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PROPERTIES OF MAPPINGS AND CONTINUA THEORY

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The present paper is an expanded version of a colloquium lecture given by the author at the University of Colorado (on May 14, 1973), and is aimed to survey some results and concepts which belong to continua theory but which also, when subject to the classification often used by topologists, could as well be included in a chapter about mappings. Most results presented here are published or forthcoming. A few observations are made that seem to be new (see 3.6 and the proof of 3.8 below). An open problem is restated, in § 2, and two new ones are raised in §§ 1 and 3.

1. **Mappings and continua.** All the spaces throughout this paper are assumed to be compact metric, and a *mapping* is understood to mean a continuous function which maps a compact metric space onto another one. By a *continuum* we mean a connected compact metric space. As usual, a mapping $f: X \to Y$ is said to be *monotone* provided $f^{-1}(y)$ is connected for each point $y \in Y$, and f is *open* provided $f(U)$ is open in Y for each open subset $U \subset X$. The following generalization of both the class of monotone mappings and the class of open mappings was introduced in 1950 by G. T. Whyburn. A mapping $f: X \to Y$ is said to be *quasi-interior* provided, for each point $y \in Y$ and each open subset $U \subset X$ such that a component of $f^{-1}(y)$ is contained in U, we have $y \in \operatorname{Int} f(U)$ (see [24, p. 9]). Another generalization of the same classes of mappings was introduced in 1964 by J. J. Charatonik who used the term "confluent" (as suggested by Professor B. Knaster). Namely, a mapping $f: X \to Y$ is called *confluent* provided, for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, we have $f(C) = K$ (see [4, p. 213]). Before studying the relationship between these two notions, we need some auxiliary propositions. By a *quasi-component* of a set A we mean the common part of all the subsets of A which are closed-open in A and contain a given point of A.

1.1. *If $f: X \to Y$ is a mapping of a compact metric space X onto a compact metric space Y and $A \subset X$, then* $\overline{f(A)} \setminus f(A) \subset f(\overline{A} \setminus A)$. (See [25, p. 147].)

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1.2. *If* $f : X \to Y$ *is a quasi-interior mapping,* $B \subset Y$ *is a connected set and* Q *is a quasi-component of* $f^{-1}(B)$, *then* $f(Q) = B$.

PROOF. Clearly, $f(Q) \subset B$. Suppose $B \subset f(Q)$ is not true and let $y_0 \in B \backslash f(Q)$ be a point. Thus $f^{-1}(y_0) \cap Q = \emptyset$ and, for each point $x \in f^{-1}(y_0)$, there exists a set $A_x \subset f^{-1}(B)$ such that A_x is closed-open in $f^{-1}(B)$, $Q \subset A_x$ and $x \notin A_x$. The sets $f^{-1}(B) \backslash A_x$ are open in $f^{-1}(B)$, and $x \in f^{-1}(B) \backslash A_x$ for $x \in f^{-1}(y_0)$. Since $f^{-1}(y_0)$ is compact, there exist points $x_1, \cdots, x_n$ of $f^{-1}(y_0)$ such that

$$f^{-1}(y_0) \subset [f^{-1}(B) \backslash A_{x_1}] \cup \cdots \cup [f^{-1}(B) \backslash A_{x_n}],$$

and, consequently, putting $A = A_{x_1} \cap \cdots \cap A_{x_n}$, we get $Q \subset A$ and $f^{-1}(y_0) \subset f^{-1}(B) \backslash A$. The set A is also closed-open in $f^{-1}(B)$. It follows [**12**, p. 145] that there exists an open subset $U \subset X$ such that $A \subset U$ and

$$f^{-1}(B) \cap (\bar{U} \backslash U) = \emptyset = \bar{U} \cap [f^{-1}(B) \backslash A], \tag{1}$$

whence $U \cap f^{-1}(y_0) = \emptyset$. Then $y_0 \in B \backslash f(U)$. But $Q \subset A \subset U$ and the non-empty set $f(Q)$ is contained in $B \cap f(U)$. The connectedness of B now implies the existence [**12**, p. 127] of a point $b_0 \in B$ such that b_0 belongs to the closures of both the sets $f(U)$ and $Y \backslash f(U)$. By 1.1 and (1), we obtain $B \cap [\overline{f(U)} \backslash f(U)] \subset B \cap f(\bar{U} \backslash U) = f[f^{-1}(B) \cap (\bar{U} \backslash U)] = \emptyset$, so that b_0 must, in fact, belong to $f(U)$. Let $x_0 \in U$ be a point with $f(x_0) = b_0$. We have $x_0 \in f^{-1}(B)$, whence $x_0 \in A$, by (1). Let C be the component of $f^{-1}(b_0)$ which contains x_0. The set A being closed-open in $f^{-1}(B)$, we conclude that $C \subset A$, and thus C is contained in U. Since f is quasi-interior, $b_0 \in \operatorname{Int} f(U)$ contrary to the fact that b_0 belongs to the closure of $Y \backslash f(U)$; the latter closure is exactly the complement of $\operatorname{Int} f(U)$. This completes the proof of 1.2.

1.3. *Each quasi-interior mapping is confluent.*

PROOF. For compact sets, the quasi-components coincide with the components [**12**, p. 169]. Hence 1.2 implies 1.3.

We say that $f : X \to Y$ is an *OM-mapping* (or an *MO-mapping*) provided there exists a compact metric space Z and there exist two mappings $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$, where g is monotone and h is open (or g is open and h is monotone, respectively). Here the symbol $\circ$ is used to denote the operation of forming the composite of two functions.

1.4. *Each MO-mapping is an OM-mapping.* (See [**20**, Corollary 3.2].)

1.5. *A mapping is quasi-interior if and only if it is an OM-mapping.* (See [**20**, Corollary 3.1].)

The notion of the confluent mapping can be localized as follows. We say that a mapping $f: X \rightarrow Y$ is *locally confluent* provided, for each point $y \in Y$, there exists a closed neighborhood V of y in Y such that $f \mid f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V (see [**9**, p. 239]).

1.6. *A mapping of a compact metric space onto a locally connected compact metric space is quasi-interior if and only if it is locally confluent.* (See [**20**, Corollary 5.2].)

From the standpoint of applications in continua theory, however, another generalization of confluent mappings seems to be even more relevant. A mapping $f: X \rightarrow Y$ is called *weakly confluent* provided, for each continuum $K \subset Y$, there exists a component C of $f^{-1}(K)$ such that $f(C) = K$ (see [**16**, p. 98]). Equivalently, a mapping is weakly confluent if and only if each continuum contained in the range space is the image of a continuum contained in the domain space. It is not difficult to find mappings which are weakly confluent but not locally confluent and which map the unit interval [0, 1] onto itself [**20**, Example 4.1] or onto the circle [**16**, p. 99]. On the other hand, there exist mappings of arc-like continua that are locally confluent but not weakly confluent (ibidem); we use the term "arc-like" as defined below in § 2. We notice that, by 1.3 and 1.6, none of those arc-like continua can be locally connected. Some mappings of arc-like continua onto arc-like continua are confluent without being quasi-interior [**20**, Example 3.6]. There exists a mapping $h: [0, 1] \rightarrow [0, 1]$ such that h is an OM-mapping and h is not an MO-mapping [**16**, p. 97]. Clearly, there exist MO-mappings that are neither monotone nor open. Thus none of the implications shown as arrows in Diagram I can be replaced by an equivalence except one, also shown, which corresponds to 1.5.

(monotone) ⇘ (weakly confluent) ⇗

(open) ⇗ $(MO) \Rightarrow (OM) \Rightarrow$ (confluent) ⇘ (locally confluent)

$\Updownarrow$

(quasi-interior)

Diagram I

The classes of mappings under discussion behave differently when subject to some operations. The confluency of mappings is not preserved by the operation of forming the union of two spaces [**20**, Example 5.6] unless rather strong restrictions are imposed upon them [**20**, Theorem 5.4] (see [**18**] for some generalizations). No such restrictions are needed, however, if the same operation is applied to the class of OM-mappings [**22**, Theorem 3].

1.7. *The composite of two OM-mappings is an OM-mapping.* (See [**20**, Theorem 2.8].)

It follows from 1.7 that, by taking the composites of a finite number of mappings that are either monotone or open, one always obtains OM-mappings. This emphasizes to some extent the significance of the class of quasi-interior mappings in continua theory (compare [**16**, p. 100]).

1.8. *The composite of two confluent (weakly confluent) mappings is confluent (weakly confluent, respectively).* (See [4, p. 214] and [**20**, Theorem 4.4].)

The composite of two MO-mappings need not be an MO-mapping, and the composite of a locally confluent mapping and a monotone mapping can be neither weakly confluent nor locally confluent [**20**, Examples 3.5 and 4.5]. We list all these facts in Table I which also includes the second interesting operation: that of forming the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of two mappings $f_i : X_i \to Y_i$ $(i = 1, 2)$, defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for $x_1 \in X_1$ and $x_2 \in X_2$.

1.9. *The product of two monotone (open) mappings is monotone (open, respectively).*

Proof. The statements are consequences of the formulae

$$(f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2),$$
$$(f_1 \times f_2)(U_1 \times U_2) = f_1(U_1) \times f_2(U_2).$$

1.10. *The product of two OM-mappings (MO-mappings) is an OM-mapping (an MO-mapping, respectively).*

Proof. If $f_i = h_i \circ g_i$, where $g_i : X_i \to Z_i$ and $h_i : Z_i \to Y_i$ $(i = 1, 2)$, then $f_1 \times f_2 = (h_1 \times h_2) \circ (g_1 \times g_2)$. Thus 1.10 follows from 1.9.

Problem I. *Is the product of two confluent (locally confluent) mappings always confluent (locally confluent, respectively)?*

We note that, according to 1.6 and 1.10, an affirmative solution of Problem I is provided in the particular case of mappings onto spaces which are locally connected. (*Added in proof*: T. Mackowiak has now solved Problem I in the negative.)

TABLE I

Mappings	monotone	open	MO	OM	confluent	weakly confluent	locally confluent
composite	+	+	−	+	+	+	−
product	+	+	+	+	?	−	?

EXAMPLE. There exists a weakly confluent mapping $f: [0, 1] \to S$ of $[0, 1]$ onto the circle S such that the product

$$f \times id: [0, 1] \times [0, 1] \to S \times [0, 1] \tag{2}$$

of f and the identity mapping of $[0, 1]$ is not weakly confluent. We consider S to be identical with the set of complex numbers having modulus one, and we define f by $f(x) = e^{4\pi i x}$ for $x \in [0, 1]$. It is not difficult to check that f so defined is weakly confluent. To see that the product (2) is not weakly confluent, let us take the arc K in the cylinder $S \times [0, 1]$, given by the formula

$$K = \{(e^{6\pi i x}, x) : x \in [0, 1]\},$$

and notice that the set $(f \times id)^{-1}(K)$ is composed of 6 components C_j, where

$$C_1 = \{(1, 0)\},$$
$$C_2 = \{((3/2)x + 1/2, x): x \in [0, 1/3]\},$$
$$C_3 = \{((3/2)x, x) : x \in [0, 2/3]\},$$
$$C_4 = \{(0, 1)\},$$
$$C_5 = \{((3/2)x - 1, x) : x \in [2/3, 1]\},$$
$$C_6 = \{((3/2)x - 1/2, x) : x \in [1/3, 1]\}.$$

Neither of the sets $(f \times id)(C_j)$ $(j = 1, \cdots, 6)$ contains both end-points $(1, 0)$ and $(1, 1)$ of K, whence no component of $(f \times id)^{-1}(K)$ is mapped onto K.

2. **A classification of continua.** A continuum X is called *tree-like* (or *arc-like*) provided, for each number $\epsilon > 0$, there exists a mapping $f: X \to Y$ such that $\operatorname{diam} f^{-1}(y) < \epsilon$ for each point $y \in Y$ and Y is a one-dimensional polyhedron containing no simple closed curve

(or Y is an arc, respectively). Sometimes the arc-like continua are also called "chainable" or "snake-like". Clearly, each arc-like continuum is tree-like, and each tree-like continuum is one-dimensional (compare [12, p. 111]).

2.1. *A continuum X is tree-like if and only if X is one-dimensional and each mapping of X into a one-dimensional polyhedron is homotopic to a constant mapping.* (See [3, pp. 74–75].)

We say that a continuum is *acyclic* provided each mapping of it into the circle is homotopic to a constant mapping. A continuum X is said to be *unicoherent* provided, for each two continua C_1 and C_2 such that $C_1 \cup C_2 = X$, the common part $C_1 \cap C_2$ is a continuum. We say X is *hereditarily unicoherent* provided each continuum contained in X is unicoherent. The next proposition follows from 2.1.

2.2. *Each tree-like continuum is one-dimensional and acyclic.*

2.3. *Each one-dimensional acyclic continuum is hereditarily unicoherent.*

Proof. Let X be a one-dimensional acyclic continuum and let $C \subset X$ be a continuum. Suppose f is a mapping of C into the circle. Since X is one-dimensional, f admits a continuous extension f^* over X [12, p. 354], and f^* must be homotopic to a constant mapping because X is acyclic. Thus f is also homotopic to a constant mapping, which means that C is acyclic. Consequently, C is unicoherent [12, p. 437] and 2.3 is proved.

$$(\text{arc-like}) \Rightarrow (\text{tree-like}) \Rightarrow \begin{pmatrix}\text{one-dimensional}\\ \text{acyclic}\end{pmatrix} \Rightarrow \begin{pmatrix}\text{hereditarily}\\ \text{unicoherent}\end{pmatrix} \Rightarrow (\text{unicoherent})$$

Diagram II

It is known that there exist one-dimensional acyclic continua which are not tree-like [3, pp. 80–82]. The so-called standard solenoid, i.e., the inverse limit of circles $S_k = S$ with bonding maps $f_k : S_{k+1} \to S_k$ defined by the formula $f_k(e^{2\pi i x}) = e^{4\pi i x}$, for $x \in [0, 1]$ and $k = 1, 2, \cdots$, makes an example of a one-dimensional hereditarily unicoherent continuum that is not acyclic. Other examples of continua can readily be found to show that, as a result, none of the implications given in Diagram II is replaceable by an equivalence. In the following statements we assume that the spaces, both the continua and their images under some mappings, are non-degenerate.

2.4. *The monotone image of an arc-like continuum is arc-like.* (See [**2**, p. 47].)

2.5. *The open image of an arc-like continuum is arc-like.* (See [**23**, Theorem 1.0].)

According to 2.4 and 2.5, the class of arc-like continua is preserved by OM-mappings. Obviously, it is not preserved by weakly confluent mappings, although certain local properties of continua are invariant under these mappings (see [**7**] and [**8**] for some results in this direction). On the other hand, a locally confluent mapping can destroy the arc-likeness of a continuum that is not locally connected (see [**16**, p. 99] and [**20**, Example 4.2]). The case of locally connected continua is, however, a trivial one since the only arc-like locally connected continuum is the arc itself and, by 1.5 and 1.6, all the locally confluent mappings of an arc are OM-mappings.

PROBLEM II. *Is the confluent image of an arc-like continuum always arc-like*? (See [**15**, p. 94].)

2.6. *The confluent image of a tree-like continuum is tree-like.* (See [**21**, p. 472].)

2.7. *The confluent image of an acyclic (one-dimensional and acyclic) continuum is acyclic (one-dimensional and acyclic, respectively).*

PROOF. Let $f : X \to Y$ be a confluent mapping of a continuum X onto a continuum Y. If X is acyclic, so is Y [**13**, p. 230]. If X is one-dimensional and acyclic, then Y is acyclic, and let us consider an arbitrary closed subset $Z \subset Y$. The mapping $f' = f \mid f^{-1}(Z)$: $f^{-1}(Z) \to Z$ is confluent too [**4**, p. 214]. Given any mapping g of Z into the circle S, the composite $h = g \circ f'$ transforms the subset $f^{-1}(Z)$ of X into S, and X being one-dimensional, h admits a continuous extension h^* over X [**12**, p. 354]. Since X is acyclic, h^* is homotopic to a constant mapping, and so is h. It follows that g is also homotopic to a constant mapping [**13**, p. 229]. We conclude that g admits a continuous extension over Y [**12**, p. 365] which implies that Y is one-dimensional [**12**, p. 354].

2.8. *The weakly confluent (or locally confluent) image of a one-dimensional acyclic continuum is one-dimensional.* (See [**7**, Theorem 1.2].)

The one-dimensionality of non-acyclic continua is not preserved by confluent mappings. In fact, each locally connected continuum can be represented as the image of the Menger universal curve under a mapping that is monotone and open [1, p. 348]. There also exist easy examples of non-acyclic continua being the images of one-dimensional acyclic continua, even of arcs or arc-like continua, under mappings that are weakly confluent or locally confluent, respectively [**16**, p. 99]. By 2.2 and 2.7, the next result implies 2.6.

2.9. *A continuous image of a tree-like continuum is tree-like if and only if it is one-dimensional and acyclic.* (See [**11**, Theorem 3.1].)

2.10. *The monotone image of a unicoherent (hereditarily unicoherent) continuum is unicoherent (hereditarily unicoherent, respectively).*

PROOF. Let $f: X \to Y$ be a monotone mapping. If $K_1, K_2 \subset Y$ are continua, the sets $f^{-1}(K_1)$, $f^{-1}(K_2)$ are connected, hence continua too, and the connectedness of their common part $C = f^{-1}(K_1) \cap f^{-1}(K_2) = f^{-1}(K_1 \cap K_2)$ yields the connectedness of the image $f(C) = K_1 \cap K_2$. Thus 2.10 follows.

Table II

Continua / Mappings	arc-like	tree-like	one-dimensional acyclic	hereditarily unicoherent	unicoherent
monotone	+	+	+	+	+
open	+	+	+	−	−
confluent	?	+	+	−	−
weakly confluent	−	−	−	−	−
locally confluent	−	−	−	−	−

The natural projection of the standard solenoid onto the first circle $S_1 = S$ is an open mapping. Since the solenoid is a hereditarily unicoherent continuum and S is not unicoherent, the analogue of 2.10 for open mappings does not hold. We summarize these observations in Table II indicating whether or not some classes of continua are preserved by certain types of mappings.

3. **Another classification of continua.** Some other classes of continua are to be distinguished by means of properties that have more local character than those discussed above. A continuum is called *regular* (or *rational*) provided it possesses a basis of open sets whose boundaries are finite (or countable, respectively).

3.1. *A continuum X is regular if and only if, for each number $\epsilon > 0$, there exists a positive integer n such that each collection of mutually disjoint subcontinua of X having diameters greater than ϵ consists of at most n elements.* (See [14, p. 132].)

The following definitions are suggested by 3.1. We say a continuum X is *finitely Suslinian* provided, for each number $\epsilon > 0$, each collection of mutually disjoint subcontinua of X having diameters greater than ϵ is finite (ibidem). A continuum is said to be *Suslinian* provided each collection of mutually disjoint non-degenerate subcontinua of it is countable. As can be verified easily, the latter two properties are, indeed, local ones (compare [10, Theorem 2.1]). A continuum is called *hereditarily locally connected* provided each subcontinuum of it is locally connected. Our next proposition follows from 3.1.

3.2. *Each regular continuum is finitely Suslinian.*

3.3. *Each finitely Suslinian continuum is hereditarily locally connected.* (See [14, p. 132].)

3.4. *Each hereditarily locally connected continuum is rational.* (See [25, p. 94].)

3.5. *Each rational continuum is Suslinian.* (See [14, p. 132].)

There exist Suslinian continua, even arc-like Suslinian continua, that are not rational (see [6, p. 178] and [14, p. 135]). Clearly, there exist many rational continua which fail to be locally connected. On the other hand, the hereditarily locally connected continua need not be finitely Suslinian [12, p. 270], and the finitely Suslinian continua need not be regular [12, p. 284]. We note, however, that each hereditarily locally connected continuum embeddable in the plane is finitely Suslinian [14, p. 132]. Consequently, neither implication comprised by Diagram III can, in general, be replaced by an equivalence.

(regular) ⇒ (finitely Suslinian) ⇒ (hereditarily locally connected) ⇒ (rational) ⇒ (Suslinian)

Diagram III

3.6. *The weakly confluent* (*or locally confluent*) *image of a regular continuum is regular.*

PROOF. Suppose X is a regular continuum and $f : X \to Y$ is a mapping. Then, for each number $\epsilon > 0$, there exists a number $\delta > 0$ such that $\operatorname{diam} A \leqq \delta$ implies $\operatorname{diam} f(A) \leqq \epsilon$ for each set $A \subset X$. If f is weakly confluent and $\boldsymbol{K}$ is a collection of mutually disjoint subcontinua of Y having diameters greater than ϵ, each continuum $K \in \boldsymbol{K}$ admits at least one component $C(K)$ of $f^{-1}(K)$ which is mapped onto K by f. Hence $\operatorname{diam} C(K) > \delta$ for $K \in \boldsymbol{K}$. The collection

$$\boldsymbol{C} = \{C(K) : K \in \boldsymbol{K}\} \tag{3}$$

consists of mutually disjoint subcontinua of X. Applying 3.1, we can select a positive integer n that depends only on δ and estimates from above the cardinality of $\boldsymbol{C}$, thus also of $\boldsymbol{K}$. We conclude that n actually depends on ϵ, and not on $\boldsymbol{K}$. By 3.1 again, Y is regular.

If f is locally confluent, there exists, for each point $y \in Y$, a closed neighborhood V of y in Y such that the mapping $g = f \mid f^{-1}(V)$ is confluent. But X being regular, X is locally connected, and so is Y. We can then find a continuum $K \subset V$ containing y in its interior. Since g is confluent, for each component C of $g^{-1}(K)$, we have $g(C) = K$, and the mapping $g \mid C$ is confluent too [4, p. 213]. But $C \subset X$ is a regular continuum, and it follows that K is regular. Thus there exist (in K) arbitrarily small open neighborhoods of y in Y whose boundaries are finite, which means that Y is regular.

3.7. *The weakly confluent* (*or locally confluent*) *image of a finitely Suslinian continuum is finitely Suslinian.*

PROOF. Suppose X is a finitely Suslinian continuum and $f : X \to Y$ is a mapping. If f is weakly confluent and $\boldsymbol{K}$ is a collection of mutually disjoint subcontinua of Y having diameters greater than $\epsilon > 0$, as in the proof of 3.6, the collection (3) is finite; so that $\boldsymbol{K}$ is also finite. Similarly, if f is locally confluent, we get, for each point $y \in Y$, a closed neighborhood $V(y)$ of y in Y such that $V(y)$ does not contain infinitely many continua which are mutually disjoint and have all diameters greater than a positive number. Now, if Y were not finitely Suslinian, there would exist a number $\epsilon_0 > 0$ and an infinite collection $\boldsymbol{K}_0$ of mutually disjoint subcontinua of Y with $\operatorname{diam} K > \epsilon_0$ for $K \in \boldsymbol{K}_0$. By the compactness of Y, we would obtain a point $y_0 \in Y$ such that each neighborhood of y_0 meets infinitely many elements of $\boldsymbol{K}_0$. Let B_1 and B_2 be open balls in Y with the center y_0 and the radii ϵ_1 and ϵ_2, respectively, such that $0 < \epsilon_2 < \epsilon_1 \leqq (1/2)\epsilon_0$, $\bar{B}_1 \subset V(y_0)$, whence $\operatorname{diam} \bar{B}_1 \leqq \epsilon_0$. The ball B_2 meets infinitely many elements of $\boldsymbol{K}_0$, say $K_1, K_2, \cdots$. Since $\operatorname{diam} K_i > \epsilon_0$, K_i is not contained in $\bar{B}_1$, and a component C_i of $\bar{B}_1 \cap K_i$ $(i = 1, 2, \cdots)$ must intersect both B_2 and the boundary of $\bar{B}_1$ [12, p. 172]. Then $C_i \subset V(y_0)$ and diam $C_i > \epsilon_1 - \epsilon_2$ for $i = 1, 2, \cdots$ contradicting the property of the neighborhoods $V(y)$. Therefore Y is finitely Suslinian.

3.8. *The monotone image of a rational continuum is rational. (See* [**25**, p. 138].)

Proof. Assume X is a rational continuum and $f: X \to Y$ is a monotone mapping. The sets $f^{-1}(y)$ are continua for $y \in Y$, and since, by 3.5, the continuum X is Suslinian, the set $B = \{y \in Y : f^{-1}(y)$ non-degenerate$\}$ is countable. Moreover, the function $f \mid f^{-1}(Y \backslash B)$ is a homeomorphism of $f^{-1}(Y \backslash B)$ onto $Y \backslash B$ [12, p. 12]. The continuum X being rational, there exists a decomposition $X = P \cup Q$ such that P is zero-dimensional and Q is countable [12, p. 285]. Thus

$$Y = f(P) \cup f(Q) = [f(P) \cap (Y \backslash B)] \cup B \cup f(Q) =$$
$$= f[P \cap f^{-1}(Y \backslash B)] \cup [B \cup f(Q)],$$

where $f[P \cap f^{-1}(Y \backslash B)]$ is zero-dimensional and $B \cup f(Q)$ is countable; hence Y is a rational continuum (ibidem).

3.9. *The open image of a rational continuum is rational.*

Proof. Assume X is a rational continuum and $f: X \to Y$ is an open mapping. There exists a basis $\boldsymbol{B}$ of open sets in X such that the boundary $\bar{G} \backslash G$ of G is countable for each set $G \in \boldsymbol{B}$. Then, clearly, the sets $f(G)$ $(G \in \boldsymbol{B})$ form a basis of open sets in Y. By 1.1, the boundary $\overline{f(G)} \backslash f(G)$ of $f(G)$ is contained in $f(\bar{G} \backslash G)$, so that it is also countable, for $G \in \boldsymbol{B}$. This completes the proof of 3.9.

Problem III. *Is the confluent (or locally confluent) image of a rational continuum always rational?*

It follows from 3.8 and 3.9 that the class of rational continua is preserved by OM-mappings. Consequently, according to 1.5 and 1.6, the class of rational locally connected continua is preserved by locally confluent mappings. The weakly confluent mappings, however, do not necessarily preserve rational continua. In fact, an arc-like non-rational continuum [**6**, p. 178] can be shown to be the continuous image of a rational continuum [**19**, Example 2], and it is known that each continuous mapping of a continuum onto an arc-like continuum is weakly confluent [**22**, Theorem 4]. Some conditions imposed upon continua guarantee that all the continuous mappings of other continua having them as range spaces are even confluent (see [5, p. 243]; see [17] for a discussion concerning weakly confluent mappings). Also, the property of being a hereditarily locally connected continuum can be expressed as a local one (compare [12, p. 269]), whence we obtain the invariance, included in Table III, of the class of hereditarily locally connected continua under both weakly confluent mappings and locally confluent mappings.

3.10. *The weakly confluent (or locally confluent) image of a Suslinian continuum is Suslinian.*

PROOF. The proof of 3.10 is a replica of that of 3.7.

Table III

Continua / Mappings	regular	finitely Suslinian	hereditarily locally connected	rational	Suslinian
monotone	+	+	+	+	+
open	+	+	+	+	+
confluent	+	+	+	?	+
weakly confluent	+	+	+	−	+
locally confluent	+	+	+	?	+

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