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Properties of Harmonic Functions which are Convex of Order β with Respect to Conjugate Points

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Abstract

Let $\mathcal{H}$ denote the class of functions f which are harmonic and univalent in the open unit disc $D = \{z : |z| < 1\}$. This paper defines and investigates a family of complex-valued harmonic functions that are orientation preserving and univalent in $\mathcal{D}$ and are related to the functions convex of order $\beta(0 \leq \beta < 1)$, with respect to conjugate points. We obtain coefficient conditions, growth result, extreme points, convolution and convex combinations for the above harmonic functions.

Keywords: harmonic functions, convex of order β with respect to conjugate points, coefficient estimates

1 Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply connected complex domain E is said to be harmonic in E if both u and v are real harmonic in E. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re\ (U)$ and $v = Im\ (V)$. Then

$$f(z) = h(z) + \overline{g(z)}$$

where h and g are, respectively, the analytic functions $(U+V)/2$ and $(U-V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by

$$J_f = |h'(z)|^2 - |g'(z)|^2.$$

The mapping $z \mapsto f(z)$ is orientation preserving and locally one-to-one in E if and only if $J_f > 0$ in E. See also Clunie and Sheil-Small [1]. The function $f = h + \overline{g}$ is said to be harmonic univalent in E if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one-to-one in E. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let $\mathcal{H}$ denote the class of functions $f = h+\overline{g}$ which are harmonic and univalent in $\mathcal{D}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0,\ b_n \geq 0, |b_1| < 1. \tag{1}$$

Also let $\overline{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^n, \quad a_n \geq 0,\ b_n \geq 0, |b_1| < 1. \tag{2}$$

Now, we define new class of functions as follows:

Definition 1.1 *Let $f \in \mathcal{H}$. Then $f \in \mathcal{HC}_c(\beta)$ is said to be harmonic convex of order β, with respect to conjugate points, if and only if, for $0 \leq \beta < 1$,*

$$Re\left\{\frac{2\left[z^2h''(z) + zh'(z) + \overline{z^2g''(z) + zg'(z)}\right]}{zh'(z) - \overline{zg'(z)} + z\overline{h'(\overline{z})} - \overline{z\overline{g'(\overline{z})}}}\right\} \geq \beta. \tag{3}$$

Also, we let $\overline{\mathcal{H}}\mathcal{C}_c(\beta) = \mathcal{HC}_c(\beta) \cap \overline{\mathcal{H}}$.

The following theorem proved by Jahangiri in [2] will be used throughout in this paper.

Theorem 1.1 *([2]) Let $f = h + \overline{g}$ with h and g of the form (1). If*

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n| \leq 1, \tag{4}$$

then f is harmonic, orientation preserving, univalent in $\mathcal{D}$ and $f \in \mathcal{HK}(\beta)$.

2 Results

We begin the results with a sufficient coefficient condition for functions in $\mathcal{HC}_c\beta)$.

Theorem 2.1 *Let $f = h + \overline{g}$ be of the form (1). If*

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n| \leq 1, \tag{5}$$

then f is harmonic, orientation preserving, univalent in $\mathcal{D}$ and $f \in \mathcal{HC}_c(\beta)$.

Proof. Since condition (5) hold, it follows from Theorem 1.1 that $f \in \mathcal{HC}(\beta)$ and hence f is harmonic, orientation preserving and univalent in $\mathcal{D}$. Now, we only need to show that if (5) holds then

$$Re \left\{ \frac{2\left[z^2h''(z) + zh'(z) + \overline{z^2g''(z) + zg'(z)}\right]}{zh'(z) - \overline{zg'(z)} + z\overline{h'(\bar{z})} - \overline{z\overline{g'(\bar{z})}}} \right\} = Re \frac{A(z)}{B(z)} \geq \beta.$$

Using the fact that $Re\ (w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it sufficies to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0, \tag{6}$$

where

$$A(z) = 2\left[z^2h''(z) + zh'(z) + \overline{z^2g''(z) + zg'(z)}\right]$$

and

$$B(z) = zh'(z) - \overline{zg'(z)} + z\overline{h'(\bar{z})} - \overline{z\overline{g'(\bar{z})}}.$$

Substituting for $A(z)$ and $B(z)$ in (6), we obtain

$$\begin{aligned}
&|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \\
&= \left|(2+2(1-\beta))z + \sum_{n=2}^{\infty} n[2n+2(1-\beta)]\ a_n\ z^n \right. \\
&\quad \left. + \sum_{n=1}^{\infty} n[2n-2(1-\beta)]\ b_n\ \overline{z}^{\ n}\right| - \left|(2-2(1+\beta))z \right. \\
&\quad \left. + \sum_{n=2}^{\infty} n[2n-2(1+\beta)]\ a_n\ z^n + \sum_{n=1}^{\infty} n[2n+2(1+\beta)]\ b_n\ \overline{z}^{\ n}\right| \\
&\geq (2+2(1-\beta))|z| - \sum_{n=2}^{\infty} n[2n+2(1-\beta)]\ |a_n|\ |z|^n \\
&\quad - \sum_{n=1}^{\infty} n[2n-2(1-\beta)]\ |b_n|\ |z|^n - (2(1+\beta)-2)|z|
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=2}^{\infty} n[2n-2(1+\beta)]\ |a_n|\ |z|^n - \sum_{n=1}^{\infty} n[2n+2(1+\beta)]\ |b_n|\ |z|^n \\
\geq\ & 4(1-\beta)|z| \left\{1-\sum_{n=2}^{\infty}\frac{n(n-\beta)}{1-\beta}|a_n| - \sum_{n=1}^{\infty}\frac{n(n+\beta)}{1-\beta}|b_n|\right\} \\
\geq\ & 0, \quad by\ (5). \quad \square
\end{aligned}
$$

The harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty}\frac{1-\beta}{n(n-\beta)}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\beta}{n(n+\beta)}\overline{y}_n \overline{z}^{\ n}, \tag{7}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (7) are in $\mathcal{HC}_c(\beta)$ since

$$\sum_{n=2}^{\infty}\frac{1-\beta}{n(n-\beta)}|a_n| + \sum_{n=1}^{\infty}\frac{1-\beta}{n(n+\beta)}|b_n| = \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1.$$

Next, we show that the bound (5) is also necessary for functions in $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$.

Theorem 2.2 *Let $f = h + \overline{g}$ with h and g of the form (2). Then $f \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n(n+\beta)}{1-\beta}|b_n| \leq 1. \tag{8}$$

Proof. In view of Theorem 2.1, we only need to show that f is not in $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ if condition (8) does not hold. We note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (2) to be in $\mathcal{HC}_c(\beta)$ is that the coefficient condition (3) to be satisfied. Equivalently, we must have

$$
\begin{aligned}
&Re\ \left\{\frac{2\left[z^2h''(z)+zh'(z)+\overline{z^2g''(z)+zg'(z)}\right]}{zh'(z)-\overline{zg'(z)}+z\overline{h'(\overline{z})}-\overline{z\overline{g'(\overline{z})}}} - \beta\right\} \\
&= Re\ \left\{\frac{z-\sum_{n=2}^{\infty}n^2\ |a_n|\ z^n - \sum_{n=1}^{\infty}n^2\ |b_n|\ \overline{z}^{\ n}}{z-\sum_{n=2}^{\infty}n\ |a_n|\ z^n + \sum_{n=1}^{\infty}n\ |b_n|\ \overline{z}^{\ n}} - \beta\right\} \\
&= Re\ \left\{\frac{(1-\beta)-\sum_{n=2}^{\infty}n(n-\beta)\ |a_n|\ z^{n-1} - \frac{\overline{z}}{z}\sum_{n=1}^{\infty}n(n+\beta)\ |b_n|\ \overline{z}^{\ n-1}}{1-\sum_{n=2}^{\infty}n\ |a_n|\ z^{n-1} + \frac{\overline{z}}{z}\sum_{n=1}^{\infty}n\ |b_n|\ \overline{z}^{\ n-1}}\right\} \\
&\geq 0.
\end{aligned}
$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$\frac{(1-\beta)-\sum_{n=2}^{\infty}n(n-\beta)\ |a_n|\ r^{n-1} - \sum_{n=1}^{\infty}n(n+\beta)\ |b_n|\ r^{n-1}}{1-\sum_{n=2}^{\infty}n\ |a_n|\ r^{n-1} + \sum_{n=1}^{\infty}n\ |b_n|\ r^{n-1}} \geq 0. \tag{9}$$

If condition (8) does not hold then the numerator in (9) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient (9) is negative. This contradicts the required condition for $f \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$ and so the proof is complete. □

The growth result for functions in $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ is discussed in the following theorem.

Theorem 2.3 *If* $f \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$ *then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\beta}{2(2-\beta)} - \frac{1+\beta}{2(2-\beta)}|b_1|\right) r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\beta}{2(2-\beta)} - \frac{1+\beta}{2(2-\beta)}|b_1|\right) r^2, \quad |z| = r < 1.$$

Proof. Let $f \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^2 \\
&= (1+|b_1|)r + \frac{1-\beta}{2(2-\beta)}\sum_{n=2}^{\infty}\left(\frac{2(2-\beta)}{1-\beta}|a_n| + \frac{2(2-\beta)}{1-\beta}|b_n|\right)\ r^2 \\
&\leq (1+|b_1|)r + \\
&\quad \frac{1-\beta}{2(2-\beta)}\sum_{n=2}^{\infty}\left(\frac{n(n-\beta)}{1-\beta}|a_n| + \frac{n(n+\beta)}{1-\beta}|b_n|\right)\ r^2 \\
&\leq (1+|b_1|)r + \frac{1-\beta}{2(2-\beta)}\left(1 - \frac{1+\beta}{1-\beta}|b_1|\right)\ r^2 \\
&= (1+|b_1|)r + \left(\frac{1-\beta}{2(2-\beta)} - \frac{1+\beta}{2(2-\beta)}|b_1|\right)\ r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)\ r^2 \\
&= (1-|b_1|)r - \frac{1-\beta}{2(2-\beta)}\sum_{n=2}^{\infty}\left(\frac{2(2-\beta)}{1-\beta}|a_n| + \frac{2(2-\beta)}{1-\beta}|b_n|\right)\ r^2 \\
&\geq (1-|b_1|)r -
\end{aligned}$$

$$\begin{aligned}&\frac{1-\beta}{2(2-\beta)}\sum_{n=2}^{\infty}\left(\frac{n(n-\beta)}{1-\beta}|a_n|+\frac{n(n+\beta)}{1-\beta}|b_n|\right)\ r^2\\ \geq\ & (1-|b_1|)r-\frac{1-\beta}{2(2-\beta)}\left(1-\frac{1+\beta}{1-\beta}|b_1|\right)\ r^2\\ =\ & (1-|b_1|)r-\left(\frac{1-\beta}{2(2-\beta)}-\frac{1+\beta}{2(2-\beta)}|b_1|\right)\ r^2. \quad \square\end{aligned}$$

The bounds given in Theorem 2.3 for the functions $f=h+\overline{g}$ of the form (2) also hold for functions of the form (1) if the coefficient condition (5) is satisfied. The upper bound given for $f\in\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ is sharp and the equality occurs for the function

$$f(z)=z+|b_1|\overline{z}+\left(\frac{1-\beta}{2(2-\beta)}-\frac{1+\beta}{2(2-\beta)}|b_1|\right)\overline{z}^{\,2},\quad |b_1|\leq\frac{1-\beta}{1+\beta}.$$

Next, we determine the extreme points of closed hulls of $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ denoted by $clco\overline{\mathcal{H}}\mathcal{C}_c(\beta)$.

Theorem 2.4 *$f\in clco\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ if and only if $f(z)=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)$ where*

$$h_1(z)=z,\ h_n(z)=z-\frac{1-\beta}{n(n-\beta)}\ z^n\ (n=2,3,...),$$

$$g_n(z)=z-\frac{1-\beta}{n(n+\beta)}\ \bar{z}^{\,n}\ (n=1,2,...),$$

$\sum_{n=1}^{\infty}(X_n+Y_n)=1,\ X_n\geq 0$ *and* $Y_n\geq 0$.

Proof. For h_n and g_n as given above, we may write

$$\begin{aligned}f(z)\ &=\ \sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)\\ &=\ \sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{1-\beta}{n(n-\beta)}X_nz^n-\sum_{n=1}^{\infty}\frac{1-\beta}{n(n+\beta)}Y_n\bar{z}^{\,n}\\ &=\ z-\sum_{n=2}^{\infty}\frac{1-\beta}{n(n-\beta)}X_nz^n-\sum_{n=1}^{\infty}\frac{1-\beta}{n(n+\beta)}Y_n\bar{z}^{\,n}.\end{aligned}$$

Then

$$\sum_{n=2}^{\infty}\frac{n(n-\beta)}{1-\beta}|a_n|+\sum_{n=1}^{\infty}\frac{n(n+\beta)}{1-\beta}|b_n|$$

$$=\ \sum_{n=2}^{\infty}\frac{n(n-\beta)}{1-\beta}\left(\frac{1-\beta}{n(n-\beta)}X_n\right)$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{n(n+\beta)}{1-\beta}\left(\frac{1-\beta}{n(n+\beta)}Y_n\right)\\
=&\ \sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n\\
=&\ 1-X_1\\
\leq&\ 1.
\end{aligned}
$$

Therefore $f \in clco\overline{\mathcal{H}}\mathcal{C}_c(\beta)$.

Conversely, suppose that $f \in clco\overline{\mathcal{H}}\mathcal{C}_c(\beta)$. Set

$$X_n = \frac{n(n-\beta)}{1-\beta}|a_n| \ , (n = 2, 3, 4, ...),$$

and

$$Y_n = \frac{n(n+\beta)}{1-\beta}|b_n| \ , (n = 1, 2, 3, ...),$$

where $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$. Then

$$
\begin{aligned}
f(z) &= h(z) + \overline{g(z)}\\
&= z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n}\\
&= z - \sum_{n=2}^{\infty}\frac{1-\beta}{n(n-\beta)}X_n z^n - \sum_{n=1}^{\infty}\frac{1-\beta}{n(n+\beta)}Y_n\bar{z}^{\,n}\\
&= z + \sum_{n=2}^{\infty}(h_n(z)-z)X_n + \sum_{n=1}^{\infty}(g_n(z)-z)Y_n\\
&= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n). \quad \square
\end{aligned}
$$

For harmonic functions $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\bar{z}^{\,n}$, we define the convolution of f and F as

$$(f \star F)(z) = z - \sum_{n=2}^{\infty}|a_n A_n|z^n - \sum_{n=1}^{\infty}|b_n B_n|\bar{z}^{\,n}. \tag{10}$$

In the next theorem, we examine the convolution properties of the class $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$.

Theorem 2.5 *For $0 \leq \alpha \leq \beta < 1$, let $f \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$ and $F \in \overline{\mathcal{H}}\mathcal{C}_c(\alpha)$. Then $(f \star F) \in \overline{\mathcal{H}}\mathcal{C}_c(\beta) \subset \overline{\mathcal{H}}\mathcal{C}_c(\alpha)$.*

Proof. Write $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\bar{z}^{\,n}$. Then the convolution of f and F is given by (10).

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$ since $F \in \overline{\mathcal{H}}\mathcal{C}_c(\alpha)$. Then we have

$$\sum_{n=2}^{\infty} n(n-\beta)|a_n||A_n| + \sum_{n=1}^{\infty} n(n+\beta)|b_n||B_n|$$

$$\leq \sum_{n=2}^{\infty} n(n-\beta)|a_n| + \sum_{n=1}^{\infty} n(n+\beta)|b_n|.$$

Therefore $(f \star F) \in \overline{\mathcal{H}}\mathcal{C}_c(\beta) \subset \overline{\mathcal{H}}\mathcal{C}_c(\alpha)$ since the right hand side of the above inequality is bounded by $1-\beta$ while $1-\beta \leq 1-\alpha$. □

Now, we determine the convex combination properties of the members of $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$.

Theorem 2.6 *The class $\overline{\mathcal{H}}\mathcal{C}_c(\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n - \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^{\,n}.$$

For $\sum_{i=1}^{\infty} c_i = 1,\ 0 \leq c_i \leq 1$, the convex combinations of f_i may be written as

$$\begin{aligned}
\sum_{i=1}^{\infty} c_i f_i(z) &= c_1 z - \sum_{n=2}^{\infty} c_1 |a_{n,1}| z^n - \sum_{n=1}^{\infty} c_1 |b_{n,1}| \bar{z}^{\,n} + c_2 z - \sum_{n=2}^{\infty} c_2 |a_{n,2}| z^n - \sum_{n=1}^{\infty} c_2 |b_{n,2}| \bar{z}^{\,n} ... \\
&= z \sum_{i=1}^{\infty} c_i - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\,n} \\
&= z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \bar{z}^{\,n}.
\end{aligned}$$

Next, consider

$$\sum_{n=2}^{\infty} \left(n(n-\beta) \left| \sum_{i=1}^{\infty} c_i |a_{n,i}| \right| \right) + \sum_{n=1}^{\infty} \left(n(n+\beta) \left| \sum_{i=1}^{\infty} c_i |b_{n,i}| \right| \right)$$

$$\begin{aligned}
&= c_1 \sum_{n=2}^{\infty} n(n-\beta)|a_{n,1}| + ... + c_m \sum_{n=2}^{\infty} n(n-\beta)|a_{n,m}| + ... \\
&\quad + c_1 \sum_{n=1}^{\infty} n(n+\beta)|b_{n,1}| + ... + c_m \sum_{n=1}^{\infty} n(n+\beta)|b_{n,m}| + ... \\
&= \sum_{i=1}^{\infty} c_i \left\{ \sum_{n=2}^{\infty} n(n-\beta)|a_{n,i}| + \sum_{n=1}^{\infty} n(n+\beta)|b_{n,i}| \right\}.
\end{aligned}$$

Now, $f_i \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$, therefore from Theorem 2.2, we have

$$\sum_{n=2}^{\infty} n(n-\beta)|a_{n,i}| + \sum_{n=1}^{\infty} n(n+\beta)|b_{n,i}| \leq 1-\beta \ .$$

Hence

$\sum_{n=2}^{\infty} \left(n(n-\beta) \left|\sum_{i=1}^{\infty} c_i |a_{n,i}|\right|\right) + \sum_{n=1}^{\infty} \left(n(n+\beta) \left|\sum_{i=1}^{\infty} c_i |b_{n,i}|\right|\right)$

$\leq (1-\beta) \sum_{i=1}^{\infty} c_i$

$= 1-\beta.$

By using Theorem 2.2 again, we have $\sum_{i=1}^{\infty} c_i f_i \in \overline{\mathcal{H}}\mathcal{C}_c(\beta)$. □

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