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PROPERTIES OF INCLUSIVE REACTIONS IN A UNITARIZED DUAL MODEL OF PRODUCTION AMPLITUDES *)

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A B S T R A C T

Inclusive cross-sections are studied in the planar dual model by first constructing a unitarized set of production amplitudes. Total cross-sections, single and many-particle spectra are then constructed and conservation laws are imposed through a set of sum rules. A weak form of the Harari-Freund conjecture for total cross-sections is derived and is easily generalized to single (and many) particle spectra. A large number of predictions are obtained for single-particle spectra, many of which are capable of experimental verification. Inclusive experiments on deuterium targets are urged in order to provide a crucial test of the model.

Using a stronger form of the Harari-Freund conjecture as an input, we obtain predictions for the way single-particle spectra approach scaling. Criteria for early scaling, and for rising and falling inclusive cross-sections, are discussed in detail.

Diffraction dissociation is included in a very natural way. Effects of interference terms, cuts and absorption are briefly discussed, and it is argued that they cannot change the general results discussed in this paper.

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1. INTRODUCTION AND DESCRIPTION OF OUR APPROACH

This paper is an expanded and extended version of a shorter note by the authors[1], in which the question of approach to scaling in inclusive hadronic reactions was studied within the dual model framework. For the convenience of the readers, the present paper is self-contained. The basic idea is to extend the successful scheme of Harari and Freund[2] (HF), which explains several features of total cross-sections, to the more complicated case of one- (and possibly many-) particle inclusive cross-sections. As in the case of total cross-sections, one expects some of the properties of the inclusive cross-section to depend upon the quantum numbers of the external particles involved and, in particular, upon which channels exhibit exotic quantum numbers.

Even for single-particle distributions, however, it is far from obvious how to extrapolate the total cross-section arguments to this more complicated situation. Basically the point is that there is no unanimously accepted way of defining what duality means for amplitudes other than two-body reactions. As pointed out by Mueller[3], single-particle spectra can be analysed in terms of six-point functions, and in such cases it is not clear *a priori* which s-type channels are dual to the various t-type channels.

This difficulty explains in part why so many different approaches to this problem have appeared in the literature[4-6].

Our point of view is that only by precisely defining a model, or a framework, for implementing duality at the many-particle amplitudes level, can one hope to reach a set of logically consistent predictions. It will then be up to experiments to decide which of these implementations of duality (if any) is satisfied by nature.

Indeed, if single-particle spectra are more complicated to study than are total cross-sections, they are also much richer. The reason is not only that one is able to study dependences on three kinematical variables instead of one, but also that a larger number of reactions are accessible experimentally, the limitations on the type of detected particles being relatively mild. Thus we do expect to have a lot to learn about theoretical models from inclusive spectra.

By now there seems to be good evidence for "scaling" [7] in inclusive hadronic reactions; however, most of the popular models[8] do predict scaling, which, in fact, can be argued to be true on very general grounds. Therefore such a property cannot be a useful way of differentiating various models. On the other hand, the way scaling is approached as energy increases seems to be extremely model-dependent, and, even more exciting, can be studied with present and near-future accelerators. The question of approach to scaling is one of those discussed in this paper. It turns out however that our predictions on this question are not always completely

clear cut, owing to some parameters which are at present unconstrained both theoretically and experimentally. Our results do, however, cast doubt on several criteria for early scaling which have appeared in the literature[4,5].

More interestingly, we find another set of predictions for single-particle spectra, which depends only on very few and reasonable assumptions within our scheme. These generalize a subset of the Harari-Freund results for total cross-sections, i.e. those which do not involve statements on the approach to asymptotia [e.g. $\sigma_{tot}(K^+p) = \sigma_{tot}(K^+n) < \sigma_{tot}(K^-p)$]. We propose to check these predictions as a crucial test of the model.

Throughout the paper it will be important to use general theoretical constraints as much as possible. In this respect, conservation of additive quantum numbers has been shown to provide rigorous constraints for integrals over inclusive cross-sections[9]. These constraints, neglected in other treatments of this problem, will be crucial in our discussion.

We will base our approach on three main theoretical ingredients:

a) model independent sum rules[9] expressing conservation of energy-momentum and of any other additive quantum number (e.g. charge);

b) description of inclusive cross-sections following the general results of the Regge-Mueller approach[3];

c) duality for multiparticle amplitudes in the form of the planar dual model with a particular way of introducing unitarity corrections[10].

The sum rules (a) have been discussed at length elsewhere[9]. They do not involve any specific assumption but are very useful in constraining models. As to assumption (b), we also do not enter into any detailed discussion since this can easily be found elsewhere[3]. It is the third point (c), that we wish to discuss here briefly (for more details see Section 3). This is also the most model-dependent of our inputs, and our point of view on this matter differs in some respects from that of other authors.

It is clear that one can approach the problem of constructing a model for inclusive cross-sections either by giving a model for production amplitudes and then summing exclusive cross-sections appropriately (first approach) or by giving a model for elastic multiparticle amplitudes and then taking appropriate discontinuities as in Mueller's analysis (second approach).

If one had a unitary S-matrix the results obtained in either way must coincide. However, one is still rather far from such an achievement and, therefore, using an approximate S-matrix, can lead to quite different results for the cross-sections in the two approaches. Which of the two ways of calculating is more appealing depends strongly on the model one is going to use. Roughly speaking, the more general (and unspecified) the model is, the more convenient Mueller's approach will be.

On the other hand, very specific models for production will lead, in the first approach, to specific (and possibly reasonable) cross-sections even if the model is non-unitary. Also, in this case, inclusive sum rules will have to be be obeyed if the production amplitudes are consistent with the corresponding conservation laws.

Instead, if the second approach is used, a model violating unitarity could lead to cross-sections in violation of the sum rules or even yield negative cross-sections. Actually, the sum rules have been shown[11] to be -- in this case -- a way of imposing unitarity and to put rigorous constraints on theoretical models.

Since the dual model in the tree approximation is not unitary but, on the other hand, appears to be a reasonable model for production amplitudes, we will prefer to calculate inclusive cross-sections from production amplitudes rather than from optical-type theorems.

This has also been our point of view in the past[1,4], while other authors (for instance, Einhorn et al. Ref. 4) have taken the second approach. Here we have two remarks to make: i) the dual model in the tree approximation has resonances of zero width. This would lead to infinite inclusive cross-sections (also in average) and is, of course, related to the need to introduce unitarity corrections to the tree approximation (this is different, however, from constructing a fully unitary S-matrix). ii) If these corrections are evaluated by the "naive" method based on the narrow resonance approximation[10], one finds in the end that the contributions to the inclusive cross-sections can be practically computed as discontinuities of dual trees and loops, as one would have guessed by starting with Mueller's approach. However, conceptually, this approach is different from those using directly four-, six-, and eight-point elastic amplitudes. For instance, we claim to know which diagrams one has to include and which would produce double counting. Furthermore, each contribution will easily be related to a particular production mechanism, and this will be very important if one wants to use experimental information of an exclusive type. Finally, since the sum rules have to be satisfied independently for each dynamically different production mechanism, their impact will be considerably enhanced.

The rest of the paper is organized as follows. Section 2 reviews, for the sake of completeness, our notations and conventions, inclusive sum rules, Mueller's Regge phenomenology, and some basic kinematics. A general result on the necessity of some rising inclusive cross-sections, as opposed to constant or decreasing total cross-sections, is reproduced. The threshold for Regge behaviour is also studied as a function of some kinematical parameters, indicating that, for some inclusive reactions, scaling has not yet been reached, even at present ISR energies. Some technical details pertaining to this section are given in

Appendix A. In Section 3 we recall how the Harari-Freund conjecture emerges in the context of planar dual models, and how this can be extended to general inclusive reactions. Production amplitudes in the dual model are unitarized in the "naive nuclear physics type" approach of Ref. 10. Such an approach is discussed in a more consistent and familiar (at least in nuclear physics) way in Appendix B. A seven-component theory of single-particle spectra is obtained with additional components describing diffractive dissociation contribution. These extra components, which admittedly had been neglected in our previous analysis[1] emerge very naturally from the general formalism of Appendix B. The whole set of components (seven plus diffraction) is shown to lead to a number of (relatively model-independent) predictions relating various inclusive cross-sections for particles in the same isospin multiplets, or for particles which are charge-conjugated of each other.

In Section 4 we check the consistency of the sum rules with the above dual theory of inclusive cross-sections. This leads to several strong constraints and to an identification of the components giving rising cross-sections. Details of the proof of consistency are confined to Appendix C. Diffraction dissociation effects, as well as the role of interference terms, are discussed in Section 5.

In Section 6 all our predictions are summarized (see also Tables 3 and 4) and comparison is made with available experimental data. We also propose some new inclusive experiments which could crucially test our model.

Section 7 deals briefly with the generalization to many-particle distributions. We can see quite generally (Appendix C) that the internal quantum number structure of the sum rules will continue to work. Section 8 contains our conclusions and some final remarks.

2. KINEMATICS, SUM RULES, AND THE REGGE-MUELLER ANALYSIS OF INCLUSIVE CROSS-SECTIONS

Let σ_{ab} denote the total cross-section for a + b → anything and $d\sigma_{ab}/dp_c \equiv 2E_c(d\sigma_{ab}/d^3\vec{p}_c)$ be the invariant inclusive cross-section for a + b → → c($\vec{p}_c$) + anything.

It is well known that the following equations relate these two inclusive quantities[9)]

$$\langle n \rangle \sigma_{ab} = \sum_c{}' \int dP_c \; \frac{d\sigma_{ab}}{dP_c} \qquad ; \; dP_c \equiv \frac{d^3\vec{P}_c}{2E_c}$$

$$(P_a + P_b)_\mu \, \sigma_{ab} = \sum_c{}' \int dP_c \cdot P_{c,\mu} \cdot \frac{d\sigma_{ab}}{dP_c} \tag{2.1}$$

$$(Q_a + Q_b)\, \sigma_{ab} = \sum_c{}' \int dP_c \; Q_c \; \frac{d\sigma_{ab}}{dP_c} \; , \tag{2.2}$$

where the summation over c in the r.h.s. is over all species of detected particles c, ⟨n⟩ is the average multiplicity of final particles produced and is, in general, an unknown function of the energy. This is in contrast with the quantities $(p_a + p_b)$ or $(Q_a + Q_b)$ of Eqs. (2.1), (2.2) which, denoting energy-momentum and a conserved internal quantum number, respectively, are known from the initial state.

Indeed Eqs. (2.1) and (2.2) follow from the conservation of energy-momentum and any additive internal quantum number Q, respectively. They can be generalized to relate the n-particle inclusive cross-section to the n + 1 particle inclusive cross-sections [the sum on the r.h.s. is over all species of the $(n+1)^{th}$ particle]:

$$\left(P_a + P_b - \sum_{i=1}^{n} P_i\right)_\mu \frac{d^{(n)}\sigma_{ab}}{dP_1 dP_2 \cdots dP_n} = \sum_{n+1} \int dP_{n+1} \, \frac{d^{(n+1)}\sigma_{ab}}{dP_1 dP_2 \cdots dP_n dP_{n+1}} \, P_{n+1,\mu} \tag{2.3}$$

$$\left(Q_a + Q_b - \sum_{i=1}^{n} Q_i\right) \frac{d^{(n)}\sigma_{ab}}{dP_1 dP_2 \cdots dP_n} = \sum_{n+1} \int dP_{n+1} \, Q_{n+1} \, \frac{d^{(n+1)}\sigma_{ab}}{dP_1 dP_2 \cdots dP_n dP_{n+1}} \; , \tag{2.4}$$

These impose constraints on the inclusive cross-sections and turn out to be very useful in the study of various high-energy production models. When n = 0, Eqs. (2.3) and (2.4) reduce to Eqs. (2.1) and (2.2) which we use in this paper.

Let us introduce Feynman's scaling variable $x = 2p_{c\parallel}/\sqrt{s}$ [$s = (p_a + p_b)^2$ is Mandelstam's variable and $p_{c\parallel}$ is $\vec{p}_c$ along $\vec{p}_a$]. Equation (2.1) becomes, with $\mu = 0$ and in the centre-of-mass frame:

$$\sigma_{ab} = \frac{1}{4} \sum_c' \int d^2 p_{c,\perp}\, dx_c \quad \frac{d\sigma_{ab}}{dp_c} \,. \tag{2.5}$$

We are interested in studying the high-energy limit of Eq. (2.5). For total cross-section, ordinary Regge theory gives:

$$\sigma_{ab} \underset{s\ \mathrm{large}}{\sim} \sum_k' C_k^{ab} (s/s_0)^{\alpha_k - 1} \sim C^{ab} + \tilde{C}^{ab} (s/s_0)^{-1/2} + \cdots , \tag{2.6}$$

where $s_0 \sim 1\ \mathrm{GeV}^2$ is the ordinary scale parameter. For $d\sigma_{ab}/dp_c$, Mueller's analysis[3] gives

$$\frac{d\sigma_{ab}}{dp_c} \sim \sum_k' f_k^{ab,c}(x, p_\perp) (s/\bar{s}_0)^{\alpha_k - 1} \simeq f^{ab,c}(x, p_\perp) + \tilde{f}^{ab,c}(x, p_\perp) (s/\bar{s}_0)^{-1/2} + \cdots \tag{2.7}$$

for every $x \neq 0$. The relation between the scale parameters s_0 and $\bar{s}_0$ will be discussed at the end of this section. It will be argued that $\bar{s}_0$ depends on x and $p_\perp$.

As far as Eq. (2.5) is concerned, we only need the fragmentation regions, the central region ($x \sim 0$) being suppressed by the energy momentum factor. For Eq. (2.2), however, the central region is important. There the behaviour of $d\sigma/dp$ is best expressed in terms of the variables

$$\begin{aligned}
&-t = -(p_a - p_c)^2 \sim s_1 = (p_a + p_c)^2 = \frac{1}{2} s (\bar{x} - x) + O(1) \longrightarrow +\infty \\
&-u = -(p_b - p_c)^2 \sim s_2 = (p_b + p_c)^2 = \frac{1}{2} s (\bar{x} + x) + O(1) \longrightarrow +\infty \\
&M^2 = (p_a + p_b - p_c)^2 \,; \quad s = (p_a + p_b)^2 \,; \quad M^2/s \longrightarrow 1 \,,
\end{aligned} \tag{2.8}$$

where

$$\bar{x} = + \sqrt{x^2 + \frac{4 m_T^2}{s}} \,; \quad m_T^2 = m_c^2 + p_{c,\perp}^2 \tag{2.9}$$

One also has

$$\frac{t \cdot u}{s} \longrightarrow m_T^2 \tag{2.10}$$

In this region a double Regge expansion of the cross-section can be used in the form:

$$\frac{d\sigma}{dp_c} = \frac{1}{s} \sum_{i,j}' (s_1)^{\alpha_i} (s_2)^{\alpha_j} f_{ij}^c (p_{c,\perp}^2) . \tag{2.11}$$

We can see that Eq. (2.11) goes smoothly into Eq. (2.7) when we let $0(1/\sqrt{s}) < |x| << 1$. In such a case, Eqs. (2.8) give, for $x > 0$,

$$s_1 \approx \frac{m_T^2}{x} \quad ; \quad s_2 \approx x s \tag{2.12}$$

which, when inserted into Eq. (2.11), gives

$$\frac{d\sigma}{dp_c} = \sum_{i,j}' s^{\alpha_j - 1} x^{\alpha_j - \alpha_i} (m_T^2)^{\alpha_i} f_{ij}^c (p_{c,\perp}^2) . \tag{2.13}$$

This corresponds to expanding $f_j(x,p_\perp^2)$ in Eq. (2.7) as

$$f_j(x, p_\perp) \underset{x \to 0^+}{\sim} \sum_i' f_{ij}(p_\perp^2) (m_T^2)^{\alpha_i} x^{\alpha_j - \alpha_i} \tag{2.14}$$

Another important piece of information that we use concerns the behaviour of $f(x,p_\perp)$ for large values of $p_\perp^2$. Consistently with experiments and with dual model calculations[12], we shall assume an exponential cut-off in $p_\perp^2$. In this case no power-type s-dependence can originate from the fact that the kinematical limit in $p_\perp$ depends on s. Effectively, transverse momenta will be finite and energy-independent.

It is important to demonstrate the consistency of inclusive sum rules with Mueller's Regge theory. Even with exponentially damped transverse momenta, there are two possible sources of spurious energy dependences in the integrals (2.1), (2.2). First the behaviour near $x = 0$ can be rather singular (if $\alpha_j \leq \alpha_i$). This could cause the sum rule integrals to acquire some unexpected power behaviour. Indeed, at $x = 0$ Eq. (2.11) gives a behaviour $s^{(\alpha_i+\alpha_j)/2}$ which does not appear in the expansion of σ_{ab} for $\alpha_i \neq \alpha_j$. In Appendix A we show that this problem does not really exist. Finally, the possibility of spurious power behaviour can originate from the fact that, near $x = \pm 1$, $d\sigma/dp$ can again be rather singular. According to Regge theory, a triple Regge behaviour[13] is expected:

$$\frac{d\sigma}{dp} \underset{x \to 1}{\longrightarrow} s^{\alpha_1 - 1} (1-x)^{\alpha_1 - \alpha_2 - \alpha_3} g_{TR}(\alpha_1, \alpha_2, \alpha_3) , \tag{2.15}$$

where α_1, α_2, α_3 are the three participating trajectories. The integral (2.5) becomes singular if $\alpha_1 - \alpha_2 - \alpha_3 \leq -1$.

In view of all these facts, it is not so obvious that the sum rules can be satisfied within a Regge description of inclusive cross-sections. In Appendix A we discuss some of these problems in detail, and we show that, in most cases, no spurious power emerges after integration. Some interesting exceptions, related to the effects of Regge cuts, are found and are briefly discussed.

Here we want to comment instead on a rather general result that can be obtained from Eq. (2.5) whenever σ_{ab} is already constant at subasymptotic energy (e.g. σ_{K^+p}).

In this case we only keep the leading term (with $\alpha = 1$) in Eq. (2.6) and, replacing Eq. (2.7) into Eq. (2.5), we obtain:

$$4C = \sum_k \left(s/s_0\right)^{\alpha_k - 1} \sum_c \int dp_{c,\perp}\, dx_c \; f_k^{ab,c}(x_c, p_{c,\perp}) \,. \tag{2.16}$$

In the r.h.s. of Eq. (2.16) the leading term ($\alpha_k = 1$) should match the l.h.s., whereas the non-leading contributions have to sum up to zero. If we can assume that, for our purposes, the various integrals in Eq. (2.16) do not depend on s, we see that the above condition reads[1,14]:

$$\sum_c \int dp_{c,\perp}\, dx_c \; f_k^{ab,c}(x_c, p_{c,\perp}) = 0 \tag{2.17}$$

for k non-leading [and not contributing in Eq. (2.6)]. Since one can prove that some $f_k^c(x_c, \underline{p})$ are positive, one concludes that other f_k^c must be negative, thus obtaining the amazing result that some inclusive cross-sections must rise to their asymptotic values, in contrast to the total cross-section case where, according to duality, only constant or decreasing cross-sections are allowed.

Establishing for which set of particles a, b, and c one should expect a rising cross-section, is one of the objectives of this paper (Section 4). We should, however, repeat that our argument depends on the assumptions that the integrals appearing in (2.15) do not develop dangerous s-dependences because of the singular behaviour of the integrand at the end points of integration. This possibility, which is unlikely to change our conclusions, is discussed in Appendix A.

In the rest of this section we want to give an estimate for the effective scale $\bar{s}_0$ in inclusive cross-sections [Eq. (2.7)]. Of course a change of $\bar{s}_0$ can be absorbed into a redefinition of $f_i(x,p_\perp)$. What we actually mean to ask is how large s has to be for the asymptotic expansion (2.7) in order to be reasonably accurate with the first couple of terms.

In the study of Regge behaviour for σ_{ab}, the scale of the incoming energy s is of order $s_0 = 1\ \text{GeV}^{-2}$. Exactly at what value of s/s_0 the Regge-pole parametrization becomes meaningful is not known. But it seems quite reasonable to put a lower bound at say $s/s_0 = 5$ (i.e. $p_{lab} \sim 2.5$ GeV/c), so that below this value, which we call threshold, Regge parametrization has lost its content.

In the double Regge (pionization) region, both s_1 and s_2 have to be large. Demanding s_1/s_0, $s_2/s_0 > 5$ for the validity of the double Regge formula, we obtain from Eqs. (2.8) and (2.10) that the threshold for Regge behaviour in the form $s^{\alpha} f(x,p_\perp^2)$ depends on x and m_T^2. For example, in $pp \to \pi + X$ at $x = 0$, $p_\perp^2 = 0$ we find the threshold to be $\sim s = 1{,}000\ \text{GeV}^2$ and decreasing with x and $p_\perp^2$. At the ISR, this slow approach to asymptotia should be observed at $x = 0$ and $p_\perp^2 \simeq 0$. Since it is difficult to measure pions with such momentum directly, this kinematical feature may be checked by looking for two-photon decays from π^0's which are at rest in the c.m. frame[15]. At finite small $x > 0$ the correction is essentially $1/x$ because the variable $s_2 \sim -u \sim xs$ has to be large (same as for $x < 0$). As we approach large x ($x > \frac{1}{2}$), M^2 becomes the critical variable $[M^2 \sim (1-x)s \sim s_2 \sim xs]$ and one has a correction factor of the type $1/1-x$, which sets the threshold for triple Regge behaviour. Of course, near $x = 1$ one easily has the ordinary Regge limit $[sd\sigma/dp = s^{2\alpha(t)-1} f(M^2,t)]$ at moderate energies ($s/s_0 > 5$).

For our purposes, where the Regge terms of order $1/s$ or lower are neglected, we should study reactions at energies higher than threshold (say, by a factor of 2 or 4). Hence any data with low laboratory momentum (below 10 GeV/c, say) should be compared with care to our theoretical predictions (obtained by going as high as we like in energy) depending on the kinematical region where the data is taken.

As we have said, some inclusive cross-sections have negative non-scaling contributions at high energy. If one imposes simple Regge phenomenology on those data and then extrapolates to low energies, some cross-sections would become negative. Hence the establishment of a threshold is important, particularly in the pionization region.

3. INCLUSIVE CROSS-SECTIONS AND PLANAR DUALITY

In this section we review and complete the dual analysis of inclusive cross-sections (ICS) originally proposed by Gordon and Veneziano[4] and further developed by one of us[4,10].

Here we shall describe our scheme in a rather pictorial and intuitive way. Formal justification of our results appears in Appendix B, which can safely be omitted by the reader interested in the more phenomenological aspects of our scheme.

As we have said in Section 1, our approach to ICS is based on a model for production amplitudes $T(a + b \to c_1 + c_2 + \ldots c_n)$ out of which all ICS can be obtained.

If one looks at the planar dual model in the tree approximation, one can see that the process $a + b \to c_1 + c_2 + \ldots c_n$ can be described in very simple physical terms. Actually, such a description closely parallels one which is used extensively and successfully in the theory of nuclear reactions, the compound nucleus model[16]. The process takes place in three successive steps. First the initial state forms a sort of "compound" intermediate state "C", which we shall describe below; secondly, the state C evolves freely in space-time in a way determined by its mass and lifetime but not by the production mechanism; finally, in the third stage C decays into the final (stable) multiparticle state again in a way which does not depend on the way it was formed.

It is one of the most important properties of planar duality that C is either a single resonance or a two-resonance state. Indeed to include more than two resonances would be double counting. Hence the production process can be of the following two types:

Type I : $$a + b \longrightarrow R \longrightarrow c_1 + c_2 + \cdots + c_n \quad ; \qquad (3.1)$$

Type II : $$a + b \longrightarrow R_1 + R_2 \longrightarrow (c_1 + c_2 \ldots + c_p) + (c_{p+1} + \cdots + c_n) \quad , \qquad (3.2)$$

where in formula (3.2) the first set of p-particles comes from the decay of R_1 and the second set from the decay of R_2. We should recall that, as a particular case of the second type of process, R_1 or R_2 (or both) can themselves be the stable particles. In the particular case $R_1 = c_1$, $R_2 = c_2$, we get the well-known (in nuclear physics) "direct process" or "potential scattering" term for $a + b \to c_1 + c_2$.

Although this physical feature has been derived from the properties of the tree diagrams, it is rather natural to assume that the three-step process will be approximately true even after the model is corrected for lack of unitarity.

According to the results of Ref. 10 for two-body reactions, unitarity will mainly modify the amplitudes by adding widths to the resonances, by introducing diffractive scattering and, finally, by correcting the production and decay amplitudes for absorption in the form of Regge cuts.

In Appendix B the analysis of Ref. 10 is completed and extended to general production amplitudes by means of the useful formalism of projection operators[16]. The result of that investigation is physically simple and appealing, and confirms that the three-step nature of the reaction does survive our unitarization procedure. The resulting production amplitudes turn out to be of the form

$$T(a+b \to c_1 + c_2 + \cdots + c_n) = T^{(1)} + T^{(2)} \tag{3.3}$$

with

$$T^{(1)}(a+b \to c_1 + c_2 + \cdots + c_n) = \sum_R T(a+b \to R)\, D_R\, T(R \to c_1 + c_2 + \cdots + c_n) \tag{3.4}$$

and

$$T^{(2)}(a+b \to c_1 + c_2 + \cdots + c_n) = \sum_P \sum_{R_1, R_2}{}' \bar{T}(a+b \to R_1 + R_2)\, D_{R_1} D_{R_2}\, T(R_1 \to c_i \cdots c_p)\, T(R_2 \to c_{i'} \cdots c_n), \tag{3.5}$$

where D_R is the physical propagator of the resonance R, containing a width and a mass shift (see Appendix B for details), and $T(a + b \to R)$, $T(R \to c_1 \ldots + c_k)$ are the couplings of R to the external stable particles.

Finally, the four-point function $\bar{T}(a + b \to R_1 + R_2)$ is itself the sum of two terms:

$$\bar{T}(a+b \to R_1 + R_2) = \bar{T}_{Real} + \bar{T}_{DD}\,, \tag{3.6}$$

where $\bar{T}_{Real}$ has the same quantum numbers and phase of the (t,u) B_4-function term and $\bar{T}_{DD}$ has pure vacuum quantum numbers in the t-channel, is assumed to be imaginary, and represents diffractive excitation of a,b into R_1, R_2. The fact that R_1 and R_2 have different quantum numbers in $\bar{T}_{Real}$ and $\bar{T}_{DD}$ makes this separation desirable, even though their contributions to ICS are not always easy to separate.

Combining Eqs. (3.3), (3.5), and (3.6), we are led to a simple three-component theory of production amplitudes, which generalizes naturally the three-component theory of two-body reactions discussed by one of us[10]. Indeed we shall write

$$T(a+b\to n) \equiv T(a+b\to c_1+c_2+\cdots+c_n) = T_{Res} + T_{Real} + T_{DD} \tag{3.7}$$

with $T_{Res} \equiv T^{(1)}$ of Eq. (3.4) and with T_{Real}, T_{DD} given by Eq. (3.5) for $T^{(2)}$ with $\bar{T}$ replaced by $\bar{T}_{Real}$ and $\bar{T}_{DD}$, respectively.

This is the basic result of Appendix B and the starting point of all our analysis in this paper. The three components are visualized in Fig. 1 for the convenience of the reader. It is from this model for production amplitudes that we now start to construct all the ICS that are needed.

The way to proceed has already been discussed elsewhere[10] and depends quite heavily on the assumption that dual resonances are individually narrow objects even if at high energy their huge density produces a smooth Regge-type behaviour. For completeness we repeat our argument in the case of total cross-sections[10]. Consider the resonance part of $T(ab \to n)$, T_{Res}. A partial cross-section due to these processes will be the cross-section to produce the resonance times the probability that the resonance decays into a specific channel (i.e. times a branching ratio). If, however, we sum over all final channels, the second factor is one and we simply get

$$\sigma_{ab}^{Res} = \sum_R T(a+b\to R)\, T^+(R\to a+b)\,. \tag{3.8}$$

In a similar way, from T_{Real} and T_{DD} we get

$$\sigma_{ab}^{Real} = \sum_{R_1,R_2} \left| \bar{T}_{Real}(a+b\to R_1+R_2) \right|^2 \tag{3.9}$$

$$\sigma_{ab}^{DD} = \sum_{R_1,R_2} \left| \bar{T}_{DD}(a+b\to R_1+R_2) \right|^2 . \tag{3.10}$$

It is now a simple matter of calculation (made in Appendix C) to prove that the internal quantum number structure of the total cross-section in Eqs. (3.9) and (3.10) is that of vacuum quantum numbers in the t-channel of the equivalent forward four-point function (there is also a contribution with vacuum q.n. in the u-channel, but that can be neglected in our kinematical limit). In conclusion, one gets, without new assumptions, what we call the weak form of Harari-Freund (WHF) conjecture

$$\sigma_{ab} = \sum_{i=1}^{2} \sigma_{ab}^{(i)} \quad ; \quad \sigma_{ab}^{(i)} > 0 \quad , \tag{3.11}$$

where $\sigma_{ab}^{(1)} \equiv \sigma_{ab}^{Res}$ is only present if there are resonances in the direct channel (ab non-exotic), whereas $\sigma_{ab}^{(2)} = \sigma_{ab}^{Real} + \sigma_{ab}^{DD}$ gives the background in the s-channel and has vacuum quantum numbers in the t-channel. These components of σ_{ab} are shown in Fig. 2. We call the strong Harari-Freund (SHF) conjecture the additional assumption that in $\sigma_{ab}^{(2)}$ there is no appreciable f_0 trajectory, i.e. only the Pomeron is present. Although experimental data certainly confirm SHF [e.g. $\sigma(K^+p) =$ = constant] for total cross-section, there seems to be problems in naively extending SHF to the case of single-particle ICS. In the rest of this section only WHF will be used, and it will be shown that several predictions of our scheme follow from that alone.

Following the argument given above for total cross-sections, one finds that for the case of one-particle ICS a larger number of components is obtained. Following our procedure for σ_{ab}, interference terms are neglected at this stage. This problem will be discussed further in Section 5, where it is shown that consistency with the sum rules forces various interference terms to be unimportant. Also, we want to use positivity as in WHF and derive several predictions from that. Comparison of these predictions with inclusive data is, of course, of crucial importance.

There are three contributions[17] to one-particle ICS from the component T_{Res} of $T(a + b \to n)$ [Eq. (3.7)]. Other four-components follow from T_{Real} to give the now familiar seven-component scheme (Ref. 4, first two papers) shown in Fig. 3. Diffraction which was previously admittedly ignored, is now incorporated in a natural way. Diffractive processes have been found to be somewhat important[6] in some regions of energy, and neglecting them completely would certainly spoil agreement with data. If one analyses the topology of single-particle ICS coming from T_{DD} in Eq. (3.7), one finds a total of $2 \times 3 = 6$ new components (the factor of 2 just comes from $a \leftrightarrow b$) indicated in Fig. 4. In some cases we will prefer to keep all these six components separated from the other seven, especially when checking consistency of the sum rules, where these six-components should match σ_{ab}^{DD} of Eq. (3.10). In other contexts, however, it is enough to consider the quantum number structure of the contribution of the ICS and, in such a case, one observes that only the components of Fig. 4 labelled as # 8 and 9, lead to a different structure, whereas the other four components, match those of Fig. 3 (the equivalence is $5' \sim 5$, $6' \sim 6$, $7'$, $7'' \sim 7$). Notice that our six-components follow rather closely the various contributions found by the authors of Ref. 6 if one includes elastic scattering, single and double excitation, and collects all possible events. When looking at the decay of a "nova", however, we distinguish the two possibilities of Figs. $5'$, $6'$ and $7'$, $7''$ (non-exotic and exotic decay of the nova), thus favouring a mesonic X^+ to decay into π^+ rather than π^-, etc.

The final result of the above analysis is then

$$\frac{d\sigma_{ab}}{dP_c} = \sum_{i=1}^{9'} \left(\frac{d\sigma_{ab}}{dP_c}\right)_i \quad ; \quad \left(\frac{d\sigma_{ab}}{dP_c}\right)_i > 0 \tag{3.12}$$

and is summarized in Table 1, both for total cross-sections and for one-particle ICS. Extension to multiparticle ICS is straightforward and will be studied in Section 7. In Table 1 we have only distinguished those components which have different internal quantum number structure. However, we also indicate (in the second column) the type of production mechanism involved. Each component has resonances in some channels and therefore is only present if those channels are not exotic. The fact that not all possible combinations of channels appear, is a consequence of planar duality. For example, a planar dual term with resonances in ab and $a\bar{c}$ must have resonances in $ab\bar{c}$, but has no resonances in $b\bar{c}$. Thus we do not have any component requiring ab, $a\bar{c}$, and $b\bar{c}$ to be non-exotic as we do not have components requiring only ab and $a\bar{c}$ non-exotic (also $ab\bar{c}$ has to be such). This explains the third column. In the fourth column we give the quantum number structure of each component with special emphasis on which channels have vacuum quantum numbers. That such structure is present follows easily from the corresponding duality diagrams but can also be obtained by direct calculation as shown in Appendix C for total cross-sections. Finally, in the last two columns we give some immediate result for the scaling and non-scaling part of each component. The scaling piece must be zero unless there are pure vacuum quantum numbers in $a\bar{a}$ (for $x < 0$) or in $b\bar{b}$ (for $x > 0$). At $x = 0$ we can have scaling only if $a\bar{a}$, $b\bar{b}$, and $c\bar{c}$ couple to the vacuum, which leaves only the seventh component. Non-scaling components (indicated with $\tilde{f}$) are always present *a priori* if we stick to the WHF conjecture. We observe, however, that from the positive definite nature of each component, positivity of $\tilde{f}_i$ only follows in those regions of phase space where $f_i = 0$.

From Table 1 a large number of predictions follow. We stress that these are only consequences of planar duality as embodied in the planar dual model and of our simple way of implementing unitarity corrections. These relations follow from the crucial positivity property of each component, which could be spoiled by absorption effects (see also Section 5). The set of predictions that we obtain are summarized in Table 3 and, following the plan of this paper, are discussed in Section 6 together with other more model-dependent predictions (dealing with approach to scaling) which make use of the SHF conjecture.

4. APPROACH TO SCALING

In Section 3 we have discussed how one can obtain predictions for one-particle ICS, just from our nine-component theory. Those predictions are parallel and extend the ones following for total cross-sections from WHF conjecture. We now look at the question of approach to scaling. This question was first raised in a stimulating paper of Chan et al.[5]. Since then a considerable number of papers have appeared[4,5] giving several criteria for early scaling. It is becoming clear, however, that discussion of these problems needs extra input information. For total cross-sections this information is provided by the so-called strong Harari-Freund (SHF) conjecture, which, in addition to the statements of WHF [Eqs. (3.11)], adds the condition that

$$\sigma_{ab}^{(2)} \text{ is energy independent.} \tag{4.1}$$

In general we shall write

$$\sigma_{ab}^{(i)} = c_{ab}^{(i)} + \tilde{c}_{ab}^{(i)} (s/s_0)^{-1/2} \quad ; \quad i = 1,2 \,. \tag{4.2}$$

Therefore SHF is the statement $\tilde{c}_{ab}^{(2)} \simeq 0$.

From Section 3 we recall Eq. (3.12), which gives:

$$\frac{d\sigma}{dp_c} = \sum_{i=1}^{9} \left(\frac{d\sigma}{dp_c}\right)_i = f_c(x,p_\perp) + \tilde{f}_c(x,p_\perp)(s/s_0)^{-1/2} \tag{4.3}$$

$$\left(\frac{d\sigma}{dp_c}\right)_i = f_{c,i}(x,p_\perp) + \tilde{f}_{c,i}(x,p_\perp)(s/s_0)^{-1/2} . \tag{4.4}$$

Some of the f_i must be zero because of quantum numbers (Table 1).

From SHF we have that if ab is exotic, $\tilde{c}_{ab} = 0$. Then, as we have already explained in Section 2, the sum rule (2.17) forces $\tilde{f}_c(x,p_\perp)$ to have different signs according to the choice of phase space and/or of c. Therefore, unlike the case for σ_{ab}, $d\sigma/dp$ must sometimes approach scaling from below.

Within the model under study we now try to give criteria for such a behaviour. In this section we find it convenient to neglect DD components in both σ_{ab} and $d\sigma_{ab}/dp_c$. This approximation will be discussed in Section 5. Here we just mention that our argument could be spoiled by a very large positive PPR vertex in components 8 and 9. This does not seem to be the case.

For ab exotic, we see from Table 1 that only components 3, 5, 6, and 7 contribute. Therefore, combining Eqs. (4.1) through (4.4) with Eqs. (2.8) and (2.5), the following conditions are obtained:

$$0 = \sum_{i=3,5,6,7}' \sum_c \int d^2P_{c,\perp}\, dx_c \quad \tilde{f}_{c,i}(x_c, P_{c\perp}) \tag{4.5}$$

$$0 = \sum_{i=3,5,6}' \sum_c Q_c \int dP_c \quad \tilde{f}_{c,i}(x_c, P_{c\perp}) \,, \tag{4.6}$$

where we have used the fact that, because of its quantum number structure, the seventh component does not contribute to Eq. (4.6).

Since $f_3 = 0$ we have $\tilde{f}_3 > 0$. Also, from the fact that $\tilde{f}_3$ has only a Regge-Regge vertex in the pionization region it follows (Table A1) that its contribution is $a(s/s_0)^{-\frac{1}{2}}$ in Eq. (4.5) and $b(s/s_0)^{-\frac{1}{2}}$ (log s ± const) in Eq. (4.6) with a and b positive. From Eq. (4.6) one then argues that f_5 and f_6 must have a negative Regge-Regge coupling in the central region and, of course, a positive Regge-Pomeron term to make the whole positive. A little more information comes from Eq. (4.5), where (using Table 1) one finds that some negative non-scaling term is required in the fragmentation regions in components 5 and 6 or in component 7 (or in all of them).

In our previous analysis[1] we had taken for simplicity $\tilde{f}_7 = 0$ (in analogy with $\tilde{c}_2 = 0$ of SHF). However, there is no real theoretical justification for this. Experiments should be able to check this point adequately. Indeed, processes such as $pp \to K^-$ or $K^+p \to K^-$ have only f_7 contributions. One should check whether these cross-sections are flat or whether they increase. Data seem to indicate an increase, but it is not clear whether one is already in the asymptotic regime or whether, because of the missing-mass quantum numbers, one is just seeing a threshold effect. By taking the consideration $(pp \to K^-) - (pp \to K)$ one isolates instead components 5+6 and can study the energy dependence of these cross-sections. Thus a rather simple set of measurements should provide the information we need.

Assume, for instance, that $\tilde{f}_7 = 0$ is confirmed. Then $(d\sigma/dp_c)_{5,6}$ must be responsible for accommodating the negative non-scaling terms. Now, component 5 has vacuum quantum numbers in $a\bar{a}$ only. Hence $f_5 = 0$ for $x > 0$ which implies $\tilde{f}_5 > 0$ for $x > 0$, because of positivity of each component. In this case ($\tilde{f}_7 = 0$) one has then $\tilde{f}_5(\tilde{f}_6) < 0$ for $x < (>) 0$. At the end of this section we shall give a physical argument for the dynamical origin of these negative components. We stress that, in any event, our predictions should always be taken as average properties over phase space, rather than local predictions.

At this point we have still to check the consistency of our scheme with the inclusive sum rules and with the condition of non-exotics in all channels. This is not at all trivial. Criteria proposed earlier for early scaling can easily be shown to be inconsistent. Here, for the purpose of concreteness, we consider a purely mesonic world. In such a case Chan and Paton[18] have proposed a way to

include internal quantum numbers by multiplying to each dual amplitude a corresponding trace factor. In an SU(3) world, particles are represented by the standard SU(3) matrices, while in an SU(2) world, particles are represented by the Pauli matrices. A Chan-Paton factor is simply the trace of a product of such matrices. For non-tree diagrams, a Chan-Paton factor is a product of such traces. Such factors are cyclic symmetric, factorizable, and exclude all exotics. Details are described in Appendix C. Here we just mention that

i) our Chan-Paton factors are symmetrized with respect to cyclic and anticyclic permutations, as suggested by duality diagrams;

ii) an SU(3) symmetric world of pseudoscalar mesons is completely consistent with all the sum rules (four-momentum, charge and hypercharge conservation) provided the mesons occur in a nonet;

iii) an SU(2) world with pions only is not self-consistent, but it becomes so if we add an extra isoscalar particle;

iv) components 1, 2, and 4 contribute to the non-scaling part of the total cross-section, while components 3, 5, 6, and 7 contribute to the scaling part of the total cross-section, as they should from their dynamical origin. Again we refer the readers to Appendix C for the details and present the result for an SU(3) world [$f_{c,i}$ = (Trace factor for c)$_i$ · f_i]:

$$\tilde{c}_1 = 3\int dP\,(\tilde{f}_1+\tilde{f}_2) = \frac{1}{4}\int d^2P_\perp\,dx\,(6\tilde{f}_1+6\tilde{f}_2+9\tilde{f}_4) \tag{4.7}$$

$$c_2 = \frac{3}{2}\int dP\,(f_5+f_6) = \frac{1}{4}\int d^2P_\perp\,dx\,(6f_5+6f_6+9f_7) \tag{4.8}$$

$$0 = \int d^2P_\perp\,dx\,(2\tilde{f}_3+3\tilde{f}_5+3\tilde{f}_6) \tag{4.9}$$

$$0 = \int dP\,(2\tilde{f}_3+\tfrac{3}{2}\tilde{f}_5+\tfrac{3}{2}\tilde{f}_6)\,, \tag{4.10}$$

where, by symmetry, the integral of component 1 equals that of 2, and the integral of component 5 equals that of 6 (we remind our readers of the notation $dp_c = d^3\vec{p}_c/2E_c$). Also we have again set $\tilde{f}_7 = 0$. Equation (4.10) and the first parts of Eqs. (4.7) and (4.8) are obtained from the charge conservation sum rule, while the other equations are obtained from the energy conservation sum rule.

Since component 4 contributes to $\tilde{c}_1$ only, $f_4 = 0$. The numerical factors in Eqs. (4.7) to (4.10) give an idea of the relative importance of the various non-scaling pieces. They will change when SU(3) ceases to be an exact symmetry. But we are confident that the general features mentioned will remain.

In the triple-Regge region, our scheme can be reduced to that of Chan et al.[5] by assuming that $\tilde{f}_5$, $\tilde{f}_6 \to 0$ as $x \to \pm 1$.

In Table 2 the contributions of various components in each physical phase-space region are indicated. It is obvious that in both the central and triple-Regge regions, Regge exchanges with intercept around zero may not be neglected as compared to other contributions [e.g. the RP term with $\alpha_R(0) \sim 0$ compared to RR terms; the $\pi\pi P$ term in triple-Regge limit]. The first column gives the Chan-Paton factors. Criteria for early scaling and for rising and falling cross-sections are gathered in Section 6.

We now give a physical argument for the occurrence of negative non-scaling terms in components 5, 6, and possibly 7. They are due to a particular way of dividing the production amplitude $T_{Real}(ab \to c_1 + \ldots + c_n)$ as shown in Fig. 5. The resonance line with a dash (5c) means excluding the possibility of there being a ground-state particle. The first term (5b) gives component 3, which is positive and not scaling. The second term (5c) gives components 5 and 7; but, having removed from a supposedly constant cross-section a positive non-scaling piece $\tilde{f}_3$, one is left with negative non-scaling components in 5 and 7. If $\tilde{f}_7 = 0$, by assumption $\tilde{f}_5 + \tilde{f}_6$ will have to exhibit negative terms in both fragmentation regions [roughly speaking, $\tilde{f}_5 + \tilde{f}_6$ should go like $-\tilde{f}_3$, see Eq. (4.9)]. Notice, however, that our dynamical argument for negative non-scaling terms does not suggest $\tilde{f}_7 = 0$ but, if anything argues in favour of $\tilde{f}_7 < 0$. This is seen by a careful comparison of momentum and charge conservation sum rules. If this is the case, one should add a negative non-scaling term in all processes. In particular, since $(d\sigma/dp)_1$ is the only component which can have a negative Pomeron Regge coupling in the central region, an increase of the cross-section at $x = 0$ in the form const $-s^{-1/4}$ would be direct evidence for $\tilde{f}_7 < 0$.

As we said before, in the absence of a decisive theoretical argument, we should leave it to experiment to decide how these negative terms are shared between $(d\sigma/dp)_{5,6,7}$.

5. POSSIBLE COMPLICATIONS OF OUR SCHEME

In previous sections we have always neglected interference terms (all our components are in fact positive definite). Also, in Section 4 we have neglected diffraction dissociation (DD) in speaking of the approach to scaling. These two possible sources of complication are discussed below. Finally, we briefly mention the problems connected with extending our arguments to baryons.

5.1 Diffractive dissociation

The presence of DD in the form of a PPR term in the triple-Regge (TR) region may give rise to a negative $s^{-\frac{1}{2}}$ contribution to σ_T (see Appendix A). This $s^{-\frac{1}{2}}$ term may be large enough to eliminate the necessity of negative non-scaling pieces in components 5, 6, and 7, and thus invalidate our criteria for the approach to scaling. We believe, however, that this is not the case.

In the unitarized dual model discussed in previous sections, the Pomeron term in σ_T is built up of two terms (Fig. 2), namely the real and the DD contribution. Both of them have vacuum quantum numbers in the t-channel. They can be studied separately, each one taken to be scaling by itself. Components 5, 6, and 7 coming from the background term should have negative non-scaling pieces, disregarding what is happening to the DD part of these components, as long as the real component scales early. This is the case if DD is relatively small.

Experimentally, DD for fixed non-asymptotic missing mass M^2 [e.g. $\pi p \to \pi N^*$ [19]] is an order of magnitude smaller than elastic scattering, which in turn contributes only a fraction of σ_T. As M^2 increases, the reaction enters the TR phase-space region and DD is identified with the PPR term (or PPP term, see below), which decreases rapidly as a power of M^2. In view of this, it may not be a bad approximation to neglect DD in the study of σ_T and $d\sigma/dp_c$ in relation to inclusive sum rules[20]. Therefore the presence of DD should not change our criteria, and has to be included only in the study of $d\sigma/dp_c$ for $|x| \sim 1$.

Recently, many groups have analysed the missing-mass data in the TR region. In particular, the data on $(d\sigma/dp)(\pi^- p \to pX)$ at $x \sim -1$ [21] and $(d\sigma/dp)(pp \to pX)$ at $|x| \sim 1$ [22] rises slightly as a function of M^2 for large M^2. The only TR term that gives a rising inclusive cross-section $d\sigma/dt\, dM^2$ as a function of M^2 at small values of t is the RRP term, which, alone fits the data quite nicely[23]. (We remind the reader about the threshold for Regge behaviour discussed earlier. Here the fit with the TR formula should be made only for M^2 large, i.e. M^2 larger than approx. 10 GeV^2.) On pure Regge phenomenology grounds, this indicates that PPP and PPR contributions are small compared with the RRP contribution. This means that the DD terms (i.e. our components 8 and 9) may be small compared with the other components. This is really not very surprising in view of recent studies

on the Pomeron as a simple pole where, if the Pomeron pole has an intercept equal to one, then it is well known that PPP has to vanish as $t = 0$ [24]. Following from this, the PPR term may have to vanish at the same point[25]. In the unitarized dual model for production amplitudes described earlier, we learn that DD emerges as a unitarity correction and it is not surprising to find it small compared with other contributions to σ_T.

Another interesting point is the proposal by Einhorn, Green, and Virasoro[4] that DD is dual to the PPP instead of to the PPR term. In this case, the PPR term may not exist at all. Their result is obtained by examining dual loop diagrams, and so far it is consistent with experiments. If this is true, our criteria for approach to scaling remains exactly valid.

Finally, we mention another open possibility. With PPP as the leading term, it is possible, within the dual model, to have the non-leading term PPR contribute with a negative sign. Then it will contribute to σ_T with a positive $s^{-\frac{1}{2}}$ term. To conclude, the only situation which may give us some worry will be a positive large PPR term. No doubt experiments will decide about this important point in the near future

5.2 Interference terms

There are three kinds of interference terms which may emerge in total cross-sections:

i) between background and DD;
ii) between resonance and background;
iii) between resonance and DD.

Interference of the first kind has Regge exchange between ab and so does not scale in high energy. This is easily recognized to be a cut term, which is real provided that DD is imaginary. Hence it does not contribute to σ_T. Interference of the second kind has quantum number structure of the form $(\bar{b})(ab\bar{a})$, which will be zero if particle b does not have vacuum quantum number. HF requires such terms to be negligible because it could otherwise destroy positivity of each component. The same holds true for the third interference term, which has the same quantum number structure as the first component (it is an absorptive, or final-state interaction correction to it), but does not need to be positive definite.

In conclusion, the no-interference picture is commonly assumed to be qualitatively correct for total cross-sections.

In the case of $d\sigma_{ab}/dp_c$, some interference terms are needed in order to yield the right signature factors (e.g. components 1,2 with component 3). Our general conclusions are unaffected by the addition of these terms. Other interference terms can, however, change our predictions considerably. Indeed Einhorn et al.[4]

propose that interference terms may provide the negative non-scaling terms demanded by the sum rules. For ab exotic one can only have interference among components 3,5,6,7 and with DD components 8,9. Typical quantum number structures which arise are:

$$(a\bar{a}c)(b\bar{b}\bar{c}),\ (a\bar{c})(c\bar{a}\bar{b}b)\ ,$$

corresponding to interference of 5 and 6 and of 9 and 3. The problem is that, when inserted in the sum rules, they give, after summation over c, $(a\bar{a}b\bar{b})$ and $(a\bar{a}\bar{b}b)$, and this is not what one needs. Indeed, a structure such as $(a\bar{a}b\bar{b})$ is that of a (t,u) four-point function, which we have already excluded for total cross-sections. The structure $(a\bar{a}\bar{b}b)$ is that of the first component of σ_{ab}, whereas we need something proportional to $(a\bar{a}) \cdot (b\bar{b})$, as in the second component.

In conclusion, we do not think that interference terms are the answer to the question of negative non-scaling terms in single-particle distributions. As we have explained in Section 4, the dynamical origin of such negative non-leading terms can be fully understood within the seven-components themselves, as coming from a particular way of dividing a process into various components. Therefore, we think that one should go ahead with no interference term and compare the results (see Section 6) with the data before making any complication of this scheme.

5.3 Baryons

We would like to mention finally what is a real short-coming of our model in its present form, i.e. the fact that it is actually made for mesons and not for baryons. As soon as two baryons and two antibaryons appear in the six-point function, exotic mesons become necessary, as pointed out by Rosner at the four-point function level. Yet duality predictions seem to be still quite good there, e.g. $\sigma(pp) \sim \sigma(pn)$ to a reasonable degree of accuracy.

The cheapest way out seems to be to include exotic mesons and baryons, with selection rules making them decouple from ordinary channels MM and MB. In this way many predictions of duality are preserved. This program has been undertaken for the six-point function by Frampton[26], and indications are that, in spite of some modifications of the scheme, most of the results are still true.

6. PREDICTIONS AND COMPARISON WITH EXPERIMENTS

For convenience, we have gathered in this section all the predictions which we have obtained. We divide them into two groups.

The first group contains the consequences of our discussion of Section 3 or, actually, of Table 1. These predictions closely parallel those obtained for total cross-sections in the weak form of the Harari-Freund conjecture (WHF), and are on more solid grounds than those coming from SHF. We recall, for completeness, that from WHF one obtains the following predictions:

$$\text{If ab exotic} \quad : \quad \sigma(ab) \text{ is charge-independent .} \tag{6.1}$$

For instance, $\sigma(K^+p) = \sigma(K^+n)$; $\sigma(pp) = \sigma(pn)$, etc. Also if ab is exotic and $\bar{a}b$ is not, one has

$$\sigma(\bar{a}b)\,,\ \sigma(a\bar{b}) > \sigma(ab)\,. \tag{6.2}$$

Examples of (6.2) are $\sigma(K^-p) > \sigma(K^+p)$; $\sigma(\bar{p}p) > \sigma(pp)$, etc. The first group of predictions generalizes formulae (6.1) and (6.2). As we shall see, the number of predictions we obtain is indeed very large even if we restrict ourselves to those predictions whose experimental verification is feasible in the near future. The set of predictions we obtain is presented in Table 3. Here we give examples of how such relations are obtained.

One set of predictions, which was already derived by one of us[4], is obtained for processes having ab, $a\bar{c}$, and $b\bar{c}$ exotic, so that only component 7 of Table 1 contributes. From the quantum number structure of component 7 one obtains that (we repeat if ab, $a\bar{c}$, $b\bar{c}$ are exotic)

$$\sigma(ab \to c) \quad \text{is independent of the charge of a, b, c .} \tag{6.3}$$

This is clearly the most extreme and simple generalization of (6.1). Examples that are suitable for experimental verification are

$$\begin{aligned}
&\sigma(pp \to K^-) = \sigma(pn \to K^-)\\
&\sigma(pp \to \bar{p}) = \sigma(pn \to \bar{p})\\
&\sigma(K^+p \to K^-) = \sigma(K^+n \to K^-)\\
&\sigma(K^+p \to \bar{p}) = \sigma(K^+n \to \bar{p})\,.
\end{aligned} \tag{6.4}$$

The importance of doing inclusive experiments on deuterium targets appears to be obvious.

Inequalities generalizing (6.2) to single-particle spectra are easy to obtain (see again Veneziano, Ref. 4) when ab, $b\bar{c}$, $a\bar{c}$ are exotic, but $\bar{a}b$ or $\bar{a}\bar{c}$ or $\bar{b}\bar{c}$, etc., are not. In other words, when ab → c has only the seventh component but $\bar{a}b \to c$, $ab \to \bar{c}$, etc., have additional components besides the seventh one. Since the seventh component contributes the same to all these reactions and <u>each component is positive definite</u>, one obtains inequalities such as

$$\begin{aligned}
&\sigma(K^- p \to K^-)\ ,\ \sigma(K^+ p \to K^+) \ > \ \sigma(K^+ p \to K^-) \\
&\sigma(\bar{p} p \to K^-)\ ,\ \sigma(pp \to K^+) \ > \ \sigma(pp \to K^-) \\
&\sigma(K^+ p \to p)\ ,\ \sigma(K^- p \to p,\bar{p}) \ > \ \sigma(K^+ p \to \bar{p}) \\
&\sigma(\bar{p} p \to p,\bar{p})\ ,\ \sigma(pp \to p) \ > \ \sigma(pp \to \bar{p})\ .
\end{aligned} \tag{6.5}$$

These inequalities have to be satisfied locally, i.e. for all values of x and $p_\perp$, and even before scaling is reached. This is clearly an important check of our positivity constraint.

A less immediate result is obtained when $ab\bar{c}$ is exotic together with another channel among ab, $a\bar{c}$, $b\bar{c}$. An example is $\sigma(\pi N \to \bar{K})$. Here only components 4, 6 and 7 contribute. If one takes the difference $\sigma(\pi p \to \bar{K}) - \sigma(\pi n \to \bar{K})$, one gets rid of both 7 and 6 and is left with 4. But component 4 has vacuum quantum numbers in $c\bar{c}$ and, therefore, one has the same quantum number structure of $(\pi p \to \pi p) - (\pi n \to \pi n)$, i.e. isospin 1 in the $\pi\pi \to N\bar{N}$ channel. Therefore

$$\begin{aligned}
&\left[\sigma(\pi^+ p \to K^-) - \sigma(\pi^+ n \to K^-)\right] = \left[\sigma(\pi^- n \to K^-) - \sigma(\pi^- p \to K^-)\right] \\
&\sigma(\pi^\circ p \to K^-) \ = \ \sigma(\pi^\circ n \to K^-)
\end{aligned} \tag{6.6}$$

and

$$\sigma(\pi p \to K^-) - \sigma(\pi n \to K^-) = \sigma(\pi p \to \overline{K^\circ}) - \sigma(\pi n \to \overline{K^\circ}) \tag{6.7}$$

which is equivalent to Eq. (6c) of Table 3.

In a similar way one obtains relations for $\sigma(KN \to \pi)$, $\sigma(NN \to \pi)$, $\sigma(\pi N \to \bar{N})$, whereas those for $\sigma(NN \to K)$ and $\sigma(KN \to \bar{N})$ also use equations such as Eq. (6.4) in order to replace K^0 or $\bar{K}^0$ with K_S or K_L.

Other relations can be obtained, for instance, in $pp \to K$, $\bar{K}$ as follows. The difference $\sigma(pp \to K) - \sigma(pp \to \bar{K})$ has only components 5 and 6. Therefore its quantum number structure is that of the resonance component of $\bar{K}N$ total cross-sections. One would then argue for the proportionality:

$$\frac{\sigma(pp\to K^+) - \sigma(pp\to K^-)}{\sigma(pp\to K^0) - \sigma(pp\to \bar{K}^0)} = \frac{\sigma(K^-p) - \sigma(K^+p)}{\sigma(\bar{K}^0p) - \sigma(K^+p)} \quad , \tag{6.8}$$

where K^0, $\bar{K}^0$ can be replaced by K_S, K_L by use of equations such as Eq. (6.4). However, the validity of Eq. (6.8) makes implicit use of the fact that there is only one function of x, $p_\perp$ associated with each component even when there could be both an F and a D coupling of mesons to $N\bar{N}$. If this is the case, the left-hand side of Eq. (6.8) is a constant (independent of x, $p_\perp$, s) and is equal to the ratio of total cross-sections on the r.h.s. if the f/d ratios are the same in the four-point and six-point functions[27]. Along the same lines a relation is derived for $\bar{K}p \to \bar{p}$ [Eq. (3f), Table 3].

For $pp \to \pi$, an interesting relation follows for the differences $\sigma(pp \to \pi^+) - \sigma(pp \to \pi^-)$ and $\sigma(pp \to \pi^+) - \sigma(pp \to \pi^0)$. Only components 5 and 6 contribute, and one gets pure I = 1 in the $\pi\pi$ channel

$$\sigma(pp\to\pi^0) = \frac{1}{2}\left[\sigma(pp\to\pi^+) + \sigma(pp\to\pi^-)\right] . \tag{6.9}$$

The same holds for $pn \to \pi$.

We now discuss other results which are supposed to hold only asymptotically, i.e. when non-scaling components have died away. As discussed in Section 2, the minimum energy corresponding to asymptotia can vary a lot from reaction to reaction and, within each reaction, is expected to depend upon x and $p_\perp$.

Asymptotically, only components 5, 6, 7, 8, and 9 are expected to contribute. Of these, 5, 7, and 8 should contribute in the fragmentation region of particle b; 6, 7, and 9 in that of particle a. A first general result is that: Any exotic fragmentation is, asymptotically, charge-independent. For instance,

$$\sigma(p \xrightarrow{\bar{K}} \bar{p}) \longrightarrow \sigma(n \xrightarrow{\bar{K}} \bar{p}) \longrightarrow \sigma(p \xrightarrow{K^+} \bar{p}) \tag{6.10}$$

In Eq. (6.10), as well as throughout Table 3, the arrow means asymptotically (scaling limit), and a $\xrightarrow{b}$ c indicates the fragmentation region of a.
In this way several asymptotic equalities and inequalities can be added to the non-asymptotic relations. Our tables, although fairly complete, do not have the pretension of exhausting all predictions that are capable of experimental verification. One case which is very well confirmed experimentally is $\sigma(\pi^+p \to \pi^-)$,

as compared to $\sigma(\pi^- p \to \pi^-)$. Asymptotically we expect the first cross-section to be equal to the second one in the proton fragmentation region, whereas in the π-fragmentation region the second cross-section should be larger so as to account for the extra components. The data at 18.5 GeV/c certainly confirm this[28]. The peak at $x \sim +1$ observed in the second reaction should be the effect of the diffractive component.

A final, rather trivial fact on which we remark is that at very high energy and $x = 0$, eventually only the seventh component survives, and therefore a lot of asymptotic equalities should hold [e.g. $(K^+ p \to K^+) = (K^- p \to K^+)$].

Before going into the other group of predictions in our scheme, namely those describing the approach to scaling, we want to stress again that the predictions listed up to here should be regarded as a very stringent test of planar duality at the multiparticle amplitudes level. The number and strength of our predictions seems to us to be rather unprecedented in the area of phenomenological duality. We believe that on the theoretical side, this type of prediction has been badly overlooked, and the emphasis has been placed too much on the question of early scaling. On the experimental side we hope that future inclusive experiments will take into account the great need we have for testing these relations.

We now go on to describe our predictions on the way scaling should be approached. These results follow from the discussion of Section 4, where the possible effects of DD on this question were neglected (in this respect, see Section 5). In Table 4 we give our results with the additional hypothesis of no non-scaling piece in the seventh component ($\tilde{f}_7 = 0$). We shall comment below on possible modifications from $\tilde{f}_7 \neq 0$. Table 4 is obtained as follows: each reaction $a + b \to c +$ anything is classified according to the exoticity of its channels ab, $a\bar{c}$, $b\bar{c}$, and $ab\bar{c}$. (Not all combinations, though, can occur physically.) For each case, the components $(d\sigma/dp_c)$ that can be present are shown in column 2. They are obtained using Table 1. Since each inclusive cross-section is a sum of the contributing components, its approach to scaling properties in both fragmentation regions are obtained. It follows immediately that if ab and $ab\bar{c}$ are both exotic but one (or both) of $a\bar{c}$ and $b\bar{c}$ are not, $d\sigma/dp_c$ will approach scaling from below in the non-exotic fragmentation region(s). Since the relative sizes of $\tilde{f}_{1,2,4}$ and $\tilde{f}_{5,6}$ are not given, the sign of the non-scaling part of some reactions cannot be predicted. The results for approach to scaling are shown in the third and fourth columns, where only the signs of the non-scaling piece are given. One should remember to add the scaling piece f_7 always and, whenever present, f_5 and f_8 for $x < 0$ and f_6, f_9 for $x > 0$.

In comparing with experimental data[29-32] (last column) we avoid data that are definitively below the threshold for high-energy Regge behaviour (see Section 2). We also emphasize that some reactions may turn out to have smaller (percentage-wise)

non-scaling contributions than one may expect from the table. This is most likely to be the case for produced pions since they have, in general, larger inclusive cross-sections; their non-scaling parts should be comparable with those of other smaller inclusive cross-sections. We consider a failure of our criteria only when an experiment has a non-scaling piece with an opposite sign as predicted. Here we turn to some specific examples and discuss them in greater detail.

1) For proton-proton scattering, all inclusive cross-sections should approach scaling from below, except for nucleons which should fall and antinucleons which go with $(d\sigma/dp)_7$. This is because in most cases only 5, 6, and 7 contribute. This should hold in both fragmentation regions and probably also for the central region where scaling takes place at a higher energy. There is evidence[31] for rising cross-sections in $\sigma(pp \to K_1)$ and $\sigma(pp \to \Lambda)$. ISR data for $pp \to \pi,K$ also seem to support this[29]. Now, $\sigma(pp \to \bar{p})$ has only a seventh component. Although for simplicity we have assumed $\tilde{f}_7 = 0$, there is no theoretical reason for this in our scheme. Data[29] seem to indicate a negative $\tilde{f}_7$; but the reaction has to produce at least three baryons plus an antibaryon, and so the rise (from 24 GeV/c to ISR energy) may be due to threshold effects.

Since the sum of all inclusive cross-sections should be a scaling function only, we expect $\sigma(pp \to N + X)$ to have a positive non-scaling piece, which seems to be the case. There may be enough data to saturate the right-hand side of the inclusive charge sum rule Eq. (2.2).

2) Although we do not know how the reactions $KN \to K + X$ approach scaling, we expect the sign of the non-scaling piece to be the same for K^+, K^0, p, and n. The same argument applies to the other blank boxes in Table 2. (This assumes that no extra complications are introduced by baryons.)

3) To check the positive non-scaling pieces of components 5 and 6, we suggest the reaction $K^+p \to \pi^- + X$ to be measured carefully in the K^+ fragmentation region. This, however, can also be changed by a non-zero $\tilde{f}_7$. Indeed, if $\tilde{f}_7 < 0$ all our minus signs in Table 4 will still be such. On the other hand, the zeros in the last row become minus and all plus signs from the sixth row and below become uncertain. We notice, however, that this is not too bad a situation from the point of view of our predictive power. This is because it will be enough to see what $\tilde{f}_7$ is in an exotic reaction (e.g. $pp \to \bar{p}$, K^-), and then the information can be used for other reactions (remember that the seventh component is the same for several different reactions).

This exhausts our list of predictions in this paper although more could be found. We think, however, that at this point the method for deriving our results should have become familiar to the readers. What seems most important now is to have, as soon as possible, experimental checks.

7. GENERALIZATION TO MANY-PARTICLE SPECTRA

So far our discussion has been concentrated on total cross-sections and single-particle inclusive cross-sections (1-PIC). Here we mention only briefly how one generalizes the model to n-PIC and, in particular, to 2-PIC. We remember that, in principle, knowledge of all inclusive cross-sections implies that of all exclusive cross-sections (See Z. Koba Ref. 9).

As far as dividing $d\sigma/dp_1 \ldots dp_n$ into components goes, it is quite a straightforward task although listing them all for large n becomes very lengthy. One starts from the three components of production amplitudes, and for each of them one takes into account all possible inequivalent ways of selecting n particles of the final state, grouping the "rest" into as few resonances as possible. Examples for 2-PIC are indicated in Fig. 6.

Once the components are listed and their quantum number structure is determined in the usual way, one is expected to derive several relations generalizing those of WHF and those listed in Table 3. For instance, one will have

$$\sigma(KN \to \overline{K}\overline{K}) \quad \text{is charge independent,} \tag{7-1}$$

as a consequence of the fact that a single component contributes, with four resonances in the missing mass. It would be interesting to list all the measurable predictions of this sort for 2-PIC, but this is not attempted here. A new feature which can be studied[33] with 2-PIC is the correlation function $c_2(p_1, p_2)$. One defines, in general,

$$\begin{aligned}
&\frac{1}{\sigma}\frac{d\sigma}{dP_1} = C_1(P_1) \\
&\frac{1}{\sigma}\frac{d\sigma}{dP_1 dP_2} = C_1(P_1)C_1(P_2) + C_2(P_1, P_2) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{1}{\sigma}\frac{d\sigma}{dP_1 dP_2 \cdots dP_n} = \prod_{i=1}^{n} C_1(P_i) + C_2(P_1,P_2)\cdot\prod_{j=3}^{n} C_1(P_j) \cdots + C_n(P_1, P_2 \ldots P_n)
\end{aligned} \tag{7.2}$$

so that the inclusive sum rule Eq. (2.3) becomes[34]

$$\left(-\sum_{i=1}^{n} P_i\right)_\mu C_n = \sum_{(n+1)} \int dP_{n+1}\; P_{n+1,\mu}\; C_{n+1} \tag{7.3a}$$

$$\text{or} \quad \left(-\sum_{i=1}^{n} \bar{X}_i\right) C_n = \frac{1}{4}\sum_{(n+1)}{}' \int d^2P_{n+1,\perp}\, dX_{n+1}\; C_{n+1}\,, \tag{7.3b}$$

where $\bar{x}_i^2 = x_i^2 + (4m_{Ti}^2/s)$ and the summation is over all species of the $(n+1)^{th}$ particle. As mentioned earlier, the scaling and the transverse momentum cut-off for $c_1(p_1)$ implies the same for all c_n and hence for all inclusive cross-sections.

In the same way as good knowledge of σ_T has allowed us to derive some properties of approach to scaling in 1-PIC, so we expect that a good knowledge of 1-PIC would enable us to study this same question at the 2-PIC level.

It should be possible for each component of the 2-PIC to select those regions of phase space where $c_2 \neq 0$, and then the sum rules (7.3) could be used to get predictions for the sign and magnitude of c_2. Again, this is not attempted in this paper. Notice that the number of sum rules relating (n-1)-PIC to n-PIC increases quickly with n; it actually goes as the number of components in (n-1)-PIC. In Appendix 3 we prove rather simply an interesting general result, namely that the internal quantum number structure of the sum rules, which worked out fine for the simpliest sum rules (n=1), continues to work for all n.

We should finally remind the reader that, in many cases, 2-PIC have even higher threshold than do 1-PIC. As a consequence, predictions for 2-PIC may be justified only at energies higher than the ones discussed in Section 6. In any case, although generalizations to larger values of n seem at present to be only of academic value, we believe that a careful study up to 2-PIC would be extremely worth while.

8. CONCLUSIONS

In Section 1 we have stressed the fact that no unambiguous generalization of duality exists from two-body reactions to multiparticle amplitudes. The existing literature on duality and inclusive reactions clearly shows how many different schemes can be set and how different the results could be.

In this paper we have adopted one (in principle, unambiguous) way of implementing duality at the n-point function level, that of planar dual models generalizing the beta-function ansatz. This restricts one's freedom considerably, but still leaves a few open doors to which modifications are important in adding unitarity corrections to the Born term. Our scheme has optimistically kept these corrections to a minimum, both for the sake of simplicity and for the sake of predictivity. The number and strength of the predictions, which are given in Section 6, is indeed quite striking. Thus it could be possible to put our scheme to a very stringent experimental test in the near future if some experiments which we propose will be performed. In spite of being simple and predictive, our scheme has a lot of structure in it: indeed it is neither a purely multiperipheral model, nor a purely diffractive one, but a combination of the two (the relative normalization being, in principle, fixed). It has therefore multiperipheral features, diffractive ones, strong damping in transverse momentum; the whole without any neglect of resonance phenomena, of quantum number effects (depression of exotics), and with a good treatment of crossing and factorization.

One would, of course, like to make this model more detailed by actually computing the various dual contributions to inclusive cross-sections. However, dual Born-terms have been constructed only for mesons and, even then, still lack a completely satisfactory spectrum (although they have passed a huge number of theoretical tests). This fact certainly suggests that one should extract from these models only general properties, and postpone detailed calculations to the (hopefully near) time when a "good" Born term will be available.

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Table 1

Components of total cross-sections and single-particle inclusive cross-sections in the planar dual model. Vacuum quantum number (VQN) in a channel means no exchange of internal quantum numbers.

Components of σ_{ab}	Production mechanism	Channels required to be non-exotic	Vacuum quantum numbers in	Non-scaling piece	Scaling piece
1	R	ab	none	$\tilde{c}_1$	$c_1 = 0$
2	B + D	none	$a\bar{a}$, $b\bar{b}$	$\tilde{c}_2 = 0$(SHF)	c_2
Components of $d\sigma_{ab}/dp_c$					
1	R	ab, $b\bar{c}$, $ab\bar{c}$	none	$\tilde{f}_1$	$f_1 = 0$
2	R	ab, $a\bar{c}$, $ab\bar{c}$	none	$\tilde{f}_2$	$f_2 = 0$
3	B	$b\bar{c}$, $a\bar{c}$, $ab\bar{c}$	none	$\tilde{f}_3$	$f_3 = 0$
4	R	ab	$c\bar{c}$	$\tilde{f}_4$	$f_4 = 0$
5	B + D	$b\bar{c}$	$a\bar{a}$	$\tilde{f}_5$	f_5, $x < 0$ $f_5 = 0$, $x > 0$
6	B + D	$a\bar{c}$	$b\bar{b}$	$\tilde{f}_6$	f_6, $x > 0$ $f_6 = 0$, $x < 0$
7	B + D	none	$a\bar{a}$, $b\bar{b}$, $c\bar{c}$	$\tilde{f}_7$	f_7
8	D	VQN in $b\bar{c}$	$b\bar{c}$, $a\bar{a}$, $\bar{b}c$	$\tilde{f}_8 = 0$, $x > 0$	$f_8 = 0$, $x > 0$
9	D	VQN in $a\bar{c}$	$a\bar{c}$, $b\bar{b}$, $\bar{a}c$	$\tilde{f}_9 = 0$, $x < 0$	$f_9 = 0$, $x < 0$

R = resonance

B = background = Real

D = Diff. Diss.

Table 2

Contribution of the nine components of $d\sigma_{ab}/dp_c$ in the various regions of longitudinal phase space ($x = 2p_{c_{//}}/\sqrt{s}$). Component 7 at $x = 0$ may be of the form PP-PR-RP±RR. Here P = Pomeron, R = Reggeon (intercept ½).

Component	Chan-Paton factors	$x \sim -1$	$x < 0$	$x \sim 0$	$x > 0$	$x \sim +1$
1	$(ab\bar{c}cb\bar{a})$	RRR	R	0	0	0
2	$(\bar{c}ab\bar{b}\bar{a}c)$	0	0	0	R	RRR
3	$(a\bar{c}b\bar{b}c\bar{a})$	RRR	R	RR	R	RRR
4	$(ab\bar{b}\bar{a})(c\bar{c})$	0	R	RR	R	0
5	$(a\bar{a})(b\bar{c}cb)$	RRP	P-R	(PR) - (RR)	R	0
6	$(b\bar{b})(a\bar{c}c\bar{a})$	0	R	(RP) - (RR)	P-R	RRP
7	$(a\bar{a})(b\bar{b})(c\bar{c})$	0	P(-R?)	PP(±...?)	P(-R?)	0
8	$(b\bar{c})(\bar{b}c)(a\bar{a})$	PPP ± PPR (?)	P ± R (?)	0	0	0
9	$(a\bar{c})(\bar{a}c)(b\bar{b})$	0	0	0	P ± R (?)	PPP ± PPR (?)

Table 3

Predictions from nine-component theory. Here $(ab \to c)$ means $\Sigma_X \sigma(ab \to c + X)$ anywhere in phase space; $(a \xrightarrow{b} c)$ means the same cross-section in the fragmentation region of particle a. A simple equality or inequality is supposedly valid even at subasymptotic energies. An arrow means "in the scaling limit". Finally, N_c means the other member of the nuclear doublet ($p_c = n$, $n_c = p$). Predictions are divided according to the "system" of particles $ab \to c$ up to charge conjugation of any of them.

1) $\boxed{NNN}$

a) $(NN \to \bar{N})$ is charge independent; $(pp \to \bar{p}) = (pn \to \bar{p})$

b) $(NN \to N) > (NN \to \bar{N})$; $(pp \to p) > (pp \to \bar{p})$

c) $(N\bar{N} \to N,\bar{N}) > (NN \to \bar{N})$; $(\bar{p}p \to p) > (pp \to \bar{p})$

d) $(\bar{p} \xrightarrow{p} p) + (\bar{p} \xrightarrow{n} p) = (\bar{p} \xrightarrow{n} n) + (\bar{p} \xrightarrow{p} n)$
$(n \xrightarrow{\bar{p}} \bar{p})$?

e) $(N \xrightarrow{N} N) - (\bar{N} \xrightarrow{N} N) \to +$ constant
f) $(N \xrightarrow{\bar{N}} N) - (N \xrightarrow{N} \bar{N}) \to +$ constant
} same

g) $(N \xrightarrow{N\bar{N}} \bar{N})$ and $(\bar{N} \xrightarrow{N} N) \to$ charge independent

2) $\boxed{NN \to K}$

a) $(NN \to \bar{K})$ is charge independent; $(pp \to K^-) = (pn \to K^-)$

b) $(NN \to K) > (NN \to \bar{K})$; $(pp \to K^+) > (pp \to K^-)$

c) $(N\bar{N} \to K,\bar{K}) > (NN \to \bar{K})$; $(\bar{p}p \to K^+,K^-) > (pp \to K^-)$

d) $(pp \to K^+) + 2(pp \to K_S) = (pn \to K^+) + 2\sigma(pn \to K_S)$

e) $\dfrac{(pp \to K^+) - (pp \to K^-)}{(pp \to K_S) - (pp \to K^-)} \stackrel{?}{=} \dfrac{(K^-p) - (K^+p)}{(K_Lp) - (K^+p)}$

f) $(N \xrightarrow{N} K) - (\bar{N} \xrightarrow{N} K) \to$ constant
g) $(N \xrightarrow{\bar{N}} K) - (N \xrightarrow{N} \bar{K}) \to$ constant
} same

h) $(N \xrightarrow{N} \bar{K})$ and $(\bar{N} \xrightarrow{N} K) \to$ charge independent

Table 3 (Cont.)

Table 3 (cont.)

3) $\boxed{KN \to N}$

a) $(KN \to \bar{N})$ is charge independent; $(K^+p \to \bar{p}) = (K^+n \to \bar{p})$

b) $(KN \to N) > (KN \to \bar{N})$; $(K^+p \to p) > (K^+p \to \bar{p})$

c) $(\bar{K}N \to N) > (KN \to \bar{N})$; $(K^-p \to p) > (K^+p \to \bar{p})$

d) $(\bar{K}N \to \bar{N}) > (KN \to \bar{N})$; $(K^-p \to \bar{p}) > (K^+p \to \bar{p})$

e) $(K^-p \to \bar{p}) + 2(K_Lp \to \bar{p}) = (K^-n \to \bar{p}) + 2(K_Ln \to \bar{p})$

f) $$\frac{(K^-p \to \bar{p}) - (K^+p \to \bar{p})}{(K_Lp \to \bar{p}) - (K^+p \to \bar{p})} \stackrel{?}{=} \frac{(K^-p) - (K^+p)}{(K_Lp) - (K^+p)}$$

g) $(N \overset{\bar{K}}{\to} N) - (N \overset{K}{\to} \bar{N}) \to +$ constant
h) $(N \overset{K}{\to} N) - (N \overset{\bar{K}}{\to} \bar{N}) \to$ constant
} same

i) $(\bar{K} \overset{N}{\to} \bar{N}) - (K \overset{N}{\to} \bar{N}) \to$ constant
j) $(K \overset{N}{\to} N) - (\bar{K} \overset{N}{\to} N) \to$ constant
} same

k) $(K \overset{N}{\to} \bar{N})$ and $(\bar{K} \overset{N}{\to} N) \to$ charge independent

l) $(N \overset{K,\bar{K}}{\to} \bar{N})$ charge independent

4) $\boxed{KN \to K}$

a) $(KN \to \bar{K})$ is charge independent; $(K^+p \to K^-) = (K^+n \to K^-)$

b) $(\bar{K}N \to K) > (KN \to \bar{K})$; $(K^-p \to K^+) > (K^+p \to K^-)$

c) $(\bar{K}N \to \bar{K}) > (KN \to \bar{K})$; $(K^-p \to K^-) > (K^+p \to K^-)$

d) $(KN \to K) > (KN \to \bar{K})$; $(K^+p \to K^+) > (K^+p \to K^-)$

e) $(N \overset{\bar{K}}{\to} K) - (N \overset{K}{\to} \bar{K}) \to +$ constant
f) $(N \overset{K}{\to} K) - (N \overset{\bar{K}}{\to} \bar{K}) \to +$ constant
} same

g) $(K \overset{N}{\to} K) - (\bar{K} \overset{N}{\to} K) \to +$ constant
h) $(\bar{K} \overset{N}{\to} \bar{K}) - (K \overset{N}{\to} \bar{K}) \to$ constant
} same

i) $(N \to \bar{K}) \to$ charge independent

j) $(K \to \bar{K})$ and $(\bar{K} \to K) \to$ charge independent

Table 3 (cont.)

Table 3 (cont.)

5) $\boxed{\bar{K}N \to \pi}$

a) $(\bar{K}N \to \pi) > (KN_C \to \bar{\pi})$; $(K^-p \to \pi^\pm) > (K^+n \to \pi^\mp)$

b) $(K^+p \to \pi^-) < (K^+n \to \pi^+)$; $(K^+n \to \pi^-) < (K^+p \to \pi^+)$

c) $\dfrac{(Kp \to \pi) - (Kn \to \pi)}{(\bar{\pi}p) - (\bar{\pi}n)}$ is independent of π, K charge

d) $(K^+ \overset{p}{\to} \pi^+) - (K^+ \overset{p}{\to} \pi^-) \to +$ constant
e) $(K^- \overset{p}{\to} \pi^-) - (K^- \overset{p}{\to} \pi^+) \to +$ constant
} same

6) $\boxed{\pi N \to K}$

a) $(\pi N \to \bar{K}) < (\bar{\pi}N_C \to K)$; $\pi^\pm p \to K^- < \pi^\mp n \to K^+$

b) $(\pi^+p \to K^-) < (\pi^-n \to K^-)$; $(\pi^+n \to K^-) < (\pi^-p \to K^-)$

c) $\dfrac{(\pi p \to \bar{K}) - (\pi n \to \bar{K})}{(\pi p) - (\pi n)}$ = independent of π, $\bar{K}$ charge

d) $(\pi^- \overset{p}{\to} K^-) - (\pi^+ \overset{p}{\to} K^-) \to +$ constant
e) $(\pi^+ \overset{p}{\to} K^+) - (\pi^- \overset{p}{\to} K^+) \to$ constant
} same

f) $(N \overset{\pi}{\to} \bar{K}) \to$ charge independent

7) $\boxed{\pi N \to N}$

a) $(\pi^+N \to N) + (\pi^-N \to N) > (\pi^+N \to \bar{N}) + (\pi^-N \to \bar{N})$

b) $(\pi^+p \to \bar{p}) + (\pi^-p \to \bar{p}) = (\pi^+n \to \bar{p}) + (\pi^-n \to \bar{p})$

c) $(N \overset{\pi}{\to} N) - (N \overset{\pi}{\to} \bar{N}) \to +$ constant

d) $(N \overset{\pi}{\to} \bar{N}) \to$ charge independent

8) $\boxed{NN \to \pi}$

a) $(N\bar{N} \to \pi) > (N_CN \to \bar{\pi})$; $(p\bar{p} \to \pi^+) > (np \to \pi^-)$

b) $2(pp \to \pi^0) = 2(pn \to \pi^0) = (pp \to \pi^+) + (pp \to \pi^-) =$
$= (pn \to \pi^+) + (pn \to \pi^-)$

9) $\boxed{\pi N \to \pi}$

a) $(\pi^+ \overset{N}{\to} \pi^+) - (\pi^+ \overset{N}{\to} \pi^-) \to (\pi^- \overset{N}{\to} \pi^-) - (\pi^+N \to \pi^-)$
$\geq 2\left[(\pi^+ \overset{N}{\to} \pi^0) - (\pi^+ \overset{N}{\to} \pi^-)\right]$

b) $(\pi^+ \overset{N}{\to} \pi^-) \to (\pi^- \overset{N}{\to} \pi^+)$

Table 4

Predictions on the approach to scaling assuming $\tilde{f}_7 = 0$ and neglecting DD effects. The numbers in the 5th column are references to experimental papers measuring the corresponding reactions.

Channels exotic	Components present	Non-scaling part of $\sigma(ab \to cx)$		Examples $(ab \to cx)$	Remarks
		frag. of a $x > 0$,	frag. of b $x < 0$		
none	all seven (plus 8 or 9)	+	+	$\pi^+N \to N, \pi^+, \pi^0, K^+$ $\pi^-N \to N, \pi^-, \pi^0, K^0$ $\bar{N}N \to \pi, K$ (28)	Components 8, 9 should be added when a = c and/or b = c $\sigma(NN \to N)$ is expected to have positive non-scaling part because $\sigma(NN \to \text{meson})$ approaches scaling from below
ab	3, 5, 6, 7, (8, 9)	+	+	$NN \to N, KN \to K, N$ (29)(30)	
$b\bar{c}$	2, 4, 6, 7, (9)		+	$K^-N \to K^-, \bar{K}^0$ $\bar{N}N \to \bar{K}$	
$a\bar{c}$	1, 4, 5, 7, (8)	+		$\pi^-p \to \pi^+, K$ $\pi^+n \to \pi^-$ $\bar{N}N \to K, \bar{K}N \to N$	
$ab\bar{c}$, ab	5, 6, 7	-	-	$KN \to \pi^0, \pi^+$ $NN \to K, \pi, \Lambda, K_1$ (29)(31)(32)	This is true independent of whether $\tilde{f}_7 = 0$ or not. Hence this provides a crucial test of the scheme
$ab\bar{c}$, $b\bar{c}$	4, 6, 7		+	$\pi N \to \bar{N}, \pi^-N \to K^-$ $\pi^+N \to \bar{K}^0$	$\tilde{f}_7 \neq 0$ may change the positive signs here to negative (or zero). This is particularly likely for reactions with components 5 plus 7 and 6 plus 7 only. It will be informative to measure $\sigma(K^+p \to \pi^-)$.
$ab\bar{c}$, $a\bar{c}$	4, 5, 7	+		$\pi^+p \to \pi^-, K^0$ (28)	
$ab\bar{c}$, ab, $b\bar{c}$	6, 7	-	+	$\Lambda N \to K^-$	
$ab\bar{c}$, ab, $a\bar{c}$	5, 7	+	-	$KN \to \pi^-$	
$ab\bar{c}$, $a\bar{c}$, $b\bar{c}$	4, 7	+	+	$\pi^+N \to K^-$ $\pi^-N \to \bar{K}^0$	
all four	7	0	0	$NN \to \bar{N}, \bar{K}$ $KN \to \bar{K}, \bar{N}$	This is assumed to be true

APPENDIX A

ENERGY DEPENDENCE OF SUM RULE INTEGRALS

In Section 2 we have discussed qualitatively why the consistency of Mueller's Regge phenomenology with the sum rules (2.1) and (2.2) is not obvious.

There are indeed two possible sources of "spurious" energy dependences after we perform the integrals appearing in those sum rules. These correspond to the regions of integration $x \sim 0$ (pionization) and $x \simeq \pm 1$. We discuss them separately:

a) Pionization region ($x \sim 0$)

Double Regge exchange is expected to hold here. Besides the Pomeron-Pomeron and Regge-Regge exchanges there are Pomeron-Regge and Regge-Pomeron terms which, at $x = 0$, behave like

$$s^{\frac{\alpha_P(0)+\alpha_R(0)}{2}-1} f(P_\perp^2) \sim s^{-1/4} f(P_\perp^2)$$

Since there is no $s^{-\frac{1}{4}}$ term in total cross-sections, consistency requires that, after integration over phase space for the detected particle c, the $s^{-\frac{1}{4}}$ term drops out. This turns out to be the case because the boundary of the double Regge exchange limit is also a function of s. To see this, a specific expression which goes smoothly from the fragmentation regions to the central region is required, i.e. from Eq. (2.11):

$$\frac{1}{s}\left[\frac{s}{s_0}\left(\bar{x}+x\right)\right]^{\alpha_1}\left[\frac{s}{s_0}\left(\bar{x}-x\right)\right]^{\alpha_2} = \frac{1}{s}\left(\frac{s_1}{s_0}\right)^{\alpha_1}\left(\frac{s_2}{s_0}\right)^{\alpha_2} \qquad (A.1)$$

where $\bar{x} = \left[x^2 + (4\, m_T^2/s)\right]^{\frac{1}{2}}$ and $s_0 \sim 1\ \mathrm{GeV}^2$.

To demonstrate the consistency, the $s^{-\frac{1}{4}}$ term must disappear if expression (A.1) is integrated from one fragmentation region to the other with $\alpha_{1,2} = 1$ and $\alpha_{2,1} = \frac{1}{2}$. This is done for both the energy and the charge sum rules [Eqs. (2.1) and (2.2)], integrating from some constant $-\varepsilon$ to zero and then from zero to ε.

For $\alpha_1 \neq \alpha_2$ we obtain

$$\int_0^\epsilon dx \left(\frac{s_1}{s_0}\right)^{\alpha_1}\left(\frac{s_2}{s_0}\right)^{\alpha_2} \simeq \frac{1}{\alpha_1-\alpha_2+1}\left(s/s_0\right)^{\alpha_1} + \\ + C_1\left[\frac{1}{\alpha_2-\alpha_1+1} - \frac{1}{\alpha_1-\alpha_2+1}\right]\left(s/s_0\right)^{\frac{\alpha_1+\alpha_2-1}{2}} \qquad (A.2)$$

and

$$\int_0^{\epsilon} \frac{dx}{x} \left(\frac{s_1}{s_0}\right)^{\alpha_1}\left(\frac{s_2}{s_0}\right)^{\alpha_2} = \frac{1}{\alpha_1-\alpha_2}\left(\frac{s}{s_0}\right)^{\alpha_1} + c_2 \frac{1}{\alpha_2-\alpha_1}\left(s/s_0\right)^{\frac{\alpha_1+\alpha_2}{2}} \qquad (A.3)$$

for Eqs. (2.1) and (2.2), respectively. The integrals for negative x are obtained by replacing $\alpha_1 \leftrightarrow \alpha_2$. For $\alpha_1 = \alpha_2$, the integrals are trivial. The results are shown in Table A.1 where terms of order lower than $s^{\frac{1}{2}}$ (i.e. $s^{-\frac{1}{2}}$ in σ_T) are neglected. The non-leading $s^{\frac{1}{4}}$ term is present in Eq. (A.2), but it exactly cancels when the integration is from $-\epsilon$ to $+\epsilon$. So consistency can be extended to this order.

We make two observations in connection with Table A.1 :

1) For the charge sum rule case the leading logarithm factor gives the multiplicity. It should be exactly cancelled when all species of particle c are summed over.

2) For the energy sum rule there is no $s^{-\frac{1}{4}}$ term. For the charge sum rule, the $s^{-\frac{1}{4}}$ term drops out only when we integrate over both positive and negative values of x.

Note that after the elimination of the $s^{-\frac{1}{4}}$ term, a term behaving like $s^{-\frac{1}{2}}$ is left with a sign opposite to the one of the leading constant term. On the other hand, in Eq. (2.1) the sign of the leading term and that of the next-to-the leading one is the same. This fact will be of some importance for the future discussion on the sign of the non-scaling contributions (see Section 4).

b) Phase space boundary ($x \sim \pm 1$)

Various unexpected energy dependences can originate from the singular behaviour of $f(x,p_\perp)$ near $x = 1$. Actually, the sum rules have to include all possible values of the missing mass in the integration region, including the elastic contribution. Therefore the physical boundary in x is $x < x_{max} = 1 - (m^2/s)$, where m^2 is the smallest value of M^2 in the missing-mass spectrum. When the behaviour near $x = 1$ is singular, one can easily obtain s-dependence from such an upper cut-off in x.

Let us consider quite generally the processes (Fig. A.1) $a + b \to a' + b'$ and $a + b + c \to a' + b' + c'$, with $s = (p_a + p_b)^2 = (p_{a'} + p_{b'})^2$, $\bar{t} = (p_a - p_{a'})^2 = (p_b - p_{b'})^2$. The non-forward inclusive sum rule[11] for this case is of the form

$$\mathrm{Disc}_s T_4(a+b\to a'+b') = \sum_c{}' \int dP_c \frac{E_c}{E_a+E_b} \mathrm{Disc}_{M^2} T_6(a+b+c\to a'+b'+c) \qquad (A.4)$$

where $M^2 = (P_a + P_b - P_c)^2$.

In the forward direction $p_a = p_{a'}$, $p_b = p_{b'}$ is equivalent to Eq. (2.1). However, it is as easy to do the analysis in this more general kinematical situation, which may turn out to be useful in some other context.

We want to check under which conditions a Regge-pole analysis of the six-point function (with no cuts) leads, through the sum rules, to Regge cuts for the four-point function. Since our analysis of the inclusive cross-sections will essentially neglect cuts, this is an important point to check.

It is quite clear that, using for Disc T_6 the same assumptions as in the forward case (like limited transverse momenta), no Regge cuts are produced through the integration region $-1 + \varepsilon < x < 1 - \varepsilon$. Indeed, in such a region we expect

$$\mathrm{Disc}_{M^2} T_6 \xrightarrow[\substack{s\to\infty \\ t_i, x \text{ fixed}}} s^{\alpha(\bar{t})} F(\bar{t}, t_1, t_2, x) \tag{A.5}$$

and the integration volume $\int dp_c$ can be written[35] as

$$\int \frac{dt_1\, dt_2\, dx}{\sqrt{J(t_1, t_2, x, \bar{t})}}$$

Hence the only regions where we can expect new cut-type behaviour is $x \sim \pm 1$. Throughout this region and including even the elastic contributions, we expect a Regge limit to hold (Fig. A.2) giving

$$\mathrm{Disc}\, T_6 \xrightarrow[\substack{(s/M^2)\to\infty \\ t_i \text{ fixed}}} (s)^{\alpha_1(t_1)+\alpha_2(t_2)} \mathrm{Disc}_{M^2} \tilde{F}(t_1, t_2, \bar{t}, M^2) \tag{A.6}$$

where $\alpha_1(t_1)$, $\alpha_2(t_2)$ are the Regge trajectories in $t_1 = (p_a - p_c)^2$, $t_2 = (p_{a'} - p_c)^2$. If, within this region, $M^2 \to \infty$ we expect triple Regge behaviour to hold:

$$\mathrm{Disc}_{M^2} \tilde{F}(t_1, t_2, \bar{t}, M^2) \to (M^2)^{\alpha(\bar{t}) - \alpha_1(t_1) - \alpha_2(t_2)} g_{TR}(t_1, t_2, \bar{t}) \tag{A.7}$$

to give [Eq. 2.15)]:

$$\mathrm{Disc}\, T_6 \xrightarrow[\substack{(s/M^2)\to\infty \\ M^2\to\infty \\ t_i \text{ fixed}}} s^{\alpha(\bar{t})} \left(\frac{M^2}{s}\right)^{\alpha(\bar{t}) - \alpha_1(t_1) - \alpha_2(t_2)} g_{TR}(t_1, t_2, \bar{t}) \tag{A.8}$$

Inserting now formula (A.6) into (A.4) we find from the region of integration $1 - \varepsilon < x < 1$, apart from unimportant factors:

$$\mathrm{Disc}_s\, T_4(a+b\to a'+b') \sim \int \frac{dt_1 dt_2}{\sqrt{J(t_1,t_2,\bar{t})}}\; s^{\alpha_1(t_1)+\alpha_2(t_2)} \int_0^{\epsilon s} d\frac{M^2}{s}\, \mathrm{Disc}_{M^2} \bar{F} \qquad \text{(A.9)}$$

The last integral in formula (A.9) has the form of a finite energy (or superconvergent) sum rule integral[36], apart from the important fact that only the right-hand cut is integrated [so-called Schwarz sum rules[37]].

Let us first consider the case where there is no fixed pole present in the $\bar{t}$-channel; then the integral is simply evaluated from Eq. (A.7) to be

$$\int_0^{\epsilon s} d\frac{M^2}{s}\, \mathrm{Disc}_{M^2} \bar{F}(t_1,t_2,t,M^2) = \frac{1}{s}\, g_{TR}(t_1,t_2,\bar{t})\, \frac{(\varepsilon s)^{\alpha(\bar{t})-\alpha_1(t_1)-\alpha_2(t_2)+1}}{\alpha(\bar{t})-\alpha_1(t_1)-\alpha_2(t_2)+1} \qquad \text{(A.10)}$$

Using Eqs. (A.10) and (A.9) we obtain:

$$\mathrm{Disc}\, T_4 \sim s^{\alpha(\bar{t})} \int \frac{dt_1 dt_2}{\sqrt{J}}\, g_{TR}(t_1,t_2,\bar{t})\, \frac{(\varepsilon)^{\alpha(\bar{t})-\alpha_1-\alpha_2+1}}{\alpha(\bar{t})-\alpha_1(t_1)-\alpha_2(t_2)+1} \;, \qquad \text{(A.11)}$$

indicating that no cut is present.

If $\alpha(\bar{t}) > \alpha_1(t_1) + \alpha_2(t_2) - 1$ throughout the integration region, the pole would be leading over the cut, but actually there is no cut at all. If $\alpha(\bar{t}) < \alpha_1(t_1) + \alpha_2(t_2) - 1$ somewhere in the integration region, the cut would dominate over the pole but is washed away by the integration over dM^2. This indicates that there have to be cancellations. Indeed if $\alpha(\bar{t}) < \alpha_1(t_1) + \alpha_2(t_2) - 1$, the sum rule (A.10) is a superconvergent one, and one has to have changes of sign in $\mathrm{Disc}_{M^2} \bar{F}$. It also seems that, in this latter case, there is a singularity at the points where $\alpha(\bar{t}) - \alpha_1(t_1) - \alpha_2(t_2) + 1 = 0$. Actually this pole is compensated by a nonsense zero in $g(t_1, t_2, \bar{t})$, which is there in the absence of fixed poles.

From the preceding discussion we see that, in some cases, we should expect a fixed pole at $J = -1$ to appear (e.g. if $\mathrm{Disc}_{M^2} F$ is positive definite for $\alpha < \alpha_1 + \alpha_2 - 1$). In this case, Eq. (A.10) is modified into:

$$\int_0^{\epsilon} d\frac{M^2}{s}\, \mathrm{Disc}_{M^2} \bar{F} = \frac{1}{s}\left[g_{TR}(t_1,t_2,\bar{t})\, \frac{(\varepsilon s)^{\alpha(\bar{t})-\alpha_1(t_1)-\alpha_2(t_2)+1}}{\alpha(\bar{t})-\alpha_1(t_1)-\alpha_2(t_2)+1} + R(t_1,t_2,\bar{t}) \right] \qquad \text{(A.12)}$$

where R is the fixed pole residue.

We see immediately that $R \neq 0$ produces a cut

$$\text{Disc}\, T_4 \sim \int \frac{dt_1 dt_2}{\sqrt{s}}\; s^{\alpha_1(t_1)+\alpha_2(t_2)-1}\, R(t_1, t_2, \bar{t}) \,, \qquad \text{(A.13)}$$

which has the standard form of an AFS cut, but includes some inelastic contributions as well. The fixed pole can be additive, in which case for the first term the above discussion still holds including the zero in g_{TR} at $\alpha - \alpha_1 - \alpha_2 + 1 = 0$. However, if the fixed pole is multiplicative, g_{TR} will not have such a zero. Both terms in the r.h.s. of Eq. (A.12) will then be singular at $\alpha - \alpha_1 - \alpha_2 + 1 = 0$, but the sum will not. At the point where pole and cut intersect, we have:

$$\int_0^{\epsilon s} dM^2/s\; \text{Disc}_{M^2} \bar{F} \underset{\alpha-\alpha_1-\alpha_2+1\sim 0}{\sim} \frac{1}{s}\, g_{TR} \log(\epsilon s) \qquad \text{(A.14)}$$

By looking carefully at the integral (A.9) one finds that Disc T_4 behaves as

$$\text{Disc}\, T_4(s,\bar{t}) \begin{cases} \to g_{TR}\, s^{\alpha(\bar{t})} \log\log s & \text{if } \log s \ll \frac{1}{\alpha(\bar{t}) - 2\alpha(\bar{t}/4)+1} \\ \to g_{TR}\, s^{\alpha(\bar{t})} \log \frac{1}{\alpha(\bar{t}) - 2\alpha(\bar{t}/4)+1} & \text{if } \log s \gg \frac{1}{\alpha(\bar{t}) - 2\alpha(\bar{t}/4)+1} \end{cases} \qquad \text{(A.15)}$$

The first behaviour, in particular, is the one that holds at the point $\alpha(\bar{t}) = \alpha_c(\bar{t})$ and represents the combined effect of the pole and the cut which coincide with a singular behaviour.

Before going into the applications of the above results needed for the rest of this paper, we remark that this approach might lead to some better understanding of Regge cuts. The appearance of fixed pole residue in the strength of the cut should be compared to the results of Lovelace and others[38] where the square of such residue appears. We remark, however, that our scheme is not in the spirit of a self-consistent determination of cuts, since we have neglected cuts in the $\bar{t}$-channel of the six-point function in the first place.

Finally, we remark that in the dual model the first two diagrams of Fig. A.3 (components 1 and 2 in Section 3) do not have fixed poles and hence do not yield cuts. A contribution leading to cuts is the third one of Fig. A.3 which is known to have an additive fixed pole [with residue $2^{-\alpha(\bar{t})-\alpha(t_1)-\alpha(t_2)-1}$] in the $\bar{t}$-channel. However, such a term does not scale, and will be neglected in our analysis of inclusive spectra.

We are now in a position to discuss, as a particular example of our general scheme, the two cases in which the cut at $\alpha_1 + \alpha_2 - 1$ can be important. Clearly, only the case $\alpha_1 \sim \frac{1}{2}$, $\alpha_2 \sim 1$ (or vice versa) and the case $\alpha_1 = \alpha_2 \sim 1$ are of any relevance when we neglect terms lower than $s^{-\frac{1}{2}}$.

The first case leads to Regge-Pomeron cuts. Such effects have been neglected for total cross-sections as far as the qualitative features of the Harari-Freund scheme are concerned (such as positivity of the two components). Hence we shall neglect such effects also at the six-point function level, i.e. we shall neglect non-diagonal contributions. More discussion on this matter can be found in Section 5.

We now turn to the case $\alpha_1 = \alpha_2 \sim 1$, i.e. to diffraction dissociation[39]. We are now in the forward direction $\bar{t} = 0$, $t_1 = t_2 = t$. If the trajectory $\alpha(\bar{t})$ is itself a Pomeron we have $\alpha(0) \sim 1$, and we end up with the familiar triple-Pomeron problem, already widely discussed in the literature[24]. One finds that Eq. (A.15) becomes inconsistent with the assumption of a leading pole and therefore demands $g_{TR}(0,0,0) \xrightarrow[\alpha_0 \to 1]{} 0$. This is still consistent with positivity because the point $\bar{t} = t = 0$ is just on the boundary of the physical region. Unfortunately it can be shown[25] that $g_{TR} \to 0$ implies that other Pomeron vertices vanish as well in the limit of $\alpha_0 \to 1$, and it has been argued[40] that the Pomeron might have to be decoupled even from total cross-sections. If this should be true, one could take the preceding discussion as suggesting that, as $\alpha(0) \to 1$, the Pomeron-Pomeron cut at $\alpha_c = 2\alpha(0) - 1$ also approaches 1 and may combine with the pole in a complex singularity. Neglecting the effect of the cut becomes unphysical in the limit $\alpha(0) \to 1$. Otherwise one can assume $\alpha(0) = 1 - \delta$ ($\delta \ll 1$).

Finally we discuss the case in which $\alpha(\bar{t})$ is not the Pomeron itself [e.g. $\bar{\alpha}(\bar{t} = 0) \sim \frac{1}{2}$]. In this case the point $\alpha(0) - 2\,\alpha_1(t) + 1 = 0$ is inside the physical region (the cut dominates over the pole). If there is a positivity requirement on this part of DD (for instance, if this is related to dissociation into resonances), then the fixed pole must be there and must erase the zero of g_{TR} at the nonsense point (i.e. it must be multiplicative). In this case we clearly have a leading cut [$\sim s^{2\alpha_p(0)-1}/\log s$] with a positive sign and a negative Regge term [$\sim -s^{\bar{\alpha}(0)}$]. Yet it is difficult to estimate the over-all sign of the non-leading term, since the rest of the fragmentation region will contribute a positive $s^{\bar{\alpha}(0)}$ term and one cannot decide which one will win.

This lack of knowledge of the DD contributions may somewhat limit the predictive power of our model. In the absence of a reliable dual model and of careful calculations of many loop diagrams, one has to rely as much as possible on experimental and other theoretical information. It is crucial to know which trajectories contribute most in the $\bar{t}$-channel when $\alpha_1 = \alpha_2 = \alpha_{\text{Pomeron}}$. This can be done by some careful analysis of the missing-mass distribution.

Table (A.1)

R = Reggeon with intercept $\frac{1}{2}$
P = Pomeron with intercept 1

Here, a, b, c, d are positive constants (i.e. independent of s); for the charge sum rule, summation over species of particle c has not been carried out yet.

	Double Regge exchange		Contribution to σ_T	
	$x > 0$	$x < 0$	$x > 0$	$x \lesssim 0$
Eq. (2.1)	P	P	1	1
	P	R	1	$s^{-\frac{1}{2}}$
	R	P	$s^{-\frac{1}{2}}$	1
	R	R	$s^{-\frac{1}{2}}$	$s^{-\frac{1}{2}}$
Eq. (2.2)	P	P	$\ln s + c$	$\ln s + c'$
	P	R	$1 - as^{-\frac{1}{4}}$	$as^{-\frac{1}{4}} - bs^{-\frac{1}{2}}$
	R	P	$as^{-\frac{1}{4}} - bs^{-\frac{1}{2}}$	$1 - as^{-\frac{1}{4}}$
	R	R	$s^{-\frac{1}{2}}(\ln s + d)$	$s^{-\frac{1}{2}}(\ln s + d)$

APPENDIX B

UNITARIZATION OF DUAL PRODUCTION AMPLITUDES

In this Appendix to Section 3 we approach the problem of unitarity corrections to dual production amplitudes along the lines followed by one of us in Ref. 10 for two-body reactions. Our "naive" unitarization scheme follows quite closely the methods successfully employed in nuclear physics in the so-called compound nucleus model of nuclear reactions[16]. More specifically, we shall make use of the projection operator technique developed by Feshbach[41], which allows a neat and simple way of dealing with problems of many discrete levels (the resonances) coupled to a continuum.

We start by recalling again that the tree dual diagrams for production amplitudes can be classified according to the two types of Figs. 1a and 1b. This suggests a useful decomposition of the Hilbert space spanned by the eigenvectors of $H^{(0)}$, the free dual Hamiltonian, through three projection operators

$$1 = |P\rangle\langle P| + |R\rangle\langle R| + |Q\rangle\langle Q| = P + R + Q$$

$$P^2 = P \; ; \; R^2 = R \; ; \; Q^2 = Q \; ; \; PQ = RQ = PR = 0 \; . \qquad \text{(B.1)}$$

P is the projection operator in the stable multiparticle states (i.e. our initial and final states), called in nuclear physics the open channels; R is the projection operator over single-particle states (i.e. one particle or one resonance); and Q spans the remaining space (two resonances, three resonances, etc.). As usual, the full Hamiltonian $H = H^{(0)} + H^{(1)}$ is projected as:

$$H = H_{PP} + H_{RR} + H_{QQ} + H_{PQ} + H_{QP} + H_{PR} + H_{RP} + H_{RQ} + H_{QR} \; , \qquad \text{(B.2)}$$

where

$$H_{PQ} = PHQ \; , \quad H_{RR} = RHR \; , \text{ etc.} \qquad \text{(B.2)}$$

We now identify the various interaction terms H_{ij} $(i,j = P,R,Q)$.

Looking at Figs. 1a and 1b, we see that all production processes in the planar dual model are described by the free Hamiltonian $H^{(0)}$ and by the following interaction terms:

1) H_{PR} and H_{RP}, which correspond to the dual vertex coupling a single resonance to the continuum (Fig. B.1a).

2) H_{PQ} and H_{QP} couple P space to the n-resonance state in a way which is the direct product of n disconnected couplings of the H_{RP} type (Fig. B.1b). Besides this, however, the two-particle state in P couples to the two-resonance state in Q through a (t,u) beta-function term as indicated in Fig. B.1c. Similarly, there would be couplings of an n-particle state in P to an n-resonance state in Q in the form of a real 2n-point function.

3) In analogy with the above connected term, there are B(-t,-u)-type interactions such as H_{PP} (P = two-particle state) and H_{QQ} (Figs. B.1d,e).

This exhausts the couplings of our Hamiltonian. We stress that, by having included in H_{PR} the whole dual amplitude, we must neglect direct H_{QR} couplings in order to avoid double counting.

With the preceding discussion in mind, we can simplify Eq. (B.2) and write:

$$H = K + V_1 + V_2 \tag{B.3}$$

with

$$K = H_{PP}^{(0)} \tag{B.4}$$

$$V_1 = H_{PP}^{(1)} + H_{PQ} + H_{QP} + H_{QQ} \tag{B.5}$$

$$V_2 = H_{PR} + H_{RP} + H_{RR}^{(0)} \tag{B.6}$$

We first solve the case of $V_2 = 0$. This is actually the whole story when the s-channel is exotic. We have

$$H = K + W_1 + W_2$$
$$W_1 = H_{PP}^{(1)} \; ; \; W_2 = H_{PQ} + H_{QP} + H_{QQ} \tag{B.7}$$

If we denote by $\chi^{(\pm)}$ the solutions of

$$(E - K - W_1)\,\chi^{(\pm)} = 0 \tag{B.8}$$

the two-potential theorm[42] gives:

$$S_{fi}^{(V_2=0)} \equiv \langle f|S|i\rangle = \langle \chi_f^{(-)} | \chi_i^{(+)} \rangle + \langle \chi_f^{(-)} | H_{PQ}\, Q\, | \phi_i^{(+)} \rangle \tag{B.9}$$

where $|\phi^{(+)}\rangle$ is the incoming wave solution of the coupled set of Lippmann-Schwinger equations:

$$(E - K - W_1)\, P\phi^{(\pm)} = H_{PQ}\, Q\phi^{(\pm)}$$
$$(E - H_{QQ})\, Q\phi^{(\pm)} = H_{QP}\, P\phi^{(\pm)} \tag{B.10}$$

Equations (B.10) can be solved by first obtaining $P\phi$ from the first equation:

$$P\phi = \chi + \frac{1}{E - K - W_1} H_{PQ}\, Q\phi \tag{B.11}$$

and then replacing it into the second. This gives:

$$Q\phi^{(+)} = \frac{1}{E - H_{QQ} - W_{QQ}} H_{QP}\, \chi^{(+)} \tag{B.12}$$

with

$$W_{QQ} = H_{QP} \frac{1}{E - H_{PP}} H_{PQ} \tag{B.13}$$

Combining Eqs. (B.12) and (B.9), one gets

$$S_{fi}^{(V_2 = 0)} = \langle \chi_f^{(-)} | \chi_i^{(+)} \rangle + \langle \chi_f^{(-)} | H_{PQ} \frac{1}{E - H_{QQ} - W_{QQ}} H_{QP} | \chi_i^{(+)} \rangle \equiv$$
$$\equiv \langle \phi_f^{(-)} | \phi_i^{(+)} \rangle . \tag{B.14}$$

This solves the problem for $V_2 = 0$. The resulting T-matrix has the structure of a Breit-Wigner formula (corrected for final-state interaction), except for the fact that the "propagator" $(E - H_{QQ} - W_{QQ})^{-1}$ is not a single-resonance state but a many-resonance state. This fact produces extra complications (with respect to the ordinary Breit-Wigner formula) because there is a direct interaction in Q-space $[H_{QQ} \neq H_{QQ}^{(0)}]$ as well as one which, through P-space, connects states in Q with a different number of resonances. However, in the approximation in which we neglect $H_{QQ}^{(1)}$ [and, for simplicity, also $H_{PP}^{(1)}$], one finds that within a narrow-resonance approximation the second term in (B.14) is simply given by the graphs of Fig. B.2, where the wiggly line with a dot denotes a renormalized propagator

$(E - M_{RR})^{-1}$, and the mass matrix M_{RR} is given by

$$M_{RR} = H^{(0)}_{RR} + H_{RP} \frac{1}{E - \bar{H}_{PP}} H_{PR} \tag{B.15}$$

with

$$\bar{H}_{PP} = H_{PP} + H_{PQ} \frac{1}{E - H_{QQ}} H_{QP} \,. \tag{B.16}$$

At this point one can add back the $H^{(1)}_{QQ}$ and $H^{(1)}_{PP}$ interactions and then obtain the whole set of diagrams of Fig. B.3, where every single pair of resonances or of stable particles is inserted with its renormalized propagator, which is the result one would naively expect. One end of the chain consists simply of an interaction like that of Fig. B.1c, whereas the other end contains both that interaction and the one of Fig. B.1b.

The final result is again indicated in Fig. B.3, where one has already summed up all diagrams with an even number of exchanges and has written the sum over the odd number of exchanges as a one-particle exchange times the previous graph. The term in Fig. B.3b has vacuum quantum numbers exchanged, whereas that of Fig. B.3a has the same quantum number structure as the (t,u) beta-function term. We recognize in these two terms what is called T_{DD} and T_{Real} in Section 3.

The third component T_{Res} is easily obtained by adding (for ab non-exotic) the coupling to R-space. Using the two-potential theorem again, we have

$$S_{fi} = \langle \phi^{(-)}_f | \phi^{(+)}_i \rangle + \langle \phi^{(-)}_f | H_{PR} \, R | \psi^{(+)}_i \rangle \,, \tag{B.17}$$

where $\psi^{(+)}_i$ is the incoming wave solution of

$$(E - K - V_1 - V_2) \, \psi^{(+)} = 0 \,. \tag{B.18}$$

Projecting Eq. (B.18) in the three spaces gives

$$\begin{aligned} (E - H_{RR}) \, R\psi &= H_{RP} \, P\psi \\ (E - H_{PP}) \, P\psi &= H_{PQ} \, Q\psi + H_{PR} \, R\psi \\ (E - H_{QQ}) \, Q\psi &= H_{QP} \, P\psi \end{aligned} \tag{B.19}$$

One can eliminate $Q\psi$ from the last of equations (B.19) and get instead of the second one:

$$(E - \bar{H}_{PP})\, P\psi = H_{PR}\, R\psi, \tag{B.20}$$

where $\bar{H}_{PP}$ is given by Eq. (B.16).

At this point the solution for $R|\psi^{(+)}\rangle$ (and therefore for S_{fi}) is the standard one

$$S_{fi} = \langle \phi_f^{(-)} | \phi_i^{(+)} \rangle + \langle \phi_f^{(-)} | H_{PR} \frac{1}{E - M_{RR}} H_{RP} | \phi_i^{(+)} \rangle, \tag{B.21}$$

where the mass matrix M_{RR} is given in Eq. (B.15).

The second term of Eq. (B.21) is to be identified with the component T_{Res} discussed qualitatively in the text. Hence Eq. (3.7) follows immediately from Eq. (B.21).

We should stress here that our procedure of unitarization is not completely conventional because of the fact that H_{RP}, as an example, has a structure (poles) unlike that of an ordinary potential. One could proceed, perhaps in a better way, by first "unitarizing" H_{RP} and then inserting the renormalized vertex instead of the bare one. The net end result should, however, coincide with our Eqs. (B.21) and (3.7), which appear to be extremely reasonable (see discussion in Section 3) from a physical point of view.

APPENDIX C

INTERNAL QUANTUM NUMBER STRUCTURE OF ICS

We give here the details of the role which the Chan-Paton factors[18] play in our scheme. We start by looking at the total cross-section, and then consider inclusive cross-sections. In both cases we consider an SU(3) world of pseudoscalar mesons, which occur in a nonet. For the moment, only the diagonal terms will be studied. We also show that the scheme can easily be generalized to the case of n-particle inclusive sum rules; hence consistency is established. We also learn that a world with only pions is not consistent. This implies the need for an isoscalar meson in an SU(2) world with pions. Finally, interference terms will be taken into consideration. We argue that they are not responsible for the negative non-scaling pieces.

Let us introduce the notations, following Chan and Paton[18]. The SU(3) λ- matrices are normalized so that $\frac{1}{2}$ Trace$(\lambda_i\bar{\lambda}_i) = (\lambda_i\bar{\lambda}_i) = 1$. Note that we have introduced a simplified notation by dropping the words "$\frac{1}{2}$ Trace". For the time being we will assume exact SU(3), so that the dual amplitudes will be the same for all particles in the same multiplet.

The prescription goes as follows: give each external particle i its SU(3) label. The desired Chan-Paton factor for the dual tree term corresponding to the ordering (1, 2, ..., n) is simply the trace $(\lambda_1\lambda_2, \ldots, \lambda_n)$. This is finally multiplied by the dual amplitude. The same prescription holds if some of the external particles are excited states. For convenience, we will sometimes write the Chan-Paton factor as (1, 2, 3, ..., n), where the label for each particle can be either an SU(3) matrix or an SU(2) matrix. They have the following properties:

1) cyclic symmetry: $(\lambda_1 \ldots \lambda_n) = (\lambda_i \ldots \lambda_n \lambda_1 \ldots \lambda_{i-1})$;

2) factorization: $(\lambda_1 \ldots \lambda_n) = \sum_{i=1}^{9} (\lambda_1 \ldots \lambda_j\lambda_i)(\bar{\lambda}_i\lambda_{j+1} \ldots \lambda_n)$;
   note that $(\lambda_i\bar{\lambda}_j) = \delta_{ij}$;

3) absence of exotics;

4) for forward elastic scattering, the trace factor is always positive semi-definite. $(\lambda_1 \ldots \lambda_n\ \bar{\lambda}_n \ldots \bar{\lambda}_1) \geq 0$;

5) absence of non-orientable loops (Shapiro, Ref. 18);

6) invariant under the insertion of planar loops.

We will also demonstrate how to write trace factors for orientable, non-planar, loop diagrams. Property (2) will be used extensively in this appendix.

Note that, in general, $(\lambda_i\lambda_j \ldots \lambda_n)$ is not equal to $(\lambda_n \ldots \lambda_j\lambda_i)$. Physically, this means that two mesons can exchange either a quark or an antiquark,

giving rise to two different duality diagrams and therefore to two different contributions to the scattering amplitude. Only when both exchanges are forbidden (i.e. exotic) should the trace factors be zero.

The reader can convince himself that factorization and all other properties of the Chan-Paton factors will be satisfied by our symmetrized trace. Actually, even for the case of no quantum numbers, a factor of two is needed. In the following we may write $(\lambda_1 \ldots \lambda_n)$, implying always $[(\lambda_1 \ldots \lambda_n) + (\lambda_n \ldots \lambda_1)]$.

In σ_T we have the two terms corresponding to the two components (see Fig. 2). One finds:

$$\sum_j' (\lambda_a \lambda_b \lambda_j)(\bar{\lambda}_j \bar{\lambda}_b \bar{\lambda}_a) = \alpha_1 (\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a) \tag{C.1}$$

$$\sum_{\ell,m}' (\lambda_a \bar{\lambda}_\ell \lambda_b \bar{\lambda}_m)(\lambda_\ell \bar{\lambda}_a \lambda_m \bar{\lambda}_b) = \alpha_2 \left[(\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) + (\lambda_a \bar{\lambda}_b)(\lambda_b \bar{\lambda}_a)\right]$$

$$\sum_{\ell,m} (\lambda_a \bar{\lambda}_\ell)(\lambda_b \bar{\lambda}_m)(\lambda_\ell \bar{\lambda}_a)(\lambda_m \bar{\lambda}_b) = \alpha_3 (\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) \tag{C.2}$$

where α_1, α_2, and α_3 are constants. The sums over j, ℓ, m are over the complete SU(3) nonet. This gives

$$\sigma_{ab} = (\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a)\, \tilde{C}_1 (s/s_0)^{-1/2} + (\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b)\, C_2 \tag{C.3}$$

If the channel ab is exotic, the first term will vanish, while the second term will always be there. The latter trace factor has vacuum quantum number exchange between a and b either in the t-channel or the u-channel, and supports the contention that it is the Pomeron term. It is easy to convince oneself that the Pomeron with unitarity corrections will still have the same quantum number structure. Note that the Chan-Paton factors are obtained by summing over the intermediate resonance states in the production amplitudes. Similarly, $d\sigma/dp_c$ has the appropriate trace factors for all of its components. For example, that of component 4 is (Fig. 3c)

$$\sum_{i,j}' (\lambda_a \lambda_b \bar{\lambda}_i \bar{\lambda}_c \bar{\lambda}_j)(\lambda_j \lambda_c \lambda_i \bar{\lambda}_b \bar{\lambda}_a) =$$

$$= \text{const.}\, (\lambda_c \bar{\lambda}_c)(\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a) + \text{non-diagonal terms.}$$

(We remind our reader that symmetrization over the cyclic and the anticyclic term in each case is implied.) The trace factors for components 1, 2, 3, 5, 6, 7, 8, and 9 are, respectively, $(\lambda_a\lambda_b\bar{\lambda}_c\lambda_c\bar{\lambda}_b\bar{\lambda}_a)$, $(\bar{\lambda}_c\lambda_a\lambda_b\bar{\lambda}_b\bar{\lambda}_a\lambda_c)$, $(\lambda_a\bar{\lambda}_c\lambda_b\bar{\lambda}_b\lambda_c\bar{\lambda}_a)$, $(\lambda_a\bar{\lambda}_a)(\lambda_b\bar{\lambda}_c\lambda_c\bar{\lambda}_b)$, $(\lambda_b\bar{\lambda}_b)(\lambda_a\bar{\lambda}_c\lambda_c\bar{\lambda}_a)$, $(\lambda_a\bar{\lambda}_a)(\lambda_b\bar{\lambda}_b)(\lambda_c\bar{\lambda}_c)$, $(\lambda_a\bar{\lambda}_c)(\lambda_c\bar{\lambda}_a)(\lambda_b\bar{\lambda}_b)$, and $(\lambda_b\bar{\lambda}_c)(\lambda_c\bar{\lambda}_b)(\lambda_a\bar{\lambda}_a)$. These are shown in Table 2. Non-diagonal terms, coming from interference between traces with different cyclic order, do not usually have the right Regge exchanges to yield an important contribution. According to our general attitude throughout the paper, they will be neglected (see Section 5).

Hence the i^{th} component should be

$$(\text{Trace factor})_i \ (f_i + \tilde{f}_i) \quad \text{for} \quad i = 1, \ldots, 9 \ .$$

Using the energy conservation inclusive sum rule Eq. (2.5), the contribution of the i^{th} component to the total cross-section is given by:

$$\sum_c' \int dP_{c,1}\, dx_c \ (\text{Trace factor for } c)_i \ \left(f_i + \tilde{f}_i (s/s_0)^{-1/2}\right) =$$

$$= \left[\sum_c (\text{Trace factor for } c)_i\right] \int dP_1\, dx \left(f_i + \tilde{f}_i (s/s_0)^{-1/2}\right) \tag{C.4}$$

The internal quantum number conservation sum-rule Eq. (2.2) gives

$$\left[\sum_c Q_c \ (\text{Trace factor for } c)_i\right] \int dP_c \left(f_i + \tilde{f}_i (s/s_0)^{-1/2}\right) \tag{C.5}$$

as a contribution to the total cross-section. The emerging trace factor will tell us to which of the two components of the total cross-section the i^{th} component of the inclusive cross-section will contribute. It will also tell us how much each of the components weighs. We now turn to these summations over trace factors. We will worry about diffractive dissociation components later. Before we begin, it is useful to note

$$\sum_{c=1}^{9}{}' \ (\lambda_9 \lambda_c \bar{\lambda}_c) = \sum_c (\lambda_9 \bar{\lambda}_c \lambda_c) = 9\sqrt{\tfrac{2}{3}}$$

$$\sum_c' \ (\bar{\lambda}_c \lambda_c \lambda_i) = \sum_c (\lambda_i \lambda_c \bar{\lambda}_c) = 0 \quad i = 1, 2 \ldots 8$$

$$\sum_i' \ (\lambda_j \lambda_i \lambda_\ell \bar{\lambda}_i) = 6 \quad \text{if } j = \ell = 9$$

$$= 0 \quad \text{otherwise} \tag{C.6}$$

We can see now what is the internal quantum number structure of Eqs. (C.4), (C.5) after summation over c. For components 1 through 7 we find respectively

$$\sum_c{}' \left(\lambda_a \lambda_b \bar{\lambda}_c \lambda_c \bar{\lambda}_b \bar{\lambda}_a\right) =$$

$$= \sum_c{}' \left(\bar{\lambda}_c \lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a \lambda_c\right) = 6\left(\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a\right). \tag{C.7}$$

$$\sum_c{}' \left(\lambda_a \bar{\lambda}_c \lambda_b \bar{\lambda}_b \lambda_c \bar{\lambda}_a\right) = \sum_{c,e}{}' \left(\bar{\lambda}_a \lambda_a \lambda_e\right)\left(\bar{\lambda}_e \bar{\lambda}_c \lambda_b \bar{\lambda}_b \lambda_c\right) =$$

$$= \sum_{\ell,j,c}{}' \left(\bar{\lambda}_a \lambda_a \lambda_e\right)\left(\lambda_c \bar{\lambda}_e \bar{\lambda}_c \lambda_j\right)\left(\bar{\lambda}_j \lambda_b \bar{\lambda}_b\right) = 4\left(\lambda_a \bar{\lambda}_a\right)\left(\lambda_b \bar{\lambda}_b\right). \tag{C.8}$$

$$\sum_c{}' \left(\lambda_a \bar{\lambda}_a \bar{\lambda}_b \lambda_b\right)\left(\lambda_c \bar{\lambda}_c\right) = 9\left(\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a\right). \tag{C.9}$$

$$\sum_c{}' \left(\lambda_a \bar{\lambda}_a \lambda_c \bar{\lambda}_c\right)\left(\lambda_b \bar{\lambda}_b\right) = \sum_c{}' \left(\lambda_b \bar{\lambda}_b \lambda_c \bar{\lambda}_c\right)\left(\lambda_a \bar{\lambda}_a\right) = 6\left(\lambda_a \bar{\lambda}_a\right)\left(\lambda_b \bar{\lambda}_b\right). \tag{C.10}$$

$$\sum_c{}' \left(\lambda_b \bar{\lambda}_b\right)\left(\lambda_a \bar{\lambda}_a\right)\left(\lambda_c \bar{\lambda}_c\right) = 9\left(\lambda_b \bar{\lambda}_b\right)\left(\lambda_a \bar{\lambda}_a\right). \tag{C.11}$$

We immediately see that components (1), (2), and (4) contribute to the resonant part $\hat{c}_1$ of the total cross-section, while components (3), (5), (6), and (7) contribute to the background part $(c_2)_B$ of the total cross-section. This is obvious from the duality diagrams, or from Table 1; but it is nice to see that the same result is obtained from the trace factor formalism.

The phase-space integral of formula (C.4) is weighed by the energy of particle c, while it is the sum over trace factor that is weighed by the quantum number Q in formula (C.5). So it is necessary to check the consistency of (C.5) also. We illustrate this with the third component of isospin I_3. Here it is useful to note that

$$2 I_3 \lambda_i = \left[\lambda_3, \lambda_i\right] = \lambda_3 \lambda_i - \lambda_i \lambda_3 ,$$

$$\sum_i{}' \left(\lambda_3^2 \lambda_i \bar{\lambda}_i\right) = \sum_i \left(\lambda_8^2 \lambda_i \bar{\lambda}_i\right) = 6 \tag{C.12}$$

For some λ_A, we have

$$\sum_c{}' \left\{ I_{3,c} \left(\lambda_A \bar{\lambda}_A \lambda_c \bar{\lambda}_c\right) + I_{3,c} \left(\bar{\lambda}_c \lambda_c \bar{\lambda}_A \lambda_A\right)\right\} =$$

$$= \frac{1}{2} \sum_{c,e}{}' \left\{ \left(\lambda_A \bar{\lambda}_A \lambda_e\right)\left(2 I_{3c} \lambda_c \bar{\lambda}_c \bar{\lambda}_e\right) + \left(\bar{\lambda}_e \bar{\lambda}_A \lambda_A\right)\left(2 I_{3c} \lambda_c \lambda_e \bar{\lambda}_c\right)\right\} =$$

$$= \frac{1}{2} \sum_{c,e}{}' \left\{ \left(\lambda_e \lambda_A \bar{\lambda}_A\right)\left(\bar{\lambda}_e [\lambda_3, \lambda_c] \bar{\lambda}_c\right) + \left(\bar{\lambda}_e \bar{\lambda}_c [\lambda_3, \lambda_c]\right)\left(\lambda_e \bar{\lambda}_A \lambda_A\right)\right\} =$$

$$= \frac{1}{2} \sum_e{}' \left\{ \left(\lambda_e \lambda_A \bar{\lambda}_A\right) \sum_c \left(\bar{\lambda}_e \lambda_3 \lambda_c \bar{\lambda}_c\right) - \right.$$

$$\left. - \left(\lambda_e \bar{\lambda}_A \lambda_A\right) \sum_c \left(\lambda_3 \bar{\lambda}_e \bar{\lambda}_c \lambda_c\right)\right\} =$$

$$= \frac{1}{2}\, 2 I_{3A} \left(\lambda_A \bar{\lambda}_A\right) \sum_c \left(\lambda_3^2 \lambda_c \bar{\lambda}_c\right) = 6 I_{3A} \left(\lambda_A \bar{\lambda}_A\right), \tag{C.13}$$

where Eq. (C.6) is used to obtain line 4 from line 3.

Replacing λ_A by $\lambda_a\lambda_b$, $\lambda_b\lambda_a$, λ_a, and λ_b, etc., we obtain for components (1) plus (2), and (5) plus (6):

$$\sum_c{}' I_{3c} \left\{ \left(\lambda_a \lambda_b \bar{\lambda}_c \lambda_c \bar{\lambda}_b \bar{\lambda}_a\right) + \left(\bar{\lambda}_c \lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a \lambda_c\right)\right\} =$$

$$= 6 \left(I_{3a} + I_{3b}\right) \left[\left(\lambda_a \lambda_b \bar{\lambda}_b \bar{\lambda}_a\right) + \left(\lambda_b \lambda_a \bar{\lambda}_a \bar{\lambda}_b\right)\right], \tag{C.14}$$

$$\sum_c I_{3c} \left\{ \left(\lambda_a \bar{\lambda}_a \lambda_c \bar{\lambda}_c\right)\left(\lambda_b \bar{\lambda}_b\right) + \left(\bar{\lambda}_c \lambda_c \bar{\lambda}_a \lambda_a\right)\left(\lambda_b \bar{\lambda}_b\right) + \right.$$

$$\left. + (a \leftrightarrow b)\right\} =$$

$$= 6 \left(I_{3a} + I_{3b}\right) \left(\lambda_a \bar{\lambda}_a\right)\left(\lambda_b \bar{\lambda}_b\right) \tag{C.15}$$

Note that formula (C.13) is not applicable to components (1) and (2) separately.

For component (3), it goes as follows:

$$\sum_c{}' \left\{ I_{3c} \left(\lambda_a \bar{\lambda}_a \lambda_c \bar{\lambda}_b \lambda_b \bar{\lambda}_c \right) + I_{3c} \left(\bar{\lambda}_c \lambda_b \bar{\lambda}_b \lambda_c \bar{\lambda}_a \lambda_a \right) \right\} =$$

$$= \frac{1}{2} \sum_{j,\ell,c}{}' \left\{ (\lambda_\ell \lambda_a \bar{\lambda}_a)(\bar{\lambda}_c \bar{\lambda}_\ell [\lambda_3, \lambda_c] \bar{\lambda}_j)(\lambda_j \bar{\lambda}_b \lambda_b) + \right.$$

$$\left. + (\lambda_\ell \bar{\lambda}_a \lambda_a)([\lambda_3, \lambda_c] \bar{\lambda}_\ell \bar{\lambda}_c \bar{\lambda}_j)(\lambda_j \lambda_b \bar{\lambda}_b) \right\} =$$

$$\frac{1}{2} \frac{2}{3} 6 \left\{ ([\lambda_3, \lambda_a] \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) + (\lambda_a \bar{\lambda}_a)([\lambda_3, \lambda_b] \bar{\lambda}_b) \right\} =$$

$$= 4 \left[I_{3a} + I_{3b} \right] (\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b), \tag{C.16}$$

where only terms with $\ell = 3$, $j = 9$, and $\ell = 9$, $j = 3$ contribute in line 2 where formula (C.6) is used. It is trivial to see that components (4), (7) do not contribute to the total cross-section in this inclusive sum rule.

We also obtain identical results using strangeness as the conserved quantum number. Since charge $Q = I_3 + (S/2)$, we have the same result for Q. Putting these together, we obtain Eqs. (4.7), (4.8), (4.9), and (4.10).

So far we have taken the DD components to be negligibly small as compared with the background terms, but it is straightforward to include them. Since DD terms have a quantum number structure which is identical to the background terms for components 5, 6, and 7, we need only discuss the last 2 components. For component 8, we have

$$\sum_c (\lambda_a \bar{\lambda}_c)(\lambda_c \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) = (\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) \tag{C.17}$$

$$\sum_c{}' I_{3,c} (\lambda_a \bar{\lambda}_c)(\lambda_c \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b) = I_{3a} (\lambda_a \bar{\lambda}_a)(\lambda_b \bar{\lambda}_b), \tag{C.18}$$

so that adding components 8 and 9 trivially, satisfies the sum rules (C.4) and (C.5).

In a world with only pions, we see that the sum over all pions in component (3) gives for the energy conservation inclusive sum rule (C.4):

$$\sum_{\pi_i}{}' (\tau_a \bar{\tau}_a \tau_i \bar{\tau}_b \tau_b \bar{\tau}_i) = (3 + I_{3a} I_{3b})(\tau_a \bar{\tau}_a)(\tau_b \bar{\tau}_b) \qquad \text{(C.19)}$$

That the r.h.s. is dependent on the third components of isospin of pions a and b is inconsistent. If an isoscalar particle is included in the sum on the l.h.s., we see that the charge dependence disappears.

The r.h.s. of (C.19) becomes $4(\tau_a\bar{\tau}_a)(\tau_b\bar{\tau}_b)$. Thus a world of pions requires also the presence of an isoscalar meson. For these trace sum rules to hold, all we need is for the phase-space integrals of different particles c to be the same; this need not imply mass degeneracy of particle c in the SU(2) world or SU(3) world.

To demonstrate the consistency for the n-particle inclusive sum rules which relates the n-particle inclusive cross-sections to (n-1)-particle inclusive cross-sections, we need to check that the following identities hold:

$$\sum_c{}' F_c \left\{(\lambda_A \bar{\lambda}_A \lambda_c \bar{\lambda}_c) + (\bar{\lambda}_c \lambda_c \bar{\lambda}_A \lambda_A)\right\} \propto F_A (\lambda_A \bar{\lambda}_A)$$

$$\sum_c{}' F_c (\lambda_A \bar{\lambda}_A)(\lambda_c \bar{\lambda}_c) \propto F_A (\lambda_A \bar{\lambda}_A)$$

$$\sum_c{}' F_c (\lambda_A \bar{\lambda}_c)(\lambda_c \bar{\lambda}_A) \propto F_A (\lambda_A \bar{\lambda}_A) \qquad \text{(C.20)}$$

$$\sum_c{}' F_c \left\{(\lambda_A \bar{\lambda}_c \lambda_B \bar{\lambda}_B \lambda_c \bar{\lambda}_A) + (\lambda_B \bar{\lambda}_c \lambda_A \bar{\lambda}_A \lambda_c \bar{\lambda}_B)\right\} \propto (F_A + F_B)(\lambda_A \bar{\lambda}_A)(\lambda_B \bar{\lambda}_B),$$

where λ_A, λ_B are in general products of λ_s. They can be considered as SU(3) representation of some resonant states; F_c is any conserved quantum number. For energy conservation, $F_c = 1$. It happens that the proof of these equations goes exactly like that of Eqs. (C.7) to (C.19). Hence we will not repeat it here.

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35) We are grateful to Mr. Kjell Hellesøe for an explicit calculation of the Jacobian.

36) M.B. Einhorn, J. Ellis and J. Finkelstein, Phys. Rev. D5, 2063 (1972).
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Figure captions

Fig. 1 : Three-component theory of production amplitudes:
a) is T_{Res}; b) is T_{Real}; c) is T_{DD}. Here, as in other figures, a wiggly line is a resonance or a stable particle and a straight line is a stable particle. The cross indicates a twisted propagator.

Fig. 2 : Components of total cross-sections. Vertical strokes indicate that a discontinuity has been taken. The last column is a shorthand notation also used in Figs. 3 and 4.

Fig. 3 : The seven components of $d\sigma_{ab}/dp_c$ arising from $T_{Res} + T_{Real}$.

Fig. 4 : The six DD components of $d\sigma_{ab}/dp_c$. Components 5′, 6′, 7′ (7″) have the same quantum number structure as components 5, 6, 7 of Fig. 3.

Fig. 5 : Dynamical origin of negative non-scaling terms in the dual model. A resonance with a dash (c) means that the ground state has been removed.

Fig. 6 : Examples of components of $d\sigma_{ab}/dp_c\, dp_d$ arising from T_{Res} (Fig. 1a).

Fig. A.1 : Graphical representation of the non-forward sum rule, Eq. (A.4).

Fig. A.2 : (a) Regge limit of the non-forward six-point function.
(b) Triple Regge limit.

Fig. A.3 : (a), (b) are components 1 and 2 of Fig. 3 and have no fixed pole;
(c) is an asymptotically vanishing contribution with a fixed pole.

Fig. B.1 : Some interaction terms in the dual Born term.

Fig. B.2 : Solution of the Lippmann-Schwinger equations neglecting various interaction terms and within the narrow resonance approximation.

Fig. B.3 : Solution of the Lippmann-Schwinger equations in the absence of coupling to R-space (e.g. for ab exotic). The narrow resonance approximation has been used.

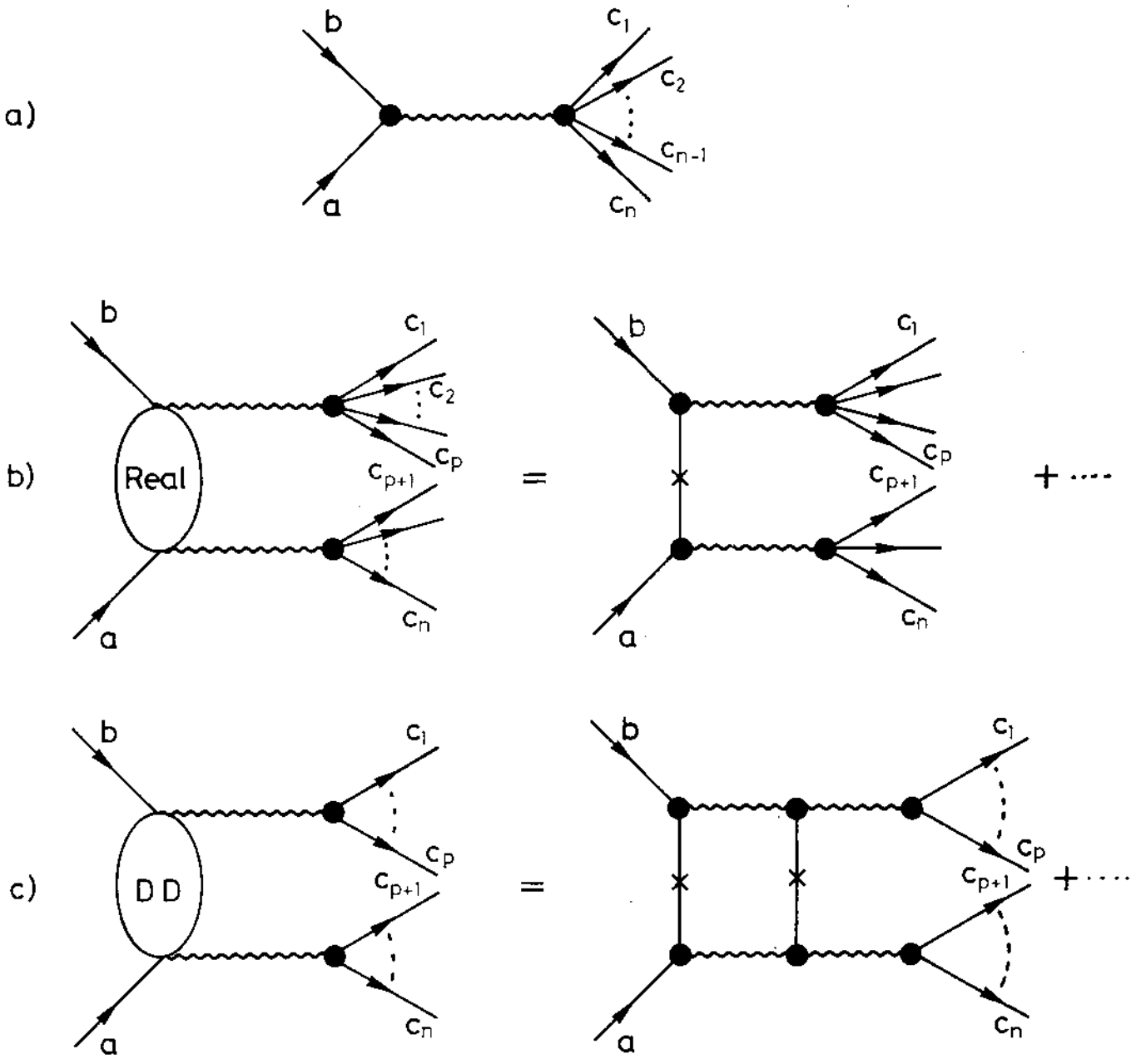


FIG.1

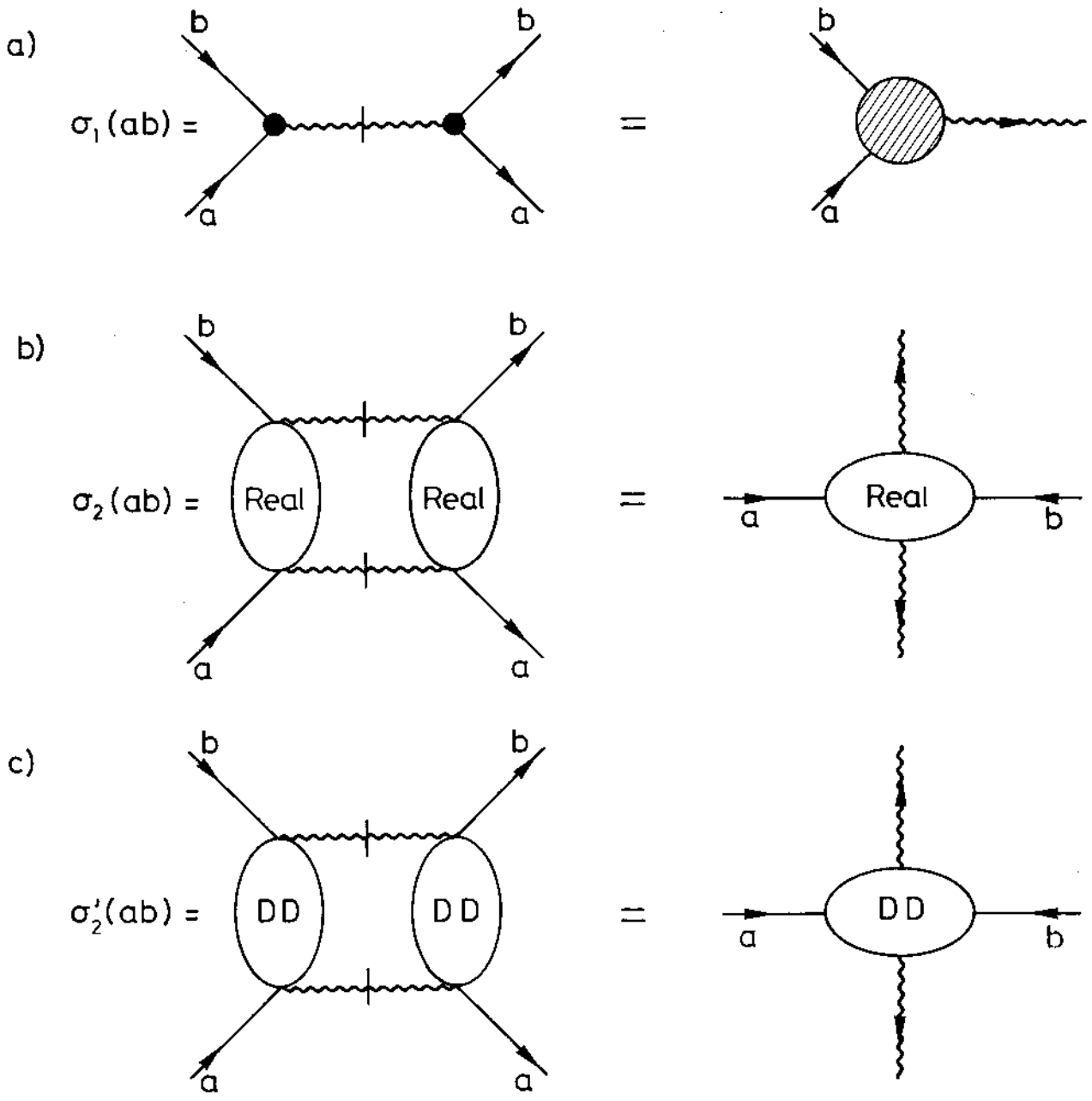

a)
$\sigma_1(ab)$ =
b
b
a
a
=
b
a
b)
$\sigma_2(ab)$ =
Real
Real
=
Real
a
b
c)
$\sigma_2'(ab)$ =
DD
DD
=
DD
a
b

FIG. 2

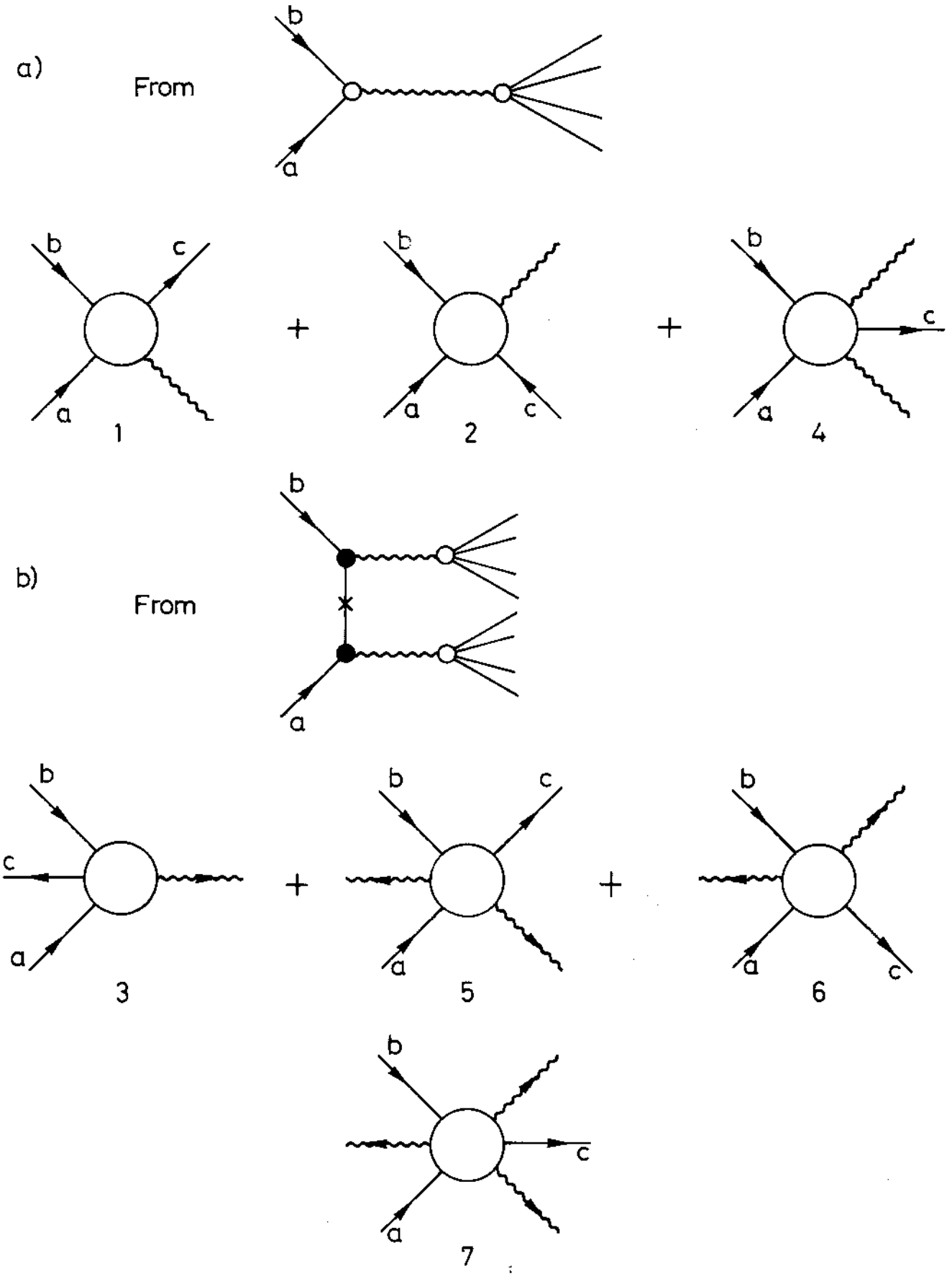


FIG. 3

c)

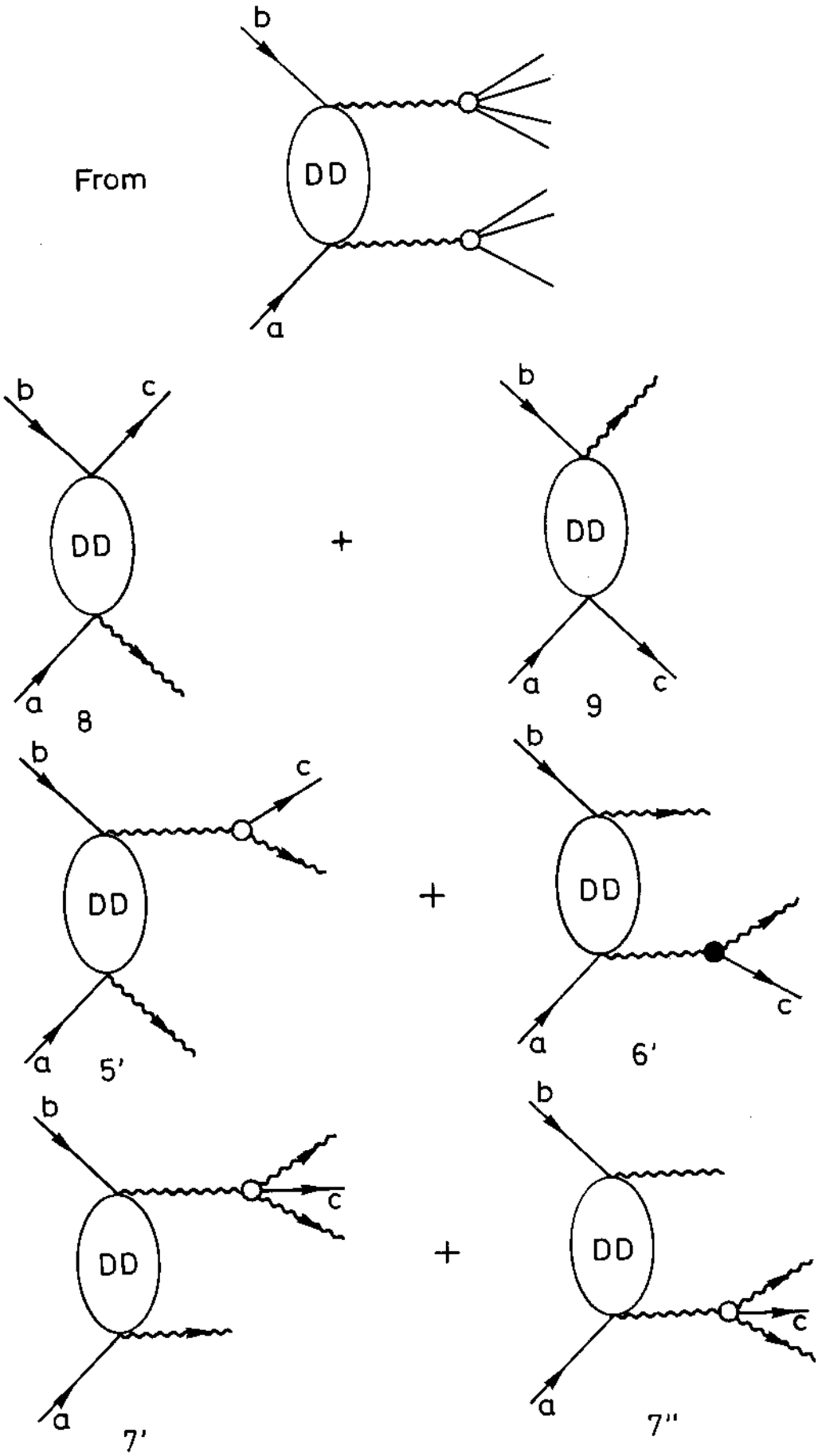
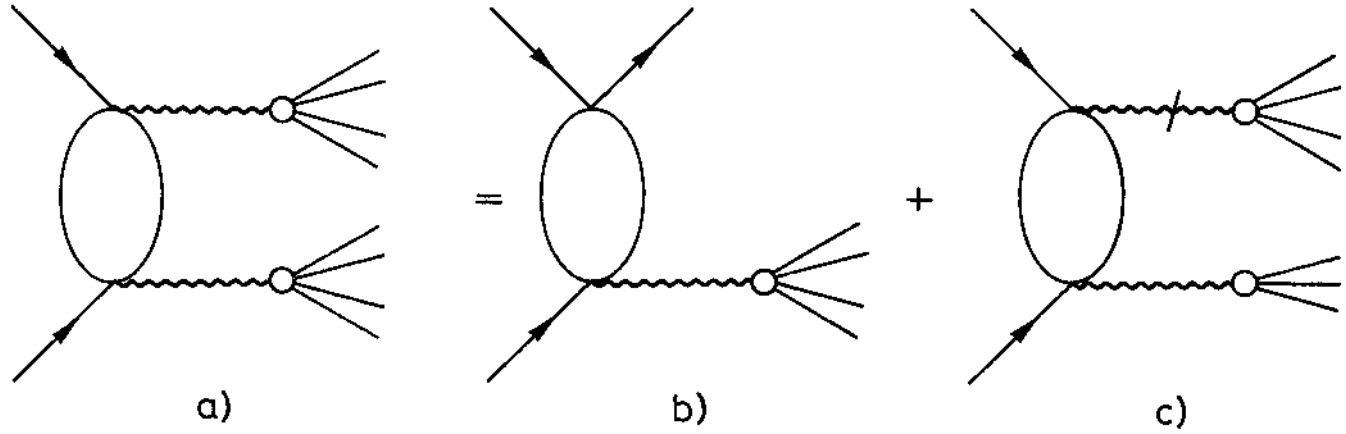
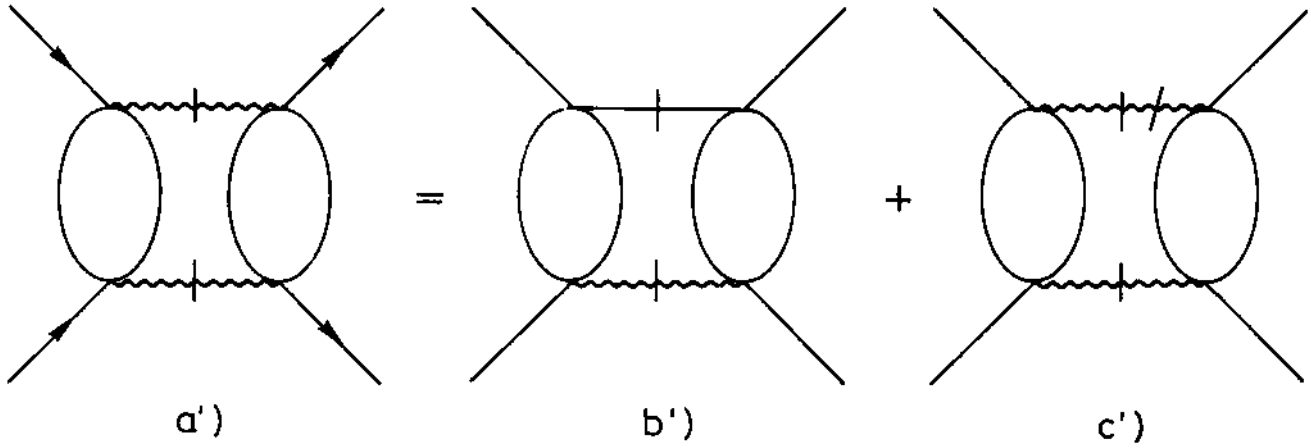


FIG. 4





$$\text{constant} = + \frac{1}{\sqrt{s}} + \left(\text{constant} - \frac{1}{\sqrt{s}}\right)$$

$$3+5+6+7 = 3 + (5+6+7)$$

FIG. 5

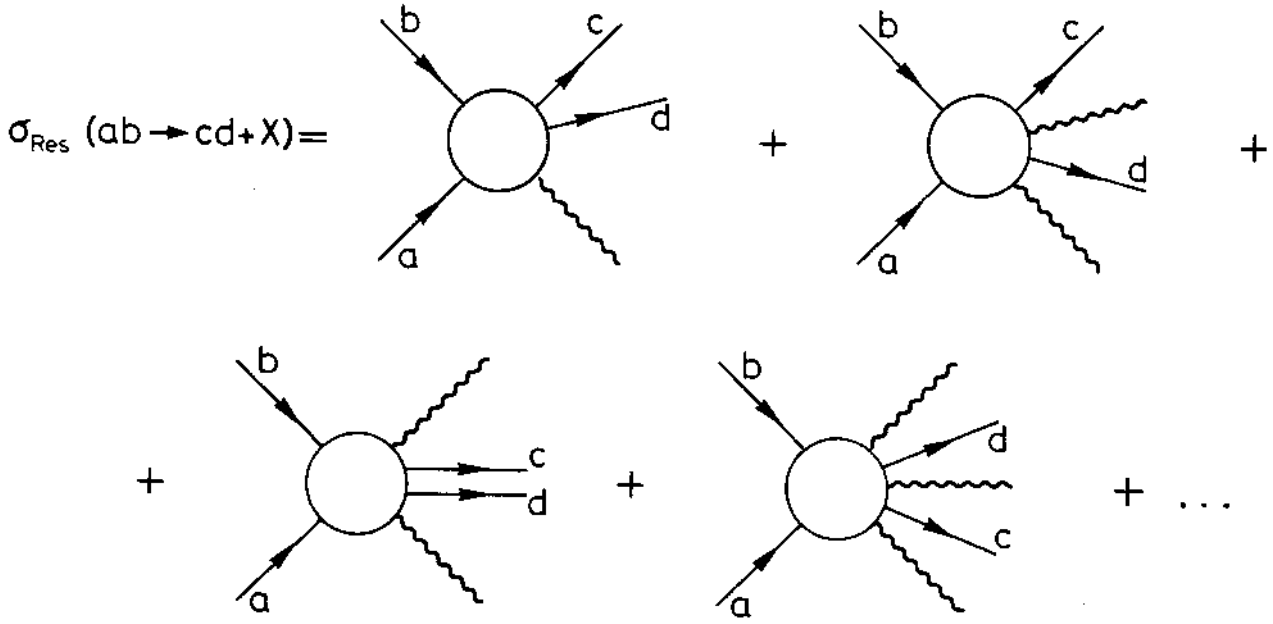


FIG.6

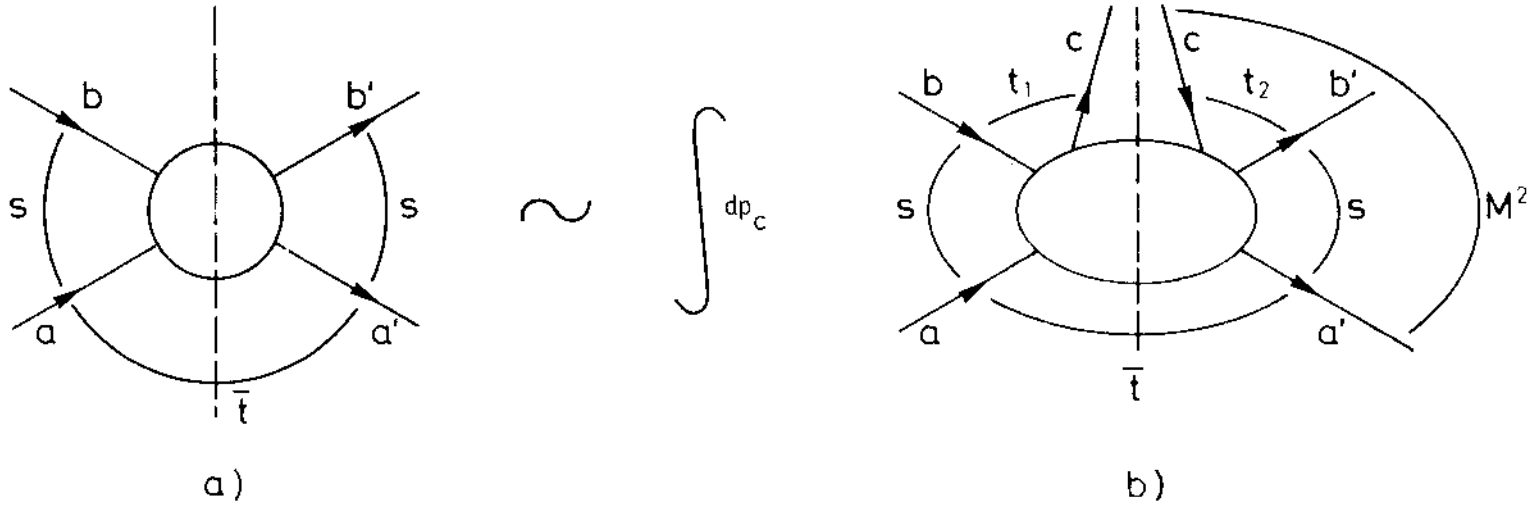
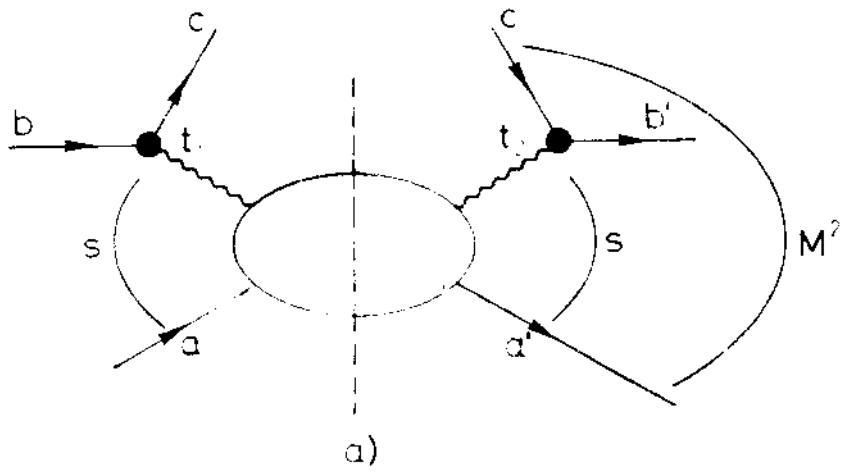
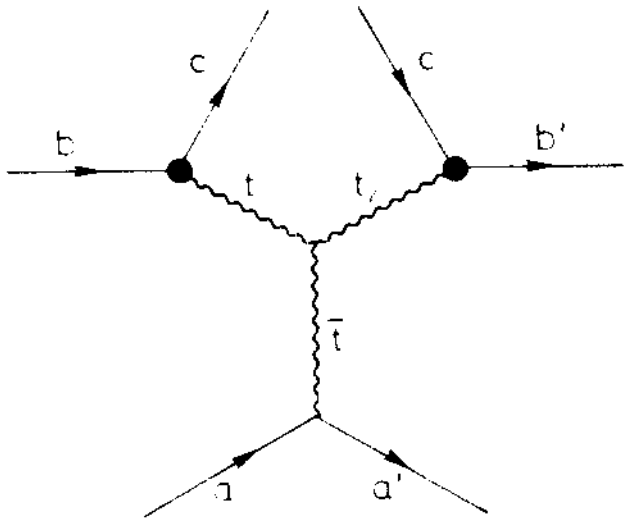


FIG. (A.1)



a)



b)

FIG (A. 2)

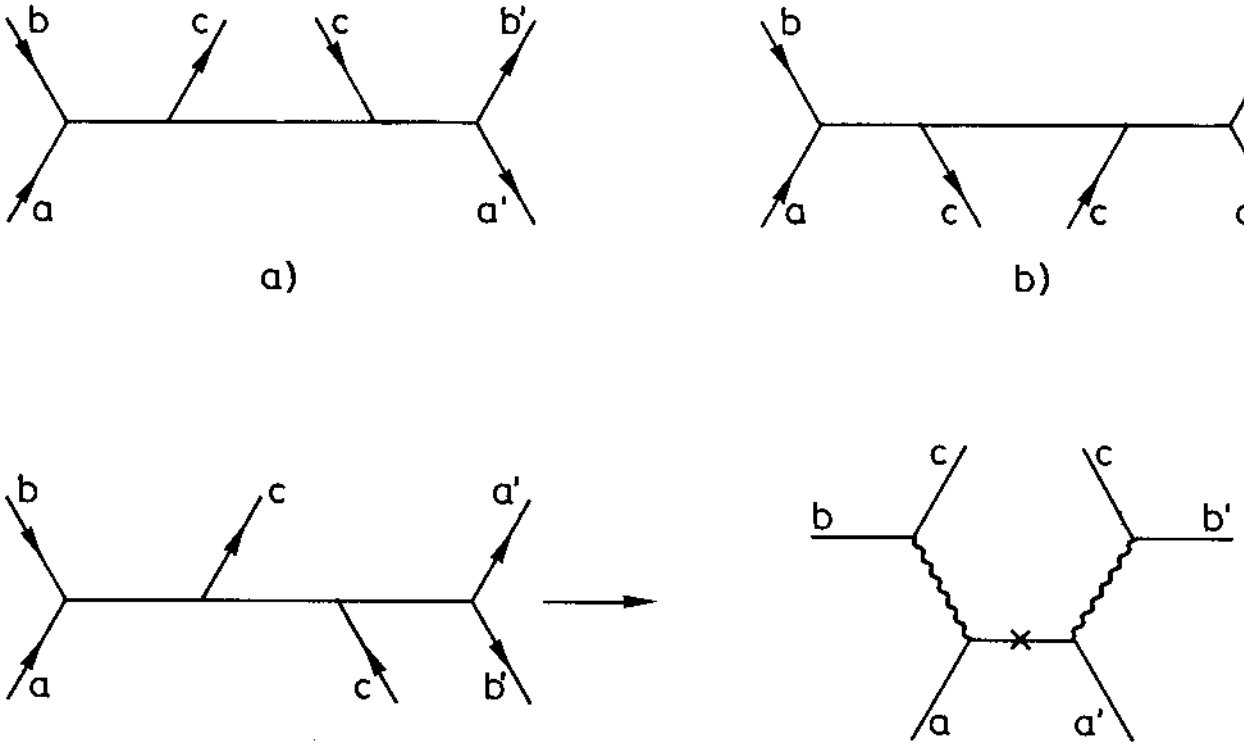


FIG.(A.3)

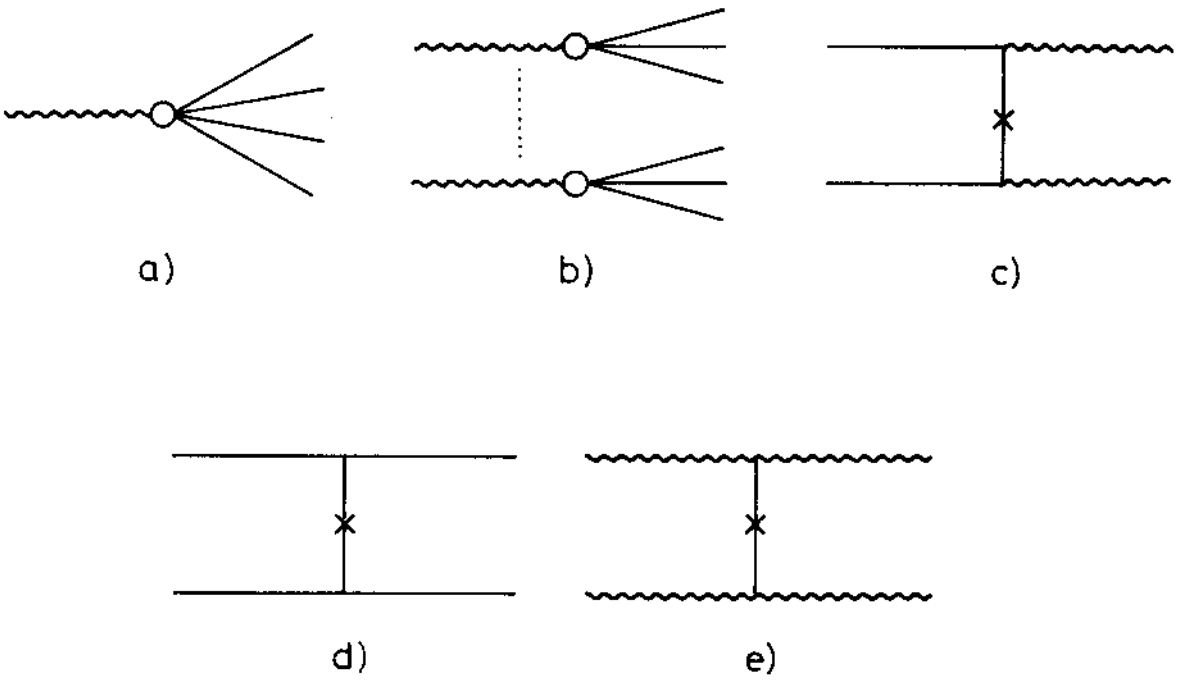


FIG. (B. 1)

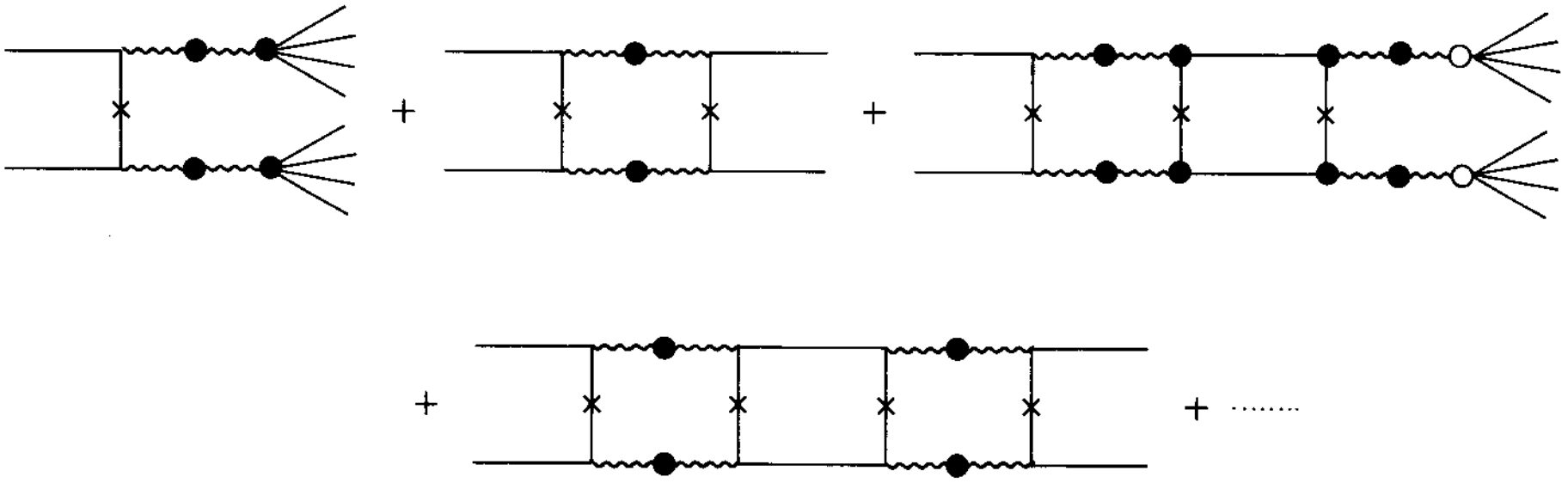


FIG. (B. 2)

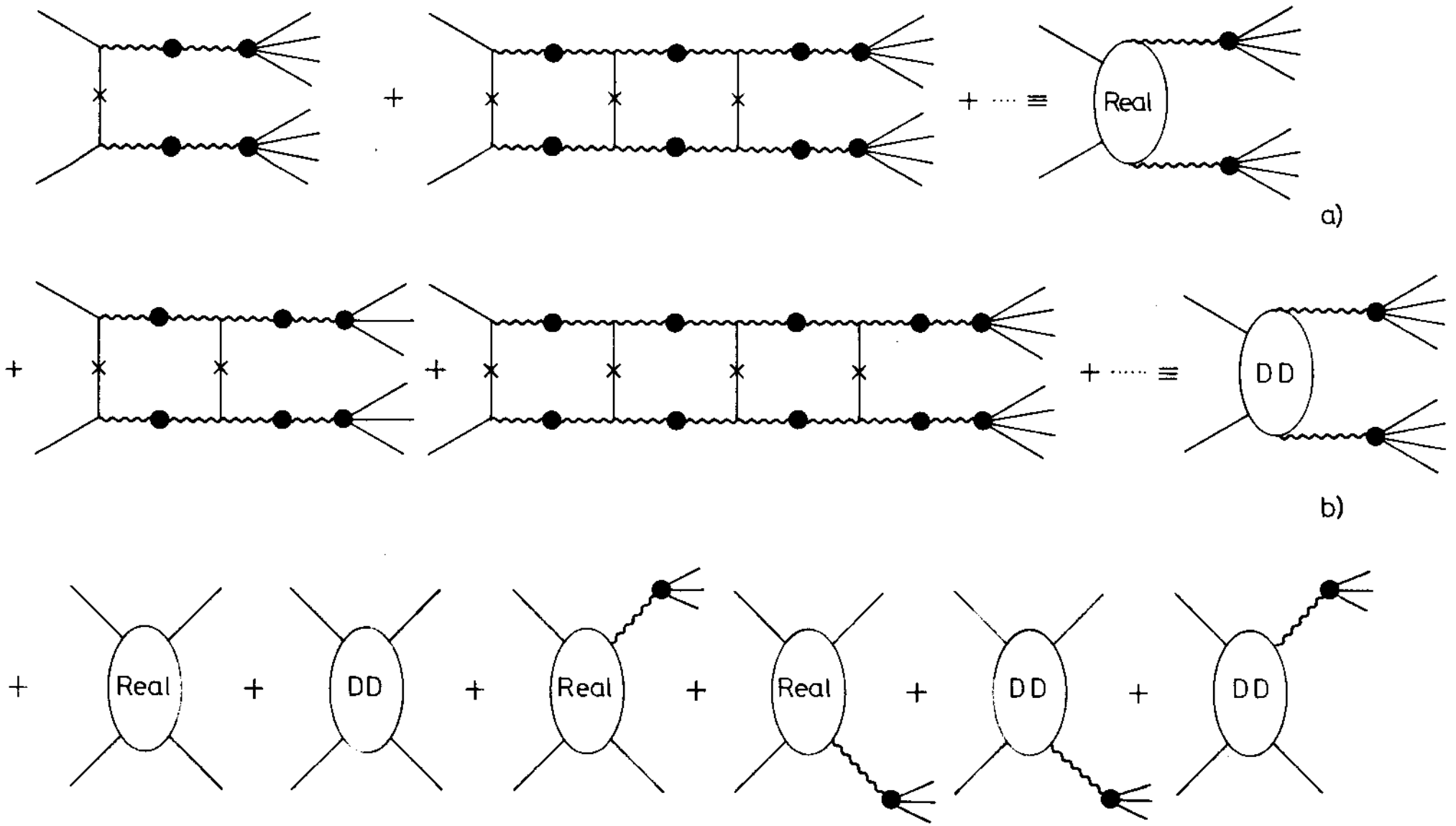


FIG.(B. 3)