PROPERTIES OF FIXED-POINT SETS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

BY

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ABSTRACT. Let C be a closed convex subset of the Banach space X. A subset F of C is called a nonexpansive retract of C if either $F = \varnothing$ or there exists a retraction of C onto F which is a nonexpansive mapping. The main theorem of this paper is that if $T : C \to C$ is nonexpansive and satisfies a conditional fixed point property, then the fixed-point set of T is a nonexpansive retract of C. This result is used to generalize a theorem of Belluce and Kirk on the existence of a common fixed point of a finite family of commuting nonexpansive mappings.

Introduction. In this paper, X always denotes a Banach space (either real or complex) and C a nonempty, closed and convex subset of X.

Most published results on nonexpansive mappings have centered on existence theorems for fixed points of nonexpansive $T : C \to X$. In this paper we initiate the study of the structure of the fixed-point set $F(T) = \{x : Tx = x\}$. In this connection it is useful to *assume* that a conditional fixed point property is satisfied:

Either T has no fixed points, or T has a fixed

(CFP)   point in every nonempty bounded closed convex

set that T leaves invariant.

It is obvious that existence theorems serve to define classes of nonexpansive mappings which satisfy (CFP). However, (CFP) holds even in contexts where no existence theorem can be hoped for. For example (Theorem 4), if C is locally weakly compact and X is strictly convex, then *every* nonexpansive $T : C \to X$ satisfies (CFP).

Our principal structure result is Theorem 2: if C is locally weakly compact, $T : C \to C$ is nonexpansive, and T satisfies (CFP), then $F(T)$ is a nonexpansive retract of C. Therefore our approach is to study the structure of $F(T)$ by studying nonexpansive retracts of C.

Although we are not primarily interested in existence results, our main structure theorem does permit us to prove an existence result (Theorem 7) under more general hypotheses than before, by a more transparent argument.

Some of the results in this paper were announced in Bruck [4] under less general hypotheses.

1. **Nonexpansive retracts.** A mapping $T : C \to X$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Our main tool is

Lemma 1. *Suppose F is a nonempty subset of the locally weakly compact, convex set C, and let $N(F) = \{f : f{:}C \to C \text{ is nonexpansive and } fx = x \text{ for all } x \in F\}$. Then $N(F)$ is compact in the topology of weak pointwise convergence.*

Proof. Fix $x_0 \in F$ and for $x \in C$ define

$$C_x = \{y \in C : \|y - x_0\| \leq \|x - x_0\|\}.$$

For each x in C and f in $N(F)$, $f(x) \in C_x$ since $f(C) \subset C$ and

$$\|fx - x_0\| = \|fx - fx_0\| \leq \|x - x_0\|.$$

Thus $N(F)$ is a subset of the Cartesian product $P = \prod_{x \in C} C_x$.

Since C is convex and locally weakly compact, each C_x is convex and weakly compact. If C_x is given the weak topology and P is given the corresponding product topology, then by Tychonoff's theorem P is compact. (The topology on P is that of weak pointwise convergence, i.e., a net $\{f_\lambda : \lambda \in \Lambda\}$ in P converges to f in P iff for each x in C the net $\{f_\lambda(x) : \lambda \in \Lambda\}$ converges weakly to $f(x)$.)

We show that $N(F)$ is closed in P. Suppose that $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $N(F)$ which converges to f in P. Then for u in F, $f_\lambda(u) \equiv u$ so $f(u) = \text{w-}\lim_\lambda f_\lambda(u) = u$. By the weak lower semicontinuity of $\|\cdot\|$, for any x, y in C,

$$\|fx - fy\| = \left\|\text{w-}\lim_\lambda (f_\lambda x - f_\lambda y)\right\| \leq \lim_\lambda \inf \|f_\lambda x - f_\lambda y\| \leq \|x - y\|$$

since each f_λ is nonexpansive.

We have shown that $f \in N(F)$, hence that $N(F)$ is closed in P. Since P is compact, therefore $N(F)$ is compact.   Q.E.D.

As a first application of Lemma 1, we have

Lemma 2. *Suppose F is a nonempty subset of C and C is locally weakly compact. Then there exists an r in $N(F)$ such that each f in $N(F)$ acts as an isometry on the range of r:*

$$\|fr(x) - fr(y)\| = \|r(x) - r(y)\|$$

for all x, y in C.

Proof. Define an order $<$ on $N(F)$ by setting $f < g$ if $\|fx - fy\| \leq \|gx - gy\|$ for all x, y in C with inequality holding for at least one pair of x, y. As usual, $f \leq g$ means $f < g$ or $f = g$. It is readily verified that $\leq$ is a partial order on $N(F)$.

For each f in $N(F)$ we define the initial segment $\text{Is}(f)$ to be $\{g \in N(F) : g \leq f\}$. The proof in Lemma 1 that $N(F)$ is closed in P can be repeated to show that $\text{Is}(f)$ is closed in $N(F)$, and hence compact.

It follows that $(N(F), \leq)$ contains a minimal element. Indeed, by Zorn's lemma it suffices to show that whenever $\{g_\lambda : \lambda \in \Lambda\}$ is *linearly* ordered by $\leq$, then there exists g in $N(F)$ with $g \leq g_\lambda$ for all λ. But if the g_λ's are linearly ordered by $\leq$, the family $\{\mathrm{Is}(g_\lambda) : \lambda \in \Lambda\}$ is linearly ordered by inclusion. Since each $\mathrm{Is}(g_\lambda)$ is compact and nonempty, there exists $g \in \bigcap_\lambda \mathrm{Is}(g_\lambda)$, i.e., there exists g in $N(F)$ with $g \leq g_\lambda$ for each λ.

Hence there exists a minimal element r of $N(F)$. It is readily verified that $f \circ r \in N(F)$ whenever $f \in N(F)$, and certainly

$$(1) \qquad \|fr(x) - fr(y)\| \leq \|r(x) - r(y)\|$$

since f is nonexpansive. If inequality holds in (1) for any pair x, y in C then $fr < r$, while $fr \in N(F)$, contradicting the minimality of r in $N(F)$. Therefore equality holds in (1) for each x, y in C.   Q.E.D.

We shall call a subset F of C a *nonexpansive retract* of C if either $F = \varnothing$ or there exists a retraction of C onto F which is a nonexpansive mapping. (We allow the empty set to avoid trivial cases in later theorems.) Nonexpansive retracts of C are characterized as follows:

Theorem 1. *Suppose C is locally weakly compact and F is a nonempty subset of C. Suppose that for each z in C there exists h in $N(F)$ such that $h(z) \in F$. Then F is a nonexpansive retract of C.*

Proof. By Lemma 2, there exists r in $N(F)$ such that each element of $N(F)$ acts as an isometry on the range of r. Since r is already nonexpansive and fixes each point of F, to show that r is a nonexpansive retraction of C onto F it suffices to show that $r(p) \in F$ for each p in C.

Applying the hypothesis of the theorem to the point $z = r(p)$, there exists h in $N(F)$ with $h(r(p)) \in F$. Let $y = h(r(p))$. By Lemma 2,

$$(2) \qquad \|h(r(p)) - h(r(y))\| = \|r(p) - r(y)\|.$$

But since $y \in F$ we have $y = h(r(y))$, while $y = h(r(p))$ by definition. It follows from (2) that $r(p) = r(y)$. Finally, $r(y) = y$ since $y \in F$. We have shown that $r(p) = y$ is a point of F for each point of C, which completes the proof that r is a nonexpansive retraction of C onto F.   Q.E.D.

2. Fixed point sets as nonexpansive retracts. Recall that $T : C \to X$ is said to satisfy the *conditional fixed point property* (CFP) if either T has *no* fixed points, or T has a fixed point in each nonempty bounded closed convex set it leaves invariant. It is an open question whether *every* nonexpansive $T : C \to X$ satisfies (CFP) when C is locally weakly compact.

Theorem 2. *If C is locally weakly compact, $T : C \to C$ is nonexpansive, and T satisfies (CFP), then $F(T)$ is a nonexpansive retract of C.*

Remark. It is a consequence of the mean ergodic theorem that if X is reflexive and $L : X \to X$ is linear with $\|L\| \leq 1$, then there exists a projection P of X onto $\mathrm{Ker}(I - L)$ with $\|P\| \leq 1$. Theorem 2 may be regarded as a nonlinear generalization of this fact.

Proof. Since the empty set is by definition a nonexpansive retract of C, we may assume $F(T) \neq \varnothing$.

Fix z in C and define $K = \{f(z) : f \in N(F(T))\}$. Now K is the image of $N(F(T))$ under the zth coordinate projection map of P onto C_z (this projection is evaluation at z), and since $N(F(T))$ is compact (Lemma 1) and the projection continuous, K must be weakly compact, hence bounded. Clearly K is nonempty. If $f, g \in N(F(T))$ and $0 \leq \lambda \leq 1$ then $\lambda f + (1 - \lambda)g$ is in $N(F(T))$, so K is convex. Therefore K is a bounded, nonempty, closed convex subset of C. Since $T \circ f \in N(F(T))$ whenever $f \in N(F(T))$, we also have $T(K) \subset K$.

But T satisfies (CFP), has a fixed point, and leaves K invariant; therefore T has a fixed point in K. That is, there exists h in $N(F(T))$ with $h(z)$ in $F(T)$.

Since this is so for each z in C, by Theorem 1, $F(T)$ is a nonexpansive retract of C.   Q.E.D.

Corollary 1. *Suppose C is closed and convex, $T : C \to C$ is nonexpansive, T satisfies* (CFP), *and the convex closure $\overline{\mathrm{co}}\, T(C)$ of the range of T is locally weakly compact. Then $F(T)$ is a nonexpansive retract of C.*

Proof. Again, we may suppose $F(T) \neq \varnothing$. Let T_0 be the restriction of T to the set $\overline{\mathrm{co}}\, T(C)$. Then T_0 maps $\overline{\mathrm{co}}\, T(C)$ into itself, and it is readily verified that T_0 satisfies (CFP) on $\overline{\mathrm{co}}\, T(C)$. By Theorem 2 there exists a nonexpansive retraction r of $\overline{\mathrm{co}}\, T(C)$ onto $F(T)$. One nonexpansive retraction of C onto $F(T)$ is $r \circ T$.   Q.E.D.

Example 1. The condition $T(C) \subset C$ in Theorem 2 is essential. We give an example of T and X where $T : C \to X$ is nonexpansive, X and X^* are uniformly convex, and T satisfies (CFP), but $F(T)$ is *not* a nonexpansive retract of C.

Let X be R^4 normed by $\|(x_1, x_2, x_3, x_4)\| = (\sum |x_i|^4)^{1/4}$. Let e_1, e_2, e_3, and e_4 be the usual unit vectors, and let C be the linear span of the vectors $e_1 + e_2$, $e_2 + e_3$, and $e_2 + e_4$. Let T be the restriction to C of the projection $(x_1, x_2, x_3, x_4) \longmapsto (x_1, x_2, x_3, 0)$. Then T is nonexpansive and $F(T)$ is the subspace spanned by $e_1 + e_2$ and $e_2 + e_3$. However, $F(T)$ is *not* a nonexpansive retract of C.

Suppose to the contrary that there *does* exist a nonexpansive retraction f of C onto $F(T)$. Fix x in $C\backslash F(T)$ and set $z = x - f(x)$, so that $z \neq 0$. For any v in $F(T)$ we have $fx - v \in F(T)$ since $F(T)$ is a linear subspace of X, so that $f(fx - v) = fx - v$. Since f is nonexpansive we therefore have

$$\|v\| = \|fx - (fx - v)\| = \|fx - f(fx - v)\| \leq \|x - (fx - v)\| = \|z + v\|.$$

Thus

(3) $$\tfrac{1}{4}\|v\|^4 \le \tfrac{1}{4}\|z + v\|^4 \quad \text{for all } v \text{ in } F(T).$$

Since $F(T)$ is a subspace, we may take $t \ne 0$, substitute $t^{-1}v$ for v in (3), multiply both sides of (3) by t^4, to obtain

(4) $$\tfrac{1}{4}\|v\|^4 \le \tfrac{1}{4}\|tz + v\|^4$$

for all v in $F(T)$ and $t \ne 0$—and obviously for $t = 0$ as well. Thus the function $\phi(t) = \tfrac{1}{4}\|tz + v\|^4$ assumes a minimum at 0. If J denotes the gradient of $\tfrac{1}{4}\|\cdot\|^4$, this means $(Jv, z) = 0$ for all v in $F(T)$. (Here (Jv, z) denotes the value of the linear functional Jv at the point z.) Therefore z is annihilated by the linear span of $J(F(T))$.

Explicitly, $J(x_1, x_2, x_3, x_4) = (x_1^3, x_2^3, x_3^3, x_4^3)$, and it is readily calculated that $J(e_1)$, $J(e_2)$, and $J(e_3)$ are contained in the linear span of $J(F(T))$. Therefore z is annihilated by Je_i for $i = 1, 2$, and 3.

But the only vectors annihilated by Je_i for $i = 1, 2$, and 3 are of the form te_4 for some real t. Since C does not contain any vectors of this form, C does not contain z. This is a contradiction which means that $F(T)$ cannot be a nonexpansive retract of C.

Remark. We cannot *repeal* the assumption $T(C) \subset C$, but we can *relax* it. Felix Browder has communicated to the author a proof of the fact that if C is bounded and $T : C \to X$ is Lipschitzian while T maps the boundary ∂C into C, then there exists λ in $(0,1)$ such that $T_\lambda = \lambda T + (1 - \lambda)I$ maps C into C. Since $F(T_\lambda) = F(T)$ for $\lambda \ne 0$, this reduces theorems about fixed points of Lipschitzian mappings which send ∂C into C to theorems about mappings which send C into C. It is not known whether T_λ must satisfy (CFP) whenever T does.

In a strictly convex space, the fixed-point set of a nonexpansive mapping is convex. This is true in a more general sense even if X is not strictly convex:

Theorem 3. *If $T : C \to C$ is nonexpansive, T satisfies* (CFP), *and $\overline{co}\, T(C)$ is locally weakly compact, then $F(T)$ is metrically convex.*

Proof. A subset F of C is said to be metrically convex if for each pair of distinct points x_0, x_1 of F there exists a point $\bar{x}$ of F, distinct from x_0 and x_1, such that $\|x_0 - x_1\| = \|x_0 - \bar{x}\| + \|\bar{x} - x_1\|$. By Corollary 1 it suffices to show that nonexpansive retracts of C are metrically convex.

Let r be a nonexpansive retraction of C onto F, and let x_0, x_1 be two distinct points of F. For t in $[0,1]$ let $x_t = (1 - t)x_0 + tx_1$ (so $x_t \in C$). Since $r(x_0) = x_0 \ne x_1 = r(x_1)$ and $t \longmapsto r(x_t)$ is continuous, there exists t in $(0,1)$ such that $r(x_t)$ is distinct from x_0 and x_1. Let $\bar{x}$ be one such $r(x_t)$. Then $\bar{x} \in F$ is distinct from x_0 and x_1, while

(5)
$$\|x_0 - x_1\| = \|r(x_0) - r(x_1)\| \le \|r(x_0) - r(x_t)\| + \|r(x_t) - r(x_1)\|$$
$$\le \|x_0 - x_t\| + \|x_t - x_1\| = \|x_0 - x_1\|$$

since x_0, x_t, and x_1 are collinear. Equality therefore holds throughout (5). In particular,

$$\|r(x_0) - r(x_1)\| = \|r(x_0) - r(x_t)\| + \|r(x_t) - r(x_1)\|.$$

But $r(x_0) = x_0$, $r(x_1) = x_1$, and $r(x_t) = \bar{x}$, hence

$$\|x_0 - x_1\| = \|x_0 - \bar{x}\| + \|\bar{x} - x_1\|.$$

Q.E.D.

Example 2. We use an example of De Marr [5] to show that the condition $T(C) \subset C$ is also essential in Theorem 3.

Let X be R^2 with the max norm, let C be the line segment joining the two points $(1, \pm 1)$, and define $T : C \to X$ by $T(x, y) = (|y|, y)$. Then T is nonexpansive and satisfies (CFP) (since T is compact), but $F(T)$ consists of the two points $(1, \pm 1)$. Thus $F(T)$ is not metrically convex.

Now we turn to concrete conditions which guarantee that T satisfies (CFP). First we introduce and recall some definitions used in the next theorem.

A set K is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of K into K has a fixed point in K; C is said to have the hereditary fixed point property for nonexpansive mappings if every nonempty bounded closed convex subset of C has the fixed point property for nonexpansive mappings. It is an open question whether the fixed point property implies the hereditary fixed point property.

We say that X has *compact faces* if the only weakly compact convex subsets of the boundary of the unit ball are strongly compact. Obviously X has compact faces if X is strictly convex. Another sufficient condition for X to have compact faces is that X satisfy (H_L):

$$x_n \to x \text{ weakly and } \|x_n\| \to \|x\| \text{ imply } x_n \to x \text{ strongly.}$$

A point x_0 of a bounded convex set K is called a nondiametral point of K if

$$\sup_{y \in K} \|x_0 - y\| < \text{diameter } K.$$

C is said to have *normal structure* (Brodskiĭ-Mil'man [3]) if each bounded closed convex subset K of C which contains more than one point contains a nondiametral point.

Theorem 4. *Suppose $T : C \to X$ is nonexpansive. Then T satisfies* (CFP) *if any one of the following is satisfied*:
  (a) *C has the hereditary fixed point property for nonexpansive mappings*;
  (b) *C is locally weakly compact and has normal structure*;
  (c) *C is locally weakly compact and X has compact faces*;
  (d) *$(I - T)(K)$ is strongly closed whenever K is a bounded closed convex set*;
  (e) *T is compact*;
  (f) *T is affine and C is locally weakly compact.*

Proofs. We may suppose that T has a fixed point. Let K be a nonempty bounded closed convex subset of C which is invariant under T.

(a) The restriction of T to K is a nonexpansive mapping of K into K, and since C has the hereditary fixed point property, T must have a fixed point in K. Q.E.D.

(b) We may rephrase the theorem of Kirk [7] as follows: if C is locally weakly compact and has normal structure, then C has the hereditary fixed point property for nonexpansive mappings. Hence (b) follows from (a). Q.E.D.

(c) We modify a proof that Edelstein [6, Proposition 3] gave for the case $X = R^n$. Fix f_0 in $F(T)$. K is weakly compact, so the set $K_0 = \{ y \in K : \|f_0 - y\| = \mathrm{dis}(f_0, K) \}$ is nonempty. We also have $K_0 = \{ y \in K : \|f_0 - y\| \leq \mathrm{dis}(f_0, K) \}$; hence K_0 is a weakly compact, convex subset of the boundary of a ball in X. Since X is assumed to have compact faces, K_0 is compact.

However,

$$\|Ty - f_0\| = \|Ty - Tf_0\| \leq \|y - f_0\| \leq \mathrm{dis}(f_0, K)$$

and $Ty \in K$ whenever $y \in K_0$, so T maps K_0 into K_0. By Schauder's theorem, T has a fixed point in K_0. In particular, T has a fixed point in K.

If X is strictly convex (the condition used by Edelstein), we do not need Schauder's theorem. For, K_0 is a single point which must be a fixed point since K_0 is invariant under T. Q.E.D.

(d) Without loss of generality we may suppose $0 \in K$. Let $\{t_n\}$ be a sequence of real numbers drawn from $(0,1)$ such that $t_n \to 1$. Then $t_n T$ is a t_n-contraction and maps K into K since K is convex and contains 0. By the Banach contraction principle $t_n T$ has a fixed point x_n in K. Since $x_n - Tx_n = (1 - t_n^{-1})x_n$ and $t_n \to 1$ while $\{x_n\}$ is bounded, we have $x_n - Tx_n \to 0$ strongly. Thus 0 lies in the closure of $(I - T)(K)$. By hypothesis this set is closed, so $0 \in (I - T)(K)$, i.e., T has a fixed point in K. Q.E.D.

(e) If T is compact, i.e. maps bounded sets into relatively strongly compact sets, then it is easy to show that $I - T$ maps closed bounded sets into closed sets. In particular, (e) follows from (d). Q.E.D.

(f) The image of a weakly compact, convex set under a continuous affine mapping is easily seen to be strongly closed. Therefore, if K is a bounded closed convex subset of C, then K is weakly compact, and since $I - T$ is affine, $(I - T)(K)$ is closed. Thus (f) follows from (d). Q.E.D.

3. **Intersections and unions of nonexpansive retracts.** Because of examples such as those already given, we have so far restricted ourselves to mappings which send C into C. We have only one result in a more general direction:

Lemma 3. *Suppose X is strictly convex and $\{T_n\}$ is a sequence of nonexpansive mappings $T_n : C \to X$. Then there exists a nonexpansive mapping $T : C \to X$ such that $F(T) = \bigcap_n F(T_n)$.*

Proof. If the T_n's do not have a common fixed point we can let T be translation by any nonzero vector; then both $F(T)$ and $\cap_n F(T_n)$ are empty.

Otherwise, let y_0 be a common fixed point of the T_n's. Let $\{\lambda_n\}$ be any sequence of positive real numbers such that $\sum_n \lambda_n = 1$ and set $T = \sum_n \lambda_n T_n$. The mapping T is well defined since

$$\|T_n x\| \leq \|T_n x - T_n y_0\| + \|T_n y_0\| \leq \|x - y_0\| + \|y_0\|;$$

thus $\sum \lambda_n T_n x$ converges absolutely for each x in C. Obviously T is nonexpansive and maps C into X.

We claim that $F(T) = \cap_n F(T_n)$. The inclusion $\cap_n F(T_n) \subset F(T)$ being trivial, we prove only the reverse inclusion.

Suppose, therefore, that $Tx = x$. Then

(6)
$$\|x - y_0\| = \|Tx - y_0\| = \|\sum \lambda_n T_n x - y_0\|$$
$$= \|\sum \lambda_n (T_n x - y_0)\| \leq \sum \lambda_n \|T_n x - y_0\|.$$

But $T_n y_0 = y_0$ and T_n is nonexpansive, so $\|T_n x - y_0\| \leq \|x - y_0\|$. Since $\sum \lambda_n = 1$, (6) implies

(7)
$$\|\sum \lambda_n (T_n x - y_0)\| = \|x - y_0\| \qquad \text{and}$$
$$\|T_n x - y_0\| = \|x - y_0\| \qquad \text{for all } n.$$

Since X is strictly convex and each $\lambda_n > 0$ while $\sum \lambda_n = 1$, (7) implies $T_n x - y_0 = T_m x - y_0$ for all m, n, i.e., $T_n x = T_m x$ for all m, n. Therefore

$$x = Tx = \sum_n \lambda_n T_n x = \sum_n \lambda_n T_m x = T_m x$$

for all m. That is, $x \in \cap_m F(T_m)$.   Q.E.D.

Remark. It is not known whether Lemma 3 is valid for uncountable families of nonexpansive mappings $T_\alpha : C \to X$. However:

Lemma 4. *Suppose C is locally weakly compact and $\{F_\alpha : \alpha \in A\}$ is a family of weakly closed nonexpansive retracts of C.*

(a) *If the family is directed by $\supset$, then $\cap_\alpha F_\alpha$ is a nonexpansive retract of C.*

(b) *If each F_α is convex and the family is directed by $\subset$, then $\text{Cl} \cup_\alpha F_\alpha$ is a nonexpansive retract of C.*

Proof of (a). Let $F = \cap_\alpha F_\alpha$. We may suppose $F \neq \varnothing$. Let $z \in C$ and define $K = \{f(z) : f \in N(F)\}$. As in the proof of Theorem 2, K is weakly compact. Since F_α is weakly closed, $K \cap F_\alpha$ is weakly compact for all α.

Furthermore, $K \cap F_\alpha$ is nonempty for each α. Indeed, if f_α denotes a nonexpansive retraction of C onto F_α, then since $F \subset F_\alpha$ we have $f_\alpha \in N(F)$; hence $f_\alpha(z) \in K$. But $f_\alpha(z) \in F_\alpha$ since f_α is a retraction of C onto F_α. We have shown that $K \cap F_\alpha \neq \varnothing$.

Finally, since the F_α's are directed by $\supset$, the family $\{K \cap F_\alpha : \alpha \in A\}$ has the finite intersection property. Since each $K \cap F_\alpha$ is weakly compact, the intersec-

tion $\bigcap_\alpha (K \cap F_\alpha)$ is nonempty, so that $K \cap F \neq \varnothing$.

Hence there exists h in $N(F)$ such that $h(z) \in F$. By Theorem 1, F is a nonexpansive retract of C.   Q.E.D.

Proof of (b). Without loss of generality we may suppose there exists an F_0 in the family of F_α's such that $F_0 \subset F_\alpha$ for all α.

Put $K = \text{weak-cl} \bigcup_\alpha F_\alpha$. For each $\alpha \in A$ define $W_\alpha = \{f \in N(F_0) : f(C) \subset K$ and $fx = x$ for all x in $F_\alpha\}$. Then $W_\alpha \neq \varnothing$ (it contains at least one nonexpansive retraction of C onto F_α).

First, $\{W_\alpha : \alpha \in A\}$ is directed by $\supset$. Indeed, if $\alpha, \beta \in A$ then there exists $\gamma \in A$ such that $F_\alpha \cup F_\beta \subset F_\gamma$ since the family is directed by $\subset$; clearly $W_\gamma \subset W_\alpha \cap W_\beta$.

In particular, $\{W_\alpha : \alpha \in A\}$ has the finite intersection property.

Second, it is readily seen that since K is weakly closed, each W_α is closed in $N(F_0)$. Therefore W_α is compact.

It follows that there exists f in $\bigcap_\alpha W_\alpha$. We claim that such an f is a nonexpansive retraction of C onto K.

f is nonexpansive because f is in $N(F)$; $f(C) \subset K$ because f is in each W_α. f fixes each point of $\bigcup_\alpha F_\alpha$ because f is in each W_α. By continuity, f fixes each point of the strong closure of $\bigcup_\alpha F_\alpha$. But $\bigcup_\alpha F_\alpha$ is convex since the F_α's are directed by $\subset$ and each F_α is convex. Therefore the strong closure of $\bigcup_\alpha F_\alpha$ is identical to K, the weak closure of $\bigcup_\alpha F_\alpha$.

We have shown that f is nonexpansive, $f(C) \subset K$, and $K \subset F(f)$. But this implies that f is a nonexpansive retraction of C onto K.   Q.E.D.

Theorem 5. *Suppose X is strictly convex, C is locally weakly compact, and $\{T_\alpha : \alpha \in A\}$ is a family of nonexpansive mappings $T_\alpha : C \to C$. Then $\bigcap_\alpha F(T_\alpha)$ is a nonexpansive retract of C.*

Proof. Without loss of generality we may suppose $\bigcap_\alpha F(T_\alpha)$ is nonempty. Let $\mathcal{G}$ be the family of sets of the form $\bigcap \{F(T_\lambda) : \lambda \in A'\}$, where A' runs through all possible finite subsets of A. By Lemma 3 such an intersection is the fixed-point set of a nonexpansive $T : C \to X$. Such an intersection is nonempty since $\bigcap_\alpha F(T_\alpha) \neq \varnothing$, and the construction of T in Lemma 3 shows that we may assume $T(C) \subset C$. By Theorem 2 and Theorem 4(c), $F(T)$ is a nonexpansive retract of C. $F(T)$ is convex because in a strictly convex space the fixed-point set of any nonexpansive mapping is convex.

Thus $\mathcal{G}$ is a family of convex nonexpansive retracts of C which are weakly closed because they are convex and strongly closed. Since $\mathcal{G}$ is obviously directed by $\supset$, Lemma 4(a) implies that $\bigcap \{F : F \in \mathcal{G}\}$ is a nonexpansive retract of C. But this is just $\bigcap_\alpha F(T_\alpha)$.   Q.E.D.

Lemma 5. *Suppose X is reflexive and F is a convex nonexpansive retract of X. Let $x_0 \in F$ and define $K(x_0, F) = \text{Cl} \bigcup_{t>0} tF + (1 - t)x_0$. Then $K(x_0, F)$ is a nonexpansive retract of X.*

Remark. $K(x_0, F)$ is the cone with vertex x_0 generated by F. If x_0 belongs to the relative interior of F, then $K(x_0, F)$ is the smallest closed flat in X which contains F. If x_0 is a boundary point of F at which F is smooth, $K(x_0, F)$ is the closed half-space which contains F whose boundary is the tangent hyperplane to F at x_0.

Proof of Lemma 5. For $t > 0$ define $F_t = tF + (1 - t)x_0$. If f is a nonexpansive retraction of X onto F, then

$$x \longmapsto t \cdot f(t^{-1}x + (1 - t^{-1})x_0) + (1 - t)x_0$$

is a nonexpansive retraction of X onto F_t; hence F_t is a nonexpansive retract of X.

On the other hand, F_t is obviously convex and $F_s \subset F_t$ when $0 < s < t$. Since X is reflexive, X is locally weakly compact, so Lemma 4(b) implies that $\mathrm{Cl} \bigcup_{t > 0} F_t$ is a nonexpansive retract of X.   Q.E.D.

Theorem 6. *Suppose X is strictly convex and reflexive. Let $\mathfrak{I}$ be the family of nonexpansive retracts of X, $\mathfrak{I}_c$ the family of cones in X which are nonexpansive retracts of X. Then $\mathfrak{I}$ is closed under arbitrary intersections and $\mathfrak{I}_c$, $\cap$ generate $\mathfrak{I}$ (i.e., any set in $\mathfrak{I}$ is the intersection of some family of sets in $\mathfrak{I}_c$).*

Proof. That $\mathfrak{I}$ is closed under arbitrary intersections follows immediately from Theorem 5, since a nonexpansive retract of X is the fixed-point set of a nonexpansive mapping of X into X.

In particular, the intersection of sets drawn from $\mathfrak{I}_c$ must be a nonexpansive retract of X. We have left to show only that every nonexpansive retract F of X is the intersection of a family of cones which are themselves nonexpansive retracts of X.

But F is convex, being the fixed-point set of a nonexpansive mapping in a strictly convex space, and it easily follows from standard separation theorems that

$$F = \bigcap \{K(x_0, F) : x_0 \in F\}.$$

By Lemma 5, each $K(x_0, F)$ is a nonexpansive retract of X. Furthermore, each $K(x_0, F)$ is a cone in X with vertex at x_0. Thus F *is* the intersection of cones which are nonexpansive retracts of X.   Q.E.D.

4. Common fixed points of commuting nonexpansive mappings. Belluce and Kirk [1] proved that if C is weakly compact and has normal structure, then any *finite* family of commuting nonexpansive mappings $T_\lambda : C \to C$ has a common fixed point. We shall improve this by proving a result that is almost the best possible.

Lemma 6. *Suppose C is weakly compact and has the hereditary fixed point property for nonexpansive mappings. Suppose F is a nonempty nonexpansive retract*

of C and T : C → C is a nonexpansive mapping which leaves F invariant. Then F(T) ∩ F is a nonempty nonexpansive retract of C.

Proof. Let r be a nonexpansive retraction of C onto F. Then $T \circ r : C \to C$ is nonexpansive and in fact maps C into F since $r(C) \subset F$ and T leaves F invariant. Since C is bounded and has the hereditary fixed point property, $F(Tr) \neq \varnothing$. By Theorem 2 and Theorem 4(a), $F(Tr)$ is a nonexpansive retract of C.

Thus the lemma is proven once we show $F(T) \cap F = F(Tr)$. The inclusion $\subset$ is trivial; to prove the reverse inclusion, suppose $Tr(x) = x$. Since $r(x) \in F$ and T leaves F invariant, we therefore have $Tr(x) \in F$, i.e. $x \in F$. But then $r(x) = x$, so that $Tr(x) = x$ implies $Tx = x$. That is, $x \in F(T) \cap F$. Q.E.D.

Theorem 7. *Suppose C is weakly compact and has the hereditary fixed point property for nonexpansive mappings. If $\{T_j : 1 \leq j \leq n\}$ is a finite family of commuting nonexpansive mappings $T_j : C \to C$ then $\bigcap_{j=1}^{n} F(T_j)$ is a nonempty nonexpansive retract of C. In particular, the set of common fixed points of the T_j's is connected.*

Proof. By induction on n. The theorem is obviously true for $n = 1$.

Suppose the theorem true for commuting families of a certain cardinality n, and suppose $\{T_j : 1 \leq j \leq n + 1\}$ is a commuting family of cardinality $n + 1$. Let $F = \bigcap_{j=2}^{n+1} F(T_j)$. By the induction hypothesis F is a nonempty nonexpansive retract of C.

We claim that F is invariant under T_1. In fact, if $x \in F$ then $T_j x = x$ for $j \geq 2$, so $T_j T_1 x = T_1 T_j x = T_1 x$. That is, $T_1 x \in F(T_j)$ for $j \geq 2$, so $T_1 x \in F$.

By Lemma 6, $F(T_1) \cap F$ is a nonempty nonexpansive retract of C. But this set is just $\bigcap_{j=1}^{n+1} F(T_j)$. Q.E.D.

It is not known whether the conclusion of Theorem 7 is valid for infinite families. Belluce and Kirk [2] have extended their theorem of [1] to infinite families, at the expense of assuming that C has "complete normal structure."

De Marr [5] proved the earliest of these theorems; we give an improved version of this result:

Theorem 8. *Suppose C is compact and $\{T_\alpha : \alpha \in A\}$ is a family of commuting nonexpansive mappings $T_\alpha : C \to C$. Then $\bigcap_\alpha F(T_\alpha)$ is a nonempty nonexpansive retract of C.*

Proof. C has the hereditary fixed point property for nonexpansive mappings, so Theorem 7 applies.

Let $\mathcal{F}$ be the family of sets of the form $\bigcap \{F(T_\lambda) : \lambda \in A'\}$, where A' runs through all possible finite subsets of A. Such an intersection is compact since it is closed in the compact set C.

Thus, by Theorem 7, $\mathcal{F}$ consists of compact, nonempty nonexpansive retracts of C. Since $\mathcal{F}$ is directed by $\supset$, Lemma 4(a) implies that $\bigcap \{F : F \in \mathcal{F}\}$ is a

nonexpansive retract of C. This intersection is nonempty since each F in $\mathcal{F}$ is compact and $\mathcal{F}$ has the finite intersection property.

The proof is concluded by noting that $\bigcap_{\alpha} F(T_{\alpha}) = \bigcap \{F : F \in \mathcal{F}\}$.   Q.E.D.

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