

PROPERTIES OF I -SUBMAXIMAL IDEAL TOPOLOGICAL SPACES

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Abstract

In [2], the notion of I -submaximal ideal topological spaces is introduced and studied. In this paper, several characterizations and further properties of I -submaximal ideal topological spaces are obtained.

1 Introduction

The concept of submaximality of general topological spaces was introduced by Hewitt [12] in 1943. He discovered a general way of constructing maximal topologies. In [3], Alas et al. proved that there can be no dense maximal subspace in a product of first countable spaces, while under Booth's Lemma there exists a dense submaximal subspace in $[0,1]^c$. It is established that under the axiom of constructibility any submaximal Hausdorff space is σ -discrete. Any homogeneous submaximal space is strongly σ -discrete if there are no measurable cardinals. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel'skiĭ and Collins [4]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is σ -discrete [4]. The notion of ideal topological spaces was studied by Kuratowski [17] and Vaidyanathaswamy [19]. In 1990, Janković and Hamlett [13] investigated further properties of ideal topological spaces. In [2], properties of I -submaximal ideal topological spaces is studied. In this paper, several characterizations and further properties of I -submaximal ideal topological spaces are obtained. It will be shown that every ideal subspace of an I -submaximal ideal topological space is I -submaximal.

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2 Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. For a subset A of a topological space (X, τ) , $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ) , respectively. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

- (1) $A \in I$ and $B \subset A$ implies $B \in I$.
- (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : P(X) \rightarrow P(X)$, called a local function [17] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : G \cap A \notin I \text{ for every } G \in \tau(x)\}$ where $\tau(x) = \{G \in \tau : x \in G\}$. A Kuratowski closure operator $Cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the \star -topology, finer than τ , is defined by $Cl^*(A) = A \cup A^*(I, \tau)$ [13]. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. For any ideal space (X, τ, I) , the collection $\{U \setminus J : U \in \tau \text{ and } J \in I\}$ is a basis for τ^* . If I is an ideal on X , then (X, τ, I) is called an ideal topological space or simply an ideal space.

Definition 1. A subset A of an ideal space (X, τ, I) is called

- (1) α - I -open [8] if $A \subset Int(Cl^*(Int(A)))$.
- (2) pre- I -open [5] if $A \subset Int(Cl^*(A))$.
- (3) semi- I -open [8] if $A \subset Cl^*(Int(A))$.
- (4) strongly β - I -open [9] if $A \subset Cl^*(Int(Cl^*(A)))$.
- (5) \star -dense [6] if $Cl^*(A) = X$.

Lemma 1. ([2]) For a subset A of an ideal space (X, τ, I) , the following properties are equivalent:

- (1) A is pre- I -open,
- (2) $A = G \cap B$, where G is open and B is \star -dense.

Lemma 2. ([1]) Let (X, τ, I) be an ideal space and $A \subset X$. Then A is α - I -open if and only if it is semi- I -open and pre- I -open.

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Definition 2. ([2]) An ideal space (X, τ, I) is called I -submaximal if every \star -dense subset of X is open.

Theorem 3. For an ideal space (X, τ, I) , the following properties are equivalent:

- (1) X is I -submaximal,
- (2) Every pre- I -open set is open,
- (3) Every pre- I -open set is semi- I -open and every α - I -open set is open.

Proof. (1) \Rightarrow (2) : It follows from Lemma 4.4 of [2].

(2) \Rightarrow (3) : Suppose that every pre- I -open set is open. Then every pre- I -open set is semi- I -open.

Let $A \subset X$ be an α - I -open set. Since every α - I -open set is pre- I -open, then by (2), A is open.

(3) \Rightarrow (1) : Let A be a \star -dense subset of X . Since $Cl^*(A) = X$, then A is pre- I -open. By (3), A is semi- I -open. Since a set is α - I -open if and only if it is semi- I -open and pre- I -open, then A is α - I -open. Thus, by (3), A is open and hence X is I -submaximal. ■

Lemma 4. ([11]) *Let (X, τ, I) be an ideal spaces and $A, B \subset X$. If A is semi- I -open and B is open, then $A \cap B$ is semi- I -open.*

Theorem 5. *For a subset A of an I -submaximal ideal space (X, τ, I) , the following are equivalent:*

- (1) A is semi- I -open,
- (2) A is strongly β - I -open.

Proof. (2) \Rightarrow (1) : Let A be a strongly β - I -open set in X . Put $H = Cl^*(A)$ and $K = A \cup (X \setminus Cl^*(A))$. We have $A = Cl^*(A) \cap K$ and $Cl^*(K) = X$. This implies that $A = H \cap K$, where H is semi- I -open and K is \star -dense. Since X is I -submaximal, then K is open. By Lemma 4, $A = H \cap K$ is semi- I -open.

(1) \Rightarrow (2) : It follows from the fact that every semi- I -open set is strongly β - I -open. ■

Theorem 6. *For an ideal space (X, τ, I) , the following properties are equivalent:*

- (1) X is I -submaximal,
- (2) For all $A \subset X$, if $A \setminus Int(A) \neq \emptyset$, then $A \setminus Int(Cl^*(A)) \neq \emptyset$.
- (3) $\tau = \{U \setminus A : U \in \tau \text{ and } Int^*(A) = \emptyset\}$.

Proof. (1) \Rightarrow (2) : Let $A \subset X$ and $A \setminus Int(A) \neq \emptyset$. Suppose that $A \setminus Int(Cl^*(A)) = \emptyset$. Then $A \subset Int(Cl^*(A))$. This implies that A is pre- I -open. Since X is I -submaximal, by Theorem 3, A is open. Thus, $A \setminus Int(A) = A \setminus A = \emptyset$. This is a contradiction.

(2) \Rightarrow (1) : Let A be a pre- I -open set. Then $A \subset Int(Cl^*(A))$.

Suppose that A is not open. Then $A \not\subseteq Int(A)$ and hence $A \setminus Int(A) \neq \emptyset$. By (2), $A \setminus Int(Cl^*(A)) \neq \emptyset$. Thus, $A \not\subseteq Int(Cl^*(A))$. This is a contradiction.

(1) \Rightarrow (3) : Suppose that $\sigma = \{U \setminus A : U \in \tau \text{ and } Int^*(A) = \emptyset\}$.

Let $G \in \tau$. Since $G = G \setminus \emptyset$ and $Int^*(\emptyset) = \emptyset$, then $\tau \subset \sigma$.

Let $G \in \sigma$. Then $G = U \setminus A$, where $U \in \tau$ and $Int^*(A) = \emptyset$. We have $G = U \cap X \setminus A$. Since $Int^*(A) = \emptyset$, then $X \setminus Int^*(A) = Cl^*(X \setminus A) = X$. Since X is I -submaximal, then $X \setminus A$ is open. Thus, G is open. Hence $\sigma \subset \tau$.

(3) \Rightarrow (1) : Let A be a pre- I -open set. By Lemma 1, $A = G \cap B$, where G is open and B is \star -dense. We have $Cl^*(B) = X$ and hence $Int^*(X \setminus B) = \emptyset$. This implies that $A = G \setminus (X \setminus B)$ and $Int^*(X \setminus B) = \emptyset$. Thus, by (3), A is open. Hence, by Theorem 3, X is I -submaximal. ■

Definition 3. ([12]) A topological space (X, τ) is called a submaximal space if each of its dense subset is open.

Theorem 7. Let $f : (X, \tau) \rightarrow (Y, \sigma, I)$ be an open surjective function. If X is submaximal, then Y is I -submaximal.

Proof. Let X be submaximal and $A \subset Y$ be a \star -dense set. Then A is dense in Y . Since $f^{-1}(A)$ is dense, then $f^{-1}(A)$ is open in X . Since f is an open surjective function, then $A = f(f^{-1}(A))$ is open. Hence, Y is I -submaximal. ■

Corollary 8. If $\prod_{i \in I} X_i$ is a submaximal product space of X_i , then X_i is I -submaximal for every $i \in I$.

Proof. It follows from the fact that for each $i \in I$, the projective function $p_i : \prod_{i \in I} X_i \rightarrow X_i$ is an open surjection. ■

Definition 4. A subset A of an ideal space (X, τ, I) is called \star -codense if $X \setminus A$ is \star -dense.

Theorem 9. For an ideal space (X, τ, I) , the following are equivalent:

- (1) X is I -submaximal,
- (2) Every \star -codense subset A of X is closed.

Proof. (1) \Rightarrow (2) : Let A be a \star -codense subset of X . Since $X \setminus A$ is \star -dense, then $X \setminus A$ is open. Thus, A is closed.

(2) \Rightarrow (1) : It is similar to that of (1) \Rightarrow (2). ■

Definition 5. A subset A of an ideal space (X, τ, I) is called

- (1) a t - I -set [8] if $\text{Int}(A) = \text{Int}(Cl^*(A))$.
- (2) semi- I -regular [16] if A is a t - I -set and semi- I -open.
- (3) an AB_I -set [16] if $A = U \cap V$, where $U \in \tau$ and V is a semi- I -regular set.

Theorem 10. For an ideal space (X, τ, I) , the following are equivalent:

- (1) X is I -submaximal,
- (2) Every pre- I -open set is an AB_I -set,
- (3) Every \star -dense set is an AB_I -set.

Proof. (1) \Rightarrow (2) : Let $A \subset X$ be a pre- I -open set. Since X is I -submaximal, by Theorem 3, A is open. It follows from Proposition 2 of [16] that A is an AB_I -set.

(2) \Rightarrow (3) : Let $A \subset X$ be a \star -dense set. Since every \star -dense set is pre- I -open, then by (2), A is an AB_I -set.

(3) \Rightarrow (1) : Let $A \subset X$ be a \star -dense set. By (3), A is an AB_I -set. Since every \star -dense set is pre- I -open, then A is pre- I -open. Since A is pre- I -open and an AB_I -set, by Proposition 4 of [16], A is open. Hence, X is I -submaximal. ■

4 Subspaces

Recall that if (X, τ, I) is an ideal topological space and A is a subset of X , then (A, τ_A, I_A) , where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$ is an ideal topological space.

Lemma 11. ([14]) *Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $B^*(\tau_A, I_A) = B^*(\tau, I) \cap A$.*

Lemma 12. ([10]) *Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $Cl_A^*(B) = Cl^*(B) \cap A$.*

Theorem 13. *If (X, τ, I) is an I -submaximal ideal space and $A \subset X$, then (A, τ_A, I_A) is I -submaximal.*

Proof. Let B be a \star -dense set in (A, τ_A, I_A) . Let $U = B \cup (X \setminus A)$. By Lemma 12, we have

$$\begin{aligned} Cl^*(U) &= Cl^*(B \cup (X \setminus A)) \supset \\ Cl^*(B) \cup Cl^*(X \setminus A) &\supset Cl_A^*(B) \cup Cl^*(X \setminus A) \\ &= A \cup Cl^*(X \setminus A) = X. \end{aligned}$$

Therefore, U is a \star -dense set in (X, τ, I) . Since X is I -submaximal, then U is open in X . Thus, $B = U \cap A$ and B is open in (A, τ_A, I_A) . Hence, (A, τ_A, I_A) is I -submaximal. ■

Definition 6. ([8]) *A subset A of an ideal space (X, τ, I) is called a B_I -set if $A = U \cap V$, where $U \in \tau$ and V is a t - I -set.*

Theorem 14. *For an ideal space (X, τ, I) , the following are equivalent:*

- (1) X is I -submaximal,
- (2) Every subset of X is a B_I -set,
- (3) Every strongly β - I -open set is a B_I -set,
- (4) Every \star -dense subset of X is a B_I -set.

Proof. (1) \Rightarrow (2) : It follows from Theorem 3.2 of [18].

(2) \Rightarrow (3) : Obvious.

(3) \Rightarrow (4) : It follows from the fact that every \star -dense subset of X is a strongly β - I -open set.

(4) \Rightarrow (1) : It follows from Theorem 3.2 of [18]. ■

5 Further Properties

Definition 7. ([7]) *An ideal space (X, τ, I) is said to be \star -extremally disconnected if \star -closure of every open subset A of X is open.*

Lemma 15. ([7]) For an ideal space (X, τ, I) , the following properties are equivalent:

- (1) X is \star -extremally disconnected,
- (2) Every semi- I -open set is pre- I -open,
- (3) The \star -closure of every strongly β - I -open subset of X is open,
- (4) Every strongly β - I -open set is pre- I -open.

Theorem 16. For an ideal space (X, τ, I) , the following properties are equivalent:

- (1) X is I -submaximal and \star -extremally disconnected,
- (2) Any subset of X is strongly β - I -open if and only if it is open.

Proof. (1) \Rightarrow (2) : Let X be I -submaximal and \star -extremally disconnected. By Lemma 15, every strongly β - I -open set is pre- I -open. By Theorem 3, every pre- I -open set is open. Thus, every strongly β - I -open set is open. The converse follows from the fact that every open set is strongly β - I -open.

(2) \Rightarrow (1) : Suppose that any subset of X is strongly β - I -open if and only if it is open. Since every strongly β - I -open set is open and so pre- I -open, by Lemma 15, X is \star -extremally disconnected. Since every pre- I -open set is open, by Theorem 3, X is I -submaximal. ■

Corollary 17. For an ideal space (X, τ, I) , if X is I -submaximal and \star -extremally disconnected, the following are equivalent for a subset $A \subset X$:

- (1) A is strongly β - I -open,
- (2) A is semi- I -open,
- (3) A is pre- I -open,
- (4) A is α - I -open,
- (5) A is open.

Proof. It follows from Theorem 16. ■

Lemma 18. ([16]) Every AB_I -set is semi- I -open in an ideal topological space (X, τ, I) .

Theorem 19. For an ideal space (X, τ, I) , if X is I -submaximal and \star -extremally disconnected, the following properties are equivalent for a subset $A \subset X$:

- (1) A is semi- I -open,
- (2) A is an AB_I -set.

Proof. (1) \Rightarrow (2) : Let A is semi- I -open. Since X is \star -extremally disconnected, by Lemma 15, every semi- I -open set is pre- I -open. Since X is I -submaximal, by Theorem 10, every pre- I -open set is an AB_I -set.

(2) \Rightarrow (1) : It follows from Lemma 18. ■

Definition 8. ([15]) A subset A of an ideal space (X, τ, I) is called weakly I -local closed if $A = U \cap V$, where $U \in \tau$ and V is a \star -closed set.

Theorem 20. For an ideal space (X, τ, I) , the following properties are equivalent:

- (1) X is I -submaximal,
- (2) Every subset of X is weakly I -local closed,
- (3) Every subset of X is a union of a \star -open subset and a closed subset of X ,
- (4) Every \star -dense subset of X is an intersection of a \star -closed subset and an open subset of X .

Proof. (1) \Rightarrow (2) : It follows from Theorem 3.2 of [18].

(2) \Leftrightarrow (3) : Let $A \subset X$. By (2), we have $X \setminus A = U \cap K$, where U is open and K is \star -closed in X . This implies that $A = (X \setminus U) \cup (X \setminus K)$, where $X \setminus U$ is closed and $X \setminus K$ is \star -open in X . The converse is similar.

(2) \Rightarrow (4) : Obvious.

(4) \Rightarrow (1) : Let $A \subset X$ be a \star -dense set. Then $A = U \cap B$, where U is open and B is \star -closed. Since $A \subset B$ and so B is \star -dense, then $\text{Int}(B) = \text{Int}(Cl^*(B)) = \text{Int}(X) = X$. Hence $B = X$ and $A = U$ is open. Thus, X is I -submaximal. ■

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