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PROPERTIES OF LEGENDRE EXPANSIONS RELATED TO SERIES OF STIELTJES AND APPLICATIONS TO $\pi - \pi$ SCATTERING

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ABSTRACT

Legendre expansions whose coefficients are those of a series of Stieltjes are considered. It is shown that the analyticity domain of a function defined by such an expansion is the cut plane and that sequences of approximants may be defined which converges to the function in this domain, with each approximant determined from a finite number of coefficients in the expansion. These approximants are related to the Padé approximants of the corresponding series of Stieltjes.

It is shown that if the coefficients satisfy a "Froissart-Gribov" type representation with positive weight, then they are also coefficients of series of Stieltjes. It follows that the above results may be applied to the $\pi - \pi$ scattering amplitude $A(s,t)$ for certain states when $0 \leq s < 4$. In particular the approximation of $A(s,t)$ in the complex t plane, when only the first few partial waves $a_\ell(s)$ are known, is discussed and also considered is the interpolation of the $a_\ell(s)$ for non-integer ℓ. Another consequence is that the $a_\ell(s)$ satisfy an infinite set of determinantal inequalities when $0 \leq s < 4$.

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1. INTRODUCTION

It is well known that the analytic properties of the Legendre series

$$f(z) = \sum_{n=0}^{\infty} f_n P_n(z) \tag{1.1}$$

are related [1] to those of the corresponding power series

$$g(z) = \sum_{n=0}^{\infty} f_n (-z)^n . \tag{1.2}$$

For example, if the radius of convergence of $g(z)$ is r so that

$$\limsup_{n\to\infty} |f_n|^{1/n} = r \tag{1.3}$$

and if $r > 1$, then the Legendre series $f(z)$ converges in an ellipse with foci at $+1$ and -1 and with semi-major axes $\frac{1}{2}(r+1/r)$.

Also for z in this ellipse $f(z)$ has the representation [1]

$$f(z) = \frac{1}{\pi}\int_0^{\pi} g\left[-z-(z^2-1)^{1/2}\cos v\right] dv . \tag{1.4}$$

This is an important relation in that it allows one to connect the problem of the analytic continuation of the Legendre expansion $f(z)$ outside its ellipse of convergence to the corresponding problem of analytically continuing the power series $g(z)$ outside its circle of convergence.

We will investigate in this paper the problem of analytically continuing $f(z)$ when $g(z)$ is a series of Stieltjes [2].

The coefficients f_n are then given by

$$f_n = \int_0^{1/r} u^n \, d\phi(u) \quad , (n=0,1,2,\ldots\ldots) \tag{1.5}$$

where $\phi(u)$ is a bounded non-decreasing function taking infinitely many values and r is the radius of convergence of $g(z)$.

In this case $g(z)$ can be analytically continued to all z in the complex plane cut from $-r$ to $-\infty$ and has the representation

$$g(z) = \int_0^{1/r} \frac{d\phi(u)}{(1+uz)} \tag{1.6}$$

Even if $r=0$ so that the upper limits of the integrals in (1.5) and (1.6) are ∞, (1.6) still gives the definition of a function which is analytic in the complex plane cut from 0 to $-\infty$. This function is formally equivalent to the power series (1.2) in the sense that the latter may be obtained by expanding $(1+uz)^{-1}$ under the integral in (1.6) in powers of z and then making the illegal interchange of summation and integration [3].

We will show in Section 2 that the Legendre expansion $f(z)$ has closely analogous properties when the f_n again have the representation (1.5). For instance, we will prove that when $r > 1$, $f(z)$ can be continued outside its ellipse of convergence to the whole complex z plane cut from $\frac{1}{2}(r+1/r)$ to ∞. For $0 \leqslant r \leqslant 1$ the Legendre expansion (1.1) does not converge for any z. However, we will be able to prove, as for the power series, that a function $f(z)$ can be defined which is formally equivalent to (1.1) in some sense, and that this function is analytic in the whole complex z plane cut from 1 to ∞.

The next topic we consider is the approximation of the Legendre expansion $f(z)$ where the first few of the coefficients f_n are known. We will define approximants to $f(z)$ by using approximations to $g(z)$ in (1.4). When $g(z)$ is a series of Stieltjes, an extremely powerful method of approximation is that due to Padé [4]. The $[N,N+j]$ Padé approximant is defined in terms of the coefficients $f_0, f_1, \ldots, f_{2N+j}$ as is described in Appendix 1 and for series of Stieltjes the following convergence theorem holds.

Theorem 1

If $g(z)$ is a series of Stieltjes with radius of convergence $r > 0$ so that the coefficients f_n have the representation (1.5), the sequence $[N, N+j]$ for $j \geq -1$, converges as $N \to \infty$ to the analytic function

$$\int_0^{r} \frac{d\phi(u)}{(1+uz)}$$

for all z in the complex plane cut from $-r$ to $-\infty$. The theorem holds when $r = 0$ so long as

$$\sum_{n=0}^{\infty} (f_n)^{-1/(2n+1)}$$

diverges. [Roughly speaking one needs $f_n \leq C(2n)!$ with C independent of n].

We show in Section 3 that an analogous sequence $f_{N,j}(z)$ of approximants to the Legendre series $f(z)$ may be defined by using a generalization of the Padé method [5),6)]. The approximant $f_{N,j}(z)$ is determined, like the $[N, N+j]$ Padé approximant, from the coefficients $f_0, f_1, \ldots, f_{2N+j}$, and sequences of the $f_{N,j}(z)$ will be shown to converge to $f(z)$ for all z in the analyticity domain of $f(z)$.

Another important property of the Padé approximants $[N, N+j]$ to a series of Stieltjes is that, given the coefficients $f_0, f_1, \ldots$ $\ldots, f_{2N+j}$ needed to construct it, upper bounds can be obtained on the error between $[N, N+j]$ and $g(z)$ for $z \geq 0$. This result has been extended by the author [7)] for the case when $r > 0$ to all z in the circle of convergence of $g(z)$ and more recently by Baker [8)] to z in the whole complex plane cut from $-r$ to $-\infty$. Analogous results hold in principle for the approximants $f_{N,j}(z)$ and the error between them and $f(z)$, as follows from the representation (1.4) for $f(z)$.

It may be thought that the properties of the Legendre expansion $f(z)$ discussed above are only of mathematical interest, but it will become obvious from Section 4 that this is not the case.

We consider there the situation when the f_n have the representation

$$f_n = \frac{1}{\pi} \int_{x_0}^{\infty} Q_n(x)\, d\psi(x), \; n = 0,1,2,\ldots \tag{1.7}$$

with $x_0 > 1$ and where $Q_n(x)$ are Legendre functions of the second kind. We will prove that if $\psi(x)$ is a real bounded non-decreasing function of x, then the f_n are coefficients of series of Stieltjes with representation (1.5) and we obtain an explicit expression for the $\phi(u)$ in (1.5) in terms of $\psi(x)$. The mathematical results of the previous sections can therefore be applied to the Legendre expansion $f(z)$ in this case.

The possibility of a physical application of our results is now seen since the form (1.7) is closely related to the Froissart-Gribov representation of partial-wave amplitudes for particle scattering, and $f(z)$ would then correspond to the total scattering amplitude.

Making use of this connection we apply our results in Section 5 to two examples of $\pi\pi$ scattering. For scattering in certain isotopic spin states which will be described later the partial wave amplitudes $a_\ell(s)$ with $\ell = 2,4,6,\ldots,$ have a representation like (1.7) with $\psi(x)$ bounded and non-decreasing when $0 \leq s \leq 4$ *). This property of the $a_\ell(s)$ is rigorous in that it is a consequence of analyticity obtainable from axiomatic field theory [9] combined with unitarity and crossing.

The $a_\ell(s)$ then have representations like (1.5) from Section 4, and it follows that there are an infinite set of determinantal inequalities between them for the above values of ℓ and s. Once again we see the strong constraints placed on the $\pi\pi$ amplitudes by the requirements of analyticity, unitarity and crossing [10].

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*) We use s, t, u to denote the usual Mandelstam variables throughout this paper.

In the last part of Section 5, we illustrate the results of Section 3 concerning the properties of the approximants $f_{N,j}(z)$ by considering a simple numerical example. We assume that the amplitudes $a_2(s)$ and $a_4(s)$ are given and then construct the approximant $f_{1,0}(z)$ in terms of them. It is shown how bounds are obtained on the error between $f_{1,0}(z)$ and $f(z)$, and numerical results are given in Table 1.

Finally we investigate in Section 6 the problem of interpolating the partial wave amplitudes $a_\ell(s)$ for non-integer values of ℓ when $0 < s < 4$. We use a method described by Basdevant, Bessis and Zinn-Justin [11] to obtain approximate interpolating functions when the first few $a_\ell(s)$ with $\ell = 2,4,\ldots$ are given. It follows from the representation (1.5) satisfied by the $a_\ell(s)$, that sequences of these approximants may be defined which converge to the unique exact interpolation for $a_\ell(s)$ [12] when $\mathrm{Re}\,\ell > 2$. These properties are illustrated by a simple numerical example where only a knowledge of $a_2(s)$ and $a_4(s)$ is assumed and the results are presented in Table 2.

2. THE ANALYTIC CONTINUATION OF THE LEGENDRE EXPANSION

As stated in the Introduction, the Legendre expansion $f(z)$ given in (1.1) converges inside an ellipse with foci at $+1$ and -1 and with semi-major axis $\frac{1}{2}(r+1/r)$ when the corresponding power series $g(z)$ given in (1.2) has radius of convergence $r > 1$. $f(z)$ is an analytic function of z in the ellipse with the representation [Eq. (1.4)]:

$$f(z) = \frac{1}{\pi} \int_0^{\pi} g\left[-z - (z^2-1)^{1/2} \cos v\right] dv.$$

The apparent cut in $f(z)$ due to the term $(z^2-1)^{\frac{1}{2}}$ is removed by the integration over v, and for convenience we take the branch of $(z^2-1)^{\frac{1}{2}}$ in (1.4) to be that which is positive when $z > 1$.

We now state and prove a theorem which gives the analytic behaviour of $f(z)$ when $g(z)$ is a series of Stieltjes.

Theorem 2

If the coefficients f_n in the definition (1.1) of $f(z)$ are those of a series of Stieltjes with radius of convergence $r > 1$, then $f(z)$ can be continued analytically outside its ellipse of convergence to the whole complex plane cut from $\frac{1}{2}(r+1/r)$ to ∞.

Proof

With the prescribed conditions on the f_n, $g(z)$ has the representation, [Eq. (1.6)]:

$$g(z) = \int_0^{1/r} \frac{d\phi(u)}{(1+uz)}$$

and is therefore analytic in the complex plane cut from $-r$ to $-\infty$. The right-hand side of (1.4) is then analytic for all z for which $-z-(z^2-1)^{\frac{1}{2}}\cos v$ does not lie on the cut $-r$ to $-\infty$ when $0 \leqslant v \leqslant \pi$. We now prove, using a geometric method *), that the domain of z for which this condition is satisfied is the whole complex plane cut from $\frac{1}{2}(r+1/r)$ to $+\infty$.

As v runs from 0 to π, $w = -z-(z^2-1)^{\frac{1}{2}}\cos v$ runs along the straight line joining $w_0 = -z-(z^2-1)^{\frac{1}{2}}$ to $w_1 = -z+(z^2-1)^{\frac{1}{2}} = 1/w_0$. Let P correspond to the former point and P' to the latter point and let Q and Q' be their respective reflections in the imaginary axis as indicated in Fig. 1. The lines PQ' and $P'Q$ pass through the origin O and

$$OP \cdot OQ' = OP \cdot OP' = 1 = OQ \cdot OP' = OA \cdot OB \qquad (2.1)$$

where A and B are the points where the circle of unit radius and centre the origin meets the real axis.

The points P, A, P', Q, B and Q' therefore all lie on a circle whose centre is on the imaginary axis. The chord PP' intersects the chord AB at the point C which must lie inside this circle because of the relative ordering of P, A, P' and B round the circumference of the circle. Hence PP' does not cut the real axis between $w=-r$ and $-\infty$ and therefore $-z-(z^2-1)^{\frac{1}{2}}\cos v$ does not lie on the cut of $g(w)$ when $0 \leqslant v \leqslant \pi$.

The above argument holds for P in any quadrant and off the real axis but cannot be used when $-\infty < z \leqslant -1$ and $+1 \leqslant z \leqslant \frac{1}{2}(r+1/r)$ since P and P' are then both on the real axis. It is however simple to verify that PP' has no point in common with the cut from $w=-r$ to $-\infty$ for these values of z.

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*) We would like to thank Dr. A. Martin for suggesting this method of proof.

We have proved that when z is in the whole complex plane cut from $\frac{1}{2}(r+1/r)$ to $+\infty$, then $w = -z-(z^2-1)^{\frac{1}{2}}\cos v$ does not lie on the cut of $g(w)$ for all $0 \leqslant v \leqslant \pi$. Hence $f(z)$ defined by (1.4) is an analytic function for these same values of z as required.

When $0 \leqslant r \leqslant 1$ the Legendre expansion (1.1) exists only in a formal sense, but the function $f(z)$ defined by (1.4) still exists and is analytic for z in a domain given by the following theorem.

Theorem 3

If the coefficients f_n in the Legendre expansion $f(z)$ are those of a series of Stieltjes $g(z)$ with radius of convergence $0 \leqslant r' \leqslant 1$, then the representation (1.4) for $f(z)$ is an analytic function of z in the whole complex plane cut from 1 to ∞.

Proof

When $0 < r' \leqslant 1$ the power series $g(z)$ can be continued analytically to the whole complex z plane cut from $-r'$ to $-\infty$ and has the representation

$$g(z) = \int_0^{1/r'} \frac{d\phi(u)}{(1+uz)} \quad . \tag{2.2}$$

Even when $r' = 0$, $g(z)$ defined by (2.2) is analytic in the cut plane and is formally equivalent to the power series (1.2).

Consider first of all the case when $\operatorname{Re} z < 0$ so that $\operatorname{Re}\left[-z-(z^2-1)^{\frac{1}{2}}\right] > 0$. In the representation (1.4) for $f(z)$ we have an integration of $g(w)$ over a line like QQ' in Fig. 2 which obviously does not intersect the cut of $g(w)$ from $-r'$ to $-\infty$. Therefore, for these values $f(z)$ given by (1.4) is an analytic function of z.

When $\text{Re}\, z \geqslant 0$, then $\text{Re}\left[-z-(z^2-1)^{\frac{1}{2}}\right] \leqslant 0$ and the integration range for $w = -z-(z^2-1)^{\frac{1}{2}}\cos v$ in (1.4) is along a line like PP' which could intersect the cut from $-r'$ to $-\infty$ since $0 \leqslant r' \leqslant 1$.

To deal with this problem we go back to the case when $\text{Re}\, z \leqslant 0$. For each such z we define the holomorphic function of w,

$$K(z,w) = (1+2zw+w^2)^{1/2} \qquad (2.3)$$

with $wK(z,w) \to 1$ for $|w| \to \infty$ and which has a cut along the straight line joining its branch points $-z \pm (z^2-1)^{\frac{1}{2}}$. Then if Γ is the contour indicated in Fig. 2 and since K only differs in sign at opposite sides of the cut we may replace (1.4) by [1]

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} K(z,w)\, g(w)\, dw. \qquad (2.4)$$

It has to be ensured that Γ avoids the singularities of $g(w)$.

We may obtain the analytic continuation of $f(z)$ as given now by (2.4) for $\text{Re}\, z < 0$ to values of z with $\text{Re}\, z \geqslant 0$ by using the method of "deforming the contour" [13]. Let P be the point $-z-(z^2-1)^{\frac{1}{2}}$ and P' be the point $-z+(z^2-1)^{\frac{1}{2}}$ with $\text{Re}\, z \geqslant 0$. The analytic continuation of (2.4) to these values of z is

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1} K_\gamma(z,w)\, g(w)\, dw \qquad (2.5)$$

where Γ_1 is the contour indicated in Fig. 2 and $K_\gamma(z,w)$ is defined like $K(z,w)$ but with its cut along a curve like γ instead of along the straight line joining P to P'.

The representation (2.5) for $f(z)$ is an analytic function of z so long as Γ_1 avoids the cut of $g(w)$ as indicated. This is possible unless z is real and $\geqslant 1$ when one of the branch points of $K_\gamma(z,w)$ lies on the cut of $g(w)$. The representations (2.4) and (2.5) therefore define a function $f(z)$ which is analytic for all z in the complex z plane cut from 1 to ∞ as required by the theorem.

The analytic function $f(z)$ defined by (2.4) is formally equivalent to the Legendre series (1.1) since we have

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\Gamma} K(z,w)\, g(w)\, dw \\
&= \sum_{n=0}^{\infty} \frac{f_n}{2\pi i}\int_{\Gamma} K(z,w)(w)^n\, dw = \sum_{n=0}^{\infty} \frac{f_n}{\pi}\int_{0}^{\pi} \left[z+(z^2-1)^{1/2}\cos v\right]^n dv \\
&= \sum_{n=0}^{\infty} f_n P_n(z). \qquad (2.6)
\end{aligned}
$$

The equivalence is only formal in that we have made an illegal interchange of summation and integration in deriving (2.6). A similar result holds for the representation (2.5) of $f(z)$.

3. APPROXIMANTS FOR $f(z)$

In this Section we investigate the problem of obtaining approximants for $f(z)$ when only the first few coefficients $f_0, f_1, \ldots, f_M$ of the Legendre expansion are known. We consider first of all the case when $r > 1$ so that for all z in the complex plane cut from $\frac{1}{2}(r+1/r)$ to ∞, [Eq. (1.4)]

$$f(z) = \frac{1}{\pi}\int_0^{\pi} g\left[-z-(z^2-1)^{1/2}\cos v\right]dv.$$

An obvious way to approximate $f(z)$ is to replace $g(w)$ in (1.4) by its Padé approximants $[N, N+j]$ and this method is particularly useful in our case since $g(w)$ is a series of Stieltjes so that the $[N, N+j]$ have known convergence properties. We may write

$$[N, N+j] = \sum_{p=1}^{N} \frac{\alpha_{p,N}}{(1+\sigma_{p,N}w)} + \sum_{q=0}^{j} \beta_{q,N} w^q \tag{3.1}$$

where the $\alpha_{P,N}$, $\beta_{q,N}$ and $\sigma_{P,N}$ can be determined from $f_0, f_1, \ldots, f_{2N+j}$.

Let $f_{N,j}(z)$ be the approximant to $f(z)$ obtained by replacing $g(w)$ by $[N, N+j]$ in (1.4). After some elementary integrations one finds that

$$f_{N,j}(z) = \sum_{p=1}^{N} \frac{\alpha_{p,N}}{[1-2\sigma_{p,N}z+\sigma_{p,N}^2]^{1/2}} + \sum_{q=0}^{j} \beta_{q,N} P_q(z). \tag{3.2}$$

The branch of $[1-2\sigma_{P,N}z+\sigma_{P,N}^2]^{\frac{1}{2}}$ used is that which is positive for $z \leq 0$. We then have the following convergence theorem.

Theorem 4

For all z in the complex plane cut from $\frac{1}{2}(r+1/r)$ to

∞, where r is the radius of convergence of the series of Stieltjes (1.2) and $r > 1$,

$$\lim_{N\to\infty} f_{N,j}(z) = f(z) \quad . \tag{3.3}$$

The convergence is uniform in any finite closed region of the complex z plane which does not include any point of the cut of $f(z)$.

Proof

The theorem follows immediately from the uniform convergence of $[N,N+j]$ to $g(w)$ in any finite closed region of the complex w plane which does not include a point of the cut of $g(w)$ and in particular for points on the line joining $-z-(z^2-1)^{\frac{1}{2}}$ to $-z+(z^2-1)^{\frac{1}{2}}$.

Corollary

The theorem holds for the case when $0 \leqslant r \leqslant 1$ for all z in the complex plane cut from 1 to ∞ so long as

$$\sum_{n=0}^{\infty} (f_n)^{-1/(2n+1)}$$

diverges.

Proof

There is no problem when $\mathrm{Re}\, z < 0$ since $f(z)$ again has the representation (1.4). Once again the approximants $f_{N,j}(z)$ may be defined by (3.2) and they converge to $f(z)$ as $N \to \infty$.

When $\mathrm{Re}\ z \geqslant 0$ and z not on the cut from 1 to ∞, $f(z)$ has the representation (2.5) and we define the corresponding approximant by

$$f_{N,j}(z) = \frac{1}{2\pi i} \int_{\Gamma} K_{\gamma}(z,w)\,[N,N+j]\, dw \tag{3.4}$$

However, by the arguments of the previous Section, the $f_{N,j}(z)$

defined in (3.4) for $\mathrm{Re}\, z \geq 0$ are just the analytic continuation of the $f_{N,j}(z)$ defined for $\mathrm{Re}\, z < 0$ in (3.2) and hence have the same functional form (3.2). The corollary then follows immediately for $\mathrm{Re}\, z \geq 0$ from the uniform convergence of $[N,N+j]$ on the contour Γ_1.

The approximant $f_{N,j}(z)$ can be calculated from the coefficients $f_0, f_1, f_2, \ldots, f_{2N+j}$. As mentioned earlier, bounds can be put on the error between $[N,N+j]$ and $g(w)$ from a knowledge of these coefficients [7),8)]. It follows from the integral forms (1.4) and (3.4) for $f(z)$ and $f_{N,j}(z)$ respectively that they also in principle have this property. We will illustrate this result by an example in Section 5, but meanwhile we present a set of inequalities satisfied by $f_{N,j}(z)$ when $r > 1$ and when $1 \leq z < \frac{1}{2}(r+1/r)$. They are

$$f_{N+1,j}(z) - f_{N,j}(z) \geq 0 \tag{3.5}$$

$$f_{N,j}(z) - f_{N-1,j+2}(z) \geq 0 \tag{3.6}$$

$$f_{N,j+1}(z) - f_{N,j}(z) \geq 0 \tag{3.7}$$

and follow immediately from the corresponding inequalities (A1.6)-(A1.8) satisfied by the approximants $[N,N+j]$ to $g(w)$ for $-r < w \leq 0$.

It may be proved from (3.5) and (3.6) that the $f_{N,j}(z)$ are lower bounds to $f(z)$ when $1 \leq z < \frac{1}{2}(r+1/r)$ and for a given even number of the coefficients the best bound is given by $f_{N,0}(z)$ and for a given odd number is $f_{N,-1}(z)$. Using (3.7) these "best" approximants satisfy the inequalities

$$f_{1,0}(z) \leq f_{2,-1}(z) \leq f_{2,0}(z) \leq \ldots \leq f_{N,-1}(z) \leq f_{N,0}(z) \leq f_{N,1}(z) \leq f(z). \tag{3.8}$$

Finally, we should mention that the approximants $f_{N,j}(z)$ defined here are essentially a particular example of generalizations to Padé approximants which have been defined recently by Gammel [5] and Baker [6].

4. THE RELATION BETWEEN THE FROISSART-GRIBOV REPRESENTATION AND SERIES OF STIELTJES

We consider in this Section the situation when the coefficients in the Legendre series (1.1) are given by [Eq. (1.7)]

$$f_n = \frac{1}{\pi}\int_{x_0}^{\infty} Q_n(x)\,d\psi(x) \quad , n=0,1,2,\ldots\ldots$$

with $x_0 > 1$ and $\psi(x)$ bounded and non-decreasing for all $x \geq x_0$. There is a close connection between these coefficients and the Froissart-Gribov representation for partial wave amplitudes which will become apparent from the next Section.

The interest of the above representation for the f_n comes from the following theorem.

Theorem 5

The coefficients f_n defined in (1.7) are coefficients of a series of Stieltjes with radius of convergence $x_0+(x_0^2-1)^{\frac{1}{2}}$.

Proof

We start from the following representation for $Q_n(x)$,

$$Q_n(x) = \int_0^{\infty} \left[x + (x^2-1)^{1/2}\cosh\theta\right]^{-n-1} d\theta \qquad (4.1)$$

which holds for all x not on the line joining -1 to $+1$. Substituting in (1.7)

$$f_n = \frac{1}{\pi}\int_0^{\infty}\int_{x_0}^{\infty}\left[x+(x^2-1)^{1/2}\cosh\theta\right]^{-n-1} d\psi(x)\,d\theta. \qquad (4.2)$$

Consider any set of real numbers $\{y_p\}$. Then, for any pair of positive numbers m, n

$$\sum_{p,q=0}^{n} f_{p+q+m}\, y_p y_q = \frac{1}{\pi}\int_0^{\infty}\int_{x_0}^{\infty}\Big\{\sum_{p,q=0}^{n}\left[x+(x^2-1)^{1/2}\cosh\theta\right]^{-p-q-m-1} y_p y_q\Big\}\, d\psi(x)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{x_0}^{\infty}\Big\{y_0 + y_1\left[x+(x^2-1)^{1/2}\cosh\theta\right]^{-1} + \cdots\cdots + y_n\left[x+(x^2-1)^{1/2}\cosh\theta\right]^{-n}\Big\}^2 \times$$

$$\times \left[x + (x^2-1)^{1/2}\cosh\theta\right]^{-m-1} d\psi(x)\, d\theta.$$

$$> 0. \tag{4.3}$$

It then follows from (4.3) [4] that if

$$D(m,n) = \begin{vmatrix} f_m & f_{m+1} & \cdots\cdots & f_{m+n} \\ f_{m+1} & & & \vdots \\ \vdots & & & \vdots \\ f_{m+n} & \cdots\cdots & & f_{m+2n} \end{vmatrix}, \tag{4.4}$$

then

$$D(m,n) > 0 \qquad m,n = 0,1,2,\ldots\ldots \tag{4.5}$$

Conditions (4.5) are both necessary and sufficient [4] for there to exist a bounded non-decreasing function $\phi(u)$ taking infinitely many values for $0 \leq u \leq \infty$ such that

$$f_n = \int_0^{\infty} u^n\, d\phi(u) \qquad n = 0,1,2\ldots\ldots \tag{4.6}$$

The f_n are therefore coefficients of series of Stieltjes as required.

From (1.7)

$$\frac{f_{n+1}}{f_n} \leq \frac{Q_{n+1}(x_0)}{Q_n(x_0)} \tag{4.7}$$

and using the asymptotic form for Q_n for large n [14],

$$\limsup_{n\to\infty}\left[\frac{f_{n+1}}{f_n}\right] \leq \frac{1}{x_0+(x_0^2-1)^{1/2}} \quad . \tag{4.8}$$

The radius of convergence of the series of Stieltjes is at least $r = x_0+(x_0^2-1)^{\frac{1}{2}}$ and so we have the representation,

$$f_n = \int_0^{1/r} u^n \, d\phi(u) \qquad n=0,1,2\ldots.. \tag{4.9}$$

Theorem 5 has therefore been proved so that the results of Sections 2 and 3 can be applied to the Legendre expansion

$$f(z) = \sum_{n=0}^{\infty} f_n P_n(z), \tag{1.1}$$

when f_n have the representation (1.7).

Yndurain has independently [15] obtained the representation (4.9) for f_n starting from (1.7). He has derived the inequalities (4.5) and noted the possible application of these results to π-π scattering amplitudes which we also discuss in the next Section.

We now obtain an expression for $\phi(u)$ in (4.9) in terms of $\psi(x)$. When the f_n have the representation (1.7),

$$g(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{1}{\pi}\int_{x_0}^{\infty} L(z,x)\, d\psi(x) \tag{4.10}$$

where

$$L(z,x) = \sum_{n=0}^{\infty} z^n Q_n(x) \quad . \tag{4.11}$$

$L(z,x)$ can be continued analytically [1] as a function z to the whole complex plane cut from $x+(x^2-1)^{\frac{1}{2}}$ to ∞ when $1 < x < \infty$ and for z on this cut [16]

$$\underset{\epsilon\to 0+}{Lt}\left[L(z+i\epsilon,x)-L(z-i\epsilon,x)\right]=2\pi i\,(1-2xz+z^2)^{-1/2}. \qquad (4.12)$$

Now from (4.9)

$$g(-z)=\int_0^{1/r}\frac{d\phi(u)}{(1-uz)}=\int_r^{\infty}\frac{d\chi(v)}{(v-z)}\,, \qquad (4.13)$$

where $d\chi(v)=-v\,d\phi(1/v)$ and $r=x_0+(x_0^2-1)^{\frac{1}{2}}$. Therefore [17]

$$\chi(v)-\chi(r)=\underset{\epsilon\to 0+}{Lt}\;\frac{1}{2i}\int_r^{v}\left\{g\left[-v'-i\epsilon\right]-g\left[-v'+i\epsilon\right]\right\}dv'$$

$$=\underset{\epsilon\to 0+}{Lt}\;\frac{1}{2\pi i}\int_r^{v}\int_{x_0}^{\infty}\left\{L(v'+i\epsilon,x)-L(v'-i\epsilon,x)\right\}d\psi(x)\,dv'. \qquad (4.14)$$

We may interchange the order of integration in (4.14) and then using (4.12) and the fact that

$$\underset{\epsilon\to 0+}{Lt}\left[L(v'+i\epsilon,x)-L(v'-i\epsilon,x)\right]=0$$

when $v'<x+(x^2-1)^{\frac{1}{2}}$,

$$\chi(v)-\chi(r)=\underset{\epsilon\to 0+}{Lt}\;\frac{1}{2i\pi}\int_{x_0}^{\frac{1}{2}(v+1/v)}\int_{x+(x^2-1)^{1/2}}^{v}\left[L(v'+i\epsilon,x)-L(v'-i\epsilon,x)\right]dv'\,d\psi(x)$$

$$=\int_{x_0}^{\frac{1}{2}(v+1/v)}\left\{\int_{x+(x^2-1)^{1/2}}^{v}\frac{dv'}{[1-2v'x+v'^2]^{1/2}}\right\}d\psi(x)$$

$$=\int_{x_0}^{\frac{1}{2}(v+1/v)}\log_e\left[\frac{v-x+(v^2-2vx+1)^{\frac{1}{2}}}{(x^2-1)^{1/2}}\right]d\psi(x). \qquad (4.15)$$

Having obtained $\chi(v)$ for a particular $d\psi(x)$, one can immediately get $d\phi(v)=-v\,d\chi(1/v)$.

APPLICATION TO π - π SCATTERING

It has been proved [18] from axiomatic field theory that for π-π scattering $\ell = 2$ and higher partial waves have a Froissart-Gribov representation. In particular, if $a_\ell^{00}(s)$ is the partial wave amplitude for $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ scattering and $a_\ell^0(s)$ is that for $\pi\pi \rightarrow \pi\pi$ scattering in the isotopic spin $T = 0$ state then,

$$a_\ell^{00}(s) = \frac{4}{(4-s)\pi} \int_4^\infty Q_\ell\left(\frac{2t}{4-s} - 1\right)\left[\frac{1}{3}A_t^0(s,t) + \frac{2}{3}A_t^2(s,t)\right]dt \tag{5.1}$$

and

$$a_\ell^0(s) = \frac{4}{(4-s)\pi} \int_4^\infty Q_\ell\left(\frac{2t}{4-s} - 1\right)\left[\frac{1}{3}A_t^0(s,t) + \frac{5}{3}A_t^2(s,t) + A_t^1(s,t)\right]dt \tag{5.2}$$

for even $\ell \geq 2$. The $A_t^T(s,t)$ are the discontinuities of the π-π amplitude in the t channel for isospin T and s, t are the usual Mandelstam variables. It follows from unitarity [9] that $A_t^T(s,t)$ is non-negative for all $t \geq 4$ when $0 \leq s \leq 4$.

Comparing (5.1) and (5.2) with (1.7), we see that there is a close connection between $a_\ell^{00}(s)$, $a_\ell^0(s)$ and f_n. It is because of this connection that we can apply the results of the previous sections to the π-π system.

Let $a_\ell(s)$ be the partial wave amplitude for either of the above mentioned π-π states and set

$$f_n = a_\ell(s) \qquad \ell = 2, 4, 6, \ldots \tag{5.3}$$

where $n = \ell/2 - 1$. Then

$$f_n = \frac{1}{\pi} \int_{x_0}^\infty Q_{2n+2}(x)\, d\psi(x) \tag{5.4}$$

with

with

$$x = \frac{2t}{4-s} - 1 \qquad ; \quad x_0 = \frac{4+s}{4-s}$$

and where

$$d\psi(x) = 2\left[\tfrac{1}{3}A_t^0(s,t) + \tfrac{2}{3}A_t^2(s,t)\right]dt \quad \text{or} \quad 2\left[\tfrac{1}{3}A_t^0(s,t) + \tfrac{5}{3}A_t^2(s,t) + A_t^1(s,t)\right]dt \tag{5.5}$$

depending on the scattering state being considered.

When $0 \leq s < 4$, $d\psi(x) \geq 0$ from the positivity of the $A_t^T(s,t)$ and hence the determinants $D(m,n)$ defined in (4.4) are again positive. We thus have the following theorem.

Theorem 6

When $0 \leq s < 4$, the partial wave amplitudes $a_\ell(s)$ for $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ scattering or $\pi\pi \rightarrow \pi\pi$ scattering in the $T = 0$ state satisfy the inequalities

$$\begin{vmatrix} a_\ell(s) & a_{\ell+2}(s) & \cdots & a_{\ell+\ell'}(s) \\ a_{\ell+2}(s) & & & \vdots \\ \vdots & & & \vdots \\ a_{\ell+\ell'}(s) & \cdots & \cdots & a_{\ell+2\ell'}(s) \end{vmatrix} > 0 \tag{5.6}$$

for all even ℓ, ℓ' with $\ell \geq 2$ and $\ell' \geq 0$.

Corollary

There exists a bounded non-decreasing function $\phi(u)$ such that

$$a_\ell(s) \equiv f_{(\ell/2-1)} = \int_0^{1/r} u^{\ell/2-1}\, d\phi(u) \tag{5.7}$$

for all even $\ell \geq 2$ where $r = \left[x_0 + (x_0^2-1)^{\frac{1}{2}}\right]^2$.

Proof

This result is an immediate consequence of (5.6) when it is seen from (5.1) and (5.2) that

$$\limsup_{\ell \to \infty} |a_\ell(s)|^{1/\ell} = \left[x_0 + (x_0^2-1)^{1/2}\right]^{-1}$$

and hence that

$$\limsup_{n \to \infty} |f_n|^{1/n} = \left[x_0 + (x_0^2-1)^{1/2}\right]^{-2} . \tag{5.8}$$

It should be noted that the most general linear combination of the $a^0_\ell(s)$ and $a^2_\ell(s)$ which corresponds to f_n with the representation (5.4) where $\psi(x)$ is bounded and non-decreasing is

$$(1+\lambda/3)\, a_\ell^0(s) + \frac{2\lambda}{3}\, a_\ell^2(s)$$

with $\lambda \geq -1$. The results of this and the following Section can be applied to these linear combinations of π-π partial wave amplitudes.

We now consider the partial wave expansion of the scattering amplitude, i.e.,

$$A(s,t) = a_0(s) + \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} (2\ell+1)\, a_\ell(s) P_\ell\left(1+\frac{2t}{s-4}\right) . \tag{5.9}$$

This converges when $0 \leq s < 4$ for all $x = 2t/(4-s)-1$ inside an ellipse with foci at $+1$ and -1 and semi-major axis $x_0 = (4+s)/(4-s)$. For these values of x the amplitude $A(s,t)$ has the representation [1]

$$A(s,t) = a_0(s) + \frac{1}{\pi}\int_0^{\pi} g\left[-x-(x^2-1)^{1/2}\cos v\right] dv \tag{5.10}$$

where

$$g(x) = \sum_{n=0}^{\infty} (4n+5) f_n x^{2n+2} = \sum_{\substack{\ell=2\\ \text{even}}}^{\infty} (2\ell+1)\, a_\ell(s)\, x^\ell . \tag{5.11}$$

We now use the representation (5.10) to prove the following theorem.

Theorem 7

When the partial wave amplitudes $a_\ell(s)$ have the representations (5.1) or (5.2), $A(s,t)$ may be continued analytically out of its ellipse of convergence to the whole complex plane cut from $x = x_0$ to ∞ and $x = -x_0$ to $-\infty$ when $0 \leq s < 4$.

Proof

Let $X = -x^2$ and define

$$G(X) = \sum_{n=0}^{\infty} f_n (-X)^n . \tag{5.12}$$

Then $G(X)$ is a series of Stieltjes with radius of convergence $r = \left[x_0 + (x_0^2 - 1)^{\frac{1}{2}}\right]^2$, and so can be continued analytically to all X in the complex plane cut from $-r$ to $-\infty$ and its continuation there is given by

$$G(X) = \int_0^{1/r} \frac{d\phi(u)}{1 + Xu} . \tag{5.13}$$

Now

$$g(x) = \sum_{n=0}^{\infty} (4n+5) f_n x^{2n+2} = x^2 \left[4 X G'(X) + 5 G(X)\right] \tag{5.14}$$

so that $g(x)$ can be analytically continued to the whole complex x plane cut from $x = \sqrt{r}$ to ∞ and $x = -\sqrt{r}$ to $-\infty$. It follows from (5.10) by using the methods of Section 2 that for $0 \leq s < 4$, $A(s,t)$ is an analytic function of x in the whole complex plane cut from $x = \frac{1}{2}(\sqrt{r} + 1/\sqrt{r}) = x_0$ to ∞ and from $x = -x_0$ to $-\infty$. Q.E.D.

Then from Theorem 7 when $0 \leq s < 4$, $A(s,t)$ is analytic in the whole complex t plane cut from 4 to ∞ and from $-s$ to $-\infty$. In fact the Froissart-Gribov representations are derived by assuming this domain of analyticity for $A(s,t)$ which can be obtained by other means [9].

We come now to the problem of approximating $A(s,t)$ from a knowledge of the first few partial wave amplitudes $a_0(s)$, $a_2(s)$, $a_4(s),\ldots,a_{2m}(s)$ say. If $m = 2N+j+1$ we can construct the $[N,N+j]$ Padé approximant to $G(X)$ which we write in the form

$$[N,N+j] = \sum_{p=1}^{N} \frac{\alpha_{p,N}}{1+\sigma_{p,N}X} + \sum_{q=0}^{j} \beta_{q,N} X^q . \qquad (5.15)$$

We obtain a corresponding approximant for $g(x)$ from (5.14) and substituting this for $g(x)$ in (5.10), we get the approximants $A_{N,j}(s,t)$ for $A(s,t)$, where

$$A_{N,j}(s,t) = a_0(s) + \sum_{p=1}^{N} \frac{\alpha_{p,N}}{\sigma_{p,N}} \left\{ \frac{1}{2}(1-\sigma_{p,N}) \left[\frac{1}{(1+2\sqrt{\sigma_{p,N}}x+\sigma_{p,N})^{3/2}} + \frac{1}{(1-2\sqrt{\sigma_{p,N}}x+\sigma_{p,N})^{3/2}} \right] - 1 \right\} +$$

$$+ \sum_{q=0}^{j} \beta_{q,N} (-1)^q (5+4q) P_{2q+2}(x) . \qquad (5.16)$$

Since the poles of $[N,N+j]$ are on the cut of $G(X)$

$$0 \leq \sigma_{p,N} \leq 1/r = [x_0 - (x_0^2-1)^{1/2}]^2 < 1 , \qquad (5.17)$$

and it follows that the poles of $A_{N,j}(s,t)$ in the complex x plane lie on the cuts, $x = x_0$ to $+\infty$ and $x = -x_0$ to $-\infty$, of the exact amplitude $A(s,t)$. The branch of

$$[1 + 2\sqrt{\sigma_{p,N}}\, x + \sigma_{p,N}]^{1/2}$$

used in (5.17) is that which is positive for x real and between these cuts.

We can now prove the following convergence theorem for the $A_{N,j}(s,t)$.

Theorem 8

For all $x = 1+2t/(s-4)$ in the complex plane cut from x_0 to ∞ and $-x_0$ to $-\infty$ and for $j \geqslant -1$,

$$\mathop{Lt}_{N\to\infty} A_{N,j}(s,t) = A(s,t). \tag{5.18}$$

The convergence is uniform in any finite closed region of the cut plane.

Proof

The amplitude $A(s,t)$ has the representation

$$A(s,t) = a_0(s) + \frac{1}{\pi}\int_0^{\pi} \left[x+(x^2-1)^{1/2}\cos v\right]^2 \Big\{-4\left[x+(x^2-1)^{1/2}\cos v\right]^2 G'\left[-(x+\sqrt{x^2-1}\cos v)^2\right] + 5G\left[-(x+\sqrt{x^2-1}\cos v)^2\right]\Big\}\, dv\,, \tag{5.19}$$

where $G(X)$ has been defined in (5.13). Since the $A_{N,j}(s,t)$ are obtained by replacing $G(X)$ by its $[N,N+j]$ approximant in (5.19) and as $G[-(x+\sqrt{x^2-1}\cos v)^2]$ is an analytic function of its argument for all x in the cut plane and all $0 \leqslant v \leqslant \pi$, the theorem follows from the uniform convergence of the $[N,N+j]$ as $N \to \infty$.

When x takes values such that

$$x_0 = \frac{4+s}{4-s} > x > 1 \qquad \text{or} \qquad -x_0 < x < -1, \tag{5.20}$$

the argument of G and G' in (5.19) satisfies,

$$0 \geqslant -\left[x+(x^2-1)^{1/2}\cos v\right]^2 \geqslant -\left[x_0+(x_0^2-1)^{1/2}\right]^2 = -r \tag{5.21}$$

for $0 \leqslant v \leqslant \pi$.

It therefore follows from the inequalities satisfied by $G(X)$, $G'(X)$, $[N,N+j]$ and $[N,N+j]'$, that when x takes values given by (5.20)

$$A_{N+1,j}(s,t) - A_{N,j}(s,t) \geq 0 \tag{5.22}$$

$$A_{N,j}(s,t) - A_{N-1,j+2}(s,t) \geq 0 \tag{5.23}$$

$$A_{N,j+1}(s,t) - A_{N,j}(s,t) \geq 0 \, . \tag{5.24}$$

As previously the best approximants to $A(s,t)$ for a given number of partial wave amplitudes are $A_{N,0}(s,t)$ and $A_{N,-1}(s,t)$ and these approximants satisfy the inequalities

$$A_{N,-1}(s,t) \leq A_{N,0}(s,t) \leq A_{N+1,-1}(s,t) \leq A(s,t). \tag{5.25}$$

We now illustrate how bounds may be put on the error between an approximant $A_{N,j}(s,t)$ and $A(s,t)$ using just those $a_\ell(s)$ needed to evaluate $A_{N,j}(s,t)$.

Consider again the case when $0 \leq s < 4$ and x satisfy (5.20) so that the argument of $G(X)$ in (5.19) lies between 0 and $-r = -[x_0 + (x_0^2 - 1)^{\frac{1}{2}}]^2$. We can use the approximants $(N,N+j)$ to $G(X)$ which have been defined by the author [7] to give a new set of approximants $B_{N,j}(s,t)$ to $A(s,t)$ through (5.10) and (5.14). We note that the explicit forms for the $B_{N,j}(s,t)$ are similar to the representations (5.16) of $A_{N,j}(s,t)$.

The definition and properties of the approximants $(N,N+j)$ to $G(X)$ are given in Appendix 3. In particular we can use the inequalities (A3.4)-(A3.7) given there to prove that for the values of s, t under consideration

$$A(s,t) \leq B_{N,0}(s,t) \leq B_{N,-1}(s,t) \leq B_{N-1,0}(s,t) \, , \tag{5.26}$$

where again the $B_{N,0}(s,t)$ and $B_{N,-1}(s,t)$ are the "best approximants" for a given number of partial wave amplitudes.

If we are given $a_0(s), a_2(s), \ldots, a_{2M}(s)$ with M even we can evaluate the "best approximants" $A_{M/2,-1}(s,t)$ and $B_{M/2-1,0}(s,t)$ and have

$$A_{M/2,-1}(s,t) \leq A(s,t) \leq B_{M/2-1,0}(s,t). \tag{5.27}$$

Therefore, the error between $A(s,t)$ and either approximant is less than the difference between the two approximants.

Consider the case when we are given $a_2(s)$ and $a_4(s)$ with $0 \leq s < 4$. The $[1,0]$ and $(0,0)$ approximants to $G(X)$ can be constructed and are given by

$$[1,0] = \frac{a_2(s)}{1 + \frac{a_4(s)}{a_2(s)} X} \tag{5.28}$$

and

$$(0,0) = \frac{a_2(s) r}{r + X} + X\left[\frac{a_2(s)}{r} - a_4(s)\right]. \tag{5.29}$$

The corresponding approximants to $A(s,t)$ are

$$A_{1,0}(s,t) = a_0(s) + \frac{[a_2(s)]^2}{a_4(s)}\left\{\frac{1}{2}\left[1 - \frac{a_4(s)}{a_2(s)}\right]\left[\frac{1}{\left(1+2\sqrt{\frac{a_4(s)}{a_2(s)}}\,x + \frac{a_4(s)}{a_2(s)}\right)^{3/2}} + \frac{1}{\left(1-2\sqrt{\frac{a_4(s)}{a_2(s)}}\,x + \frac{a_4(s)}{a_2(s)}\right)^{3/2}}\right] - 1\right\} \tag{5.30}$$

and

$$\begin{aligned} B_{0,0}(s,t) = a_0(s) + r a_2(s)\Bigg\{-1 - \frac{\sqrt{r}}{2}\left[\frac{1}{(r-2\sqrt{r}x+1)^{1/2}} + \frac{1}{(r+2\sqrt{r}x+1)^{1/2}}\right] + \\ + \frac{(\sqrt{r}-x)r}{(r-2\sqrt{r}x+1)^{3/2}} + \frac{(\sqrt{r}+x)r}{(r+2\sqrt{r}x+1)^{3/2}}\Bigg\} - \\ - 9\left[\frac{a_2(s)}{r} - a_4(s)\right] P_4(x) \quad . \end{aligned} \tag{5.31}$$

They give upper and lower bounds to $A(s,t)$ when $0 \leq s \leq 4$ for x real and $1 \leq |x| \leq x_0$ i.e., $4 \geq t \geq 4-s$ and $0 \geq t \geq -s$.

An alternative lower bound to $A(s,t)$ for the above values of x is obtained by taking the truncated Legendre expansion which is in fact the approximant $A_{0,1}(s,t)$, i.e.,

$$A_{0,1}(s,t) = a_0(s) + 5a_2(s)P_2\left(1+\frac{2t}{s-4}\right) + 9a_4(s)P_4\left(1+\frac{2t}{s-4}\right). \tag{5.32}$$

But from (5.25)

$$A(s,t) \geq A_{1,0}(s,t) \geq A_{0,1}(s,t) \tag{5.33}$$

so we get a better approximant than the truncated Legendre expansion by using $A_{1,0}(s,t)$.

An alternative upper bound to $A(s,t)$ is obtained by using the property [18]

$$a_\ell(s) \leq a_4(s) \frac{Q_\ell\left(\frac{4+s}{4-s}\right)}{Q_4\left(\frac{4+s}{4-s}\right)} \quad ; \; 0 < s < 4 \text{ and } \ell = 4,6,8\ldots \tag{5.34}$$

Then

$$A(s,t) \leq a_0(s) + 5a_2(s)P_2\left(1+\frac{2t}{s-4}\right) + \frac{a_4(s)}{Q_4\left(\frac{4+s}{4-s}\right)} \sum_{\substack{\ell=4 \\ \text{even}}}^{\infty} (2\ell+1) Q_\ell\left(\frac{4+s}{4-s}\right) P_\ell\left(1+\frac{2t}{s-4}\right)$$

$$= a_0(s) + 5a_2(s)P_2\left(1+\frac{2t}{s-4}\right) + \frac{a_4(s)}{Q_4\left(\frac{4+s}{4-s}\right)} \left\{ -5Q_2\left(\frac{4+s}{4-s}\right)P_2\left(1+\frac{2t}{s-4}\right) - Q_0\left(\frac{4+s}{4-s}\right) + \frac{16-s^2}{4(s+t)(4-t)} \right\}. \tag{5.35}$$

We denote the right-hand side of (5.35) by $C(s,t)$ and then

$$A(s,t) \leq C(s,t). \tag{5.36}$$

A simple numerical example is considered to illustrate the inequalities (5.27), (5.33) and (5.36). Let

$$a_4(s) = \frac{1}{2}\, a_2(s)\, \frac{Q_4\left(\frac{4+s}{4-s}\right)}{Q_2\left(\frac{4+s}{4-s}\right)} \tag{5.37}$$

i.e., half its maximum possible value for given $a_2(s)$, and for convenience we take $a_2(s) = 1$ and $a_0(s) = 0$. In Table 1 we give the corresponding values of $A_{0,1}(s,t)$, $A_{1,0}(s,t)$, $B_{0,0}(s,t)$ and $C(s,t)$ for $4 > t > 2$ when $s = 2$.

It is seen that the above inequalities are well satisfied. In particular the approximant $A_{1,0}(s,t)$ is significantly better than the truncated Legendre expansion $A_{0,1}(s,t)$ especially near the singularity at $t = 4$. The approximant $B_{0,0}(s,t)$ is rather poor near $t = 4$ but it should be remembered that this is only the lowest order approximant of the set. The inequalities (5.26) guarantee that the upper bounds provided by the $B_{N,j}(s,t)$ will improve as more of the partial wave amplitudes $a_\ell(s)$ are used.

6. INTERPOLATION OF $a_\ell(s)$ FOR NON-EVEN INTEGER ℓ

The relations (5.1) and (5.2) give the unique [12] interpolation of $a^{0}_{\ell}(s)$ and $a^{00}_{\ell}(s)$ to non-even integer ℓ when Re $\ell > 2$. In this Section we look at the problem of approximating these interpolating functions when only the first few of the $a^{0}_{\ell}(s)$ or $a^{00}_{\ell}(s)$ are known and we will follow a method due to Basdevant, Bessis and Zinn-Justin [11].

Once again let $a_\ell(s)$ be the amplitude $a^{0}_{\ell}(s)$ or $a^{00}_{\ell}(s)$ depending on the scattering state under consideration. Let

$$\psi(z) = \sum_{n=1}^{\infty} a_{2n}(s) z^{n} . \tag{6.1}$$

Then from (5.7) we have the representation

$$\psi(z) = z \int_0^{1/r} \frac{d\phi(u)}{1 - uz} \tag{6.2}$$

so that $\psi(z)$ is analytic in the complex z plane cut from r to ∞.

Following the above authors we write the interpolating function $a(\ell, s)$ for the $a_\ell(s)$ in the form

$$a(\ell, s) = \frac{1}{2\pi i} \int_C \frac{\psi(z)\, dz}{z^{\ell/2+1}} \qquad \mathrm{Re}\, \ell > 2 , \tag{6.3}$$

where C is a contour encircling the negative axis in the clockwise direction lying completely in the analyticity domain of $\psi(z)$. This form has the correct analytic and asymptotic behaviour for Re $\ell > 2$ and it is easy to check that one recovers the amplitudes $a_\ell(s)$ when $\ell = 4, 6, 8, \ldots$

Suppose we are given $a_2(s), a_4(s), \ldots, a_{4N}(s)$. Then we can evaluate the $[N,N-1]$ approximant to

$$\int_0^{1/r} \frac{d\phi(u)}{(1-uz)}$$

and it may be written in the form

$$[N,N-1] = \sum_{p=1}^{N} \frac{\alpha_{P,N}}{1-\sigma_{P,N} z} \tag{6.4}$$

with $0 \leq \sigma_{P,N} \leq 1/r < 1$. Substituting the corresponding approximant for $\psi(z)$ in (6.3) we obtain an approximant $a_N(\ell, s)$ for the interpolating function $a(\ell, s)$ which is given by

$$a_N(\ell,s) = \int_C \frac{z[N,N-1]}{z^{\ell/2+1}} dz = \sum_{p=1}^{N} \alpha_{P,N}\, \sigma_{P,N}^{\ell/2-1}, \quad Re\,\ell > 2. \tag{6.5}$$

From the convergence of the $[N,N-1]$ as $N \to \infty$ it follows that

$$\lim_{N\to\infty} a_N(\ell,s) = a(\ell,s) \qquad Re\,\ell > 2. \tag{6.6}$$

Also, when $\ell = 2,4,\ldots,4N$,

$$a_N(\ell,s) = \sum_{p=1}^{N} \alpha_{P,N}\, \sigma_{P,N}^{\ell/2-1} = a_\ell(s) \tag{6.7}$$

since the coefficients of powers of z in the expansion of

$$\int_0^{1/r} \frac{d\phi(u)}{(1-uz)}$$

and $[N,N-1]$ have to match up to the $(2N-1)^{th}$ power. Therefore, the approximant $a_N(\ell, s)$ coincides with the exact partial waves $a_\ell(s)$ for these values of ℓ. We illustrate this result by a simple numerical example. Suppose the $A_t^T(s,t)$ in (5.1) and (5.2) are for $0 \leq s < 4$ proportional to $\delta(t-4)$. Then

$$a_\ell(s) = a_2(s)\; Q_\ell\left(\tfrac{4+s}{4-s}\right) \Big/ Q_2\left(\tfrac{4+s}{4-s}\right) \qquad \ell \geq 2. \tag{6.8}$$

We construct the $[1,0]$ approximant to $\psi(z)$ from $a_2(s)$ and $a_4(s)$ getting the interpolating function

$$a_1(\ell,s) = a_2(s)\left[\frac{Q_4\left(\frac{4+s}{4-s}\right)}{Q_2\left(\frac{4+s}{4-s}\right)}\right]^{\ell/2-1} \qquad \mathrm{Re}\,\ell > 2. \qquad (6.9)$$

In Table 2 we compare the values of $a_1(\ell,s)$ and $a(\ell,s)$ for $s=2$ and $2 \leq \ell \leq 10$ using once again the normalization $a_2(s)=1$. It can be seen that $a_1(\ell,s)$ is a good approximation to $a(\ell,s)$ for $2 \leq \ell \leq 4$, and gives a good extrapolation, accurate to within 10%, up to $\ell = 6$. Even when $\ell = 10$ $a_1(\ell,s)$ is of the same order as $a(\ell,s)$.

7. CONCLUSIONS

We have studied the properties of the Legendre expansion $f(z)$ given in (1.1) when the coefficients f_n are those of a series of Stieltjes $g(z)$. The key tool for doing this was the relation (1.4) giving $f(z)$ in terms of $g(z)$. This allowed us to use the properties of $g(z)$, which have been studied in great detail, to investigate the corresponding properties for $f(z)$.

In fact the properties of $f(z)$ obtained are closely analogous to those of $g(z)$. For instance in each case the domain of analyticity is the whole complex plane cut along part of the negative real axis. Also there is a method of approximation for $f(z)$ corresponding to the powerful Padé method used to approximate $g(z)$. It would seem to us to be very worth while to study further the relations between $f(z)$ and $g(z)$.

That the mathematical results in Sections 2 and 3 have an immediate physical application followed from Section 4 where it was shown that if the f_n have a "Froissart-Gribov" type representation (1.7) with $\psi(x)$ bounded and non-decreasing, then they are also coefficients of a series of Stieltjes. It was shown in Section 5 that the partial wave amplitudes for $\pi^0\pi^0 \to \pi^0\pi^0$ scattering and $\pi\pi \to \pi\pi$ scattering in isotopic spin $T = 0$ state and related states do have such a representation, so the results of Sections 2 to 4 could be applied to them and the corresponding scattering amplitudes.

An important result was the derivation of a new infinite set of determinantal inequalities (5.6) for the above partial wave amplitudes satisfied when $0 \leq s < 4$. These indicate once again how unitarity, analyticity and crossing impose strong constraints on the π-π amplitudes.

Two other topics investigated were in Section 5 the approximation of the π-π scattering amplitude given the first few partial waves and in Section 6 the interpolation of the partial wave amplitudes for non-even integer ℓ using the same information. The simple numerical examples considered indicate that these methods of approximation could prove very useful.

These applications to π-π system can only be made when $0 \leq s < 4$ since it is only for these values of s that the partial wave amplitudes $a_\ell(s)$ have the representation (1.7) with $\psi(x)$ having the required properties. It would be useful if one could extend our results to other values of s. A possible tool one might use to do this is a generalization of the Padé method suggested recently by Villani and Prosperi [19], which takes into account the analytic properties of $a_\ell(s)$. However, no convergence theorems have been proved yet except for a special class of potential scattering problems.

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APPENDIX 1

The $[N,N+j]$ Padé approximant to the series

$$f(z) = \sum_{n=0}^{\infty} f_n (-z)^n$$

is defined in the following way. It is the ratio of one polynomial $P(z)$ of degree $N+j$ divided by another $Q(z)$ of degree N and the polynomials are determined by equating coefficients of z in the relation

$$Q(z) \sum_{n=0}^{\infty} f_n (-z)^n - P(z) = A z^{2N+j+1} + B z^{2N+j+2} + \ldots \tag{A1.1}$$

and by the normalization

$$Q(0) = 1.0 \tag{A1.2}$$

Thus to determine the $[N,N+j]$ approximant one needs to know $f_0, f_1, f_2, \ldots, f_{2N+j}$. We may write the $[N,N+j]$ approximant in the form

$$[N,N+j] = \sum_{p=1}^{N} \frac{\alpha_{p,N}}{1+\sigma_{p,N} z} + \sum_{q=0}^{j} \beta_{q,N} z^q \tag{A1.3}$$

(the last term being absent for $j=-1$), where from (A1.1)

$$\sum_{p=1}^{N} \alpha_{p,N} (-\sigma_{p,N})^{\ell} + \beta_{\ell,N} = f_\ell \qquad 0 \leq \ell \leq j \tag{A1.4}$$

$$\sum_{p=1}^{N} \alpha_{p,N} (-\sigma_{p,N})^{\ell} = f_\ell \qquad j < \ell \leq 2N+j. \tag{A1.5}$$

When $f(z)$ is a series of Stieltjes with the representation

$$f(z) = \int_0^{1/r} \frac{d\phi(u)}{(1+uz)}$$

where $\phi(u)$ is bounded and non-decreasing, we have the convergence theorem described in Section 1. The $[N,N+j]$ satisfy a certain set of inequalities [4] when $z \geqslant 0$ and when $-r < z \leqslant 0$ derivatives of $[N,N+j]$ as well as $[N,N+j]$ itself satisfy a similar system of inequalities [7]. The latter sets are used in this paper and are

$$\left\{ [N+1,N+j+1]^{(n)} - [N,N+j]^{(n)} \right\} (-1)^n \geqslant 0 \tag{A1.6}$$

$$\left\{ [N,N+j+1]^{(n)} - [N,N+j]^{(n)} \right\} (-1)^n \geqslant 0 \tag{A1.7}$$

$$\left\{ [N,N+j]^{(n)} - [N-1,N+j+1]^{(n)} \right\} (-1)^n \geqslant 0 \tag{A1.8}$$

$$\left\{ f^{(n)}(z) - [N,N+j]^{(n)} \right\} (-1)^n \geqslant 0 \, . \tag{A1.9}$$

APPENDIX 2

We will construct here a sequence of coefficients f_n which have the representation (4.9) with $\phi(u)$ bounded and non-decreasing but which do not have the representation (1.7) with $\Psi(x)$ bounded and non-decreasing.

Consider the function

$$g_0(z) = \sum_{n=0}^{\infty} f_n z^n = \int_{y_0}^{\infty} \frac{\rho_0(y)\,dy}{(y-z)} \tag{A2.1}$$

where

$$\rho_0(y) = \frac{1}{y^4}(y-y_0)\left[y-a(x_1)\right]^2; \quad a(x) = x+(x^2-1)^{1/2} \tag{A2.2}$$

We take $x_1 > x_0 = \frac{1}{2}(y_0+1/y_0)$ with $y_0 > 1$. Then

$$f_n = \int_{y_0}^{\infty} \frac{\rho_0(y)\,dy}{y^{n+1}} = \int_0^{1/y_0} u^n \, d\phi(u) \tag{A2.3}$$

with

$$\frac{d\phi(u)}{du} = \frac{1}{u}\rho_0\left(\frac{1}{u}\right) = (1-uy_0)\left[1-u\,a(x_1)\right]^2. \tag{A2.4}$$

Therefore $\phi(u)$ is a bounded non-decreasing function in the interval $[0, 1/y_0]$ so that f_n have the representation (4.9).

We consider now the function $g(z) = 2zg_0'(z)+g_0(z)$. Then

$$g(z) = \sum_{n=0}^{\infty}(2n+1)f_n z^n = \int_{y_0}^{\infty} \frac{\rho(y)\,dy}{(y-z)} \tag{A2.5}$$

with

$$\rho(y) = 2y\rho_0'(y) + \rho_0(y) = 2y^{1/2}\frac{d}{dy}\left[y^{1/2}\rho_0(y)\right]. \qquad \text{(A2.6)}$$

The Legendre expansion

$$f(z) = \sum_{n=0}^{\infty} (2n+1) f_n P_n(z) = \int_{x_0}^{\infty} \frac{\sigma(x)\,dx}{(x-z)} \quad , \qquad \text{(A2.7)}$$

where [20]

$$\sigma(x) = \frac{1}{\pi}\int_{y_0}^{a(x)} (2xy-1-y^2)^{-1/2}\rho(y)\,dy \qquad \text{(A2.8)}$$

and $x_0 = \frac{1}{2}(y_0 + 1/y_0)$. Substituting from (A2.6) for $\rho(y)$,

$$\sigma(x_1) = \underset{\epsilon\to 0+}{Lt}\ \frac{1}{\pi}\int_{y_0}^{a(x_1)-\epsilon} 2y^{1/2}\left[2x_1y-1-y^2\right]^{-1/2}\frac{d\left[y^{1/2}\rho_0(y)\right]}{dy}\,dy$$

$$= \underset{\epsilon\to 0+}{Lt}\ \frac{1}{\pi}\left[2y^{1/2}(2x_1y-1-y^2)^{-1/2}\,y^{1/2}\rho_0(y)\right]_{y_0}^{a(x_1)-\epsilon}$$

$$- \underset{\epsilon\to 0+}{Lt}\ \frac{1}{\pi}\int_{y_0}^{a(x_1)-\epsilon} y^{1/2}\rho_0(y)\left[2x_1y-1-y^2\right]^{-3/2}\left\{\frac{1}{y^{1/2}}(2x_1y-1-y^2) - y^{1/2}(2x_1-2y)\right\}dy$$

$$= -\underset{\epsilon\to 0+}{Lt}\ \frac{1}{\pi}\int_{y_0}^{a(x_1)-\epsilon} \rho_0(y)\left[2x_1y-1-y^2\right]^{-3/2}(y^2-1)\,dy \ . \qquad \text{(A2.9)}$$

The limit of the integral as $\epsilon \to 0+$ exists and since $\rho_0(y) > 0$ for $y > y_0 > 1$ this limit is positive. Therefore $\sigma(x_1) < 0$ and as $\sigma(x)$ is a continuous function of x at $x = x_1$ as may be proved from (A2.8), $\sigma(x)$ is strictly negative in some interval containing $x = x_1$.

Suppose now that f_n has the representation (1.7) with $\psi(x)$ bounded and non-decreasing. Then

$$f(z) = \sum_{n=0}^{\infty} (2n+1) \int_{x_0}^{\infty} P_n(z)\, Q_n(x)\, dx = \int_{x_0}^{\infty} \frac{d\psi(x)}{(x-z)} . \tag{A2.10}$$

Comparing with (A2.7) we find that

$$\frac{d\psi(x)}{dx} = \sigma(x) \qquad\qquad x \geqslant x_0 . \tag{A2.11}$$

But this gives a contradiction in the small interval about $x = x_1$, when the left-hand side is positive and the right-hand side is negative. Therefore the f_n do not have the representation (1.7) with $\psi(x)$ bounded and non-decreasing.

APPENDIX 3

We define here the approximants $(N,N+j)$ to the series of Stieltjes $G(X)$ given by

$$G(X) = \int_0^{1/r} \frac{d\phi(u)}{(1+Xu)} = \sum_{n=0}^{\infty} f_n(-X)^n \quad . \tag{5.13}$$

They are obtained by writing [7]

$$G(X) = \frac{f_0 r}{(r+X)} + X\left[XK'(X) + K(X)\right] \tag{A3.1}$$

where $K(X)$ is again a series of Stieltjes with radius of convergence r. Its power series expansion is

$$K(X) = \sum_{n=0}^{\infty} k_n (-X)^n \tag{A3.2}$$

where

$$k_n = \frac{1}{(n+1)} \left\{ f_0/r^{n+1} - f_{n+1} \right\} \quad . \tag{A3.3}$$

The $(N,N+j)$ approximants to $G(X)$ are defined to be those approximants obtained by replacing $K(X)$ in (A3.1) by its $[N,N+j]$ Padé approximant. The convergence of the $(N,N+j)$ to $G(X)$ in the cut plane follows from the convergence of the $[N,N+j]$. From the inequalities (A1.6)-(A1.9) we can, using (A3.1), derive the corresponding inequalities for the $(N,N+j)$ when $-r < X \leq 0$. They are

$$\left\{(N+1,N+j+1)^{(n)} - (N,N+j)^{(n)}\right\}(-1)^n \leq 0 \tag{A3.4}$$

$$\left\{(N,N+j)^{(n)} - (N-1,N+j+1)^{(n)}\right\}(-1)^n \leq 0 \tag{A3.5}$$

$$\left\{(N,N+j+1)^{(n)} - (N,N+j)^{(n)}\right\}(-1)^n \leq 0 \tag{A3.6}$$

$$\left\{ G^{(n)}(X) - (N,N+j)^{(n)} \right\} (-1)^n \leq 0 \quad . \tag{A3.7}$$

Therefore $(N,N+j)$ gives an upper bound to $G(X)$ and $(N,N+j)'$ a lower bound to $G'(X)$ for these values of X.

Alternative approximants giving upper bounds to $G(X)$ have been defined by Baker [8] and they give in the examples considered better upper bounds than the $(N,N+j)$. However, for applications to the $\pi\pi$ scattering amplitude $A(s,t)$, we also need lower bounds to $G'(X)$ and we have to use the $(N,N+j)'$ since Baker has not proved any theorems on the derivatives of these alternative approximants.

TABLE 1

Bound functions for the scattering amplitude $A(s,t)$ when $s = 2$

t	$A_{0,1}(s,t)$	$A_{1,0}(s,t)$	$C(s,t)$	$B_{0,0}(s,t)$
2.2	8.709	8.728	8.755	8.854
2.5	15.80	15.94	16.16	16.95
2.8	25.25	25.83	26.87	30.62
3.1	37.54	39.34	43.42	57.91
3.4	53.22	58.00	73.44	128.7
3.7	72.94	84.38	155.1	437.5

TABLE 2

$a(\ell, s)$ and its approximant

ℓ	$a(\ell, s)$	$a_1(\ell, s)$
2.0	1.00E+0	1.00E+0
2.5	3.81E-1	3.87E-1
3.0	1.47E-1	1.50E-1
3.5	5.73E-2	5.80E-2
4.0	2.24E-2	2.24E-2
4.5	8.85E-3	8.69E-3
5.0	3.50E-3	3.36E-3
5.5	1.39E-3	1.30E-3
6.0	5.55E-4	5.04E-4
7.0	8.88E-5	7.55E-5
8.0	1.43E-5	1.13E-5
9.0	2.33E-6	1.69E-6
10.0	3.81E-7	2.54E-7

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14) See for example E.W. Hobson "Theory of spherical and ellipsoidal harmonics", New York, p. 305 (1955).

15) F. Yndurain, private communication.

16) See Eq. (A1.3) of Ref. 1).

17) If v and r are points of discontinuity of χ, then $\chi(v)-\chi(r)$ should be replaced in equations (4.14) and (4.15) by

$$\frac{1}{2}\left[\chi(v+)+\chi(v-)-\chi(r+)-\chi(r-)\right].$$

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20) See Eq. (C.5) of Ref. 1).

FIGURE CAPTIONS

Figure 1

Integration paths in the definition of $f(z)$ for Theorem 2.

Figure 2

Integration paths in the definition of $f(z)$ for Theorem 3.

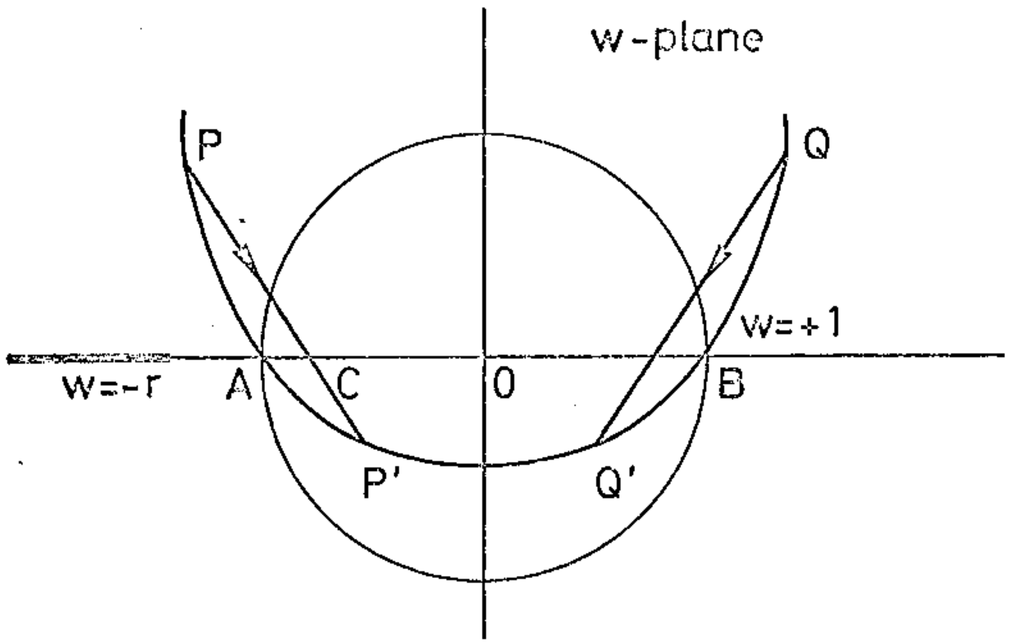

FIG.1

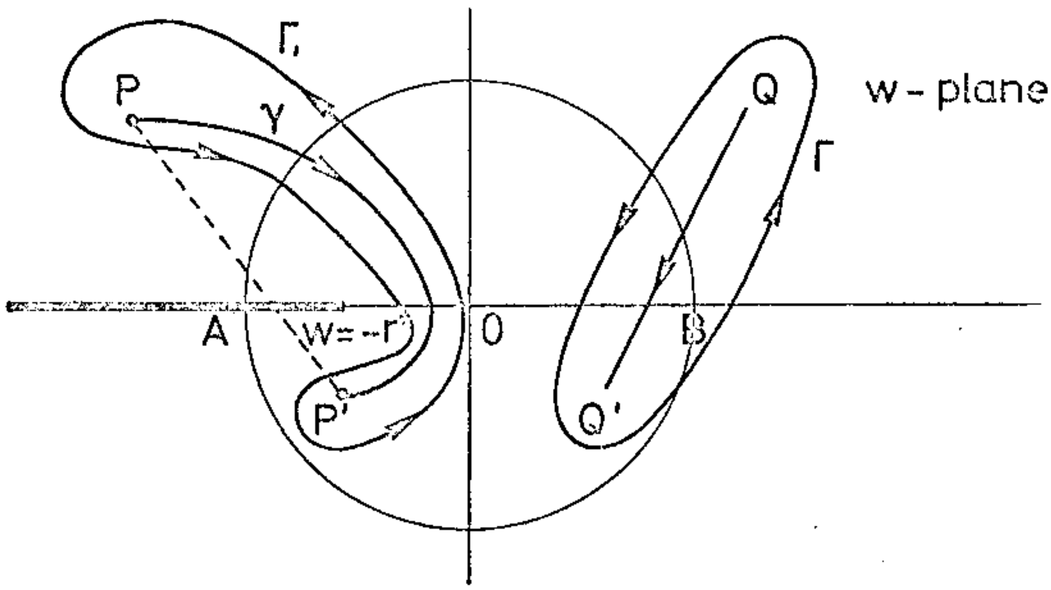

FIG. 2