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Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions

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Abstract

The authors introduce the concepts of m-invex set, generalized (s, m)-preinvex function, and explicitly (s, m)-preinvex function, provide some properties for the newly introduced functions, and establish new Hadamard-Simpson type integral inequalities for a function of which the power of the absolute of the first derivative is generalized (s, m)-preinvex function. By taking different values of the parameters, Hadamard-type and Simpson-type integral inequalities can be deduced. Furthermore, inequalities obtained in special case present a refinement and improvement of previously known results.

Keywords: Integral inequalities of Hadamard-Simpson type, Hölder's inequality, (s, m)-preinvex function.
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1. Introduction and Preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, \infty)$, and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space and $\mathbb{R}^n_+$ denotes an n-dimensional nonnegative vector space. The nonnegative real numbers are denoted by $\mathbb{R}_0 = [0, \infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$. Let us firstly recall some definitions of various convex functions.

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Definition 1.1 ([7])**.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if f is nonnegative and for all $x, y \in I$, $\lambda \in (0,1)$ we have that

$$f\big(\lambda x + (1-\lambda)y\big) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

Definition 1.2 ([6])**.** For some $(s,m) \in (0,1]^2$, a function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense if for every $x, y \in [0,b]$ and $\lambda \in (0,1]$ we have that

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.3 ([1])**.** A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see [1, 33] and the references therein.

Definition 1.4 ([1])**.** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$, for every $x, y \in K$, the η-path $P_{x\nu}$ joining the points x and $\nu = x + \eta(y,x)$ is defined by

$$P_{x\nu} = \big\{z | z = x + t\eta(y,x),\ t \in [0,1]\big\}.$$

Definition 1.5 ([22])**.** The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0,1]$ we have that

$$f\Big(x + t\eta(y,x)\Big) \le (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.6 ([13])**.** Let $K \subseteq \mathbb{R}_0$ be an invex set with respect to η. A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to η, if for all $x, y \in K$, $t \in [0,1]$ and some fixed $s \in (0,1]$ we have that

$$f\Big(x + t\eta(y,x)\Big) \le (1-t)^s f(x) + t^s f(y).$$

The following inequality is remarkable in the literature as Simpson type inequality, which plays a fundamental and important role in analysis. In particular, it is well applied in numerical integration.

Theorem 1.7 ([5])**.** *Let $f : [a,b] \to \mathbb{R}$ be a four-times continuously differentiable mapping on (a,b) with $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \tag{1.1}$$

Now it is time to recall some inequalities of Hadamard type and Simpson type for the kinds of convex functions mentioned above that have been developed in recent decades.

Theorem 1.8 ([26])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + f(b) + 4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)| + |f'(b)|\Big]. \end{aligned} \tag{1.2}$$

Theorem 1.9 ([4, 10]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}\Big(|f'(a)|+|f'(b)|\Big) \tag{1.3}$$

and

$$\left|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.4}$$

Theorem 1.10 ([5]). *Let $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $||f'||_1 = \int_a^b |f'(x)|dx < \infty$, then we have the inequality*

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\big(\frac{a+b}{2}\big)\Big]\right| \le \frac{1}{3}||f'||_1(b-a)^2. \tag{1.5}$$

Theorem 1.11 ([2, 25]). *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have that*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big) \tag{1.6}$$

and

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.7}$$

Theorem 1.12 ([28]). *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $q > 1$, $q \ge r$, $s \ge 0$ and $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\eta(a, b) \neq 0$, we have that*

$$\begin{aligned}
&\left|f\left(\frac{2b+\eta(a,b)}{2}\right) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\mathrm{d}x\right| \\
&\le \frac{|\eta(a,b)|}{4}\left\{\left(\frac{1}{r+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-r-1}\right)^{1-\frac{1}{q}}\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-s-1}\right)^{1-\frac{1}{q}}\left[\frac{(s+3)|f'(a)|^q+(s+1)|f'(b)|^q}{2(s+2)}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 1.13 ([28]). *Under the conditions of* Theorem 1.12*, when $r = s = 0$, the following inequality holds*

$$\begin{aligned}
&\left|f\left(\frac{2b+\eta(a,b)}{2}\right) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\mathrm{d}x\right| \\
&\le \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\frac{|\eta(a,b)|}{4}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{1.8}$$

Currently, Hadamard-type and Simpson-type inequalities concerning different kinds of convex functions remain attractive topics for many scholars in the field of convex analysis. For further information about this topic, the reader may refer to [3, 8, 9, 11, 16, 17, 18, 19, 20, 21, 23, 24, 27, 29, 30, 31] and references cited therein.

In the recently published articles[12] by Latif et al., based on the differentiable (α, m)-preinvex functions, they established Hadamard-type integral inequalities, and in the paper [23] Qaisar et al. also found some Simpson-type inequality for differentiable (α, m)-convex functions. Motivated by this idea and based on our previous works [14, 15, 32], in the present paper, the next section we introduce new concepts, to be referred as the m-invex, the generalized (s,m)-preinvex function, and the explicitly (s,m)-preinvex function respectively, and then we give some interesting properties for the newly introduced functions. Section 3 will derive an integral identity with two parameters for a differentiable mapping, then explore new Hadamard-Simpson-type integral inequalities for generalized (s,m)-preinvex functions. Some inequalities obtained in special case present a refinement and improvement of previously known results.

2. New definitions and properties

As one can see, the definitions of the (s,m)-convex, s-preinvex, Godunova-Levin functions have similar forms. This observation leads us to generalize these varieties of convexity. Firstly, the so-called 'm-invex ', may be introduced as follows.

Definition 2.1. A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + \lambda\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $\lambda \in [0,1]$.

Example 2.2. Let $m = \frac{1}{4}$ and $X = [-\pi/2, 0) \bigcup (0, \pi/2]$

$$\eta(y,x,m) = \begin{cases} m\cos(y-x), & \text{if } x \in (0,\pi/2], y \in (0,\pi/2]; \\ -m\cos(y-x), & \text{if } x \in [-\pi/2,0), y \in [-\pi/2,0); \\ m\cos(x), & \text{if } x \in (0,\pi/2], y \in [-\pi/2,0); \\ -m\cos(x), & \text{if } x \in [-\pi/2,0), y \in (0,\pi/2], \end{cases}$$

then X is an m-invex set with respect to η for $\lambda \in [0,1]$ and $m = \frac{1}{4}$. It is obvious that X is not a convex set.

Remark 2.3. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example, in the above Example 2.2, when $m = 1$, then the m-invex set degenerates an invex set on X.

We next give new definitions, to be referred to as generalized (s,m)-preinvex function and explicitly (s,m)-preinvex function respectively.

Definition 2.4. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For $f : K \to \mathbb{R}$ and some fixed $s, m \in (0,1]$, if

$$f\Big(mx + \lambda\eta(y,x,m)\Big) \leq m(1-\lambda)^s f(x) + \lambda^s f(y) \tag{2.1}$$

is valid for all $x, y \in K$, $\lambda \in [0,1]$, then we say that $f(x)$ is a generalized (s,m)-preinvex function with respect to η.

The function $f(x)$ is said to be strictly generalized (s,m)-preinvex function on K with respect to η, if a strict inequality holds on (2.1) for any $x, y \in K$ and $x \neq y$.

Remark 2.5. In Definition 2.4, it is worthwhile to note that generalized (s,m)-preinvex function is an (s,m)-convex function on K with respect to $\eta(y,x,m) = y - mx$.

Definition 2.6. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For $f : K \to \mathbb{R}$ and some fixed $s, m \in (0,1]$, if $\forall\lambda \in (0,1)$, $\forall x, y \in K$ and $f(x) \neq f(y)$, we have

$$f\Big(mx + \lambda\eta(y,x,m)\Big) < m(1-\lambda)^s f(x) + \lambda^s f(y), \tag{2.2}$$

then we say that $f(x)$ is an explicitly (s,m)-preinvex function with respect to η.

Example 2.7. Let $f(x) = -|x|$, $s = 1$, and

$$\eta(y,x,m) = \begin{cases} y - mx, & \text{if } x \geq 0, y \geq 0; \\ y - mx, & \text{if } x \leq 0, y \leq 0; \\ mx - y, & \text{if } x \geq 0, y \leq 0; \\ mx - y, & \text{if } x \leq 0, y \geq 0. \end{cases}$$

Then $f(x)$ is a generalized $(1,m)$-preinvex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \times (0,1] \to \mathbb{R}$ and some fixed $m \in (0,1]$. However, it is obvious that $f(x) = -|x|$ is not a convex function on $\mathbb{R}$. By letting $x = 1, y = 2, \lambda = \frac{1}{2}$, we have $f(x) = -1 \neq -2 = f(y)$ and

$$f\Big(mx + \lambda\eta(y,x,m)\Big) = f\Big(m + \frac{1}{2}\eta(2,1,m)\Big) = -(\frac{1}{2}m + 1) = m(1-\lambda)^s f(x) + \lambda^s f(y).$$

Thus, f is not also an explicitly (s,m)-preinvex function on $\mathbb{R}$ with respect to η for $s = 1$ and some fixed $m \in (0,1]$.

According to the above definitions, we now derive some interesting properties of the generalized (s,m)-preinvex function and the explicitly (s,m)-preinvex function as follows.

The proof of propositions 2.8, 2.9, and 2.10 are straightforward.

Proposition 2.8. *If K_i, $i \in I = \{1, 2, \cdots, n\}$ is a family of m-invex sets in $\mathbb{R}^n$ with respect to the same $\eta : \mathbb{R}^n \times \mathbb{R}^n \times (0,1] \to \mathbb{R}$ for same fixed $m \in (0,1]$, then the intersection $\bigcap_{i \in I} X_i$ is an m-invex set.*

Proposition 2.9. *If $f_i : K \subseteq \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \cdots, n)$ are generalized (s,m)-preinvex (explicitly (s,m)-preinvex) functions with respect to the same $\eta : K \times K \times (0,1] \to \mathbb{R}$ for same fixed $s, m \in (0,1]$, then the function*

$$f = \sum_{i=1}^{n} a_i f_i, a_i \geq 0, (i = 1, 2, \cdots, n)$$

is also a generalized (s,m)-preinvex (explicitly (s,m)-preinvex) functions on K with respect to the same η for fixed $s, m \in (0,1]$.

Proposition 2.10. *If $f_i : K \subseteq \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \cdots, n)$ are generalized (s,m)-preinvex (explicitly (s,m)-preinvex) functions and with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}$ for same fixed $s, m \in (0,1]$, then the function*

$$f = \max\{f_i, i = 1, 2, \cdots, n\}$$

is also a generalized (s,m)-preinvex (explicitly (s,m)-preinvex) function on K with respect to the η for fixed $s, m \in (0,1]$.

In Proposition 2.11 we prove that combination of a generalized (s,m)-preinvex function with a positively homogenous and nondecreasing function is generalized (s,m)-preinvex with respect to η on K for fixed $s, m \in (0,1]$.

Proposition 2.11. *Let K be a nonempty m-invex set in $\mathbb{R}^n$ with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$, $f : K \to \mathbb{R}$ be a generalized (s,m)-preinvex function with respect to η for some fixed $s, m \in (0,1]$, and let $g : W \to \mathbb{R}$ ($W \subseteq \mathbb{R}$) be a positively homogenous and nondecreasing function, where $rang(f) \subseteq W$. Then the composite function $g(f)$ is a generalized (s,m)-preinvex function with respect to η on K for fixed $s, m \in (0,1]$.*

Proof. Since f is a generalized (s,m)-preinvex function, then for all $x, y \in K$

$$f\Big(mx + \lambda\eta(y,x,m)\Big) \leq m(1-\lambda)^s f(x) + \lambda^s f(y)$$

holds for any $\lambda \in [0,1]$. Since g is a positively homogenous and nondecreasing function, then

$$\begin{aligned} g\Big(f\big(mx + \lambda\eta(y,x,m)\big)\Big) &\leq g\Big(m(1-\lambda)^s f(x) + \lambda^s f(y)\Big) \\ &= m(1-\lambda)^s g\big(f(x)\big) + \lambda^s g\big(f(y)\big), \end{aligned}$$

which follows that $g(f)$ is a generalized (s,m)-preinvex function with respect to η on K for some fixed $s, m \in (0,1]$. □

Proposition 2.12. *If $g_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \cdots, n)$ are generalized (s,m)-preinvex functions with respect to η for same fixed $m, s \in (0,1]$, then the set $M = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, \cdots, n\}$ is an m-invex set.*

Proof. Since $g_i(x)$, $(i = 1, 2, \cdots, n)$ are generalized (s,m)-preinvex functions, then for all $x, y \in \mathbb{R}^n$

$$g_i\Big(mx + \lambda\eta(y,x,m)\Big) \leq m(1-\lambda)^s g_i(y) + \lambda^s g_i(x), \quad i = 1, 2, \cdots, n$$

holds for any $\lambda \in [0,1]$. When $x, y \in M$, we know $g_i(x) \leq 0$ and $g_i(y) \leq 0$, from the above inequality, it yields that

$$g_i\Big(mx + \lambda\eta(y,x,m)\Big) \leq 0, \quad i = 1, 2, \cdots, n.$$

That is, $mx + \lambda\eta(y,x,m) \in M$. Hence, M is an m-invex set. □

Proposition 2.13. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a generalized (s,m)-preinvex function with respect to $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \times (0,1] \to \mathbb{R}_+$ for some fixed $m, s \in (0,1]$. Assume that f is monotone decreasing, η is monotone increasing regarding m for fixed $x, y \in \mathbb{R}_+$, and $m_1 \leq m_2 (m_1, m_2 \in (0,1])$. If f is a generalized (s, m_1)-preinvex function on $\mathbb{R}_+$ with respect to η, then f is a generalized (s, m_2)-preinvex function on $\mathbb{R}_+$ with respect to η.*

Proof. Since f is a generalized (s, m_1)-preinvex function, then for all $x, y \in \mathbb{R}_+$

$$f\Big(m_1 x + \lambda\eta(y,x,m_1)\Big) \leq m_1(1-\lambda)^s f(x) + \lambda^s f(y).$$

Combining the conditions f is monotone decreasing, η is monotone increasing regarding m for fixed $x, y \in \mathbb{R}_+$, and $m_1 \leq m_2$, it follows that

$$f\Big(m_2 x + \lambda\eta(y,x,m_2)\Big) \leq f\Big(m_1 x + \lambda\eta(y,x,m_1)\Big)$$

and

$$m_1(1-\lambda)^s f(x) + \lambda^s f(y) \leq m_2(1-\lambda)^s f(x) + \lambda^s f(y).$$

Following the above two inequalities, we have that

$$f\Big(m_2 x + \lambda\eta(y,x,m_2)\Big) \leq m_2(1-\lambda)^s f(x) + \lambda^s f(y).$$

Hence, f is also a generalized (s, m_2)-preinvex function on $\mathbb{R}_+$ with respect to η for fixed $s \in (0,1]$, which completes the proof. □

Proposition 2.14. *Let K be a nonempty m-invex set in $\mathbb{R}^n$ with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$, and $f_i : K \to \mathbb{R} (i \in I = \{1, 2, \cdots, n\})$ be a family of real-valued functions which are explicitly (s,m)-preinvex functions with respect to the same η for same fixed $s, m \in (0,1]$ and bounded from above on K. Then the function $f(x) = \sup\{f_i(x), i \in I\}$ is also an explicitly (s,m)-preinvex function on K with respect to the same η for fixed $s, m \in (0,1]$.*

Proof. Since each $f_i(x)(i \in I)$ is an explicitly (s,m)-preinvex function with respect to the same η for some fixed $s, m \in (0,1]$, we have for each $i \in I$

$$f_i\Big(mx + \lambda\eta(y,x,m)\Big) < m(1-\lambda)^s f_i(x) + \lambda^s f_i(y), \forall x, y \in K, \lambda \in (0,1).$$

Therefore, for each $i \in I$,

$$f_i\Big(mx + \lambda\eta(y,x,m)\Big) < m(1-\lambda)^s \sup_{i\in I} f_i(x) + \lambda^s \sup_{i\in I} f_i(y), \forall x, y \in K, \lambda \in (0,1).$$

Taking sup of the left-hand side of the above equation, we obtain

$$\sup_{i\in I} f_i\Big(mx + \lambda\eta(y,x,m)\Big) < m(1-\lambda)^s \sup_{i\in I} f_i(x) + \lambda^s \sup_{i\in I} f_i(y), \forall x, y \in K, \lambda \in (0,1).$$

That is, $f(x) = \sup\{f_i(x), i \in I\}$ is also an explicitly (s,m)-preinvex function on K with respect to the same η for fixed $s, m \in (0,1]$. □

Proposition 2.15 shows that a local minimum of an explicitly (s,m)-preinvex function over an m-invex set is a global one under some conditions.

Proposition 2.15. *Let K be a nonempty m-invex set in $\mathbb{R}^n$ with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$, and $f : K \to \mathbb{R}$ be an explicitly (s,m)-preinvex function with respect to η for some fixed $s, m \in (0,1]$. And let fixed $s, m \in (0,1]$ satisfy $m(1-\lambda)^s + \lambda^s \leq 1$ for $\forall \lambda \in (0,1)$. If $\bar{x} \in K$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in K$, then $\bar{x}$ is a global one.*

Proof. Suppose that $\bar{x} \in K$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in K$. Then there is an ε-neighborhood $N_\varepsilon(\bar{x})$ around $\bar{x}$ such that

$$f(\bar{x}) \leq f(x), \forall x \in K \cap N_\varepsilon(\bar{x}). \tag{2.3}$$

If $\bar{x}$ is not global minimum of $f(x)$ on K, then there exists an $x^* \in K$ such that

$$f(x^*) < f(\bar{x}).$$

By the explicit (s,m)-preinvexly of $f(x)$ and the condition $m(1-\lambda)^s + \lambda^s \leq 1$,

$$f\Big(m\bar{x} + \lambda\eta(x^*, \bar{x}, m)\Big) < m(1-\lambda)^s f(\bar{x}) + \lambda^s f(x^*) < [m(1-\lambda)^s + \lambda^s] f(\bar{x}) < f(\bar{x})$$

for all $0 < \lambda < 1$. For a sufficiently small $\lambda > 0$, it follows that

$$m\bar{x} + \lambda\eta(x^*, \bar{x}, m) \in K \cap N_\varepsilon(\bar{x}),$$

which is a contradiction to (2.3). This completes the proof. □

By Proposition 2.15, we can conclude that explicitly (s,m)-preinvex functions constitute an important class of generalized convex functions in mathematical programming. The function in Example 2.7 is not an explicitly (s,m)-preinvex function with respect to η based on Proposition 2.15.

3. Hadamard-Simpson type integral inequalities

For establishing our new integral inequalities of Hadamard-Simpson type for generalized (s,m)-preinvex function, we need the following key integral identity, which will be used in the sequel.

Lemma 3.1. *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \to \mathbb{R}$ is a differentiable function, f' is integrable on $[ma, ma+\eta(b,a,m)]$, and $k, t \in \mathbb{R}$, then for each $x \in [ma, ma+\eta(b,a,m)]$ we have that*

$$\begin{aligned} &tf(ma) + (1-k)f\Big(ma+\eta(b,a,m)\Big) + (k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x \\ &= \eta(b,a,m)\left[\int_0^{\frac{1}{2}}(\lambda-t)f'\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda + \int_{\frac{1}{2}}^1(\lambda-k)f'\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda\right]. \end{aligned} \tag{3.1}$$

Proof. Set

$$J = \eta(b,a,m)\left[\int_0^{\frac{1}{2}}(\lambda-t)f'\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda + \int_{\frac{1}{2}}^1(\lambda-k)f'\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda\right].$$

Since $a, b \in K$ and K is an m-invex set with respect to η, for every $\lambda \in [0,1]$ and some fixed $m \in (0,1]$, we have $ma + \lambda\eta(b,a,m) \in K$. Integrating by parts yields

$$\begin{aligned} J &= \eta(b,a,m)\Bigg\{\frac{1}{\eta(b,a,m)}\left[(\lambda-t)f\Big(ma+\lambda\eta(b,a,m)\Big)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda\right] \\ &\quad + \frac{1}{\eta(b,a,m)}\left[(\lambda-k)f\Big(ma+\lambda\eta(b,a,m)\Big)\Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 f\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda\right]\Bigg\} \\ &= \left(\frac{1}{2}-t\right)f\left(ma+\frac{\eta(b,a,m)}{2}\right) + tf(ma) - \int_0^{\frac{1}{2}} f\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda \\ &\quad + (1-k)f\Big(ma+\eta(b,a,m)\Big) - \left(\frac{1}{2}-k\right)f\left(ma+\frac{\eta(b,a,m)}{2}\right) - \int_{\frac{1}{2}}^1 f\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda \\ &= tf(ma) + (1-k)f\Big(ma+\eta(b,a,m)\Big) + (k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right) - \int_0^1 f\Big(ma+\lambda\eta(b,a,m)\Big)\mathrm{d}\lambda. \end{aligned}$$

Let $x = ma + \lambda\eta(b,a,m)$, then $\mathrm{d}x = \eta(b,a,m)\mathrm{d}\lambda$ and we have

$$J = tf(ma) + (1-k)f\Big(ma+\eta(b,a,m)\Big) + (k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x,$$

which is required. □

Remark 3.2. clearly, if $m = 1$, $\eta(b,a,1) = b-a$ and applying $t = \frac{1}{6}$, $k = \frac{5}{6}$ in Lemma 3.1, then we obtain Lemma 2.1 in [23].

In what follows, we establish another refinement of the Simpson's inequality for generalized (s,m)-preinvex functions in the second sense.

Theorem 3.3. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \to \mathbb{R}_0$ for some fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function, $|f'|$ is a generalized (s,m)-preinvex function on A for some fixed $s, m \in (0,1]$, and let $k, t \in \mathbb{R}$, then for each $x \in [ma, ma+\eta(b,a,m)]$ the following inequality holds:*

$$\begin{aligned} &\left|tf(ma) + (1-k)f\Big(ma+\eta(b,a,m)\Big) + (k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \\ &\leq \Big|\eta(b,a,m)\Big|\Big[m\nu_1|f'(a)| + \nu_2|f'(b)|\Big], \end{aligned} \tag{3.2}$$

where

$$v_1 = \frac{2(1-t)^{s+2} + 2(1-k)^{s+2} + \big[2(k+t)(s+2) - 2(s+3)\big]\frac{1}{2^{s+2}} + (ts+2t-1)}{(s+1)(s+2)}$$

and

$$v_2 = \frac{2t^{s+2} + 2k^{s+2} + \big[2(s+1) - 2(s+2)(k+t)\big]\frac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)}.$$

Proof. Since $ma + \lambda\eta(b,a,m) \in A$ for every $\lambda \in [0,1]$ and some fixed $m \in (0,1]$, by Lemma 3.1 and the generalized (s,m)-preinvexity of $|f'|$ on A, we have

$$\begin{aligned}
&\left| tf(ma) + (1-k)f\Big(ma+\eta(b,a,m)\Big) + (k-t)f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x \right| \\
&\le \big|\eta(b,a,m)\big|\left[\int_0^{\frac{1}{2}} |\lambda - t|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda\right] \\
&\le \big|\eta(b,a,m)\big|\left\{ m\int_0^{\frac{1}{2}} |\lambda-t|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda + \int_0^{\frac{1}{2}} |\lambda-t|\lambda^s|f'(b)|\mathrm{d}\lambda \right. \\
&\quad \left. + m\int_{\frac{1}{2}}^1 |\lambda-k|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|\lambda^s|f'(b)|\mathrm{d}\lambda \right\} \\
&= \big|\eta(b,a,m)\big|\left\{ m\left[\int_0^{\frac{1}{2}} |\lambda-t|(1-\lambda)^s\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|(1-\lambda)^s\mathrm{d}\lambda\right]\big|f'(a)\big| \right. \\
&\quad \left. + \left[\int_0^{\frac{1}{2}} |\lambda-t|\lambda^s\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|\lambda^s\mathrm{d}\lambda\right]\big|f'(b)\big| \right\}.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} |\lambda-t|(1-\lambda)^s\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|(1-\lambda)^s\mathrm{d}\lambda \\
&= \frac{2(1-t)^{s+2} + 2(1-k)^{s+2} + \big[2(k+t)(s+2) - 2(s+3)\big]\frac{1}{2^{s+2}} + (ts+2t-1)}{(s+1)(s+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} |\lambda-t|\lambda^s\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda-k|\lambda^s\mathrm{d}\lambda \\
&= \frac{2t^{s+2} + 2k^{s+2} + \big[2(s+1) - 2(s+2)(k+t)\big]\frac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)},
\end{aligned}$$

the desired inequality (3.2) is established. □

Direct computation yields the following corollaries.

Corollary 3.4. *Under the conditions of* Theorem 3.3*,*

(1) *if* $\eta(b,a,m) = b - ma$*,* $m = 1$*,* $t = \frac{1}{6}$*, and let* $k = \frac{5}{6}$*, we have*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + f(b) + 4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)| + |f'(b)|\Big];
\end{aligned} \tag{3.3}$$

(2) *If* $\eta(b,a,m) = b - ma$, $s = m = 1$, *and let* $t = k = \dfrac{1}{2}$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}\Big(|f'(a)| + |f'(b)|\Big). \tag{3.4}$$

(3) *Let* $m = 1$, *if* $\eta(b,a,1)$ *degenerates* $\eta(b,a)$, $s = 1$, *and let* $t = k = \dfrac{1}{2}$, *we have*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big). \tag{3.5}$$

Remark 3.5. Inequality (3.3) is the same as inequality of (1.2) presented by Sarikaya in [26]. Inequality (3.4) is the same as inequality of (1.3) established by Dragomir in [4]. Inequality (3.5) is the same as inequality of (1.6) given by Barani in [2]. Thus, inequality (3.2) is a generalization of these Simpson-type and Hadamard-type inequalities.

Corollary 3.6. *The upper bound of the midpoint inequality for the first derivative is developed as follows:*

(1) *By putting* $f(ma) = f\Big(ma + \eta(b,a,m)\Big) = f\Big(ma + \frac{\eta(b,a,m)}{2}\Big)$ *in inequality* (3.2), *we have*

$$\left|f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \le \big|\eta(b,a,m)\big|\Big[m\nu_1|f'(a)| + \nu_2|f'(b)|\Big], \tag{3.6}$$

where v_1 *and* v_2 *are defined in* Theorem 3.3.

(2) *If* $\eta(b,a,m) = b - ma$, $m = s = 1$, $t = \dfrac{1}{6}$, *and let* $k = \dfrac{5}{6}$ *in the above inequality* (3.6), *it yields that*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{5(b-a)}{72}\Big[|f'(a)| + |f'(b)|\Big]. \tag{3.7}$$

(3) *Let* $m = 1$, *if* $\eta(b,a,1)$ *degenerates* $\eta(b,a)$, $s = 1$, $t = \dfrac{1}{6}$, *and let* $k = \dfrac{5}{6}$ *in the above inequality* (3.6), *we have*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \le \frac{5|\eta(b,a)|}{72}\Big(|f'(a)| + |f'(b)|\Big). \tag{3.8}$$

Remark 3.7. It is noted that the above midpoint inequality (3.7) is better than the inequality (1.4) presented by Kirmaci in [10]; Apparently, the result of inequality (3.8) also has a better result compared with inequality (1.7) presented by Sarikaya in [25].

We continue with

Theorem 3.8. *Let* f *be defined as in* Theorem 3.3 *with* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. *If* $|f'|^q$ *is a generalized* (s,m)*-preinvex function on* A *for some fixed* $s, m \in (0,1]$ *and let* $k, t \in \mathbb{R}$, *then for each* $x \in [ma, ma + \eta(b,a,m)]$ *the following inequality holds:*

$$\begin{aligned}
&\left|tf(ma) + (1-k)f\Big(ma + \eta(b,a,m)\Big) + (k-t)f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \\
&\le \frac{|\eta(b,a,m)|}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left\{\left[t^{p+1} + (\frac{1}{2} - t)^{p+1}\right]^{\frac{1}{p}}\left[m\Big(1-\big(\tfrac{1}{2}\big)^{s+1}\Big)\Big|f'(a)\Big|^q + (\frac{1}{2})^{s+1}\Big|f'(b)\Big|^q\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[(k - \frac{1}{2})^{p+1} + (1-k)^{p+1}\right]^{\frac{1}{p}}\left[m(\frac{1}{2})^{s+1}\Big|f'(a)\Big|^q + \Big(1 - (\frac{1}{2})^{s+1}\Big)\Big|f'(b)\Big|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.9}$$

Proof. Since $ma + \lambda\eta(b, a, m) \in A$ for every $\lambda \in [0, 1]$ and some fixed $m \in (0, 1]$, by Lemma 3.1 and the famous Hölder's integral inequality, we have

$$\begin{aligned}
&\left|tf(ma) + (1-k)f\Big(ma + \eta(b,a,m)\Big) + (k-t)f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \\
&\le \big|\eta(b,a,m)\big|\left[\int_0^{\frac{1}{2}} |\lambda - t|\Big|f'\Big(ma + \lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda + \int_{\frac{1}{2}}^1 |\lambda - k|\Big|f'\Big(ma + \lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda\right] \\
&\le \big|\eta(b,a,m)\big|\left\{\left(\int_0^{\frac{1}{2}} |\lambda - t|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}} \Big|f'\Big(ma + \lambda\eta(b,a,m)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 |\lambda - k|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1 \Big|f'\Big(ma + \lambda\eta(b,a,m)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Also, making use of the generalized (s, m)-preinvexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|tf(ma) + (1-k)f\Big(ma + \eta(b,a,m)\Big) + (k-t)f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \\
&\le \big|\eta(b,a,m)\big|\left\{\left(\int_0^{\frac{1}{2}} |\lambda - t|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}} \Big(m(1-\lambda)^s\big|f'(a)\big|^q + \lambda^s\big|f'(b)\big|^q\Big)\mathrm{d}\lambda\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 |\lambda - k|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1 \Big(m(1-\lambda)^s\big|f'(a)\big|^q + \lambda^s\big|f'(b)\big|^q\Big)\mathrm{d}\lambda\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Direct calculation yields that

$$\int_0^{\frac{1}{2}} |\lambda - t|^p\mathrm{d}\lambda = \frac{t^{p+1} + \left(\frac{1}{2} - t\right)^{p+1}}{p+1} \quad \textit{and} \quad \int_{\frac{1}{2}}^1 |\lambda - k|^p\mathrm{d}\lambda = \frac{\left(k - \frac{1}{2}\right)^{p+1} + (1-k)^{p+1}}{p+1}.$$

Similarly, we have

$$\int_0^{\frac{1}{2}} (1-\lambda)^s\mathrm{d}\lambda = \int_{\frac{1}{2}}^1 \lambda^s\mathrm{d}\lambda = \frac{1 - \left(\frac{1}{2}\right)^{s+1}}{s+1} \quad \textit{and} \quad \int_0^{\frac{1}{2}} \lambda^s\mathrm{d}\lambda = \int_{\frac{1}{2}}^1 (1-\lambda)^s\mathrm{d}\lambda = \frac{\left(\frac{1}{2}\right)^{s+1}}{s+1}.$$

Therefore, combining the above four equalities, this leads to the desired result. The statement in Theorem 3.8 is proved. □

Corollary 3.9. *Under the condition of* Theorem 3.8*,*

(1) *when $s = 1$, we have*

$$\begin{aligned}
&\left|tf(ma) + (1-k)f\Big(ma + \eta(b,a,m)\Big) + (k-t)f\left(ma + \frac{\eta(b,a,m)}{2}\right) - \frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(x)\mathrm{d}x\right| \\
&\le \frac{|\eta(b,a,m)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{\left[t^{p+1} + (\frac{1}{2} - t)^{p+1}\right]^{\frac{1}{p}}\left[\frac{3m|f'(a)|^q}{4} + \frac{|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[(k - \frac{1}{2})^{p+1} + (1-k)^{p+1}\right]^{\frac{1}{p}}\left[\frac{m|f'(a)|^q}{4} + \frac{3|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right\};
\end{aligned} \tag{3.10}$$

(2) *Let $m = 1$, if $\eta(b, a, 1)$ degenerates $\eta(b, a)$, $k = 1$, and let $t = 0$ in inequality* (3.10)*, we can get*

$$\begin{aligned}&\left|f\left(a+\frac{\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\&\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{|\eta(b,a)|}{4}\left[\left(\frac{3}{4}|f'(a)|^{q}+\frac{1}{4}|f'(b)|^{q}\right)^{\frac{1}{q}}+\left(\frac{1}{4}|f'(a)|^{q}+\frac{3}{4}|f'(b)|^{q}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.11}$$

Remark 3.10. By substituting $p = \dfrac{q}{q-1}$ into inequality (3.11) and exchanging a and b , we can deduce the inequality (1.8).

In the following corollary, we have the midpoint inequality for powers in terms of the first derivative.

Corollary 3.11. *By substituting $f(ma) = f\Big(ma + \eta(b, a, m)\Big) = f\Big(ma + \frac{\eta(b,a,m)}{2}\Big)$, $t = \dfrac{1}{6}$, and $k = \dfrac{5}{6}$ into inequality* (3.9)*, we have*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(x)\mathrm{d}x-f\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|\\&\leq\frac{|\eta(b,a,m)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{1}{6}\right)^{p+1}+\left(\frac{1}{3}\right)^{p+1}\right]^{\frac{1}{p}}\\&\quad\times\left\{\left[\frac{3m|f'(a)|^{q}}{4}+\frac{|f'(b)|^{q}}{4}\right]^{\frac{1}{q}}+\left[\frac{m|f'(a)|^{q}}{4}+\frac{3|f'(b)|^{q}}{4}\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.12}$$

In the following theorem, we obtain another form of Simpson type inequality for powers in term of the first derivative.

Theorem 3.12. *Let f be defined as in* Theorem 3.3*. If the mapping $|f'|^q$ for $q \geq 1$ is generalized (s, m)-preinvex on A for some fixed $s, m \in (0, 1]$ and let $k, t \in \mathbb{R}$, then for each $x \in [ma, ma + \eta(b, a, m)]$ the following inequality holds:*

$$\begin{aligned}&\left|tf(ma)+(1-k)f\Big(ma+\eta(b,a,m)\Big)+(k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right)-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(x)\mathrm{d}x\right|\\&\leq\big|\eta(b,a,m)\big|\left\{\left(t^{2}-\frac{1}{2}t+\frac{1}{8}\right)^{1-\frac{1}{q}}\left[m\xi_{1}|f'(a)|^{q}+\xi_{2}|f'(b)|^{q}\right]^{\frac{1}{q}}\right.\\&\quad\left.+\left(k^{2}-\frac{3}{2}k+\frac{5}{8}\right)^{1-\frac{1}{q}}\left[m\xi_{3}|f'(a)|^{q}+\xi_{4}\big|f'(b)\big|^{q}\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.13}$$

where

$$\xi_1 = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts + 4t - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_2 = \frac{2t^{s+2} + (s + 1 - 2ts - 4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_3 = \frac{2(1-k)^{s+2} + (2ks + 4k - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

and

$$\xi_4=\frac{2k^{s+2}+(s+1-2ks-4k)\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}.$$

Proof. Since $ma+\lambda\eta(b,a,m)\in A$ for every $\lambda\in[0,1]$ and some fixed $m\in(0,1]$, by Lemma 3.1 and power-mean integral inequality, it follows that

$$\begin{aligned}
&\left|tf(ma)+(1-k)f\Big(ma+\eta(b,a,m)\Big)+(k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right)-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(x)\mathrm{d}x\right|\\
&\le\big|\eta(b,a,m)\big|\left[\int_0^{\frac12}|\lambda-t|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda+\int_{\frac12}^1|\lambda-k|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|\mathrm{d}\lambda\right]\\
&\le\big|\eta(b,a,m)\big|\left\{\left(\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_0^{\frac12}|\lambda-t|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac1q}\right.\\
&\quad\left.+\left(\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_{\frac12}^1|\lambda-k|\Big|f'\Big(ma+\lambda\eta(b,a,m)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac1q}\right\}.
\end{aligned}$$

Using the generalized (s,m)-preinvexity of $|f'|^q$, we have that

$$\begin{aligned}
&\left|tf(ma)+(1-k)f\Big(ma+\eta(b,a,m)\Big)+(k-t)f\left(ma+\frac{\eta(b,a,m)}{2}\right)-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(x)\mathrm{d}x\right|\\
&\le\big|\eta(b,a,m)\big|\left\{\left(\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_0^{\frac12}|\lambda-t|\Big(m(1-\lambda)^s\big|f'(a)\big|^q+\lambda^s\big|f'(b)\big|^q\Big)\mathrm{d}\lambda\right]^{\frac1q}\right.\\
&\quad\left.+\left(\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_{\frac12}^1|\lambda-k|\Big(m(1-\lambda)^s\big|f'(a)\big|^q+\lambda^s\big|f'(b)\big|^q\Big)\mathrm{d}\lambda\right]^{\frac1q}\right\}.
\end{aligned}$$

By simple calculations, we can get

$$\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda=t^2-\frac12t+\frac18,\quad\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda=k^2-\frac32k+\frac58,\tag{3.14}$$

$$\int_0^{\frac12}|\lambda-t|(1-\lambda)^s\mathrm{d}\lambda=\frac{t(s+2)-1+2(1-t)^{s+2}+(2ts+4t-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},\tag{3.15}$$

$$\int_0^{\frac12}|\lambda-t|\lambda^s\mathrm{d}\lambda=\frac{2t^{s+2}+(s+1-2ts-4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},\tag{3.16}$$

$$\int_{\frac12}^1|\lambda-k|(1-\lambda)^s\mathrm{d}\lambda=\frac{2(1-k)^{s+2}+(2ks+4k-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},\tag{3.17}$$

and

$$\int_{\frac12}^1|\lambda-k|\lambda^s\mathrm{d}\lambda=\frac{2k^{s+2}+(s+1-2ks-4k)\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}.\tag{3.18}$$

Thus, our desired result can be obtained by combining equalities (3.14)-(3.18), and the proof is completed. □

Corollary 3.13. *Let f be defined as in* Theorem 3.12*, if $s = 1$, $t = \frac{1}{6}$, and $k = \frac{5}{6}$, the inequality holds for extended m-preinvex functions:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma)+4f\left(ma+\frac{\eta(b,a,m)}{2}\right)+f\left(ma+\eta(b,a,m)\right)\right]-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(x)\mathrm{d}x\right|\\&\leq\left|\eta(b,a,m)\right|\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left[\left(\frac{61m}{1296}|f'(a)|^q+\frac{29}{1296}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{29m}{1296}|f'(a)|^q+\frac{61}{1296}|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned} \tag{3.19}$$

In particular, let $m = 1$, if $\eta(b, a, 1)$ degenerates $\eta(b, a)$ in inequality (3.19)*, the inequality holds for convex function. If $|f'(x)| \leq Q$, $\forall x \in I$, we can deduce that*

$$\left|\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(x)\mathrm{d}x\right|\leq\frac{5(b-a)^2}{36}Q. \tag{3.20}$$

Remark 3.14. It is observed that the inequality (3.20) gives an improvement for the inequality (1.3) with the integral interval length $b - a \geq \frac{1}{2}$. Thus, Theorem 3.12 and its consequences generalize the main results in [5].

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