Let $0 < x < 1$ and $0 < k < 1$. Then the following inequalities for the product $\Gamma_k(x)\Gamma_k(1-x)$ hold

$$\frac{(\frac{2}{k})^{1-\frac{1}{k}}\Gamma(1/k)}{1-x} < \Gamma_k(x)\Gamma_k(1-x) < \frac{(\frac{2}{k})^{1-\frac{1}{k}}\Gamma(1/k)(2^{\frac{1}{k}-1}-1)}{1-x}. \tag{4.8}$$

Proof By setting $y = k+1-x$ instead of y in (4.1) we obtain

$$2^{1-\frac{1}{k}} < B_k(x, k+1-x) < 1-2^{1-\frac{1}{k}}. \tag{4.9}$$

Using (1.5) the inequalities (4.9) become

$$2^{1-\frac{1}{k}} < \frac{\Gamma_k(x)\Gamma_k(k+1-x)}{\Gamma_k(k+1)} < 1-2^{1-\frac{1}{k}}. \tag{4.10}$$

From (1.4) we obtain easily

$$\Gamma_k(k+1-x) = (1-x)\Gamma_k(1-x)$$

and

$$\Gamma_k(k+1) = \Gamma_k(1) = k^{\frac{1}{k}-1}\Gamma(\frac{1}{k}).$$

From (4.10) using the above equalities we obtain the inequalities (4.8).

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