PROPERTIES OF FINITE RANK OPERATORS THAT ARISE IN APPROXIMATION OF INTEGRAL OPERATORS RELATED TO LINEAR DYNAMICAL SYSTEMS

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Abstract. In this paper we present an integral operator that arises in the context of linear dynamical systems, which describes the time evolution of the state probability density function. We propose a finite rank approximation to this integral operator and show that this finite rank operator converges in norm to the integral operator. We discuss Markov chains arising from this finite rank approximation, and show that the eigenvalues of the transition matrices of these Markov chains converge to the eigenvalues of the integral operator as the number of divisions in the state-discretization is increased.

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1. Introduction. This work is part of a program of study related to the approximation of discrete-state systems by linear continuous-state systems. The motivation for this approximation is the computational complexity of discrete-state algorithms when the number of states becomes large (e.g. the Viterbi algorithm), as opposed to the simplicity of the continuous state counterparts (e.g. the Kalman filter). We are interested in having a characterisation of the class of discrete-state systems which are amenable to such an approximation, and as part of this characterisation we have began by studying the inverse process. Specifically, we have investigated the properties of discrete-state models that arise in the state-discretization of continuous-state linear systems. In particular, this paper examines a finite rank approximation to transition operators for continuous-state systems. For the continuous-state systems studied here, the transition operator is given by an integral operator. As a result of the finite rank approximation, a state-discretization is performed, yielding a discrete-state Markov chain. For the latter, the transition operator is given by a transition matrix. The limiting properties of the eigenvalues and eigenvectors of these transition matrices are shown to be related to the ones of the continuous-state transition operator.

In order to simplify the notation and to facilitate the presentation of the main ideas, we present here the case of first order continuous state systems. These results can be extended to the general case in a straightforward manner [1], although the notation in that case is far more complicated. Related work can be found in [3], [5] and [8] in a more general setting. However, since our work, as reported here, is restricted to linear continuous-state systems, we are able to provide more specific information about the structure of the discrete-state approximation.

The paper is organised as follows. In section 2 we present an integral operator that describes the time evolution of the state probability density function for continuous-state linear systems. Eigenvalues and eigenfunctions of this operator are also presented. In section 3 we propose a finite rank approximation to this integral operator. The proposed approximation consists of a partition of the state space into

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subintervals and a piecewise constant approximation of the kernel of the integral operator. We prove that this approximation converges to the integral operator in norm. In section 4 we relate the finite rank approximation to Markov chains obtained as a result of the state space discretization. In particular, we show that the eigenvalues of the transition matrix of the Markov chain and those of the finite rank operator are the same. In section 5 we show that the eigenvalues of the transition matrices have limiting points, as the state discretization becomes finer and finer, in a subset of the eigenvalues of the integral operator. We illustrate this convergence by an example and in section 6 we present the conclusions.

2. Integral operators in linear dynamical systems. We consider a first order linear autoregressive model

$$x_{t+1} = ax_t + bv_t, \tag{2.1}$$

where $t \in \mathbb{Z}^+$ represents a discrete-time index, $x_t, x_{t+1} \in \mathbb{R}$ are the states at times t and $t+1$ respectively. $v_t \in \mathbb{R}$ is an independent and identically distributed (iid) sequence of random variables with finite variance, and the initial state x_0 is assumed to have arbitrary distribution save that it has finite variance and is independent of v_t for all t. The coefficients a and b are assumed to satisfy: $0 \neq |a| < 1$ and $b \neq 0$.

We will denote the probability density function (pdf) of a random variable y, evaluated at the point of the real line y_1, by $f_y(y_1)$. We will assume that all the pdf's of interest have compact support inside the interval $[-A, A]$ of the real line, where A is arbitrarily large. This assumption is necessary in the proofs to follow. From an engineering perspective, it covers most distributions of interest. Other common distributions can be approximated arbitrarily closely by a distribution with compact support. We also assume that the pdf's belong to $\mathcal{L}^2([-A, A])$.

The pdf of x_{t+1} can be expressed in terms of those of x_t and v_t (for example by using the auxiliary variables method, see e.g. [7]) by the following integral operator:

$$f_{x_{t+1}}(x) = \int_{-\infty}^{+\infty} k(x,y) f_{x_t}(y)dy \triangleq (Tf_{x_t})(x), \tag{2.2}$$

where the kernel is given by $k(x,y) = f_v(\frac{x-ay}{b})/|b|$ (we omit the notation $(\cdot)_t$ in f_{v_t} since v_t is iid). Eq. (2.2) defines a convex operator on the convex space of pdf's. Since we are interested in spectral properties, we work with the extension of this operator to the linear space $\mathcal{L}^2([-A, A])$. Note that this extension is the linear operator T on $\mathcal{L}^2([-A, A])$ defined by (2.2).

Since v_t is iid and $|a| < 1$, there exists a *stationary pdf*, denoted f_{x_∞}, satisfying:

$$f_{x_\infty} = Tf_{x_\infty}, \tag{2.3}$$

in Eq. (2.2) (see for example Theorems 2.3 and 2.7 in [6] for a proof of the more general case where the coefficients in the autoregressive model are random).

We now concentrate on the spectral properties of the operator T which are, as shown in the following sections, directly related to the spectrum of the Markov chains resulting from the state discretization. The following lemma gives some of these properties.

Lemma 2.1. *The complex numbers λ_T and the functions v_T given by:*

$$\lambda_T = a^n, \tag{2.4}$$

$$V_T(x) = \frac{d^n f_{x_\infty}(x)}{dx^n}, \tag{2.5}$$

are eigenvalues and eigenfunctions of the operator T, where n is an arbitrary nonnegative integer.

Proof. Note first that Eq. (2.2) can be written as the convolution:

$$(Tf_{x_t})(x) = \frac{1}{|ab|} f_v\left(\frac{x}{b}\right) * f_{x_t}\left(\frac{x}{a}\right), \tag{2.6}$$

and hence, by (2.3) we have

$$f_{x_\infty}(x) = (Tf_{x_\infty})(x) = \frac{1}{|ab|} f_v\left(\frac{x}{b}\right) * f_{x_\infty}\left(\frac{x}{a}\right). \tag{2.7}$$

Then, the operator T applied to the n^{th} derivative of f_{x_∞} can be evaluated as follows:

$$\begin{aligned}
\left(T\frac{d^n f_{x_\infty}}{dx^n}\right)(x) &= \frac{1}{|ab|} f_v\left(\frac{x}{b}\right) * \left(\left.\frac{\partial^n f_{x_\infty}(y)}{\partial y^n}\right|_{y=\frac{x}{a}}\right) \\
&= \frac{1}{|ab|} f_v\left(\frac{x}{b}\right) * a^n \frac{d^n f_{x_\infty}\left(\frac{x}{a}\right)}{dx^n} \\
&= a^n \frac{d^n}{dx^n}\left(\frac{1}{|ab|} f_v\left(\frac{x}{b}\right) * f_{x_\infty}\left(\frac{x}{a}\right)\right) \\
&= a^n \frac{d^n f_{x_\infty}(x)}{dx^n},
\end{aligned} \tag{2.8}$$

where the second equality follows from the chain rule, the third equality from linearity of the convolution and the last from Eq. (2.7). □

REMARK 2.1. Since the operator T is compact ([2]), the set of eigenvalues of T is countable and $\lambda_T = 0$ is the only possible point of accumulation of the set ([4]). From this, we conjecture that the set given by (2.4) is the set of all the eigenvalues of T.

3. Finite rank approximation. The integral operator T defined by Equation (2.2) provides the time evolution of the continuous-state probability density functions. We are interested now in obtaining a finite rank approximation to this operator. Since we have assumed that all the pdf's of interest have compact support inside the interval $[-A, A]$ we will restrict our study to this bounded interval, and write the operator T as:

$$(Tf)(x) = \int_{-A}^{A} k(x,y)f(y)dy, \tag{3.1}$$

for $f \in \mathcal{L}^2([-A, A])$.

To discretize the region $[-A, A]$ of the state space we use the idea of *refinements* ([5], [8]), i.e. we divide $[-A, A]$ into N subintervals of length $2A/N$ denoted

$e_1, e_2, \cdots, e_N$. We also use $k_{i,j}$ to denote the scaled integral of $k(x,y)$ over the cell $e_i \times e_j$:

$$k_{i,j} = \left(\frac{N}{2A}\right)^2 \int_{e_i}\int_{e_j} k(x,y)dydx, \tag{3.2}$$

and define a *piecewise-constant* function on $[-A, A] \times [-A, A]$ by

$$n_N^k(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{N} k_{i,j}\mathcal{X}_{e_i}(x)\mathcal{X}_{e_j}(y), \tag{3.3}$$

namely, a scaled piecewise-constant approximation of the kernel, averaged over a finite element. In Eq. (3.3), $\mathcal{X}_{e_i}(\cdot)$ is the indicator function:

$$\mathcal{X}_{e_i}(x) = \begin{cases} 1 & \text{if } x \in e_i \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

and analogously for $\mathcal{X}_{e_j}(\cdot)$.

We now define the *finite rank approximation* T_N of the operator T as the integral operator with kernel $n_N^k(\cdot,\cdot)$. Direct computation from (3.2) and (3.3) shows that the image of the operator T_N, applied to $f \in \mathcal{L}^2([-A, A])$, is constant over each cell e_i taking the value:

$$\begin{aligned}(T_N f)(x) &\triangleq \int_{-A}^{A} n_N^k(x,y)f(y)dy \\ &= \frac{N}{2A}\sum_{j=1}^{N}\left[\left(\int_{e_i}\int_{e_j} k(x,y)dydx\right)\left(\frac{N}{2A}\int_{e_j} f(z)dz\right)\right] \quad ; \text{ for } x \in e_i. \end{aligned} \tag{3.5}$$

Lemma 3.1. *T_N converges to T in the natural norm of operators on $\mathcal{L}^2([-A, A])$.*

Proof. From the Hölder inequality it can be shown from (3.1) and (3.5) that:

$$||T - T_N|| \leq \left(\int_{-A}^{A}\int_{-A}^{A} |k(x,y) - n_N^k(x,y)|^2 dydx\right)^{\frac{1}{2}}. \tag{3.6}$$

If the function k is continuous, it is uniformly continuous in the compact interval $[-A, A] \times [-A, A]$. It follows that the functions n_N^k converge uniformly to k. Thus, in the case of continuous k, it follows from the inequality $||\cdot||_2 \leq 2A||\cdot||_\infty$ that the right-hand side of Equation (3.6) converges to zero as the number of divisions N tends to infinity. For an arbitrary $k \in \mathcal{L}^2([-A, A] \times [-A, A])$, we can approximate k in the 2-norm by a continuous function g (i.e. $||k - g||_2 \to 0$). For any N, the corresponding discrete approximations n_N^k and n_N^g, given by (3.2) and (3.3), satisfy:

$$||n_N^k - n_N^g||_2^2 = \int_{-A}^{A}\int_{-A}^{A} \sum_{i,j} \mathcal{X}_{e_i}(x)\mathcal{X}_{e_j}(y)|k_{i,j} - g_{i,j}|^2 dydx$$

$$= \sum_{i,j}\left(\frac{2A}{N}\right)^2 \left|\left(\frac{N}{2A}\right)^2 \int_{-A}^{A}\int_{-A}^{A}(k(x,y) - g(x,y))\mathcal{X}_{e_i}(x)\mathcal{X}_{e_j}(y)dydx\right|^2 \tag{3.7}$$

$$\leq \sum_{i,j}\left(\frac{N}{2A}\right)^2 ||(k-g)\mathcal{X}_{e_i}\mathcal{X}_{e_j}||_1^2.$$

Now, writing the identity $(k-g)\mathcal{X}_{e_i}\mathcal{X}_{e_j} \equiv [(k-g)\mathcal{X}_{e_i}\mathcal{X}_{e_j}][\mathcal{X}_{e_i}\mathcal{X}_{e_j}]$, and applying the Hölder inequality gives

$$\begin{aligned} ||n_N^k - n_N^g||_2^2 &\leq \sum_{i,j} \left(\frac{N}{2A}\right)^2 ||(k-g)\mathcal{X}_{e_i}\mathcal{X}_{e_j}||_2^2 ||\mathcal{X}_{e_i}\mathcal{X}_{e_j}||_2^2 \\ &= \sum_{i,j} ||(k-g)\mathcal{X}_{e_i}\mathcal{X}_{e_j}||_2^2 = ||k-g||_2^2. \end{aligned} \tag{3.8}$$

Since we have already seen that $||n_N^g - g||_2 \to 0$ for the continuous function g, and since from (3.8) we have

$$\begin{aligned} ||n_N^k - k||_2 &\leq ||n_N^k - n_N^g||_2 + ||n_N^g - g||_2 + ||g-k||_2 \\ &\leq 2||k-g||_2 + ||n_N^g - g||_2, \end{aligned} \tag{3.9}$$

we deduce that $||n_N^k - k||_2 \to 0$. Therefore, we conclude from (3.6) that the operator T_N converges to T in norm. □

4. State-discretization: Markov chains. From the approximation of the last section we can perform a state-discretization of system (2.1), and define a Markov chain (see definitions in, for example, [5] and [7]) on the resulting discrete state space, as follows.

The probability vectors of the Markov chain are given by:

$$\begin{aligned} \mathbf{p}_t &\triangleq (P\{x_t \in e_1\}, \cdots, P\{x_t \in e_N\})^T \\ \text{where } P\{x_t \in e_j\} &= \int_{e_j} f_{x_t}(x)dx. \end{aligned} \tag{4.1}$$

The transition matrix between time t and $t+1$, $\mathbf{Q}_t = \{q_{ij}^{(t)}\}$, $1 \leq i,j \leq N$, is defined as:

$$q_{ij}^{(t)} \triangleq P\{x_{t+1} \in e_i | x_t \in e_j\} = \int_{e_i} f_{x_{t+1}}(x | x_t \in e_j)dx \tag{4.2}$$

where $f_{x_{t+1}}(\,\cdot\,|x_t \in e_j)$ is the conditional probability of the random variable x_{t+1} given that the random variable x_t belongs to the subinterval e_j (see e.g. [7] for this notation). Notice that we have made explicit the dependence of $\mathbf{Q}_t$ on t. If this is not the case, we say that the Markov chain has stationary or homogeneous transition probabilities, and denote by $\mathbf{Q}$ the corresponding transition matrix. The pdf $f_{x_{t+1}}$ in (4.2) is computed from Eq. (2.2). Also, we simplify the computation of the matrix $\mathbf{Q}_t$ by approximating the pdf's $f_{\mathbf{x}_t}$ by functions $\tilde{f}_{\mathbf{x}_t}$ that assume piecewise-constant values in each subset e_i. This approach has the advantage that the transition matrices are stationary (see the independence from t in the last expression of Eq. (4.3)). Using the definition of the matrix $\mathbf{Q}$ in Eq. (4.2), $f_{\mathbf{x}_{t+1}}$ defined by (2.2) and the properties of conditional probabilities and of the piecewise-constant approximation $\tilde{f}_{\mathbf{x}_k}$, it can be shown that the elements of the transition matrix are:

$$
\begin{aligned}
q_{ij} &= \int_{e_i} \left(\int_{-A}^{A} k(x,y) \tilde{f}_{x_t}(y|x_t \in e_j) dy \right) dx \\
&= \int_{e_i} \int_{e_j} k(x,y) \left(\frac{\tilde{f}_{x_t}(y)}{\int_{e_j} \tilde{f}_{x_t}(z) dz} \right) dy dx \\
&= \frac{N}{2A} \int_{e_i} \int_{e_j} k(x,y) dy dx,
\end{aligned}
\tag{4.3}
$$

where we have omitted the reference to t in q_{ij} since $k(\cdot,\cdot)$ is independent of t. Notice from (3.2) and (4.3) that q_{ij} is a scaled version of the values k_{ij} which, in turn, generate $n_N^k(\cdot,\cdot)$, the piecewise-constant approximation of the kernel $k(\cdot,\cdot)$.

Lemma 4.1. *The eigenvalues of the operator T_N and those of the matrix $\mathbf{Q}$ are the same, and the eigenfunctions of T_N are given by the piecewise-constant held versions of the eigenvectors of $\mathbf{Q}$, i.e. functions of the form:*

$$
f_{\mathbf{v_Q}}(x) = v_{\mathbf{Q}i} \quad \textit{if} \quad x \in e_i, \quad i = 1, 2, \cdots, N,
$$

where $f_{\mathbf{v_Q}}$ represents an eigenfunction of T_N and $\{v_{\mathbf{Q}i}\}$; $1 \leq i \leq N$; is an eigenvector of the matrix $\mathbf{Q}$.

Proof. From (4.3) we have that the i^{th} element of the product of the matrix $\mathbf{Q}$ and a vector $\mathbf{p} = \{p_j\}$; $1 \leq j \leq N$; is given by:

$$
(\mathbf{Qp})_i = \frac{N}{2A} \sum_{j=1}^{N} \left[\left(\int_{e_i} \int_{e_j} k(x,y) dy dx \right) p_j \right]. \tag{4.4}
$$

Let $\mathbf{v_Q} = \{v_{\mathbf{Q}i}\}$; $1 \leq i \leq N$; be an eigenvector of the matrix $\mathbf{Q}$ which corresponds to the eigenvalue $\lambda_\mathbf{Q}$, i.e. $\mathbf{Q}$, $\mathbf{v_Q}$ and $\lambda_\mathbf{Q}$ satisfy: $\mathbf{Qv_Q} = \lambda_\mathbf{Q} \mathbf{v_Q}$. Now, consider a function $f_{\mathbf{v_Q}}$ constructed by holding each value of $\mathbf{v_Q}$ constant over the corresponding subset (i.e. $f_{\mathbf{v_Q}} = \sum_i v_{\mathbf{Q}i} \mathcal{X}_{e_i}$, with $\mathcal{X}_{e_i}$ given by (3.4)). Then, noticing that $\frac{N}{2A} \int_{e_j} f_{\mathbf{v_Q}}(z) dz = v_{\mathbf{Q}j}$, it can be seen from (3.5) and (4.4) that $T_N f_{\mathbf{v_Q}}$ is constant over each subset e_i with value:

$$
\begin{aligned}
(T_N f_{\mathbf{v_Q}})(x) &= (\mathbf{Qv_Q})_i \; ; \text{ for } x \in e_i \\
&= \lambda_\mathbf{Q} v_{\mathbf{Q}i} \; ; \text{ for } x \in e_i \\
&= \lambda_\mathbf{Q} f_{\mathbf{v_Q}}(x) \; ; \text{ for } x \in e_i
\end{aligned}
\tag{4.5}
$$

Hence $(T_N f_{\mathbf{v_Q}})(x) = \lambda_\mathbf{Q} f_{\mathbf{v_Q}}(x)\ \forall x$, and we conclude that $\lambda_\mathbf{Q}$ is an eigenvalue of T_N whose corresponding eigenfunction is $f_{\mathbf{v_Q}}$. □

REMARK 4.1. In the next section, the functions $f_{\mathbf{v_Q}}$ are illustrated for an example. In figure 5.3 we show four eigenfunctions of T_N for different values of N, the number of subintervals in the state discretization.

5. Limiting properties of the eigenvalues of the transition matrices. We denote by λ_T the eigenvalues of the operator T and by $\lambda_{\mathbf{Q}}(N)$ the eigenvalues of the transition matrix $\mathbf{Q}$ obtained with a partition of N cells. Notice that the spectrum of $\mathbf{Q}$, $\sigma(\mathbf{Q})$, has N points. We are interested in the accumulation (or limit) points of these sets as $N \to \infty$. The following theorem provides a connection between these accumulation points and the eigenvalues of the operator T.

Theorem 5.1. *The set of non-zero limit points, as $N \to \infty$, of eigenvalues $\lambda_{\mathbf{Q}}(N)$ is a subset of the set of eigenvalues λ_T of the operator T.*

Proof. In Lemma 4.1 we proved that the eigenvalues of $\mathbf{Q}$ are the same as the eigenvalues of the operator T_N defined in (3.5). Thus, we need to establish that the set of non-zero limit points of the eigenvalues of the operator T_N is a subset of the set of eigenvalues of the operator T. For this purpose, let's suppose that $\lambda \neq 0$ is a limit point of a sequence of eigenvalues λ_N of T_N, then we have to show that λ is an eigenvalue of T. In Lemma 3.1 we proved that the operator T_N converges to T in the natural norm of operators on $\mathcal{L}^2([-A, A])$. Hence, $(\lambda_N I - T_N)$ converges to $(\lambda I - T)$ in norm. Since the set of invertible operators is open ([9] Theorem 10.12), if $(\lambda_N I - T_N)$ is not invertible (i.e. λ_N belongs to the spectrum of T_N) for all N then $(\lambda I - T)$ is not invertible (i.e. λ belongs to the spectrum of T). Since the operator T is compact ([2]), and for a compact linear operator every spectral value, with the possible exclusion of zero, is an eigenvalue ([4] Theorem 8.4-4), the theorem is proved. $\square$

EXAMPLE 5.1. In order to illustrate the convergence of eigenvalues and eigenvectors let's consider the model in (2.1) with $a = 0.5$, $b = 1$, $v_t \sim Uniform[-2, 2]$ and iid, and $x_0 \sim Uniform[-10, 10]$ and independent of v_t for all t.

Notice from (2.4) and (2.5), that the eigenvalues and eigenfunctions of the operator T defined by (3.1) are given, respectively, by:

$$\lambda_T = 0.5^n,$$
$$V_T(x) = \frac{d^n f_{x_\infty}(x)}{dx^n},$$

where f_{x_∞} is the stationary pdf which satisfies $f_{x_\infty} = T f_{x_\infty}$. In figure 5.1 the stationary pdf f_{x_∞} and its three first derivatives are shown. The eigenfunctions are ordered according to the magnitude of the corresponding eigenvalue (i.e. the first eigenfunction corresponds to the larger eigenvalue, etc.).

Since all distributions in this example have compact support equal or contained in the interval $[-10, 10]$ of the real line, the region in the state space to be discretized is chosen as the interval $[-10, 10]$. This interval is divided into N subsets of size $20/N$ and a Markov chain is defined on the discrete state space $\{e_1, e_2, \cdots, e_N\}$. The transition probability matrix $\mathbf{Q}$ is computed using Equation (4.3).

In figure 5.2 the eight larger eigenvalues of the matrices $\mathbf{Q}$ corresponding to different values of N (denoted $\lambda_{\mathbf{Q}}(N)$) are shown on a logarithmic scale. Notice that they converge to the eigenvalues 0.5^n corresponding to the continuous-state operator T:

$$\frac{log(\lambda_{\mathbf{Q}}(N))}{log(0.5)} \to \frac{log(0.5^n)}{log(0.5)} \equiv n\ ; \ \text{ for } n = 0, 1, \cdots, 7 \tag{5.1}$$

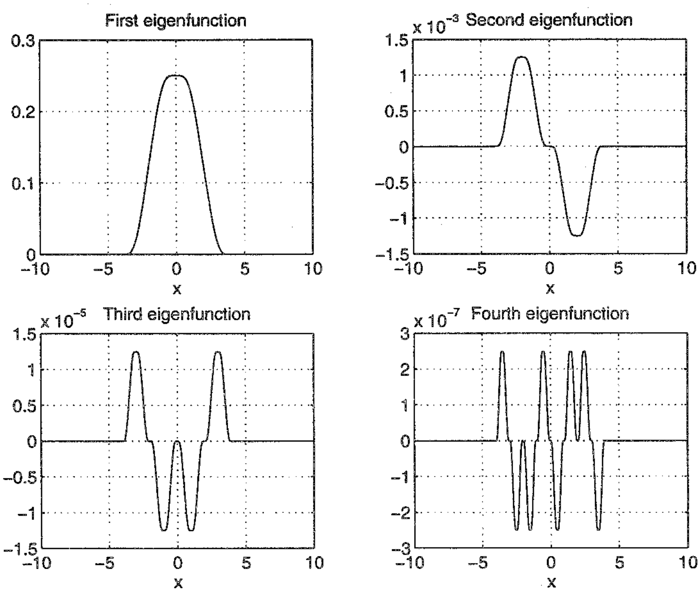

FIG. 5.1. *Eigenfunctions of the operator T corresponding to the four eigenvalues of larger magnitude (1, 0.5, 0.25 and 0.125 respectively).*

as $N \to \infty$. Notice also that the number of points N needed for convergence of the successive eigenvalues is exponentially increasing. The reason for this can be found in figure 5.1. The successive eigenfunctions have peaks, the number of which increases proportionally to the powers of two of the eigenfunction order. Therefore, in order to have a faithful representation of them with the eigenvectors of the $\mathbb{Q}$ matrices, the number of turning points of the eigenvectors increases as powers of two. In general, for a particular N, unless an eigenfunction of T_N approximates one of T, we cannot expect the corresponding eigenvalues to be close approximations.

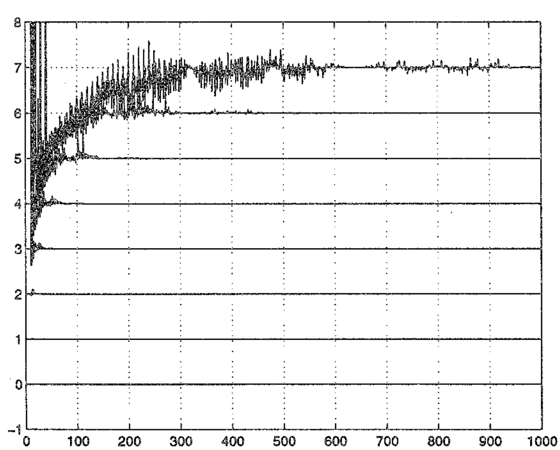

FIG. 5.2. *Eight larger eigenvalues of matrices $\mathbb{Q}$ on logarithmic scale $log(\lambda_{\mathbb{Q}}(N))/log(0.5)$ as a function of the number of subintervals N.*

In figure 5.3 the eigenfunctions corresponding to the four eigenvalues of larger modulus of the operators T_N for $N = 10, 100, 200$ and 300 are shown to illustrate their convergence to the eigenfunctions of T depicted in figure 5.1. The eigenfunctions are ordered, as in figure 5.1, according to the magnitude of the corresponding eigenvalue.

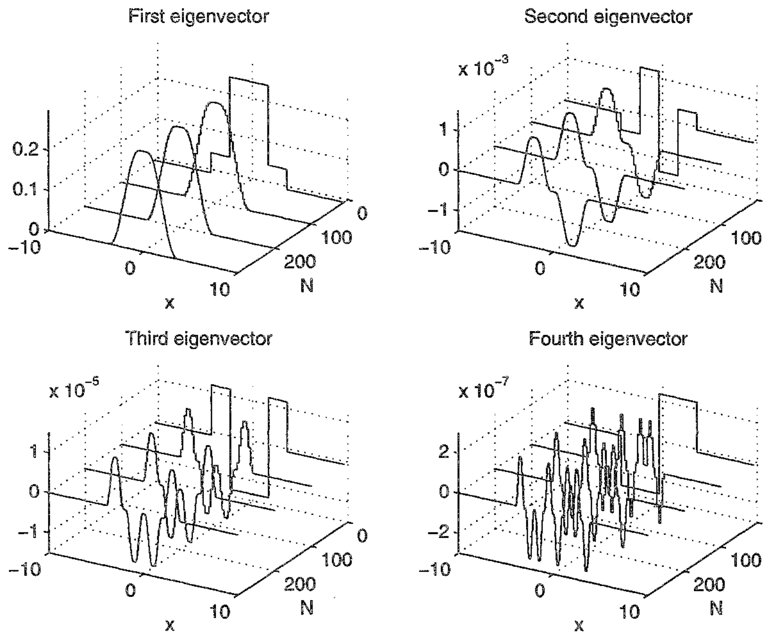

FIG. 5.3. *Eigenfunctions of the finite rank operators T_N corresponding to the four eigenvalues of larger magnitude for $N = 10$, 100, 200 and 300 subintervals.*

6. Conclusions.

An integral operator that represents the time evolution of the state probability density function for linear dynamical systems, together with its eigenvalues and eigenvectors, has been presented. A finite rank approximation to the integral operator has been discussed and it has been shown that the finite rank operators converge in norm to the integral operator. Markov chains arising from the finite rank approximation have also been presented. It has been shown that the eigenvalues of the transition matrices of the Markov chains converge to the eigenvalues of the integral operator as the number of divisions in the state-discretization is increased. This latter fact has been illustrated by an example. The results presented can be extended to the case of higher order systems and this has been reported in [1].

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