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PROPERTIES OF DISTRIBUTIONS OF RANDOM VARIABLES WITH INDEPENDENT DIFFERENCES OF CONSECUTIVE ELEMENTS OF THE OSTROGRADSKIĬ SERIES

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Abstract. Several metric relations for representations of real numbers by the Ostrogradskiĭ type 1 series are obtained. These relations are used to prove that a random variable with independent differences of consecutive elements of the Ostrogradskiĭ type 1 series has a pure distribution, that is, its distribution is either purely discrete, or purely singular, or purely absolutely continuous. The form of the distribution function and that of its derivative are found. A criterion for discreteness and sufficient conditions for the distribution spectrum to have zero Lebesgue measure are established.

Introduction

By the Ostrogradskiĭ algorithms, any real number $x \in [0;1]$ can be represented as follows:

$$\frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_1 q_2 \cdots q_n} + \cdots$$

where q_n are positive integers such that $q_{n+1} > q_n$ for any $n \in \mathbf{N}$, or

$$\frac{1}{q_1} - \frac{1}{q_2} + \cdots + \frac{(-1)^{n-1}}{q_n} + \cdots$$

where q_n are again positive integers such that $q_{n+1} \geq q_n(q_n+1)$ for any $n \in \mathbf{N}$.

The problems related to algorithms of sign alternating series expansion of a number were investigated by M. V. Ostrogradskiĭ not very long before his death and were not published. His short notes on the problem were discovered in 1951 in the manuscript section of the Academy of Sciences of Ukraine and further deciphered by E. Ya. Remez in [1]. In that paper, the author points out at a certain analogy between the Ostrogradskiĭ series and a continued fraction and devotes much attention to the application of the Ostrogradskiĭ series for finding approximate solutions of algebraic equations. Sierpiński [2] studied similar problems independently of Ostrogradskiĭ (in the paper [2], there are several algorithms for series expansion of a real number; two of these algorithms give the Ostrogradskiĭ series expansion). Pierce probably also worked on the problem (the book [4, p. 10] mentions, referring to [3], the Pierce algorithm for sign alternating series expansion of a real number; the result is the Ostrogradskiĭ type 1 series). Gnedenko mentions two algorithms due to Ostrogradskiĭ in editor's remarks to the book [5] noting that there had been no detailed studies of these series by the time of writing (1961).

There exist other papers dealing with applications of the Ostrogradskiĭ series. Let us mention some of them. The paper [6, pp. 91–96] establishes a link of the Ostrogradskiĭ

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algorithms with the algorithm of continued fraction expansion of a number. Some generalizations of these algorithms related to branching continued fractions are also obtained there. In [7], p-adic analogs of the Euclid algorithm and the Ostrogradskiĭ algorithm are used for constructing p-adic continued fractions, and unimprovable rates of convergence of the corresponding convergents to the real number are obtained. The same author "combines" in [8] the Engel algorithm and the Ostrogradskiĭ algorithm for constructing an algorithm of representation of real numbers by series whose convergence rate is higher than that of the Engel series and that of the Ostrogradskiĭ series. The paper [9] should perhaps be considered the first to contain a metric theory of numbers represented by the Ostrogradskiĭ series. In this paper, the first Ostrogradskiĭ algorithm is studied and estimates for the error of the nth approximation are found. A generalization of the Ostrogradskiĭ algorithm for approximations in Banach spaces is also proposed in [9]. In the paper [10], an algorithm for sign alternating series expansion of a number is introduced leading, under a certain choice of the parameters, to the Lüroth series, Engel series, and Ostrogradskiĭ series (though the latter series are not studied in that paper). We also note that the algorithm of the $\tilde{Q}_\infty$-representation (see [11] or [12]) can give, under a certain choice of the set $\tilde{Q}_\infty$, the sign alternating series expansions of the Lüroth type.

In this paper, we study a random variable such that the consecutive terms in the expansion of this variable in the Ostrogradskiĭ type 1 series have differences that are independent random variables. The main question in the study of this variable is to determine the structure of its distribution; if it is singular, then the structure of this singular distribution must be established. By the Lebesgue theorem, any distribution function admits a unique representation in the form

$$F(x) = \alpha_1 F_d(x) + \alpha_2 F_{ac}(x) + \alpha_3 F_s(x) \tag{1}$$

where F_d is a discrete distribution function, F_{ac} is an absolutely continuous distribution function, F_s is a singular distribution function, $\alpha_k \geq 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Representation (1) is called the *structure of the distribution function* (or the *structure of the distribution*). Any singular distribution function can be represented as follows:

$$F(x) = \gamma_1 F^S(x) + \gamma_2 F^C(x) + \gamma_3 F^K(x) \tag{2}$$

where F^S, F^C, and F^K are an S-type, a C-type, and a K-type distribution function, respectively, $\gamma_k \geq 0$, and

$$\gamma_1 + \gamma_2 + \gamma_3 = 1.$$

Representation (2) is called the *structure of the singular distribution function* (or the *structure of the singular distribution*) [11, p. 74]. Solving the problem of the structure of a distribution (or that of the structure of a singular distribution) consists in determining the numbers α_1, α_2, α_3, and the functions F_d, F_{ac}, F_s (γ_1, γ_2, γ_3, and F^S, F^C, F^K, respectively).

Recall also that random variables having Q-, Q_∞-, $\tilde{Q}_\infty$-representations have already been studied, as well as those represented by a continued fraction or by an Ostrogradskiĭ type 2 series whose elements are either independent random variables or form a Markov chain (see [11]). Besides the problem of the structure of the distribution, fractal properties of these random variables have been studied, that is, fractal properties of the distribution spectrum and distribution support.

This paper contains four sections. In Section 1, an Ostrogradskiĭ type 1 series and its approximant numbers are defined and some lemmas describing certain properties of the approximant numbers are given, as well as a theorem stating that a real number can be represented by an Ostrogradskiĭ type 1 series and that this representation is unique. An $\bar{O}_1$-representation of a number is introduced together with the cylindric sets corresponding to the $\bar{O}_1$-representation of a number.

In Section 2, the set $C[\bar{O}_1, \{V_k\}]$ is introduced containing all numbers x having $\bar{O}_1$-representations whose elements take values in the sets $V_1, V_2, \ldots, V_k, \ldots,$ respectively. In particular, some sufficient conditions for this set to have zero Lebesgue measure are obtained.

In Section 3, we consider a random variable whose elements of the $\bar{O}_1$-representation are independent. The forms of the distribution function of this variable and its derivative are obtained, a criterion for the distribution to be discrete is proved, and sufficient conditions for being Cantor-type singular are established. Section 4 contains a proof of the fact that the $\bar{O}_1$-elements of a uniformly distributed random variable cannot be independent and cannot form a homogeneous Markov chain.

1. Representation of numbers by the Ostrogradskiĭ type 1 series

Definition 1. An expression of the form

$$q_0 + \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_1 q_2 \cdots q_n} + \cdots \tag{3}$$

is called an *Ostrogradskiĭ type 1 series*, which is written for brevity as

$$O_1(q_0; q_1, q_2, \ldots, q_n, \ldots)$$

where q_0 is an integer, $q_1, q_2, q_3, \ldots$ are positive integers and $q_{k+1} > q_k$ for any $k \in \mathbf{N}$. The numbers q_k are called the *elements of the Ostrogradskiĭ type 1 series.*

It is clear that any finite partial sum of series (3) is a rational number as the result of a finite number of rational operations with rational numbers. An infinite series of the form (3) is absolutely convergent under the above assumptions imposed on q_k (which can be readily checked by the d'Alembert criterion of convergence of positive series) and is therefore a finite real number (which is irrational by Theorem 1).

Definition 2. A number having the form

$$\frac{A_k}{B_k} = O_1(q_0; q_1, q_2, \ldots, q_k) = q_0 + \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k}$$

is called an *approximant number of order k* of the Ostrogradskiĭ type 1 series.

The following results can readily be proved [13].

Lemma 1 (The law of creation of the approximant numbers)**.** *For any positive integer k, we have*

$$A_k = A_{k-1} q_k + (-1)^{k-1},$$
$$B_k = B_{k-1} q_k = q_1 q_2 \cdots q_k$$

(assuming that $A_0 = q_0$, $B_0 = 1$).

Lemma 2. *The approximant numbers of even orders form an increasing sequence, while the approximant numbers of odd orders form a decreasing sequence. Moreover, each approximant number of an odd order is greater than any approximant number of an even order.*

Theorem 1 (M. V. Ostrogradskiĭ)**.** *Each real number x can be represented by Ostrogradskiĭ series* (3)*. Moreover, if x is irrational, this representation is unique and expression* (3) *contains infinitely many terms; if x is rational, it can be represented in the form* (3) *with a finite number of terms in two different ways:*

$$O_1(q_0; q_1, q_2, \ldots, q_n),$$
$$O_1(q_0; q_1, q_2, \ldots, q_n - 1, q_n).$$

The elements $q_1, q_2, q_3, \dots$ of the Ostrogradskiĭ series expansion of a number

$$x \in (0;1)$$

can be calculated applying the *first Ostrogradskiĭ algorithm* to the number x:

$$\begin{array}{ll} 1 = q_1 x + \alpha_1, & 0 \le \alpha_1 < x, \\ 1 = q_2 \alpha_1 + \alpha_2, & 0 \le \alpha_2 < \alpha_1, \\ \dots\dots\dots\dots\dots\dots & \\ 1 = q_n \alpha_{n-1} + \alpha_n, & 0 \le \alpha_n < \alpha_{n-1}, \\ \dots\dots\dots\dots\dots\dots & \end{array}$$

The representation of a number x in the form (3) is also called the *O_1-representation* of x and the numbers q_k are called the *O_1-elements* of x.

Let us switch from the O_1-representation of $x \in (0;1)$ to an $\bar{O}_1$-representation by setting

$$g_1 = q_1, \qquad g_{n+1} = q_{n+1} - q_n \quad \text{for an arbitrary } n \in \mathbf{N}.$$

The representation of the number x as

$$x = \frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \dots + \frac{(-1)^{n-1}}{g_1(g_1+g_2)\cdots(g_1+g_2+\dots+g_n)} + \cdots$$

is called the *$\bar{O}_1$-representation* of the number x and is written for brevity as

$$x = \bar{O}_1(0; g_1, g_2, \dots, g_n, \dots).$$

The numbers g_k are called the *$\bar{O}_1$-elements* of x. It is clear that each g_k can take any positive integer value.

We define a *cylindric segment of range n with base* $(s_1, s_2, \dots, s_n)$ corresponding to the $\bar{O}_1$-representation of numbers as the set $\bar{\Delta}_{s_1 s_2 \dots s_n}$ of all numbers $x \in [0;1]$ admitting the $\bar{O}_1$-representation satisfying

$$g_1 = s_1, \qquad g_2 = s_2, \qquad \dots, \qquad g_n = s_n.$$

We define a *cylindric interval of range n with base* $(s_1, s_2, \dots, s_n)$ corresponding to the $\bar{O}_1$-representation of numbers as the set $\bar{\nabla}_{s_1 s_2 \dots s_n}$ of all numbers $x \in [0;1]$ for which there is no $\bar{O}_1$-representations failing to satisfy

$$g_1 = s_1, \qquad g_2 = s_2, \qquad \dots, \qquad g_n = s_n.$$

The motivation for the terms "segment" and "interval" is given in the following assertion.

Lemma 3. *A range n segment $\bar{\Delta}_{s_1 s_2 \dots s_n}$ is equal to the segment (closed interval)*

$$\big[\bar{O}_1(0; s_1, s_2, \dots, s_{n-1}, s_n + 1); \bar{O}_1(0; s_1, s_2, \dots, s_{n-1}, s_n)\big]$$

if n is odd, and to the segment

$$\big[\bar{O}_1(0; s_1, s_2, \dots, s_{n-1}, s_n); \bar{O}_1(0; s_1, s_2, \dots, s_{n-1}, s_n + 1)\big]$$

if n is even. A range n interval $\bar{\nabla}_{s_1 s_2 \dots s_n}$ is equal to the (open) interval having the same endpoints as the interval $\bar{\Delta}_{s_1 s_2 \dots s_n}$.

Lemma 4. *The cylindric sets have the following properties:*

(1) $\bar{\nabla}_{s_1 s_2 \dots s_{n-1} s} \subset \bar{\nabla}_{s_1 s_2 \dots s_{n-1}}$ *for any admissible* $s_1, s_2, \dots, s_{n-1}$, s.

(2) $\bar{\nabla}_{s_1 s_2 \dots s_n} \cap \bar{\nabla}_{t_1 t_2 \dots t_n} = \varnothing$ *if and only if there exists a number*

$$k \in \{1, \dots, n\}$$

such that $s_k \neq t_k$.

(3) $\bar\Delta_{s_1 s_2 \dots s_{n-1}} = \bigcup_{s=1}^{\infty} \bar\Delta_{s_1 s_2 \dots s_{n-1} s}$, *and moreover*

$$\bar\Delta_{s_1 s_2 \dots (s_n+1)} \le \bar\Delta_{s_1 s_2 \dots s_n} \quad \text{if } n \text{ is odd, and}$$
$$\bar\Delta_{s_1 s_2 \dots s_n} \le \bar\Delta_{s_1 s_2 \dots (s_n+1)} \quad \text{if } n \text{ is even.}$$

(For two sets A and B, the notation $A \le B$ means that $a \le b$ for all $a \in A$ and $b \in B$.)

(4) *The length of a range n interval $\bar\nabla_{s_1 s_2 \dots s_n}$ equals*

$$\left|\bar\nabla_{s_1 s_2 \dots s_n}\right| = \frac{1}{s_1(s_1+s_2)\dots(s_1+s_2+\dots+s_n)(s_1+s_2+\dots+s_n+1)} = \frac{B_{n-1}}{B_n(B_n+B_{n-1})}$$

where B_n is the denominator of the approximant number of order n.

2. Some problems of the metric theory of numbers represented by the Ostrogradskiĭ type 1 series

The metric number theory studies measures of numeric sets whose elements (symbols, numbers), represented in some system, have certain properties. Different representations of numbers generate different metric theories (the metric theory of n-adic expansions, that of Q-representations [11], that of continued fractions [5]).

Representations of numbers by the Ostrogradskiĭ series enable us to develop another metric theory.

Take an arbitrary sequence $\{V_k\}$ of subsets of the set of positive integers and consider the set $C[\bar O_1, \{V_k\}]$ of all numbers $x \in [0;1]$ admitting $\bar O_1$-representation whose elements satisfy the following conditions:

$$g_1(x) \in V_1, \qquad g_2(x) \in V_2, \qquad \dots, \qquad g_k(x) \in V_k, \qquad \dots .$$

Lemma 5. *The set $C[\bar O_1, \{V_k\}]$ can be represented as follows:*

$$C[\bar O_1, \{V_k\}] = \bigcap_{k=1}^{\infty} \bigcup_{\substack{v_1 \in V_1 \\ \dots \\ v_k \in V_k}} \bar\Delta_{v_1 v_2 \dots v_k}.$$

Proof. Indeed, a number x_0 belongs to the family $C[\bar O_1, \{V_k\}]$ if and only if for any positive integer k, there exists a cylindric segment $\bar\Delta_{v_1 v_2 \dots v_k}$ of range k containing the point x_0 and such that $v_1 \in V_1$, $v_2 \in V_2$, $\dots$, $v_k \in V_k$. □

Theorem 2. *Let*

$$V_k = \left\{ v_1^{(k)}, v_2^{(k)}, \dots, v_{N_k}^{(k)} \right\}.$$

If all sets V_k are finite and moreover $\underline{\lim}_{k\to\infty} (N_1 N_2 \cdots N_k)/(k+1)! = 0$, then

$$\lambda\left(C[\bar O_1, \{V_k\}]\right) = 0.$$

Proof. Write

$$S_k = \bigcup_{\substack{v_1 \in V_1 \\ \dots \\ v_k \in V_k}} \bar\Delta_{v_1 v_2 \dots v_k}.$$

Then the above lemma implies that $C[\bar O_1, \{V_k\}] = \bigcap_{k=1}^{\infty} S_k$. Each set S_k is the union of $N_1 N_2 \cdots N_k$ intervals with no common interior point, whose lengths satisfy the inequality

$$\left|\bar\Delta_{v_1 v_2 \dots v_k}\right| \le \frac{1}{(k+1)!}.$$

Therefore

$$\lambda\left(S_k\right)=\sum_{\substack{v_1\in V_1\\ \cdots\\ v_k\in V_k}}\left|\bar{\Delta}_{v_1v_2\dots v_k}\right|\le\frac{N_1N_2\cdots N_k}{(k+1)!}.$$

Moreover, we have $S_1\supset S_2\supset\cdots\supset S_k\supset S_{k+1}\supset\cdots$. Then

$$\lambda\left(C[\bar{O}_1,\{V_k\}]\right)=\lim_{k\to\infty}\lambda\left(S_k\right)\le\varliminf_{k\to\infty}\frac{N_1N_2\cdots N_k}{(k+1)!}=0,$$

since Lebesgue measure is continuous. This completes the proof of the theorem. □

3. Random variables with independent differences of consecutive elements of the Ostrogradskiĭ type 1 series

Consider the following random variable:

$$\xi=\bar{O}_1(0;\eta_1,\eta_2,\dots,\eta_k,\dots)\tag{4}$$

whose $\bar{O}_1$-elements η_k are independent random variables taking values $1,2,\dots,m,\dots$ with probabilities p_{1k}, p_{2k}, …, p_{mk}, …, respectively, that is,

$$\mathsf{P}\left\{\eta_k=m\right\}=p_{mk},\qquad p_{mk}\ge 0,\ \sum_{m=1}^{\infty}p_{mk}=1\quad\text{for all } k\in\mathbf{N}.$$

Since the random variable ξ is the sum of infinitely many terms, it can only take irrational values.

It is clear that the distribution of the random variable ξ is completely determined by the numbers p_{mk}.

The following assertion can readily be proved.

Lemma 6. *The distribution function F_ξ of the random variable ξ is as follows:*

$$F_\xi(x)=1-\sum_{j=1}^{g_1(x)-1}p_{j1}+\sum_{k\ge 2}(-1)^{k-1}\left(1-\sum_{j=1}^{g_k(x)-1}p_{jk}\right)\prod_{i=1}^{k-1}p_{g_i(x)i}\quad \textit{for } 0<x\le 1\tag{5}$$

where $g_k(x)$ is the kth $\bar{O}_1$-element of the number x. The sum is finite or infinite depending on whether or not the number x is rational.

Proof. The distribution function of a random variable is defined by

$$F_\xi(x)=\mathsf{P}\left\{\xi<x\right\}.$$

First, suppose the number

$$x=\bar{O}_1(0;g_1(x),g_2(x),\dots,g_k(x),\dots)$$

is irrational. Since the event $\{\xi<x\}$ is represented as the union of disjoint events:

$$\begin{aligned}\{\xi<x\}=&\{\eta_1>g_1(x)\}\cup\{\eta_1=g_1(x),\eta_2<g_2(x)\}\cup\cdots\\&\cup\{\eta_1=g_1(x),\dots,\eta_{2k-2}=g_{2k-2}(x),\eta_{2k-1}>g_{2k-1}(x)\}\\&\cup\{\eta_1=g_1(x),\dots,\eta_{2k-1}=g_{2k-1}(x),\eta_{2k}<g_{2k}(x)\}\cup\cdots,\end{aligned}$$

we have

$$F_\xi(x) = 1 - \sum_{j=1}^{g_1(x)} p_{j1} + \sum_{j=1}^{g_2(x)-1} p_{j2} \cdot p_{g_1(x)1} + \cdots$$
$$+ \left(1 - \sum_{j=1}^{g_{2k-1}(x)} p_{j,2k-1}\right) \prod_{i=1}^{2k-2} p_{g_i(x)i} + \sum_{j=1}^{g_{2k}(x)-1} p_{j,2k} \prod_{i=1}^{2k-1} p_{g_i(x)i} + \cdots .$$

The latter expression is easily reduced to the form (5).

If the number x is rational and such that $x = \bar{O}_1(0; g_1(x), g_2(x), \ldots, g_{2k}(x))$, then the event $\{\xi < x\}$ is the union of the first $2k$ events identical to those in the previous case. Therefore $F_\xi(x)$ is now represented in the form (5) and the sum contains $2k$ terms.

If $x = \bar{O}_1(0; g_1(x), g_2(x), \ldots, g_{2k-1}(x))$, then

$$\begin{aligned} \mathsf{P}\{\xi < x\} &= \mathsf{P}\{\eta_1 > g_1(x)\} + \mathsf{P}\{\eta_1 = g_1(x), \eta_2 < g_2(x)\} + \cdots \\ &\quad + \mathsf{P}\{\eta_1 = g_1(x), \ldots, \eta_{2k-2} = g_{2k-2}(x), \eta_{2k-1} \geq g_{2k-1}(x)\} \\ &= 1 - \sum_{j=1}^{g_1(x)} p_{j1} + \sum_{j=1}^{g_2(x)-1} p_{j2} \cdot p_{g_1(x)1} + \cdots \\ &\quad + \left(1 - \sum_{j=1}^{g_{2k-1}(x)-1} p_{j,2k-1}\right) \prod_{i=1}^{2k-2} p_{g_i(x)i}. \end{aligned}$$

The latter expression is reduced to (5) containing $2k-1$ terms.

It remains to show that the values taken by the function F_ξ for different representations of a rational number are the same. Indeed,

$$\begin{aligned} &F_\xi\left(\bar{O}_1(0; g_1(x), g_2(x), \ldots, g_k(x), 1)\right) \\ &\quad = 1 - \sum_{j=1}^{g_1(x)-1} p_{j1} - \cdots + (-1)^{k-1}\left(1 - \sum_{j=1}^{g_k(x)-1} p_{jk}\right) \prod_{i=1}^{k-1} p_{g_i(x)i} + (-1)^k \prod_{i=1}^{k} p_{g_i(x)i} \\ &\quad = 1 - \sum_{j=1}^{g_1(x)-1} p_{j1} - \cdots + (-1)^{k-1}\left(1 - \sum_{j=1}^{g_k(x)} p_{jk}\right) \prod_{i=1}^{k-1} p_{g_i(x)i} \\ &\quad = F_\xi\left(\bar{O}_1(0; g_1(x), g_2(x), \ldots, g_k(x)+1)\right). \qquad \square \end{aligned}$$

Remark. The proof of Lemma 6 makes it clear that if all columns of the matrix $\|p_{mk}\|$ are the same (that is, η_k are identically distributed) and positive, equality (5) is the $\tilde{Q}_\infty$-representation of the number $F_\xi(x)$ if $\tilde{Q}_\infty = \{p_{11}, p_{21}, \ldots, p_{m1}, \ldots\}$, and $g_k(x)$ are the $\tilde{Q}_\infty$-elements of the number $F_\xi(x)$. In other words, if

$$x = \bar{O}_1(g_1(x), g_2(x), \ldots, g_k(x), \ldots),$$

then

$$F_\xi(x) = \Delta_{g_1(x)g_2(x)\ldots g_k(x)\ldots}$$

(the latter expression is a symbolic form of the $\tilde{Q}_\infty$-representation of the number $F_\xi(x)$; see [11] and [12]).

Lemma 7. *If the derivative of the function F_ξ exists at a point x such that*

$$x = \bar{O}_1(0; g_1(x), g_2(x), \ldots, g_k(x), \ldots),$$

then

$$F'_\xi(x) = \lim_{n\to\infty} B_n\big(g_1(x) + g_2(x) + \cdots + g_n(x) + 1\big) \prod_{i=1}^{n} p_{g_i(x)i}$$

where B_n is the denominator of the approximant number of order n.

Proof. Indeed, if the derivative exists at the point x, then

$$F'_\xi(x) = \lim_{\substack{x'<x<x'' \\ x''-x'\to 0}} \frac{F_\xi(x'')-F_\xi(x')}{x''-x'} = \lim_{n\to\infty} \frac{P\left\{\xi \in \bar\Delta_{g_1(x)g_2(x)\dots g_n(x)}\right\}}{\left|\bar\Delta_{g_1(x)g_2(x)\dots g_n(x)}\right|}$$

$$= \lim_{n\to\infty} B_n\big(g_1(x)+g_2(x)+\cdots+g_n(x)+1\big)\prod_{i=1}^{n} p_{g_i(x)i}. \qquad \square$$

Theorem 3. *The random variable ξ has a discrete distribution if and only if*

$$\prod_{k=1}^{\infty} \max_m \{p_{mk}\} > 0. \tag{6}$$

Let $p_{g'_k k} = \max_m \{p_{mk}\}$ for all $k \in \mathbf{N}$. If the random variable is discrete, the atoms of the distribution of ξ are those and only those $x \in [0;1]$ that differ from

$$x_0 = \bar O_1\big(0; g'_1, g'_2, \dots, g'_k, \dots\big)$$

only in a finite number of $\bar O_1$-elements $g_k(x)$ satisfying $p_{g_k(x)k} > 0$.

Proof. Note that x is an atom of the distribution of ξ if

$$\prod_{k=1}^{\infty} p_{g_k(x)k} > 0.$$

Necessity. Let the random variable ξ have a discrete distribution and let x be an atom of the distribution of ξ. Assume that the infinite product in (6) diverges to zero. Then

$$\mathsf{P}\{\xi = x\} = \prod_{k=1}^{\infty} p_{g_k(x)k} \le \prod_{k=1}^{\infty} \max_m \{p_{mk}\} = 0$$

contradicting the assumption that x is an atom of the distribution. Therefore this assumption is false proving the necessity.

Sufficiency. Suppose that (6) holds. Then it is clear that x_0 and all x that differ from x_0 by a finite number of $\bar O_1$-elements $g_k(x)$ satisfying $p_{g_k(x)k} > 0$ are atoms of the distribution of ξ. Let us show that this distribution is discrete.

Assume that $x_j^{(m)} = \bar O_1(0; g_1, g_2, \dots, g_m, g'_{m+1}, \dots, g'_k, \dots)$ is an arbitrary atom among those atoms whose $\bar O_1$-elements coincide with the $\bar O_1$-elements of x_0 beginning with the $(m+1)$th element. Then

$$\mathsf{P}\left\{\xi = x_j^{(m)}\right\} = p_{g_1 1} p_{g_2 2} \cdots p_{g_m m} \prod_{k=m+1}^{\infty} p_{g'_k(x)k},$$

$$\mathsf{P}\left\{\xi \in \left\{x_j^{(m)}\right\}\right\} = \sum_{\substack{g_1 :\, p_{g_1 1}>0 \\ \cdots \\ g_m :\, p_{g_m m}>0}} \left(p_{g_1 1} p_{g_2 2} \dots p_{g_m m} \prod_{k=m+1}^{\infty} p_{g'_k(x)k}\right)$$

$$= \sum_{g_1 :\, p_{g_1 1}>0} p_{g_1 1} \sum_{g_2 :\, p_{g_2 2}>0} p_{g_2 2} \cdots \sum_{g_m :\, p_{g_m m}>0} p_{g_m m} \prod_{k=m+1}^{\infty} p_{g'_k(x)k}$$

$$= \prod_{k=m+1}^{\infty} p_{g'_k(x)k},$$

since all these sums are equal to 1.

The set $D = \bigcup_{m=1}^{\infty} \{x_j^{(m)}\}$ is at most countable, since it is a countable union of at most countable sets.

Let us calculate $\mathsf{P}\{\xi \in D\}$. Since $\{x_j^{(m)}\} \subseteq \{x_j^{(m+1)}\}$, we have

$$\mathsf{P}\{\xi \in D\} = \lim_{m\to\infty} \mathsf{P}\left\{\xi \in \left\{x_j^{(m)}\right\}\right\} = \lim_{m\to\infty} \prod_{k=m+1}^{\infty} p_{g'_k k} = 1$$

by continuity of the probability. The latter limit equals 1 by properties of convergent infinite products.

Therefore $\mathsf{P}\{\xi \in D\} = 1$, that is, the random variable ξ is concentrated on a set which is at most countable. By definition, this means that the distribution of ξ is discrete. The theorem is proved. □

The following result follows from Theorem 3.

Corollary 1. *The distribution of the random variable ξ is continuous if and only if the infinite product in* (6) *equals* 0.

Theorem 4. *The distribution of the random variable ξ is pure, that is, the distribution of ξ is either purely discrete, or purely singular, or purely absolutely continuous.*

Proof. By Theorem 3, it is sufficient to prove that if the distribution of ξ is continuous, then it cannot be a mixture of a singular distribution and an absolutely continuous distribution, that is, the distribution of ξ is either purely singular or purely absolutely continuous.

Let $x = \bar{O}_1(0; g_1(x), g_2(x), \dots, g_n(x), \dots)$ and let $t_1, \dots, t_n$ be a fixed set of positive integers. Put

$$\bar{\Delta}_{t_1\dots t_n}(x) = \bar{O}_1(0; t_1, \dots, t_n, g_{n+1}(x), g_{n+2}(x), \dots)$$

and

$$\bar{\Delta}_{t_1\dots t_n}(E) = \left\{u \colon u = \bar{\Delta}_{t_1\dots t_n}(x), x \in E\right\},$$

$$T_n(E) = \bigcup_{t_1,\dots,t_n} \bar{\Delta}_{t_1\dots t_n}(E), \qquad T(E) = \bigcup_n T_n(E)$$

for any set E of the interval $[0;1]$.

Consider the event $A = \{\xi \in T(E)\}$. Since η_k are independent, the event A, being generated by the sequence of random variables η_k, is independent of all σ-algebras $\mathcal{B}_m$ generated by $\eta_1, \dots, \eta_m$. Therefore A is a tail event. Then the Kolmogorov 0-1 law gives that either $\mathsf{P}(A) = 0$ or $\mathsf{P}(A) = 1$.

Since $T(E) \supset E$, the inequality $\mathsf{P}\{\xi \in E\} > 0$ implies

$$\mathsf{P}\{\xi \in T(E)\} \geq \mathsf{P}\{\xi \in E\} > 0$$

and hence $\mathsf{P}\{\xi \in T(E)\} = 1$.

Only one of the following two cases may occur:

1. There exists a set E such that $\lambda(E) = 0$ and $\mathsf{P}\{\xi \in E\} > 0$.
2. For an arbitrary set E satisfying $\lambda(E) = 0$, we have $\mathsf{P}\{\xi \in E\} = 0$.

In the first case, the equality $\lambda(E) = 0$ implies that $\lambda(T(E)) = 0$, meaning that there exists a set $T(E)$ satisfying $\lambda(T(E)) = 0$ and $\mathsf{P}\{\xi \in T(E)\} = 1$. Therefore the distribution of ξ is purely singular by definition.

In the second case, the distribution function of ξ has the N-property, which is equivalent to its absolute continuity [15]. □

Recall [11, p. 66] that the *spectrum* S_ξ of the distribution of a random variable ξ (of the distribution function F_ξ) is defined as the set of growth points of F_ξ, that is,

$$S_\xi = \left\{x \colon F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) > 0 \text{ for all } \varepsilon > 0\right\}.$$

Lemma 8. *The spectrum S_ξ of the distribution of the random variable ξ is a subset of the set $C[\bar{O}_1, \{V_k\}]$ where $V_k = \{v \colon p_{vk} > 0\}$. The spectrum differs from the set $C[\bar{O}_1, \{V_k\}]$ by a set which is at most countable.*

Proof. It is sufficient to justify the following two inclusions in order to prove the theorem:

1. $S_\xi \subseteq C[\bar{O}_1, \{V_k\}]$.
2. Irrational points of $C[\bar{O}_1, \{V_k\}]$ belong to S_ξ.

1. Let $x = \bar{O}_1(0; g_1(x), \dots, g_k(x), \dots)$ be an irrational number belonging to the spectrum S_ξ. Assume that $x \notin C[\bar{O}_1, \{V_k\}]$, that is, there exists a number k_0 such that $p_{g_{k_0}(x)k_0} = 0$. Then there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset \bar{\Delta}_{g_1(x)g_2(x)\dots g_{k_0}(x)}$ and therefore

$$\mathsf{P}\left\{\xi \in (x-\varepsilon, x+\varepsilon)\right\} \le \mathsf{P}\left\{\xi \in \bar{\Delta}_{g_1(x)g_2(x)\dots g_{k_0}(x)}\right\} = \prod_{k=1}^{k_0} p_{g_k(x)k} = 0$$

contradicting the fact that $x \in S_\xi$. Hence our assumption is false and $x \in C[\bar{O}_1, \{V_k\}]$.

Let x be a rational number belonging to the spectrum S_ξ. The random variable ξ cannot take rational values and therefore $\mathsf{P}\{\xi = x\} = 0$. Then for any $\varepsilon > 0$ the interval $(x-\varepsilon, x+\varepsilon)$ contains at least one irrational number $y \in S_\xi$ (since otherwise we would have

$$\mathsf{P}\left\{\xi \in (x-\varepsilon, x+\varepsilon)\right\} = 0$$

contradicting the inclusion $x \in S_\xi$). By the part of the lemma proved above, we have $y \in C[\bar{O}_1, \{V_k\}]$. Therefore for any $\varepsilon > 0$ the interval $(x-\varepsilon, x+\varepsilon)$ contains at least one number $y \in C[\bar{O}_1, \{V_k\}]$, that is, x is a limit point of $C[\bar{O}_1, \{V_k\}]$. Since the set $C[\bar{O}_1, \{V_k\}]$ is closed, we have $x \in C[\bar{O}_1, \{V_k\}]$ proving the first inclusion.

2. Let x be an irrational number belonging to $C[\bar{O}_1, \{V_k\}]$. For any fixed $\varepsilon > 0$ one can find a positive integer k_0 such that $\bar{\nabla}_{g_1(x)g_2(x)\dots g_{k_0}(x)} \subset (x-\varepsilon, x+\varepsilon)$. Then

$$\mathsf{P}\left\{\xi \in (x-\varepsilon, x+\varepsilon)\right\} \ge \mathsf{P}\left\{\xi \in \bar{\nabla}_{g_1(x)g_2(x)\dots g_{k_0}(x)}\right\} = \prod_{k=1}^{k_0} p_{g_k(x)k} > 0,$$

whence $x \in S_\xi$. The lemma is proved. □

Lemma 9. *The probability that the random variable ξ belongs to the set*

$$C\left[\bar{O}_1, \{V_k'\}\right], \quad \textit{where } V_k' \subseteq V_k = \{v \colon p_{vk} > 0\},$$

is given by

$$\mathsf{P}\left\{\xi \in C\left[\bar{O}_1, \{V_k'\}\right]\right\} = \prod_{k=1}^{\infty} L_k = \prod_{k=1}^{\infty} (1 - \mathit{\Gamma}_k)$$

where $L_k = \sum_{j \in V_k'} p_{jk}$ and $\mathit{\Gamma}_k = \sum_{j \in V_k \setminus V_k'} p_{jk}$.

Proof. By Lemma 5, the set $C\left[\bar{O}_1, \{V_k'\}\right]$ can be represented as follows:

$$C\left[\bar{O}_1, \{V_k'\}\right] = \bigcap_{k=1}^{\infty} \Bigg(\bigcup_{v_1 \in V_1'} \cdots \bigcup_{v_k \in V_k'} \bar{\Delta}_{v_1 \dots v_k} \Bigg).$$

Since

$$\bigcup_{v_1 \in V_1'} \cdots \bigcup_{v_k \in V_k'} \bigcup_{v_{k+1} \in V_{k+1}'} \bar{\Delta}_{v_1 \dots v_k v_{k+1}} \subseteq \bigcup_{v_1 \in V_1'} \cdots \bigcup_{v_k \in V_k'} \bar{\Delta}_{v_1 \dots v_k}$$

and

$$\mathsf{P}\Big\{\xi \in \bigcup_{v_1\in V_1'} \bigcup_{v_2\in V_2'} \cdots \bigcup_{v_k\in V_k'} \bar{\Delta}_{v_1 v_2 \dots v_k}\Big\} = \sum_{v_1\in V_1'} \sum_{v_2\in V_2'} \cdots \sum_{v_k\in V_k'} p_{v_1 1} p_{v_2 2} \dots p_{v_k k}$$
$$= \sum_{v_1\in V_1'} p_{v_1 1} \sum_{v_2\in V_2'} p_{v_2 2} \cdots \sum_{v_k\in V_k'} p_{v_k k}$$
$$= L_1 L_2 \cdots L_k,$$

we have

$$\mathsf{P}\left\{\xi \in C\left[\bar{O}_1, \{V_k'\}\right]\right\} = \lim_{k\to\infty} L_1 L_2 \cdots L_k = \prod_{k=1}^{\infty} L_k$$

by the continuity of the probability. □

The following result is a corollary of Lemma 8 and Theorems 2 and 3.

Theorem 5. *If the infinite product in* (6) *diverges to* 0, *the* k*th column of the matrix* $\|p_{mk}\|$ *contains* N_k *positive elements, and*

$$\varliminf_{k\to\infty} \frac{N_1 N_2 \cdots N_k}{(k+1)!} = 0,$$

then the distribution of the random variable ξ *is of the Cantor type (that is, the spectrum* S_ξ *of the random variable* ξ *has Lebesgue measure* 0*).*

4. Random variables with differences of consequent elements of the Ostrogradskiĭ type 1 series forming a Markov chain

Consider the following random variable:

$$\tilde{\xi} = \bar{O}_1(0; \eta_1, \eta_2, \dots, \eta_k, \dots) \tag{7}$$

whose $\bar{O}_1$-elements η_k are random variables forming a Markov chain with initial probabilities $p_1, p_2, \dots, p_m, \dots$ and transition matrix $\|p_{ij}\|$, that is,

$$\mathsf{P}\{\eta_1 = m\} = p_m, \qquad p_m \ge 0, \qquad \sum_{m=1}^{\infty} p_m = 1$$

and

$$\mathsf{P}\{\eta_{k+1} = j \mid \eta_k = i\} = p_{ij}, \quad p_{ij} \ge 0, \qquad \sum_{j=1}^{\infty} p_{ij} = 1 \quad \text{for all } i \in \mathbf{N}.$$

The proof of the next result is similar to that of Lemma 6.

Lemma 10. *The distribution function* $F_{\tilde{\xi}}$ *of the random variable* $\tilde{\xi}$ *has the following form:*

$$\begin{aligned} F_{\tilde{\xi}}(x) = 1 - &\sum_{j=1}^{g_1(x)-1} p_j \\ &+ \sum_{k\ge 2} (-1)^{k-1} \Big(1 - \sum_{j=1}^{g_k(x)-1} p_{g_{k-1}(x) j}\Big) p_{g_1(x)} \prod_{i=1}^{k-2} p_{g_i(x) g_{i+1}(x)} \\ &\text{for } 0 < x \le 1 \end{aligned} \tag{8}$$

where $g_k(x)$ *is the* k*th* $\bar{O}_1$*-element of the number* x*. The number of terms in the sum is finite or infinite depending on whether or not the number* x *is rational.*

Proof. Let us prove the statement for an irrational number x. Since the event

$$\left\{\tilde{\xi} < x\right\}$$

can be represented as a union of disjoint events (see the proof of Lemma 6) and since

$$\begin{aligned}
&\mathsf{P}\left\{\eta_1 = g_1(x), \dots, \eta_{2k-2} = g_{2k-2}(x), \eta_{2k-1} > g_{2k-1}(x)\right\}\\
&\quad = \mathsf{P}\left\{\eta_1 = g_1(x)\right\} \cdots \mathsf{P}\left\{\eta_{2k-1} > g_{2k-1}(x) \mid \eta_{2k-2} = g_{2k-2}(x)\right\}\\
&\quad = p_{g_1(x)} \prod_{i=1}^{2k-3} p_{g_i(x)g_{i+1}(x)} \left(1 - \sum_{j=1}^{g_{2k-1}(x)} p_{g_{2k-2}(x)j}\right),\\
&\mathsf{P}\left\{\eta_1 = g_1(x), \dots, \eta_{2k-1} = g_{2k-1}(x), \eta_{2k} < g_{2k}(x)\right\}\\
&\quad = \mathsf{P}\left\{\eta_1 = g_1(x)\right\} \cdots \mathsf{P}\left\{\eta_{2k} < g_{2k}(x) \mid \eta_{2k-1} = g_{2k-1}(x)\right\}\\
&\quad = p_{g_1(x)} \prod_{i=1}^{2k-2} p_{g_i(x)g_{i+1}(x)} \sum_{j=1}^{g_{2k}(x)-1} p_{g_{2k-1}(x)j},
\end{aligned}$$

we have

$$\begin{aligned}
F_{\tilde{\xi}}(x) = 1 - \sum_{j=1}^{g_1(x)} p_j + \sum_{j=1}^{g_2(x)-1} p_{g_1(x)j} \cdot p_{g_1(x)} + \cdots\\
+ \left(1 - \sum_{j=1}^{g_{2k-1}(x)} p_{g_{2k-2}(x)j}\right) p_{g_1(x)} \prod_{i=1}^{2k-3} p_{g_i(x)g_{i+1}(x)}\\
+ \sum_{j=1}^{g_{2k}(x)-1} p_{g_{2k-1}(x)j} \cdot p_{g_1(x)} \prod_{i=1}^{2k-2} p_{g_i(x)g_{i+1}(x)} + \cdots .
\end{aligned}$$

The latter expression is easily reduced to (8). □

Theorem 6. *The distribution of the random variable $\tilde{\xi}$ has atoms if and only if there exists a set $(g_1, g_2, \dots, g_k, \dots)$ such that*

$$p_{g_1} \prod_{k=1}^{\infty} p_{g_k g_{k+1}} > 0.$$

Proof. Indeed, the number $x = \bar{O}_1(0; g_1, g_2, \dots, g_k, \dots)$ is an atom of the distribution of $\tilde{\xi}$ if and only if

$$\begin{aligned}
\mathsf{P}\left\{\tilde{\xi} = x\right\} &= \mathsf{P}\left\{\eta_1 = g_1, \eta_2 = g_2, \dots, \eta_k = g_k, \dots\right\}\\
&= p_{g_1} \prod_{k=1}^{\infty} p_{g_k g_{k+1}} > 0. \qquad \square
\end{aligned}$$

We conclude by proving that a uniformly distributed random variable cannot be of type (4) or (7).

Theorem 7. *If the distribution of a random variable τ is uniform on $[0; 1]$, then the $\bar{O}_1$-elements $g_k(\tau)$ of this random variable are dependent. Moreover they cannot form a Markov chain.*

Proof. Let us first prove by contradiction that $g_k(\tau)$ cannot be independent.

Since τ is uniformly distributed on $[0; 1]$, any Borel set B, in particular, any interval

$[a;b]$, satisfies

$$\mathsf{P}\left\{\tau \in B\right\} = \lambda(B)$$

where λ denotes Lebesgue measure.

Assume (for a contradiction) that $g_k(\tau)$ are independent random variables taking values 1, 2, ..., m, ..., with probabilities p_{1k}, p_{2k}, ..., p_{mk}, ..., respectively. Then

$$\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_k}\right\} = \prod_{i=1}^{k} p_{g_i i}$$

and

$$\frac{\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_{k-1} g}\right\}}{\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_{k-1}}\right\}} = p_{gk} \quad \text{for all } k \text{ and } g_1, \ldots, g_{k-1}, g.$$

The expression for the length of the interval $\bar{\Delta}_{g_1 g_2 \ldots g_k}$ gives

$$\frac{\lambda\left(\bar{\Delta}_{g_1 g_2 \ldots g_{k-1} g}\right)}{\lambda\left(\bar{\Delta}_{g_1 g_2 \ldots g_{k-1}}\right)} = \frac{g_1 + \cdots + g_{k-1} + 1}{(g_1 + \cdots + g_{k-1} + g)(g_1 + \cdots + g_{k-1} + g + 1)}.$$

Therefore the equality

$$\frac{g_1 + \cdots + g_{k-1} + 1}{(g_1 + \cdots + g_{k-1} + g)(g_1 + \cdots + g_{k-1} + g + 1)} = p_{gk}$$

should hold for all k and $g_1, \ldots, g_{k-1}$, g. However,

$$\frac{\lambda\left(\bar{\Delta}_{1g}\right)}{\lambda\left(\bar{\Delta}_{1}\right)} \neq \frac{\lambda\left(\bar{\Delta}_{2g}\right)}{\lambda\left(\bar{\Delta}_{2}\right)},$$

showing that our assumption is false and the random variables $g_k(\tau)$ cannot be independent.

Assume now that the random variables $g_k(\tau)$ form a homogeneous Markov chain with initial probabilities $p_1, p_2, \ldots, p_m, \ldots$ and transition matrix $\|p_{ij}\|$. Then

$$\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_k}\right\} = p_{g_1} \prod_{i=1}^{k-1} p_{g_i g_{i+1}}$$

and

$$\frac{\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_{k-1} g}\right\}}{\mathsf{P}\left\{\tau \in \bar{\Delta}_{g_1 g_2 \ldots g_{k-1}}\right\}} = p_{g_{k-1} g} \quad \text{for all } k \text{ and } g_1, \ldots, g_{k-1}, g.$$

Therefore the equality

$$\frac{g_1 + \cdots + g_{k-1} + 1}{(g_1 + \cdots + g_{k-1} + g)(g_1 + \cdots + g_{k-1} + g + 1)} = p_{g_{k-1} g}$$

should hold for all k and $g_1, \ldots, g_{k-1}$, g leading us, as before, to the conclusion that the random variables $g_k(\tau)$ cannot form a homogeneous Markov chain. The theorem is proved. □

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