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Properties of Binary Relations

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Summary. The paper contains definitions of some properties of binary relations: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, strong connectedness, and transitivity. Basic theorems relating the above mentioned notions are given.

The terminology and notation used here have been introduced in the following articles: [1], [2], and [3]. For simplicity we adopt the following convention: X will have the type set; x, y, z will have the type Any; P, R will have the type Relation. We now define several new predicates. Let us consider R, X. The predicate

R is_reflexive_in X is defined by $x \in X$ **implies** $\langle x, x\rangle \in R$.

The predicate

R is_irreflexive_in X is defined by $x \in X$ **implies not** $\langle x, x\rangle \in R$.

The predicate

R is_symmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ **implies** $\langle y, x\rangle \in R$.

The predicate

R is_antisymmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ & $\langle y, x\rangle \in R$ **implies** $x = y$.

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The predicate

$$R \text{ is\_asymmetric\_in } X$$

is defined by

$$x \in X \ \& \ y \in X \ \& \ \langle x, y \rangle \in R \textbf{ implies not } \langle y, x \rangle \in R.$$

The predicate

$$R \text{ is\_connected\_in } X$$

is defined by

$$x \in X \ \& \ y \in X \ \& \ x \neq y \textbf{ implies } \langle x, y \rangle \in R \textbf{ or } \langle y, x \rangle \in R.$$

The predicate

$$R \text{ is\_strongly\_connected\_in } X$$

is defined by

$$x \in X \ \& \ y \in X \textbf{ implies } \langle x, y \rangle \in R \textbf{ or } \langle y, x \rangle \in R.$$

The predicate

$$R \text{ is\_transitive\_in } X$$

is defined by

$$x \in X \ \& \ y \in X \ \& \ z \in X \ \& \ \langle x, y \rangle \in R \ \& \ \langle y, z \rangle \in R \textbf{ implies } \langle x, z \rangle \in R.$$

We now state several propositions:

(1) $$R \text{ is\_reflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds } \langle x, x \rangle \in R,$$

(2) $$R \text{ is\_irreflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds not } \langle x, x \rangle \in R,$$

(3) $$R \text{ is\_symmetric\_in } X$$
$$\textbf{iff for } x, y \textbf{ st } x \in X \ \& \ y \in X \ \& \ \langle x, y \rangle \in R \textbf{ holds } \langle y, x \rangle \in R,$$

(4) $$R \text{ is\_antisymmetric\_in } X$$
$$\textbf{iff for } x, y \textbf{ st } x \in X \ \& \ y \in X \ \& \ \langle x, y \rangle \in R \ \& \ \langle y, x \rangle \in R \textbf{ holds } x = y,$$

(5) $$R \text{ is\_asymmetric\_in } X$$
$$\textbf{iff for } x, y \textbf{ st } x \in X \ \& \ y \in X \ \& \ \langle x, y \rangle \in R \textbf{ holds not } \langle y, x \rangle \in R,$$

(6) $$R \text{ is\_connected\_in } X$$
$$\textbf{iff for } x, y \textbf{ st } x \in X \ \& \ y \in X \ \& \ x \neq y \textbf{ holds } \langle x, y \rangle \in R \textbf{ or } \langle y, x \rangle \in R,$$

(7) $$R \text{ is\_strongly\_connected\_in } X$$
$$\textbf{iff for } x, y \textbf{ st } x \in X \ \& \ y \in X \textbf{ holds } \langle x, y \rangle \in R \textbf{ or } \langle y, x \rangle \in R,$$

(8) R is_transitive_in X **iff** **for** x,y,z
st $x \in X$ & $y \in X$ & $z \in X$ & $\langle x, y\rangle \in R$ & $\langle y, z\rangle \in R$ **holds** $\langle x, z\rangle \in R$.

We now define several new predicates. Let us consider R. The predicate

R is_reflexive is defined by R is_reflexive_in field R.

The predicate

R is_irreflexive is defined by R is_irreflexive_in field R.

The predicate

R is_symmetric is defined by R is_symmetric_in field R.

The predicate

R is_antisymmetric is defined by R is_antisymmetric_in field R.

The predicate

R is_asymmetric is defined by R is_asymmetric_in field R.

The predicate

R is_connected is defined by R is_connected_in field R.

The predicate

R is_strongly_connected is defined by R is_strongly_connected_in field R.

The predicate

R is_transitive is defined by R is_transitive_in field R.

We now state a number of propositions:

(9) R is_reflexive **iff** R is_reflexive_in field R,

(10) R is_irreflexive **iff** R is_irreflexive_in field R,

(11) R is_symmetric **iff** R is_symmetric_in field R,

(12) R is_antisymmetric **iff** R is_antisymmetric_in field R,

(13) R is_asymmetric **iff** R is_asymmetric_in field R,

(14) R is_connected **iff** R is_connected_in field R,

(15) R is_strongly_connected **iff** R is_strongly_connected_in field R,

(16) R is_transitive **iff** R is_transitive_in field R,

(17) R is_reflexive **iff** $\triangle$ field $R \subseteq R$,

(18) R is_irreflexive **iff** $\triangle$ (field R) $\cap R = \emptyset$,

(19) R is_antisymmetric_in X **iff** $R \setminus \triangle X$ is_asymmetric_in X,

(20) R is_asymmetric_in X **implies** $R \cup \triangle X$ is_antisymmetric_in X,

(21) R is_antisymmetric_in X **implies** $R \setminus \triangle X$ is_asymmetric_in X,

(22) R is_symmetric & R is_transitive **implies** R is_reflexive ,

(23) $\triangle X$ is_symmetric & $\triangle X$ is_transitive ,

(24) $\triangle X$ is_antisymmetric & $\triangle X$ is_reflexive ,

(25) R is_irreflexive & R is_transitive **implies** R is_asymmetric ,

(26) R is_asymmetric **implies** R is_irreflexive & R is_antisymmetric ,

(27) R is_reflexive **implies** $R\tilde{}$ is_reflexive ,

(28) R is_irreflexive **implies** $R\tilde{}$ is_irreflexive ,

(29) R is_reflexive **implies** dom R = dom ($R\tilde{}$) & rng R = rng ($R\tilde{}$),

(30) R is_symmetric **iff** $R = R\tilde{}$,

(31) P is_reflexive & R is_reflexive **implies** $P \cup R$ is_reflexive & $P \cap R$ is_reflexive ,

(32) P is_irreflexive & R is_irreflexive
implies $P \cup R$ is_irreflexive & $P \cap R$ is_irreflexive ,

(33) P is_irreflexive **implies** $P \setminus R$ is_irreflexive ,

(34) R is_symmetric **implies** $R\tilde{}$ is_symmetric ,

(35) P is_symmetric & R is_symmetric
implies $P \cup R$ is_symmetric & $P \cap R$ is_symmetric & $P \setminus R$ is_symmetric ,

(36) R is_asymmetric **implies** $R\tilde{}$ is_asymmetric ,

(37) P is_asymmetric & R is_asymmetric **implies** $P \cap R$ is_asymmetric ,

(38) P is_asymmetric **implies** $P \setminus R$ is_asymmetric ,

(39) R is_antisymmetric **iff** $R \cap (R\tilde{}) \subseteq \triangle$ (dom R),

(40) R is_antisymmetric **implies** $R\tilde{}$ is_antisymmetric ,

(41) $$P \text{ is\_antisymmetric} \textbf{ implies } P \cap R \text{ is\_antisymmetric} \;\&\; P \setminus R \text{ is\_antisymmetric},$$

(42) $$R \text{ is\_transitive} \textbf{ implies } R\check{}\,\text{is\_transitive},$$

(43) $$P \text{ is\_transitive} \;\&\; R \text{ is\_transitive} \textbf{ implies } P \cap R \text{ is\_transitive},$$

(44) $$R \text{ is\_transitive} \textbf{ iff } R \cdot R \subseteq R,$$

(45) $$R \text{ is\_connected} \textbf{ iff } [\!:\text{field } R, \text{field } R:\!] \setminus \triangle(\text{field } R) \subseteq R \cup R\check{},$$

(46) $$R \text{ is\_strongly\_connected} \textbf{ implies } R \text{ is\_connected} \;\&\; R \text{ is\_reflexive},$$

(47) $$R \text{ is\_strongly\_connected} \textbf{ iff } [\!:\text{field } R, \text{field } R:\!] = R \cup R\check{}.$$

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