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Properties of Polynomial Identity Quantized Weyl Algebras

Jesse S. F. Levitt
Louisiana State University and Agricultural and Mechanical College

PROPERTIES OF POLYNOMIAL IDENTITY QUANTIZED WEYL ALGEBRAS

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Jesse S. F. Levitt
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Abstract

In this work on Polynomial Identity (PI) quantized Weyl algebras we begin with a brief survey of Poisson geometry and quantum cluster algebras, before using these as tools to classify the possible centers of such algebras in two different ways. In doing so we explicitly calculate the formulas of the discriminants of these algebras in terms of a general class of central polynomial subalgebras. From this we can classify all members of this family of algebras free over their centers while proving that their discriminants have the properties of effectiveness and local domination. Applying these results to the family of tensor products of PI quantized Weyl algebras we solve the automorphism and isomorphism problems.

Chapter 1
Introduction

This dissertation concerns properties of the quantizations of Weyl algebras that are Polynomial Identity (PI) algebras as modules over their centers and how tools from Poisson geometry and quantum cluster algebras can be used to classify them via their discriminants. In this chapter we will primarily concern ourselves with describing the final results of this dissertation and how they fit into current mathematical programs being carried out by the community. Chapter 2 will concern itself with background on quantized Weyl algebras, Poisson geometry, quantum cluster algebras, and noncommutative discriminants. Explicit determinations of the centers of these algebras will be given in Chapter 3 reflecting one of the main findings of this dissertation, a classification of those PI quantized Weyl algebras free over their centers. We return to covering some preparatory material in Chapter 4 where we describe the Poisson structure of these centers and their Poisson prime elements. Chapter 5 contains two proofs covering most of the second main result of this dissertation, first in terms Poisson geometry and then in terms of quantum cluster algebras. We conclude in Chapter 6 by addressing the solution of the automorphism and isomorphism questions for PI quantized Weyl algebras and their tensor products. A final appendix includes a generalization of the results of Chapter 5.

1.1 Quantized Weyl Algebras

Definition 1.1.1. Let T be a commutative integral domain with $n \in \mathbb{Z}_+$. We denote by $T^\times$ the group of units of T. Given $E := (\epsilon_1, \ldots, \epsilon_n) \in (T^\times)^n$, along with the multiplicatively skew-symmetric matrix $B := (\beta_{jk}) \in M_n(T^\times)$, where

$\beta_{jk}\beta_{kj} = 1$ for all $j \neq k$ and $\beta_{jj} = 1$ for all $j \in [1, n]$ where $[1, n] := \{1, \ldots, n\}$. Then the multiparameter quantized Weyl algebra $A_n^{E,B}(T)$ is the unital associative T-algebra with generators

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n$$

and relations

$$\begin{aligned}
y_j y_k &= \beta_{jk} y_k y_j, && \forall j, k, \\
x_j x_k &= \epsilon_j \beta_{jk} x_k x_j, && j < k, \\
x_j y_k &= \beta_{kj} y_k x_j, && j < k, \\
x_j y_k &= \epsilon_k \beta_{kj} y_k x_j, && j > k, \\
x_j y_j - \epsilon_j y_j x_j &= 1 + \sum_{i=1}^{j-1} (\epsilon_i - 1) y_i x_i, && \forall j.
\end{aligned} \tag{1.1}$$

Many different points of view have been used to study quantized Weyl algebras of this and related definitions. In [19, 29] they were approached via quantum groups and Hecke type quantizations, while [6, 20, 21, 26] studied the structure of their prime spectra and representations. The automorphism and isomorphism problems, addressing what the group of automorphisms is and when two algebras are isomorphic, are also covered in the works of [3, 15, 22, 27, 32, 33], while [14] studies their homological and ring theoretic dimensions. But most of these results concern generic cases rather than that of specifically PI algebras.

Let us begin by considering sequences of positive integers $\chi_1 < \chi_2 < \cdots < \chi_n$. Under each sequence, the algebra $A_n^{E,B}(T)$ has an $\mathbb{N}$-filtration defined by

$$\deg x_j = \deg y_j = \chi_j.$$

This defines the associated graded algebra $\operatorname{gr} A_n^{E,B}(T)$ as the connected $\mathbb{N}$-graded skew polynomial algebra with generators $\overline{x}_1, \overline{y}_1, \ldots, \overline{x}_n, \overline{y}_n$, where each generator

has degree $\chi_1, \chi_1, \ldots, \chi_n, \chi_n$, and with relations as defined in (1.1) excepting the final relation which becomes

$$\overline{x}_j \overline{y}_j = \epsilon_j \overline{y}_j \overline{x}_j, \quad \forall j.$$

For $\operatorname{gr} A_n^{E,B}(T)$ to be a PI algebra it can be directly verified using these relations and the standard identity polynomial that it is necessary and sufficient that each $\epsilon_j, \beta_{jk} \in T^\times$ be a root of unity. Under this condition $\operatorname{gr} A_n^{E,B}(T)$ is module finite over its center. Similarly, the algebra $A_n^{E,B}(T)$ is module finite over its center if and only if $\epsilon_j \neq 1$ for all j in addition to the above conditions in the case when $\operatorname{char} T = 0$.

For this reason we will assume for the remainder of this dissertation that

$$\epsilon_j, \beta_{jk} \in T^\times \text{ are roots of unity and } \epsilon_j \neq 1 \text{ for all } j. \tag{1.2}$$

1.2 A Summary of Results

Chan, Young and Zhang [11], based on the works [9, 5], have proposed the closer study of families, $\mathcal{F}$, of filtered PI algebras A with skew polynomial structures on their associated graded algebras $\operatorname{gr} A$. They suggest the following approach:

1. Classify the algebras A in $\mathcal{F}$ where $\operatorname{gr} A$ is free over its center, denoted $\mathcal{Z}(\operatorname{gr} A)$. Then, for those algebras:
2. Describe the center $\mathcal{Z}(A)$ in terms of $\mathcal{Z}(\operatorname{gr} A)$.
3. Explicitly determine a formula for the discriminant $d(A/\mathcal{Z}(A))$.
4. Classify $\operatorname{Aut}(A)$ and determine related properties of A when applicable.

This dissertation approaches the family of quantized Weyl algebras, assuming (1.2), in this manner. To begin, we let

$$\epsilon_j = \exp(2\pi i m_j/d_j), \quad \beta_{jk} = \exp(2\pi i m_{jk}/d_{jk}) \tag{1.3}$$

for some $m_j, d_j, d_{jk} \in \mathbb{Z}_+$, and $m_{jk} \in \mathbb{N}$ for $j \leq k$ such that $\gcd(m_j, d_j) = \gcd(m_{jk}, d_{jk}) = 1$, while requiring that $d_{jk} = d_{kj}$, and $m_{jk} = -m_{kj}$ to ensure the multiplicative skew-symmetry of B. Note that this permits $\beta_{jk} = 1$ (when $m_{jk} = 0$) for any choice of j, k. Let $D(E, B) = \operatorname{lcm}(d_j, d_{jk}, 1 \leq j, k \leq n)$.

A priori, T may not contain i, so we interpret the imaginary exponents in (1.3) as follows. Assume that T contains a $D(E, B)$-th primitive root of unity, denoted by $\exp(2\pi i/D(E, B))$, where $\operatorname{char} T \nmid D(E, B)$. All imaginary exponents in (1.3) then correspond to powers of this element in $T^\times$.

Chan, Young and Zhang solved part (1) of this approach for arbitrary skew polynomial algebras by means of subtle conditions from number theory on the defining matrix for these algebras [11, Theorem 0.3]. Unfortunately, checking these conditions is nontrivial even in the instance of a specific square integer matrix.

For PI quantized Weyl algebras this dissertation covers a more explicit determination covering the first part of this approach. Furthermore, we prove that $\mathcal{Z}(A_n^{E,B}(T)) \cong \mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$:

Theorem 1.2.1. *Let T be an integral domain satisfying the conditions in* (1.3).

(i) The canonical map $\operatorname{gr}\colon A_n^{E,B}(T) \to \operatorname{gr} A_n^{E,B}(T)$ *induces a T-algebra isomorphism* $\mathcal{Z}(A_n^{E,B}(T)) \cong \mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$ *if* $\epsilon_j - 1 \in T^\times$ *for all j.*

(ii) For all integral domains T, the following are equivalent for the quantized Weyl algebra $A_n^{E,B}(T)$:

(a) The algebra $\operatorname{gr} A_n^{E,B}(T)$ *is free over its center;*

(b) The center $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$ *is a polynomial algebra;*

(c) The algebra $A_n^{E,B}(T)$ *is free over its center;*

(d) The center $\mathcal{Z}(A_n^{E,B}(T))$ *is a polynomial algebra;*

(e) $d_j | d_l$ *and* $d_{jk} | d_l$ *for all* $j \leq l$ *and* $k \in [1, n]$.

When these conditions are satisfied, then

$$\mathcal{Z}(A_n^{E,B}(T)) = T[x_j^{d_j}, y_j^{d_j}, 1 \leq j \leq n]. \tag{1.4}$$

Additionally, Theorem 3.1.1 describes $\mathcal{Z}(A^{E,B}(T))$ for all PI quantized Weyl algebras $A_n^{E,B}(T)$. An initial family that satisfies the conditions in Theorem 1.2.1 is the uniparameter case with $\epsilon_1 = \ldots = \epsilon_n$ and $\beta_{jk} = 1$ for all j, k.

Following the outlined program, we investigate further the algebras $A_n^{E,B}(T)$ which satisfy the conditions of Theorem 1.2.1 (ii), working through parts (2) and (3). This investigation follows two different paths, one based on deformation theory and Poisson geometry, while the other uses quantum cluster algebras. Under the restrictions in Theorem 1.2.1 (ii), $D(E, B)$ is then

$$D(E, B) = d_n. \tag{1.5}$$

To establish our Poisson algebra we let

$$X_j := x_j^{d_j}, Y_j := y_j^{d_j} \in \mathcal{Z}(A_n^{E,B}(T)) \tag{1.6}$$

and then define the sequence of elements $Z_0, \ldots, Z_n \in \mathcal{Z}(A_n^{E,B}(T))$ where $Z_0 := 1$ and

$$Z_j := -(1 - \epsilon_j)^{d_j} Y_j X_j + Z_{j-1}^{d_j/d_{j-1}} \quad \text{for } \ j \in [1, n]. \tag{1.7}$$

Conveniently we find in Proposition 4.1.2 that $Z_j = z_j^{d_j}$ for normal elements

$$z_j := 1 + (\epsilon_1 - 1) y_1 x_1 + \cdots + (\epsilon_j - 1) y_j x_j = [x_j, y_j] \in A_n^{E,B}(T).$$

The algebras $A_n^{E,B}(T)$ may be considered algebras over $T[q^{\pm 1}]$, for an indeterminant q specialized at a certain value of ε. The nature of this specialization induces a canonical Poisson algebra structure on $\mathcal{Z}(A_n^{E,B}(T))$, turning $A_n^{E,B}(T)$ into a Poisson

order over its center [8]. We describe this induced Poisson structure on $\mathcal{Z}(A_n^{E,B}(T))$ and the Poisson prime elements of this algebra. With this information we compute the discriminant of $A_n^{E,B}(T)$ over $\mathcal{Z}(A_n^{E,B}(T))$, establishing that it is both dominating and effective. Definitions are described in further detail in Chapter 2, but we note here that throughout this dissertation, two elements t_1 and t_2 of any integral domain T, will be denoted $t_1 =_{T^\times} t_2$ when t_1 and t_2 are associates.

Theorem 1.2.2. *Let T be an integral domain. In the setting of* (1.3)*, assume that the conditions in Theorem 1.2.1(ii) are satisfied and that* $\operatorname{char} T \nmid d_n = D(E, B)$.

(i) The induced Poisson structure on $\mathcal{Z}(A_n^{E,B}(T))$ is given by (4.2)*. If $T = \mathbb{C}$, then the only Poisson prime elements of $\mathcal{Z}(A_n^{E,B}(T))$ up to associates are $Z_1, \ldots, Z_n$.*

(ii) The discriminant of $A_n^{E,B}(T)$ over its center is given by

$$\begin{aligned} d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) &=_{T^\times} \eta Z_1^{N^2(d_1-1)/d_1} \ldots Z_n^{N^2(d_n-1)/d_n} \\ &=_{T^\times} \eta z_1^{N^2(d_1-1)} \ldots z_n^{N^2(d_n-1)}, \end{aligned}$$

where $N := d_1 \ldots d_n$ and $\eta := \left(N \prod_{j=1}^n ([d_j - 1]_{\epsilon_j}!)\right)^{N^2} \in T$, see (5.2) *for an alternate expression for η.*

(iii) Let $A_{n_1}^{E_1,B_1}(T), \ldots, A_{n_l}^{E_l,B_l}(T)$ be a collection of quantized Weyl algebras satisfying the conditions in Theorem 1.2.1 (ii) and A be their tensor product over T. If $\operatorname{char} T \nmid D(E_1, B_1) \ldots D(E_l, B_l)$*, then the discriminant $d(A, \mathcal{Z}(A))$ is locally dominating and effective in the sense of [9] and [5].*

This theorem was proved in [11] for the case when $n = n_1 = \ldots = n_l = 1$.

Following convention, we set $[k]_q := (1-q^k)/(1-q)$ and $[k]_q! = [1]_q \dots [k]_q$. In Theorem 1.2.2 (ii) we compute the discriminant using the trace $\mathrm{tr}\colon A_n^{E,B}(T) \to \mathcal{Z}(A_n^{E,B}(T))$ conventionally associated to the embedding $A_n^{E,B}(T) \hookrightarrow M_{N^2}(\mathcal{Z}(A_n^{E,B}(T)))$ via $\mathcal{Z}(A_n^{E,B}(T))$-bases of $A_n^{E,B}(T)$.

In Theorem A.1.2 a general formula for the discriminant

$$d(A_n^{E,B}(T), T[x_j^{L_j}, y_j^{L_j}, 1 \le j \le n])$$

is proven for any collection of central elements $x_j^{L_j}, y_j^{L_j} \in \mathcal{Z}(A_n^{E,B}(T))$ without assuming that the conditions in Theorem 1.2.1 (ii) are satisfied. This formula Is left for the Appendix because of its rather complicated expression.

In writing two proofs of Theorem 1.2.2 (ii) based on distinct approaches, we note that both proofs not only yield the explicit formulæ for the discriminants, but they also signify the factors of the discriminants with different meanings. Each approach uses irreducible factors from two different algebras: the first is calculated from the factors in the centers of the PI quantized Weyl algebras, while the second with the factors in the whole PI quantized Weyl algebras:

1. The first proof uses the geometry of the induced Poisson structure from specialization. The factors of this discriminant are precisely the unique Poisson primes $Z_1, \dots Z_n$ from Theorem B (i). This proof can be applied to base rings T of arbitrary characteristics (under certain change of base and filtration arguments), even though pieces in the Poisson geometric setting would require $T = \mathbb{C}$.

2. The second proof is derived via quantum cluster algebra structures on the quantized Weyl algebras. There the irreducible factors of the discriminants on the whole of the quantized Weyl algebras $A_n^{E,B}(T)$ are the frozen variables in the quantum cluster algebras.

Combining Theorem 1.2.1 and Theorem 1.2.2 with the results from [9, 5], we solve the automorphism and isomorphism questions for tensor products on this family of quantized Weyl algebras $A_n^{E,B}(T)$ restricted by the conditions of Theorem 1.2.1 (ii).

Theorem 1.2.3. *Let $A = A_{n_1}^{E_1,B_1}(T) \otimes_T \cdots \otimes_T A_{n_l}^{E_l,B_l}(T)$ for a collection of quantized Weyl algebras over an integral domain T, satisfying the conditions in Theorem 1.2.1 (ii). Assume that $\operatorname{char} T \nmid D(E_1, B_1) \dots D(E_l, B_l)$ and recall (1.5).*

(i) If $\phi \in \operatorname{Aut}_T(A)$, then the following hold:

(1) There exists $\sigma \in S_l$ such that $n_i = n_{\sigma(i)}$ and $\phi(A_{n_i}^{E_i,B_i}(T)) = A_{n_{\sigma(i)}}^{E_{\sigma(i)},B_{\sigma(i)}}(T)$ for all $i \in [1, l]$.

(2) For a given $i \in [1, l]$, denote the standard generators of $A_{n_i}^{E_i,B_i}(T)$ and $A_{n_{\sigma(i)}}^{E_{\sigma(i)},B_{\sigma(i)}}(T)$ by $x_1, y_1, \dots, x_n, y_n$ and $x'_1, y'_1, \dots, x'_n, y'_n$ where $n = n_i = n_{\sigma(i)}$. There exist scalars $\mu_1, \nu_1, \dots, \mu_n, \nu_n \in T^\times$ and a sequence $(\tau_1, \dots, \tau_n) \in \{\pm 1\}^n$ such that

$$\begin{aligned} \phi(x_j) &= \mu_j x'_j, \quad \phi(y_j) = \nu_j y'_j, && \text{if } \tau_j = 1, \\ \phi(x_j) &= \mu_j y'_j, \quad \phi(y_j) = \nu_j x'_j, && \text{if } \tau_j = -1. \end{aligned}$$

The scalars satisfy the following equalities for $B_i = (\beta_{jk})$, $B_{\sigma(i)} = (\beta'_{jk})$, $E_i = (\epsilon_1, \dots, \epsilon_n)$ and $E_{\sigma(i)} = (\epsilon'_1, \dots, \epsilon'_n)$:

$$\mu_j \nu_j = \tau_j \prod_{\substack{1 \le k \le j \\ \tau_k = -1}} \epsilon_k^{-1}, \quad \text{and} \quad \epsilon'_j = \epsilon_j^{\tau_j}, \quad \forall j, \tag{1.8}$$

$$\beta'_{jk} = \begin{cases} \beta_{jk}^{\tau_j}, & \text{if } \tau_k = 1, \\ (\epsilon_j \beta_{jk})^{-\tau_j}, & \text{if } \tau_k = -1, \end{cases} \qquad \forall j < k. \tag{1.9}$$

(ii) Every map ϕ on the x- and y-generators of A with the above properties extends to a T-linear automorphism of A.

(iii) The algebra A is strongly Zariski cancellative and LND^H*-rigid, see §2.3.1 for terminology.*

The special case where $n_1 = \ldots = n_l = 1$ was proven in [11].

Theorem 1.2.3 solves the isomorphism problem for all tensor products of collections of quantized Weyl algebras $A_n^{E,B}(T)$ that satisfy the conditions of Theorem 1.2.1 (ii). In particular, for two such tensor products A and B with the algebra isomorphism $\psi\colon A \xrightarrow{\cong} B$, then for $\phi := \psi \otimes \psi^{-1} \in \mathrm{Aut}_T(A \otimes_T B)$, we have that the theorem classifies the automorphisms of $A \otimes_T B$. In this way all isomorphisms $\psi\colon A \xrightarrow{\cong} B$ can be found. The theorem can also classify automorphisms of a single quantized Weyl algebra $A_n^{E,B}(T)$ or the isomorphisms between two quantized Weyl algebras $\psi\colon A_{n_1}^{E_1,B_1}(T) \xrightarrow{\cong} A_{n_2}^{E_2,B_2}(T)$ which all satisfy the conditions of Theorem 1.2.1 (ii) by specialization. These specializations are outlined and stated in detail in Sect. 6.3.

Chapter 2
Background and Definitions

This chapter covers several of the important ideas required to understand the tools used throughout this dissertation and their origins. We begin in § 2.1 with a general overview of Weyl algebras motivating the existence of their quantizations. From there we continue on to a brief review of Poisson algebras and their related properties in § 2.2.

Before we get down to this we give a formal definition of PI algebras.

Definition 2.1. *An algebra, A, may be referred to as a* polynomial identity (PI) algebra *if it there exists a monic polynomial $f \in \mathbb{Z}\langle x_1, \ldots, x_n\rangle$, where $f(r_1, \ldots, r_n) = 0$ for any $a_1, \ldots, a_n \in A$.*

Example 2.0.1. Any polynomial algebra over a field $\mathbb{K}$ with n-indeterminates is a PI algebra with f given by $f(x_1, x_2) = x_1x_2 - x_2x_1$.

2.1 Weyl Algebras

The *first Weyl algebra* arises naturally in mathematics and physics as the ring of differential operators with polynomial coefficients in one variable. In this context we can consider the ring $\mathbb{C}[x, \delta_x]$ where the product rule gives the relation $\delta_x(x \cdot f(x)) = x(\delta_x f(x)) + f(x)$. On the level of operators, this leads to the identity

$$\delta_x \cdot x = x \cdot \delta_x + 1.$$

This quickly generalizes to n-dimensional Weyl algebras where

$$A_n := \mathbb{K}\langle x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}\rangle / < \partial_{x_i}x_j - x_i\partial_{x_j} - \delta_{ij} >$$

where δ_{ij} is the Kronecker delta. To allow for more general setups and divest some of the notational conventions, the family of operators $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is replaced by the indeterminates $\{y_1, \ldots, y_n\}$. These suggested that the rewritten relations, $y_i x_j = x_i y_j + \delta_{ij}$, could be generalized by skewing the multiplicative commutation, so that $y_i x_j = q_{ij} x_i y_j + \delta_{ij}$, and changing the additive deformation, so that $y_i x_j = x_i y_j + a_{ij}$ for $a_{ij} \in \mathbb{K}$. Our approach in Definition 1.1.1 combines these two generalization approaches in a careful balance to ensure that the resulting algebras exist.

2.2 Poisson Algebras and Related Tools

Here we review background material on Poisson structures for algebras, especially those obtained via specialization. Followed by an overview of recent work on discriminants of noncommutative algebras, their relations to Poisson geometry, and their applications to the automorphism and isomorphism problems for algebras. We quickly build up to this material with the following definitions.

Definition 2.2.1. A k-vector space, $\mathfrak{g}$, equipped with a bilinear bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a *Lie algebra* as long as the bracket operation satisfies the following two additional properties:

1. that it is *reflexive* so that $[x, x] = 0$ for all $x \in \mathfrak{g}$;

2. that it satisfies the *Jacobi identity* where
   $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The bracket $[\cdot, \cdot]$ above is referred to as the Lie bracket. Lie algebras are not necessarily associative algebras, but given an associative Lie algebra, the following can occur.

Definition 2.2.2. An associative Lie algebra $(P, \{\cdot, \cdot\})$ is called a *Poisson algebra*, when the Lie bracket $[\cdot, \cdot]$ is also a derivation, which is to say that for $x, y, z \in P$

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

The bracket $\{\cdot,\cdot\}$ of a Poisson algebra is referred to as the Poisson bracket and traditionally is denoted with curly braces as above.

2.2.1 Poisson Structures from Specializations

Begin with a commutative integral domain T and let A be a $T[q^{\pm 1}]$-algebra that is a torsion free $T[q^{\pm 1}]$-module. For any $\epsilon \in T^{\times}$, the specialization of A at ϵ is given by the T-algebra $A_\epsilon := A/(q-\epsilon)A$. The canonical projection $\sigma\colon A \to A_\epsilon$ is then a homomorphism of T-algebras and specializing induces a canonical Poisson structure on $\mathcal{Z}(A_\epsilon)$ as well as a lifting of the hamiltonian derivations of $\mathcal{Z}(A_\epsilon)$ to derivations of A_ϵ:

(i) We define the canonical structure of the Poisson algebra on $\mathcal{Z}(A_\epsilon)$ as below. For each $z_1, z_2 \in \mathcal{Z}(A_\epsilon)$, choose a $c_i \in \sigma^{-1}(z_i)$ and set

$$\{z_1, z_2\} \coloneqq \sigma\big((c_1c_2 - c_2c_1)/(q-\epsilon)\big).$$

A straightforward series of calculations will show that the RHS remains independent of the choice of preimages c_i. We then have $\{z_1, z_2\} \in \mathcal{Z}(A_\epsilon)$ as

$$[\{z_1, z_2\}, \sigma(a)] = \sigma([[c_1, c_2], a]/(q-\epsilon)) = 0, \quad \forall a \in A.$$

(ii) For each $z \in \mathcal{Z}(A_\epsilon)$, one can then construct [12, 25] lifts of the hamiltonian derivation $x \mapsto \{z, x\}$ of $(\mathcal{Z}(A_\epsilon), \{.,.\})$ to derivations of A_ϵ as follows. We again choose $c \in \sigma^{-1}(z)$ and set

$$\partial_c(\sigma(a)) \coloneqq \sigma\big([c, a]/(q-\epsilon)\big).$$

Here, $[c, a]/(q-\epsilon)$ is a proper element of A as $[\sigma(c), \sigma(a)] = 0$, and thus $[c, a] \in \ker\sigma = (q-\epsilon)A$. Again a straightforward series of calculations will show that the RHS does not depend on the choice of preimage of $\sigma(a)$ and that ∂_c is a derivation of A_ϵ. Importantly, derivations corresponding to distinct preimages of z differ by

an inner derivation of A_ϵ:

$$\partial_c - \partial_{c'} = \operatorname{ad} \sigma((c-c')/(q-\epsilon)), \quad \forall c, c' \in \sigma^{-1}(z).$$

Brown and Gordon axiomatized the above situation in [8] as the notion of Poisson order. In this language, the constructions in (i) and (ii) define A_ϵ as a Poisson C_ϵ-order for each Poisson subalgebra C_ϵ of $\mathcal{Z}(A_\epsilon)$ where A_ϵ is a finite rank C_ϵ-module.

2.2.2 Poisson Prime Elements

Let $(P, \{.,.\})$ be a Poisson algebra which is also an integral domain as an algebra. We will require the following definitions.

Definition 2.2.3. (i) An element $a \in P$ is called *Poisson normal* if there exists a Poisson derivation ∂ of A such that

$$\{a, x\} = a\partial(x) \quad \forall x \in P.$$

Equivalently, $a \in P$ is Poisson normal if and only if the ideal (a) is Poisson.

(ii) An element $p \in P$ is called *Poisson prime* if it is a prime element of the commutative algebra P which is Poisson normal. Equivalently, $p \in P$ is Poisson prime if and only if (p) is a nonzero prime and Poisson ideal of P.

Critically for our purposes below, when P is the coordinate ring of a smooth complex affine Poisson variety W, then a given $f \in \mathbb{C}[W]$ is Poisson prime if and only if f is both prime and its zero locus $\mathcal{V}(f)$ is a union of symplectic leaves of W, see [30, Remark 2.4 (iii)].

2.3 Discriminants and Their Relations to Poisson Algebras

We begin our examination of discriminants in relation to Poisson algebras with the following setup. Consider A, an associative algebra with C a subalgebra of the center of A, $\mathcal{Z}(A)$. A C-valued trace on A is a C-linear map $\operatorname{tr}\colon A \to C$ such that

$$\operatorname{tr}(xy) = \operatorname{tr}(yx) \quad \forall x, y \in A.$$

When A is a free C-module of finite rank, one can define [31, 9] a discriminant $d(A/C) \in C$ in the following manner. Given any two C-bases $\mathcal{B}$ and $\mathcal{B}'$ of A, let

$$d_N(\mathcal{B}, \mathcal{B}' : \text{ tr}) := \det\left([\text{tr}(bb')]_{b\in\mathcal{B}, b'\in\mathcal{B}'}\right),$$

where $N = |\mathcal{B}| = |\mathcal{B}'|$. When $\mathcal{B}_1$ and $\mathcal{B}'_1$ are any two other C-bases of A, then by [10, Eq. (1.10.1)] we have that

$$d_N(\mathcal{B}_1, \mathcal{B}'_1 : \text{ tr}) =_{C^\times} d_N(\mathcal{B}, \mathcal{B}' : \text{ tr}).$$

One can define the discriminant of A over C as

$$d(A/C) :=_{C^\times} d_N(\mathcal{B}, \mathcal{B}' : \text{tr})$$

for any C-bases $\mathcal{B}$ and $\mathcal{B}'$ of A. While, more generally, when A is a finite rank C-module which is not necessarily free, there are descriptions of what *discriminant* and *modified discriminant ideals* of A over C can be,[31, 9, 10]. They are defined as the ideals of C generated by the elements of the form $d_N(\mathcal{B}, \mathcal{B} : \text{tr})$ and $d_N(\mathcal{B}, \mathcal{B}' : \text{tr})$, respectively. We define

$$d_N(\mathcal{B} : \text{tr}) := d_N(\mathcal{B}, \mathcal{B} : \text{tr}).$$

For A free and of finite rank over the subalgebra $C \subset \mathcal{Z}(A)$, there exists a natural C-valued trace map on A, frequently called the *internal* trace of A. Any choice of C-basis of A defines an embedding $A \hookrightarrow M_N(C)$, with N the rank of A over C. This natural internal trace of A is merely the composition of this embedding and the standard trace, $\text{tr} : M_N(C) \to C$.

In conjunction with the material in § 2.2.1 the following theorem connects our work with discriminants with Poisson algebras.

Theorem 2.3.1. *[30, Theorem 3.2] Let A be a $\mathbb{K}[q^{\pm 1}]$-algebra for a field $\mathbb{K}$ of characteristic 0 which is a torsion free $\mathbb{K}[q^{\pm 1}]$-module and $\epsilon \in \mathbb{K}^\times$. Assume that*

the specialization $A_\epsilon := A/(q-\epsilon)R$ *is a free module of finite rank over a Poisson subalgebra* C_ϵ *of its center and that* C_ϵ *is a unique factorization domain as a commutative algebra.*

(i) Let $\mathrm{tr}\colon A_\epsilon \to C_\epsilon$ *be a trace map which commutes with all derivations* ∂ *of* A_ϵ *such that* $\partial(C_\epsilon) \subseteq C_\epsilon$*. The corresponding discriminant*$d(A_\epsilon/C_\epsilon)$ *either equals* 0 *or*

$$d(A_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^{m} p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C_\epsilon$*.*

(ii) The internal trace coming from the freeness of A_ϵ *as a* C_ϵ*-module commutes with all derivations* ∂ *of* A_ϵ *such that* $\partial(C_\epsilon) \subseteq C_\epsilon$*.*

2.3.1 Applications of Discriminants

Beyond providing a useful tool for understanding Poisson algebras in particular, discriminants play an important role in algebraic number theory, algebraic geometry and combinatorics [18, 34]. In noncommutative algebra they are useful for the study of orders [31]. Of late, they were found to have high utility in the study of the automorphism and isomorphism problems for PI algebras and in the Zariski cancellation problem for noncommutative algebras [5, 9, 10].

Here we will review some of terminology and results in Ceken–Palmieri–Wang–Zhang [9], Makar-Limanov [28], and Bell–Zhang [5]. To do so, we fix a unital T-algebra A, before choosing a generating set $x_1, \ldots, x_n$ of A, such that $\{1, x_1, \ldots, x_n\}$ is T-linearly independent. We define $\mathcal{F}_i A$ to be the filtration of A where $\deg x_i = 1$, $\forall i$, i.e., $\mathcal{F}_k(A) := (T.1 + Tx_1 + \cdots + Tx_n)^k$. Here, we further define:

Definition 2.3.2. (i) [9, Definition 2.1 (1)] An element $f \in A$ is called *locally dominating* if for every $\phi \in \mathrm{Aut}_T(A)$:

1. $\deg \phi(f) \geq \deg f$ and

2. $\deg \phi(f) > \deg f$ if $\deg(\phi(x_i)) > \deg x_i$ for at least one i,

where the degrees are computed with respect to the filtration $\mathcal{F}_k A$.

(ii) [5, Definition 5.1 (2)] An element $f \in A$ is called *effective* if A has (a possibly different) filtration $F_k A$ whose associated graded algebra is a connected $\mathbb{N}$-graded domain with the following property. For every testing $\mathbb{N}$-filtered PI T-algebra S, whose associated graded is a domain, and for every testing subset $\{y_1, \ldots, y_n\} \subset S$, satisfying

1. $\{1, y_1, \ldots, y_n\}$ is T-linearly independent and

2. $\deg_S y_j \geq \deg_A x_j$ for all j and $\deg_S y_i > \deg_A x_i$ for some i (with respect to the filtration $F_k A$),

f has a lift $f(x_1, \ldots, x_n)$ in the free algebra $T\langle x_1, \ldots x_n\rangle$ such that either $f(y_1, \ldots, y_n) = 0$ or $\deg_S f(y_1, \ldots, y_n) > \deg_A f$.

We note that a stronger notion of dominating elements of algebras was used in [9].

Definition 2.3.3. (i) An algebra A is called *cancellative*, if $A[t] \cong B[t]$ for an algebra B implies $A \cong B$.

(ii) An algebra A is called *strongly cancellative* if, for all $k \geq 1$, $A[t_1, \ldots, t_k] \cong B[t_1, \ldots, t_k]$ for an algebra B implies $A \cong B$.

Denote $\mathrm{LND}(A)$ to be the T-module of locally nilpotent derivations of A. Then, the *Makar-Limanov invariant* [28] of A is defined to be

$$\mathrm{ML}(A) := \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Ker}\, \partial,$$

with $\mathrm{ML}(A) := A$ if $\mathrm{LND}(A) = 0$. An algebra A will be called [5, 28] *LND-rigid* when $\mathrm{ML}(A) = A$, and *strongly LND-rigid* when $\mathrm{ML}(A[t_1, \ldots, t_k]) = A$ for all $k \in \mathbb{Z}_+$.

A *locally nilpotent higher derivation* of A will be a sequence $\partial := (\partial_0 = \mathrm{id}, \partial_1, \ldots)$ of T-endomorphisms of A where

$$a \mapsto \sum_{j=0}^{\infty} \partial_j(a) t^k$$

becomes a well-defined $T[t]$-algebra automorphism of $A[t]$. This requires that for all $a \in A$, $\partial_j(a) = 0$ for sufficiently large j. It then follows that ∂_1 must be a derivation of A. $\mathrm{LND}^H(A)$ then denotes the set of such elements, ∂. The *higher Makar-Limanov invariant* [5] of A is defined by

$$\mathrm{ML}^H(A) := \bigcap_{\partial \in \mathrm{LND}^H(A)} \mathrm{Ker}\, \partial, \quad \text{where} \quad \mathrm{Ker}\, \partial = \bigcap_{j \geq 1} \mathrm{Ker}\, \partial_j.$$

We call an algebra A, *strongly* LND^H*-rigid*, when $\mathrm{ML}^H(A[t_1, \ldots, t_k]) = A$ for all $k \in \mathbb{Z}_+$ and when T is an extension of $\mathbb{Q}$, then A is LND-rigid if and only if it is LND^H-rigid, as described in [5, Remark 2.4 (a)].

Theorem 2.3.4. *Assume that A is a T-algebra which is a free module over $\mathcal{Z}(A)$ of finite rank.*

(i) [9, Theorem 2.7] If the discriminant $d(A/\mathcal{Z}(A))$ is locally dominating with respect to the filtration $\mathcal{F}_k A$ associated to a set of generators $\{x_1, \ldots, x_n\}$, then every $\phi \in \mathrm{Aut}_T(A)$ is affine in the sense that $\phi(\mathcal{F}_1 A) = \mathcal{F}_1 A$.

(ii) [5, Theorem 5.2] If A is a domain and the discriminant $d(A, \mathcal{Z}(A))$ is effective, then A is strongly LND^H*-rigid. If, in addition, A has finite Gelfand-Kirillov dimension, then A is strongly cancellative.*

We again note that a stronger cancellation property than the one in Theorem 2.3.4 (ii) was proved in [5, Theorem 5.2]

2.4 Quantum Cluster Algebras

The final major tool to be used within this dissertation are Cluster algebras, which were first introduced by Fomin and Zelevinsky in [13]. The quantum cluster algebra was then first defined by Berenstein and Zelevinsky in [7] before later being further generalized in [23]. One motivation for naming quantum cluster algebras as they are is that they have, in general, infinitely many localizations that are isomorphic to quantum tori whose generators are related by cluster mutations. The whole quantum cluster algebra is then generated by the union of all such generators.

The second proof of Theorem 1.2.2 (ii) in § 5.2 relies on evaluating the discriminant of a quantum cluster algebra in terms of the discriminants of its corresponding quantum tori which are more straightforward to compute. This yields a formula for the discriminant as a product of frozen cluster variables.

Motivation for this approach can be found from the following insight. Given ϵ_j that are not roots of unity, the quantized Weyl algebras $A_n^{E,B}(T)$ define symmetric CGL extensions, and thus by [24, Theorem 8.2] there exist quantum cluster algebra structures on $A_n^{E,B}(T)$ when T is a field. Focusing instead on the PI case, we will only need two quantum clusters whose intersection is precisely the set of frozen variables.

Proposition 2.4.1. *Let T be an integral domain, $n \in \mathbb{Z}_+$, and $\epsilon_j, \beta_{jk} \in T^\times$, $\epsilon_j \neq 1$ be such that $\epsilon_j - 1 \in T^\times$ for all j.*

(i) The localization of $A_n^{E,B}(T)[y_j^{-1}, 1 \leq j \leq n]$ is isomorphic to the mixed skew-polynomial/quantum torus algebra over T with generators

$$y_j^{\pm 1}, z_j, \quad j \in [1, n]$$

and relations

$$y_j y_k = \beta_{jk} y_k y_j, \quad z_j z_k = z_k z_j, \quad z_j y_k = \epsilon_k^{\delta_{k \leq j}} y_k z_j, \quad j, k \in [1, n]. \tag{2.1}$$

(ii) The localization of $A_n^{E,B}(T)[x_j^{-1}, 1 \leq j \leq n]$ is isomorphic to the mixed skew-polynomial/quantum torus algebra over T with generators $x_j^{\pm 1}$, z_j, $j \in [1,n]$ and relations

$$x_j x_k = \epsilon_j \beta_{jk} x_k x_j, \;\; j < k; \;\; z_j z_k = z_k z_j, \;\; z_j x_k = \epsilon_k^{-\delta_{k \leq j}} x_k z_j, \;\; j,k \in [1,n].$$

Proof. Here part (i) follows from the definition of $A_n^{E,B}(T)$ and from (3.1) that the elements $y_j, z_j \in A_n^{E,B}(T)$ must satisfy the stated relations. As the generators x_j of $A_n^{E,B}(T)$ can be redefined in terms of the elements $y_j^{\pm 1}, z_j$ by

$$x_j = (\epsilon_j - 1)^{-1} y_j^{-1} (z_j - z_{j-1}),$$

now that y_j is invertible, the isomorphism follows. Part (ii) follows in the exact same manner considering y_j in terms of $x_j^{\pm 1}, z_j$ and the remaining relations. □

We note as before that the quantum clusters from parts (i) and (ii) of Proposition 2.4.1 correspond to those constructed in [23, Theorem 1.2]. Starting from the CGL extension presentations of $A_n^{E,B}(T)$ associated to the adjunction of their generators in the orders

$$y_n, \ldots, y_1, x_1, \ldots, x_n \quad \text{and} \quad x_n, \ldots, x_1, y_1, \ldots, y_n,$$

respectively. While a useful motivating viewpoint, a technical difference arises as Proposition 2.4.1 allows the scalars $\epsilon_1, \ldots, \epsilon_n$ to be roots of unity other than 1, while the general result of [23] requires that each of $\epsilon_1, \ldots, \epsilon_n$ not be roots of unity.

Chapter 3
Centers of Quantized Weyl Algebras

Having established the building blocks of quantized Weyl algebras and other tools, we describe the centers of those that are PI. In particular, we will classify those PI quantized Weyl algebras free over their centers by relating this property to other properties of the algebras and their associated graded algebras. In calculating the centers of $A_n^{E,B}(T)$ and $\operatorname{gr} A_n^{E,B}(T)$ we assume the relations (1.3). To determine the grading we fix positive integers $\chi_1 < \chi_2 < \cdots < \chi_n$, and use the $\mathbb{N}$-filtration on our algebras $A_n^{E,B}(T)$ from §1.1 where $\deg x_j = \deg y_j = \chi_j$. We denote the filtered components by $F_j A_n^{E,B}(T)$, for $j \in \mathbb{N}$. The canonical map $\operatorname{gr}\colon A_n^{E,B}(T) \to \operatorname{gr} A_n^{E,B}(T)$ is determined by

$$r \mapsto r + F_{j-1}A_n^{E,B}(T) \quad \text{for } \ r \in F_j A_n^{E,B}(T),\ r \notin F_{j-1}A_n^{E,B}(T).$$

Following the notation from §1.1, we have

$$\overline{x}_j = \operatorname{gr}(x_j),\ \overline{y}_j = \operatorname{gr}(y_j).$$

Then, as previously mentioned in §1.2 the elements

$$z_j := 1 + (\epsilon_1 - 1)y_1x_1 + \cdots + (\epsilon_j - 1)y_jx_j = [x_j, y_j] \in A_n^{E,B}(T), \quad j \in [1, n]$$

are normal. In fact,

$$z_jx_k = \epsilon_k^{-\delta_{k\le j}} x_kz_j, \quad z_jy_k = \epsilon_k^{\delta_{k\le j}} y_kz_j, \quad j, k \in [1, n]. \tag{3.1}$$

Setting $z_0 := 1$, the last relation from (1.1) becomes

$$x_jy_j = \epsilon_j y_jx_j + z_{j-1},\ \ z_{j-1}x_j = x_jz_{j-1},\ y_jz_{j-1} = z_{j-1}y_j, \quad j \in [1, n]. \tag{3.2}$$

It then follows that

$$x_j^{d_j} y_j = y_j x_j^{d_j}, \;\; y_j^{d_j} x_j = x_j y_j^{d_j} \tag{3.3}$$

and hence that, $x_j^{d_j}, y_j^{d_j} \in A_n^{E,B}(T)$ normalize all generators x_k, y_k.

3.1 Proof of Theorem 1.2.1 (i)

To begin we define the set of elements

$$C(E,B) := \{(b_1, a_1, \ldots, b_n, a_n) \in \mathbb{N}^{2n} \mid \;\; d_j | (b_j - a_j), \forall j \in [1,n], \\ \sum_j \frac{(b_j - a_j) m_{jk}}{d_{jk}} + (a_k + \cdots + a_n)\frac{m_k}{d_k} \in \mathbb{Z}, \forall k \in [1,n]\}.$$

Which enables the following explicit description of Theorem 1.2.1 (i). We also define the extension of T, $T' := T[(\epsilon_j - 1)^{-1}, 1 \leq j \leq n]$.

Theorem 3.1.1. *Let T be an integral domain. Assume the setting of* (1.3)*. The centers of* $\operatorname{gr} A_n^{E,B}(T)$ *and* $A_n^{E,B}(T')$ *are given by*

$$\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T)) = \operatorname{Span}_T \left\{ \prod \overline{y}_j^{b_j - a_j} (\overline{y}_j \overline{x}_j)^{a_j} \right\} = \operatorname{Span}_T \left\{ \prod \overline{y}_j^{b_j} \overline{x}_j^{a_j} \right\}, \tag{3.4}$$

$$\mathcal{Z}(A_n^{E,B}(T')) = \operatorname{Span}_{T'} \left\{ \prod y_j^{\max\{b_j - a_j, 0\}} z_j^{\min\{b_j, a_j\}} x_j^{\max\{a_j - b_j, 0\}} \right\}, \tag{3.5}$$

whose spans range over $(b_1, a_1, \ldots, b_n, a_n) \in C(E,B)$.

The T'-algebras $\mathcal{Z}(A_n^{E,B}(T'))$ and $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T'))$ are isomorphic.

Conveniently, the theorem implies that for all integral domains T, the center of $A_n^{E,B}(T)$ is given by

$$\mathcal{Z}(A_n^{E,B}(T)) = \operatorname{Span}_{T'} \left\{ \prod y_j^{\max\{b_j - a_j, 0\}} z_j^{\min\{b_j, a_j\}} x_j^{\max\{a_j - b_j, 0\}} \right\} \bigcap A_n^{E,B}(T)$$

avoiding added unpleasantries from T'. The spanning set here consists of monomials of 2 factors which are powers of the normal elements $y_j^{d_j}$, z_j, $x_j^{d_j}$ of $A_n^{E,B}(T)$.

Proof. We begin by defining a generic central element $r := \overline{y}_1^{b_1} \overline{x}_1^{a_1} \ldots \overline{y}_n^{b_n} \overline{x}_n^{a_n}$ From (3.1) we have that the elements z_j are normal requiring

$$(\operatorname{gr} z_j)(\operatorname{gr} z_{j-1})^{-1} r = \epsilon_j^{b_j - a_j} r (\operatorname{gr} z_j)(\operatorname{gr} z_{j-1})^{-1},$$

implying that $d_j|(b_j - a_j)$ by the grading. As r must be normal, the identity $r\overline{y}_k = \overline{y}_k r$ forces

$$\sum_j \frac{(b_j - a_j)m_{jk}}{d_{jk}} + (a_k + \cdots + a_n)\frac{m_k}{d_k} \in \mathbb{Z}.$$

establishing the inclusion $\subseteq$ in (3.4). In the opposite direction, we have that $\mathrm{gr}(z_j) = (\epsilon_j - 1)\overline{y}_j\overline{x}_j$ where the elements of the set $\{\overline{x}_j, \overline{y}_j \mid j \in [1, n]\}$ generate $\mathrm{gr}\, A_n^{E,B}(T)$ and thus $\supseteq$ holds as well.

It is straightforward to then see the inclusion $\subseteq$ in (3.5) follows from the fact that $\mathrm{gr}(\mathcal{Z}(A_n^{E,B}(T'))) \subseteq \mathcal{Z}(\mathrm{gr}\, A_n^{E,B}(T'))$. Then $\supseteq$ is proved by a series of calculations stemming from the fact that z_j, $x_j^{d_j}$ and $y_j^{d_j}$ are normal elements of $A_n^{E,B}(T)$, these must then normalize the generators x_k, y_k, enabling the application of the first four relations in (1.1) along with (3.3).

The isomorphism of T'-algebras $\mathcal{Z}(A_n^{E,B}(T')) \cong \mathcal{Z}(\mathrm{gr}\, A_n^{E,B}(T'))$ is determined in a canonical manner by $r \mapsto \mathrm{gr}\, r$. Here r ranges over the T'-basis of $\mathcal{Z}(A_n^{E,B}(T'))$ expressed by the RHS of (3.5). The fact that the elements $x_j^{d_j}$, z_j and $y_j^{d_j}$ normalize each other, along with the congruences in (3.4) and (3.5) complete the proof. $\square$

3.2 Proof of Theorem 1.2.1 (ii)

The proof of Theorem 1.2.1 (ii) is requires a more elaborate setup. To begin with, the fact that (a) $\Leftrightarrow$ (b) in Theorem 1.2.1 (ii) is a general fact for skew polynomial algebras as determined in [10, Lemma 2.3].

Next, we demonstrate the sequence of implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (e) $\Rightarrow$ (c). When (c) is satisfied, $A_n^{E,B}(T')$ is a free module over $\mathcal{Z}(A_n^{E,B}(T'))$, and an examination of the leading terms of its basis in conjunction with the fact that $\mathcal{Z}(A_n^{E,B}(T')) \cong \mathcal{Z}(\mathrm{gr}\, A_n^{E,B}(T'))$ (Theorem 3.1.1), proves that (a) holds.

Again using [10, Lemma 2.3], (a) $\Rightarrow$ (b) and when (b) is satisfied, then $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$ is a polynomial algebra over T with generators $\overline{x}_1$, $\overline{y}_1, \ldots, \overline{x}_n$, $\overline{y}_n$. Thus,

$$\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T)) = T[\overline{x}_1^{g_1}, \overline{y}_1^{h_1}, \ldots, \overline{x}_n^{g_n}, \overline{y}_n^{h_n}]$$

for a a collection of powers $g_1, h_1, \ldots, g_n, h_n \in \mathbb{Z}_+$. By its explicit description of the centers, Theorem 3.1.1 implies that

$$(\overline{y}_{j-1}\overline{x}_{j-1})^c(\overline{y}_j\overline{x}_j)^{d_j} \in \mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$$

for all $c \in \mathbb{N}$ such that $d_k|(c + d_j)$ for $k < j$. Focusing on the second term, $(\overline{y}_j\overline{x}_j)^{d_j} \in T[\overline{x}_1^{g_1}, \overline{y}_1^{h_1}, \ldots, \overline{x}_n^{g_n}, \overline{y}_n^{h_n}]$ by assumption it follows that $g_j|d_j$ and $h_j|d_j$. Additionally, the last commutation relation in (1.1) gives that $d_j|g_j$ and $d_j|h_j$ and thus we have that $g_j = h_j = d_j$ for all $j \in [1, n]$. Furthermore,

$$\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T)) = T[\overline{x}_1^{d_1}, \overline{y}_1^{d_1}, \ldots, \overline{x}_n^{d_n}, \overline{y}_n^{d_n}].$$

Since $\overline{x}_j^{d_j}, \overline{y}^{d_j} \in \mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$, the first and fourth relations in (1.1) yield

$$\beta_{jk}^{d_j} = 1 \text{ for } k \in [1, n], \quad (\epsilon_k\beta_{kj})^{d_j} = 1 \text{ for } k < j.$$

Thus, $\epsilon_k^{d_j} = 1$ for $k < j$ and $d_{jk}|d_j$ for all k and $d_k|d_j$ for all $k < j$, establishing that (b) $\Rightarrow$ (e).

Next by assuming (e) we will establish the following

$$C(E, B) = \{(a_1, b_1, \ldots, a_n, b_n) \in \mathbb{N}^{2n} \mid d_j|a_j \text{ and } d_j|b_j, \ \forall j\}. \tag{3.6}$$

The first part of the definition of $C(E, B)$, requires that $d_j \mid (a_j - b_j)$ and by (e), $d_{jk} \mid d_j$, $\forall k$. The second part of the definition of $C(E, B)$ becomes $d_k \mid (a_k + \cdots + a_n)$, $\forall k$. Since $d_k \mid d_j$ for $k < j$ by starting with $k = n$ and working towards $k = 1$ one may deduce that this condition is equivalent to: $d_k|a_k$ for all k. Then we may

use that Theorem 3.1.1 and (3.6) imply (1.4), and thus (c) is true. Hence (e) $\Rightarrow$ (1.4) $\Rightarrow$ (c).

We conclude this by establishing we show that (e) $\Rightarrow$ (d) $\Rightarrow$ (b). It follows that (1.4) $\Rightarrow$ (d) and we earlier proved that (e) $\Rightarrow$ (1.4), so (e) $\Rightarrow$ (d).

Assuming instead that (d) is satisfied. It then follows from (3.5) that

$$\mathcal{Z}(A_n^{E,B}(T')) \cong \mathcal{Z}(A_n^{E,B}(T)) \otimes_T T',$$

describing $\mathcal{Z}(A_n^{E,B}(T'))$ as a polynomial algebra over T'. As $\mathcal{Z}(A_n^{E,B}(T')) \cong \mathcal{Z}(\operatorname{gr} A_n^{E,B}(T'))$ as T'-algebras by Theorem 3.1.1, we find that $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T'))$ is a polynomial algebra over T'. For the center of a skew polynomial ring to be a polynomial algebra depends on its structure constants rather than the base ring. Therefore $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$ is a polynomial algebra over T, so (d) $\Rightarrow$ (b) which completes the proof. $\square$

Example 3.2.1. If we examine the case where $n = 2$, with $\beta_{jk} = 1$ for all j, k. Then,

$$\begin{aligned} C(E,B) = \big(&\mathbb{Z}(d_1,0,0,0) + \mathbb{Z}(0,d_1,0,0) + \\ &+ \mathbb{Z}(-d_2,-d_2,d_2,0) + \mathbb{Z}(0,0,0,d_2)\big) \cap \mathbb{N}^4. \end{aligned}$$

Were $d_1 \nmid d_2$, then we have that $\mathcal{Z}(\operatorname{gr} A_2^{E,B}(T))$ and $\mathcal{Z}(A_2^{E,B}(T))$ are not a polynomial rings.

Chapter 4
The Poisson Structure of $\mathcal{Z}(A_n^{E,B}(T))$

In this chapter we describe the Poisson structure induced on the centers of the PI quantized Weyl algebras by characterizing them as specializations of general quantizations at roots of unity. This characterization will allow the results of [30] described in §2.3 to be brought to bear on the questions at hand. Once the Poisson structure is established we define the exact Poisson primes of $\mathcal{Z}(A_n^{E,B}(T))$. The material here extends the background introduced in Chapter 2, enabling proofs of Theorem 1.2.2 (i) & (ii) in Chapter 5 to follow.

4.1 The Induced Poisson Structure

We begin by defining $A_n^{E,B}(T)_q$, the algebra over $T[q^{\pm 1}]$ with generators $\widetilde{x}_j$, $\widetilde{y}_j$ and relations following (1.1) substituting $q^{d_n m_j/d_j}$ for each ϵ_j and $q^{d_n m_{jk}/d_{jk}}$ for each β_{jk}, with q a general indeterminate and $A_n^{E,B}(T)_q$ a domain. Now define $\epsilon := \exp(2\pi i/d_n) \in T^\times$, giving $\epsilon_j = \epsilon^{d_n m_j/d_j}$ and $\beta_{jk} = \epsilon^{d_n m_{jk}/d_{jk}}$. Then $A_n^{E,B}(T)$ is the specialization of $A_n^{E,B}(T)_q$ at $q = \epsilon$ as proven in the following lemma.

Lemma 4.1.1. *For any integral domain T, there is a unique T-algebra homomorphism $\sigma\colon A_n^{E,B}(T)_q \to A_n^{E,B}(T)$ such that $\sigma(\widetilde{x}_j) = x_j$, $\sigma(\widetilde{y}_j) = y_j$, $\sigma(q) = \epsilon$. Its kernel is $\ker\sigma = (q-\epsilon)A_n^{E,B}(T)_q$. The map σ realizes $A_n^{E,B}(T)$ as the specialization of $A_n^{E,B}(T)_q$ at $q = \epsilon$.*

The proof, while straightforward is not particularly enlightening and has been omitted.

With the map σ well-defined, we may proceed to use this specialization to induce a Poisson structure on $\mathcal{Z}(A_n^{E,B}(T))$. We begin by recalling (1.6) and (1.7) where $X_j, Y_j, Z_j \in \mathcal{Z}(A_n^{E,B}(T))$ are defined.

Proposition 4.1.2. *Let T be an integral domain and assume that the conditions in Theorem 1.2.1 (ii) are satisfied. Then*

$$\sigma(\widetilde{z}_j)^{d_j} = z_j^{d_j} = Z_j \in \mathcal{Z}(A_n^{E,B}(T)). \tag{4.1}$$

The induced Poisson bracket on $\mathcal{Z}(A_n^{E,B}(T))$ is given by

$$\begin{aligned}
\{X_j, Y_j\} &= m_j d_j d_n \epsilon^{-1}(X_j Y_j - (1-\epsilon_j)^{-d_j} Z_{j-1}), && \forall j, \\
\{Y_j, Y_k\} &= \frac{m_{jk} d_j d_k d_n}{d_{jk}} \epsilon^{-1} Y_j Y_k, && \forall j, k, \\
\{X_j, Y_k\} &= -\frac{m_{jk} d_j d_k d_n}{d_{jk}} \epsilon^{-1} X_j Y_k, && j < k, \\
\{X_j, X_k\} &= d_j d_k d_n \left(\frac{m_j}{d_j} + \frac{m_{jk}}{d_{jk}} \right) \epsilon^{-1} X_j X_k, && j < k, \\
\{X_j, Y_k\} &= d_j d_k d_n \left(\frac{m_k}{d_k} + \frac{m_{kj}}{d_{kj}} \right) \epsilon^{-1} X_j Y_k, && j > k
\end{aligned} \tag{4.2}$$

satisfying

$$\begin{aligned}
\{Z_j, X_k\} &= -\delta_{k \leq j} d_j d_k d_n \epsilon^{-1} Z_j X_k, \\
\{Z_j, Y_k\} &= \delta_{k \leq j} d_j d_k d_n \epsilon^{-1} Z_j Y_k, \quad \{Z_j, Z_k\} = 0, \quad \forall j, k.
\end{aligned} \tag{4.3}$$

Proof. Beginning with the equation (3.2), we find that

$$z_j^{d_j} = ((\epsilon_j - 1) y_j x_j + z_{j-1})^{d_j} = (\epsilon_j - 1)^{d_j} \epsilon_j^{d_j(d_j-1)/2} y_j^{d_j} x_j^{d_j} + \sum_{i=1}^{d_j - 1} t_i y_j^i x_j^i z_{j-1}^{d_j - i} + z_{j-1}^{d_j}$$

for some $t_i \in T$. One can verify this directly by using the q-binomial identities, but a slicker approach notes that $\mathcal{Z}(A_n^{E,B}(T))$ is generated by $x_k^{d_k}, y_k^{d_k}$, $k \in [1, n]$ and since $z_j^{d_j} \in \mathcal{Z}(A_n^{E,B}(T))$, $t_i = 0$ for all i. Therefore,

$$\sigma(\widetilde{z}_j)^{d_j} = z_j^{d_j} = (\epsilon_j - 1)^{d_j} \epsilon_j^{d_j(d_j-1)/2} y_j^{d_j} x_j^{d_j} + z_{j-1}^{d_j}.$$

As

$$\epsilon_j^{1+\cdots+(d_j-1)} = (-1)^{d_j - 1} \tag{4.4}$$

holds, (4.1) follows from the definition (1.7) of Z_j by induction on j.

From the first relation in (3.2) we can get

$$\widetilde{x}_j^{d_j}\widetilde{y}_j^{d_j} - q^{m_j d_j d_n}\widetilde{y}_j^{d_j}\widetilde{x}_j^{d_j} = \sum_{i=0}^{d_j-1} t_i \widetilde{y}_j^i \widetilde{x}_j^i \widetilde{z}_{j-1}^{d_j-i}$$

$$\text{for} \quad t_0 = \prod_{k=1}^{d_j}(1 + q^{d_n m_j/d_j} + \cdots + q^{(k-1)d_n m_j/d_j}), t_i \in T[q^{\pm 1}].$$

Then, since (1.4) implies $\sigma(t_i \widetilde{y}_j^i \widetilde{x}_j^i \widetilde{z}_{j-1}^{d_j-i}/(q-\epsilon)) = 0$ when $0 < i < d_j$ we have

$$\{X_j, Y_j\} = \sigma\left(\frac{x_j^{d_j} y_j^{d_j} - y_j^{d_j} x_j^{d_j}}{q-\epsilon}\right) = m_j d_j d_n \epsilon^{-1}(X_j Y_j - (1-\epsilon_j)^{-d_j} Z_{j-1})$$

.

Having taken care of the first relation, the remaining Poisson brackets between $\{X_j, Y_k \mid j,k \in [1,n], j \neq k\}$ follow from the defining relations of $A_n^{E,B}(T)_q$. Similarly, the Poisson brackets in (4.3) follow the normalizing identities in $A_n^{E,B}(T)_q$

$$\widetilde{z}_j\widetilde{x}_k = q^{-\delta_{k\leq j} d_n/d_k}\widetilde{x}_k\widetilde{z}_j, \quad \widetilde{z}_j\widetilde{y}_k = q^{\delta_{k\leq j} d_n/d_k}\widetilde{y}_k\widetilde{z}_j, \quad \widetilde{z}_j\widetilde{z}_k = \widetilde{z}_k\widetilde{z}_j,$$

mirroring (3.1) and holding for all j, and k. □

4.2 Poisson Prime Elements of $\mathcal{Z}(A_n^{E,B}(T))$

Having established the Poisson structure of $\mathcal{Z}(A_n^{E,B}(T))$, we can now classify its Poisson primes. Restricting ourselves to the conditions of interest delineated in Theorem 1.2.1 (ii) and to the case where $T = \mathbb{C}$, we denote by π the Poisson structure on $\mathrm{Spec}\mathcal{Z}(A_n^{E,B}(\mathbb{C})) \cong \mathbb{A}^{2n}$ corresponding to the Poisson bracket above in Proposition 4.1.2. Following the definitions (1.6)–(1.7) of the elements $X_j, Y_j, Z_j \in \mathcal{Z}(A_n^{E,B}(\mathbb{C}))$ laid out in Chapter 2 we find the following.

Proposition 4.2.1. *For $T = \mathbb{C}$ and under the conditions in Theorem 1.2.1 (ii), the following hold:*

(i) The Poisson structure π is symplectic on the complement of

$$(\cup_j \mathcal{V}(X_j)) \cup (\cup_j \mathcal{V}(Z_j)). \tag{4.5}$$

(ii) The Poisson prime elements of $(\mathcal{Z}(A_n^{E,B}(T)), \{.,.\})$ are $Z_1, \ldots, Z_n$.

Proof. We begin by letting the complement of (4.5) in $\mathbb{A}^{2n}$ be labeled W. The recursion (1.7) is linear and thus the elements $Z_j \in \mathcal{Z}(A_n^{E,B}(\mathbb{C}))$ must be irreducible and hence prime. Similarly, the functions $\{X_j, Z_j \mid j \in [1, n]\}$ thus form a coordinate chart on W.

Then (i) is established by the fact that $\{X_j, X_k\} \in \mathbb{C}[X_1, \ldots, X_n]$, $\{Z_j, Z_k\} = 0$ and $\{Z_j, X_k\} = -\delta_{k \leq j} d_j d_k d_n Z_j X_k$.

To establish (ii) requires modestly more work. The brackets in Proposition 4.1.2 demonstrate that $Z_1, \ldots, Z_n$ are Poisson prime elements of $(\mathcal{Z}(A_n^{E,B}(T)), \{.,.\})$. While Proposition 4.1.2 simultaneously shows that the elements X_j are not Poisson normal, and cannot then be Poisson prime. Given $f \in \mathcal{Z}(A_n^{E,B}(T))$ to be any other Poisson prime element, then, as W is symplectic and the zero locus of a Poisson prime element consists of a union of symplectic leaves (§2.2.2), $\mathcal{V}(f)$ would intersect W nontrivially forcing $\mathcal{V}(f) \supset W$. But this is then a contradiction, so no other Poisson prime element $f \in \mathcal{Z}(A_n^{E,B}(T))$ can exist. □

Pleasantly, the combination of Proposition 4.1.2 and Proposition 4.2.1 prove Theorem 1.2.2 (i).

Chapter 5
Two Proofs of Theorem 1.2.2 (ii)

This chapter consists of two distinct proofs of Theorem 1.2.2 (ii). While previous work [17, 23, 24] has shown that there are close connections between Poisson geometry and cluster algebras, the two proofs here should not necessarily be seen as a continuation of this idea. In fact, as mentioned previously, the two proofs describes the irreducible factors of distinct algebras. The first proof focuses on the Poisson primes of $\mathcal{Z}(A_n^{E,B}(T))$. While the second proof uses the cluster algebra structure of the entire algebra $A_n^{E,B}(T)$.

5.1 A Proof via Poisson Geometry

We begin our first proof via Poisson geometry by denoting $\mathrm{tr}\colon A_n^{E,B}(T) \to \mathcal{Z}(A_n^{E,B}(T))$, the internal trace associated to the natural embeddings $A_n^{E,B}(T) \hookrightarrow M_{N^2}(\mathcal{Z}(A_n^{E,B}(T)))$ by $\mathcal{Z}(A_n^{E,B}(T))$-bases of $A_n^{E,B}(T)$, where $N = d_1 \dots d_n$. Here

$$\mathcal{B} := \{x_1^{l_1'} y_1^{l_1} \dots x_n^{l_n'} y_n^{l_n} \mid l_j, l_j' \in [0, d_j - 1]\}$$

defines a $\mathcal{Z}(A_n^{E,B}(T))$-basis of $A_n^{E,B}(T)$. The goal is to show that

$$\begin{aligned} \epsilon^m d_{N^2}(\mathcal{B} : \mathrm{tr}) &= \eta Z_1^{N^2(d_1-1)/d_1} \dots Z_n^{N^2(d_n-1)/d_n} \\ &= \eta z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)} \end{aligned} \tag{5.1}$$

for some $m \in \mathbb{Z}$ determined by E and B, with $\eta \in T$ is the scalar defined by Theorem 1.2.2 (ii), and $\epsilon = \exp(2\pi\sqrt{-1}/d_n) \in T^\times$ as in §4.1. Here

$$\eta = \left(N^2(1-\epsilon_1)^{-d_1+1} \dots (1-\epsilon_n)^{-d_n+1}\right)^{N^2}, \tag{5.2}$$

because

$$(1-\epsilon_j)^{d_j-1} \prod_{i=1}^{d_j-2} (1 + \epsilon_j + \dots + \epsilon_j^i) = \prod_{i=1}^{d_j-1} (1 - \epsilon_j^i) = d_j. \tag{5.3}$$

Importantly, $d_j(1-\epsilon_j)^{-d_j+1} \in T^\times$. Now as $A_n^{E,B}(T)$ is defined over $\mathbb{Z}[\epsilon]$ with $\mathcal{B} \subset A_n^{E,B}(\mathbb{Z}[\epsilon])$, it will suffice to prove (5.1) for $T = \mathbb{Z}[\epsilon]$, and thus we assume that $T = \mathbb{Z}[\epsilon]$.

Using the filtration of $A_n^{E,B}(T)$ from §1.1, and following [10, Proposition 4.10], we have that

$$\operatorname{gr} d_{N^2}(\mathcal{B} : \operatorname{tr}) = d_{N^2}(\operatorname{gr} \mathcal{B} : \operatorname{tr}).$$

But by Theorem 1.2.1 (i) $\operatorname{gr}\mathcal{B}$ is a $\mathcal{Z}(\operatorname{gr} A_n^{E,B}(T))$-basis of $\operatorname{gr} A_n^{E,B}(T)$, and so the right hand side of the equation uses the internal trace of $\operatorname{gr} A_n^{E,B}(T)$ relative to this basis. We can then apply [10, Proposition 2.8] as $\operatorname{gr} A_n^{E,B}(T)$ is a skew polynomial algebra, to find that

$$\epsilon^{m'} d_{N^2}(\operatorname{gr}\mathcal{B} : \operatorname{tr}) = N^{2N^2}(\overline{x}_1\overline{y}_1)^{N^2(d_1-1)} \dots (\overline{x}_n\overline{y}_n)^{N^2(d_n-1)}$$

for some $m' \in \mathbb{Z}$. As $\operatorname{gr} z_j = (\epsilon_j - 1)\overline{x}_j\overline{y}_j$, (5.2) implies that

$$\epsilon^{m'} \operatorname{gr} d_{N^2}(\mathcal{B} : \operatorname{tr}) = \pm\eta \operatorname{gr}(z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)}), \tag{5.4}$$

where $\pm = (-1)^{N^2(d_1+\dots+d_n-n)}$, a power of ϵ by (4.4). The process of working with T is simplified by first passing from $T = \mathbb{Z}[\epsilon]$ to its field of fractions. Thus to prove Theorem 1.2.2 (ii), it is suffices to cover the case where $T = \mathbb{Q}(\epsilon)$ and, by extension, the case where $T = \mathbb{C}$. We therefore assume that $T = \mathbb{C}$. Then combining Theorem 2.3.1 and Proposition 4.2.1 (ii), yields

$$d(A_n^{E,B}(\mathbb{C})/\mathcal{Z}(A_n^{E,B}(\mathbb{C}))) =_{\mathbb{C}^\times} z_1^{s_1} \dots z_n^{s_n} \tag{5.5}$$

for some $s_1, \dots, s_n \in \mathbb{N}$, but (5.4) shows that $s_j = N^2(d_j - 1)$ for each $j \in [1, n]$. This resolves (5.1) for $T = \mathbb{C}$. The first and second equalities are related by Proposition 4.1.2, where $Z_j = z_j^{d_j}$, and second equality for $T = \mathbb{C}$ directly follows by combining (5.5) and (5.4). □

Remark 5.1.1. We note that a nice conceptual proof of the fact that the discriminant formula in Theorem 1.2.2 (ii) does not depend on the matrix B comes from the use of 2-cocycle twists [1]. To begin, one may denote by 1_n the $n \times n$ matrix filled with 1's. Then $A_n^{E,1_n}(T)$ and $A_n^{E,B}(T)$ are $\mathbb{Z}^n$-graded by

$$\deg y_j = -\deg x_j = e_j,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. The algebra $A_n^{E,B}(T)$ can be obtained from $A_n^{E,1_n}(T)$ by the 2-cocycle twist [1] via the cocycle

$$\gamma \colon \mathbb{Z} \times \mathbb{Z} \to T^\times, \quad \gamma(e_j, e_k) = \gamma(e_k, e_j)^{-1} \coloneqq \sqrt{\beta_{jk}}, \;\; j < k.$$

When $A_n^{E,B}(T)$ is of the type required for Theorem 1.2.1 (ii), then so is $A_n^{E,1_n}(T)$. Both algebras have centers generated by $x_j^{d_j}$, $y_j^{d_j}$, with the twist on $\mathcal{Z}_n^{E,1_n}(T)$ trivial, which is to say that it leaves the product invariant with respect to the twist. By checking the degrees, one can verify that the trace of a homogeneous element of $A_n^{E,1_n}(T)$ is also invariant under the twist implying that the two discriminants $d(A_n^{E,B}(T), \mathcal{Z}(A_n^{E,B}(T)))$ and $d(A^{E,1_n}(T), \mathcal{Z}(A_n^{E,1_n}(T)))$ are equal. Similar arguments can establish the independence of the discriminant formula in Theorem A.1.2 on the entries of the matrix B.

5.2 A Proof via Quantum Cluster Algebras

As above we will show that it is sufficient to prove the theorem in the case when $\epsilon_j - 1 \in T^\times$ for all j, and then proceed to do so. Starting with an arbitrary integral domain T, let

$$T' \coloneqq T[(\epsilon_j - 1)^{-1}, 1 \leq j \leq n].$$

Assuming that the theorem is valid for T', by (5.2), we have that

$$d(A_n^{E,B}(T')/\mathcal{Z}(A_n^{E,B}(T'))) =_{(T')^\times} N^{2N^2} z_1^{N^2(d_1-1)} \ldots z_n^{N^2(d_n-1)}.$$

Therefore,

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} \nu N^{2N^2} z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)},$$

for some $\nu \in T'$. Again following [9, Propositions 2.8 and 4.10] and using the same argument as on (5.4),

$$\operatorname{gr} d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} \eta(\operatorname{gr} z_1)^{N^2(d_1-1)} \dots (\operatorname{gr} z_n)^{N^2(d_n-1)}$$

with respect to the filtration originally defined in §1.1. This implies that $\nu N^{2N^2} =_{T^\times} \eta$ and

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} \eta z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)}.$$

Henceforth we assume that $\epsilon_j - 1 \in T^\times$ for all j. Let $A(y,z,T)$ be the skew polynomial algebra over T with generators y_j, z_j, for $j \in [1,n]$ and relations (2.1). Then Proposition 2.4.1 (i) implies that

$$A_n^{E,B}(T)[y_j^{-d_j}, 1 \le j \le n] \cong A(y,z,T)[y_j^{-d_j}, 1 \le j \le n]$$

and we label this algebra A. As $\mathcal{Z}(A_n^{E,B}(T)) = T[x_j^{d_j}, y_j^{d_j}, 1 \le j \le n]$, in conjunction with (1.7) and (4.1) we find that

$$\mathcal{Z}(A) = T[y_j^{\pm d_j}, z_j^{d_j}, 1 \le j \le n] \quad \text{and} \quad \mathcal{Z}(A(y,z,T)) = T[y_j^{d_j}, z_j^{d_j}, 1 \le j \le n].$$

Now both A and $A(y,z,T)$ are free over their centers, and since A is a central localization of both $A_n^{E,B}(T)$ and $A(y,z,T)$, the internal trace $\operatorname{tr}\colon A \to \mathcal{Z}(A)$ is an extension of the internal traces $\operatorname{tr}\colon A_n^{E,B}(T) \to \mathcal{Z}(A_n^{E,B}(T))$ and $\operatorname{tr}\colon A(y,z,T) \to \mathcal{Z}(A(y,z,T))$. Furthermore,

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{\mathcal{Z}(A)^\times} d(A/\mathcal{Z}(A)) =_{\mathcal{Z}(A)^\times} d(A(y,z,T)/\mathcal{Z}(A(y,z,T))), \tag{5.6}$$

so by [10, Proposition 2.8]

$$d(A(y,z,T)/\mathcal{Z}(A(y,z,T))) =_{T^\times} N^{2N^2} z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)}$$

noting that $N \in T^\times$.

From (5.6) we have that

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} N^{2N^2} y_1^{h_1 d_1} \dots y_n^{h_n d_n} z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)}$$

for a given $h_j \in \mathbb{Z}$. While Proposition 2.4.1 (ii) implies that

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} N^{2N^2} x_1^{g_1 d_1} \dots x_n^{g_n d_n} z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)}$$

for some $g_j \in \mathbb{Z}$. As $A_n^{E,B}(T)$ is a domain and can be characterized as an iterated skew polynomial extension, we have that

$$y_1^{h_1 d_1} \dots y_n^{h_n d_n} =_{T^\times} x_1^{g_1 d_1} \dots x_n^{g_n d_n},$$

which can only occur when $g_j = h_j = 0$ as

$$\{y_1^{l_1} \dots y_n^{l_n} x_1^{l'_1} \dots x_n^{l'_n} \mid l_j, l'_j \in \mathbb{N}\}$$

defines a T-basis of $A_n^{E,B}(T)$. Therefore

$$d(A_n^{E,B}(T)/\mathcal{Z}(A_n^{E,B}(T))) =_{T^\times} N^{2N^2} z_1^{N^2(d_1-1)} \dots z_n^{N^2(d_n-1)},$$

and Theorem 1.2.2 (ii) is now proven in a second manner. □

Remark 5.2.1. Chan, Young and Zhang [11] in their proof of the case $n = 1$ of Theorem 1.2.2 (ii) also implicitly utilize quantum cluster algebras, despite their proofs apparent differences. They generate a quantum torus using the elements y_1 and $y_1^{-1} z_1$. Whereas in the above proof one of the clusters is defined by the cluster variables y_1, z_1, defining the same quantum tori in both proofs.

Chapter 6

Automorphisms and Isomorphisms of PI Quantized Weyl Algebras

This final dissertation chapter focuses on the study of tensors of quantized Weyl algebras, in doing so it completes the proofs of Theorem 1.2.2 (iii) and Theorem 1.2.3. In doing so we first revisit the conditions of effective and locally dominating functions covered in §2.3.1, defining two new filtrations on our algebras, distinct from those outlined in §1.1 in order to facilitate working with these definitions.

6.1 Discriminant Properties of PI Quantized Weyl Algebras

To begin our consideration of tensors of quantized Weyl algebras $A_{n_1}^{E_1,B_1}(T), \ldots, A_{n_l}^{E_l,B_l}(T)$ we begin with the following definitions. Let

$$\mathcal{Z}(A) \supset Z(A) \coloneqq \bigcup_{i}^{l} \{Z_j \in \mathcal{Z}(A_{n_i}^{E_i,B_i}(T))\}$$

be the union of the Poisson prime elements over the center of each quantized Weyl algebra while

$$A \supset z(A) \coloneqq \bigcup_{i}^{l} \{z_j \in A_{n_i}^{E_i,B_i}(T)\}$$

be the union of all normal elements of each quantized Weyl algebra and $x(A) \subset A$ and $y(A) \subset A$ define the collections of all x and y-generators of A. Then Theorem 1.2.2 (iii) can be completed by proving the proposition below.

Proposition 6.1.1. *Let $A_{n_1}^{E_1,B_1}(T), \ldots, A_{n_l}^{E_l,B_l}(T)$ be a set of quantized Weyl algebras over an integral domain T of characteristic 0, satisfying the conditions in Theorem A (ii). Let A be their tensor product over T. The discriminant $d(A, \mathcal{Z}(A))$ is*

(i) locally dominating and

(ii) effective.

Since

$$d(A/\mathcal{Z}(A)) = d(A_{n_1}^{E_1,B_1}(T)/\mathcal{Z}(A_{n_1}^{E_1,B_1}(T)))\dots d(A_{n_l}^{E_l,B_l}(T), \mathcal{Z}(A_{n_l}^{E_l,B_l}(T))), \quad (6.1)$$

Proposition 6.1.1 (i) then follows by Lemma 6.1.2 below and Theorem 1.2.2 (ii).

Lemma 6.1.2. *Let* $\phi \in Aut_T(A)$.

(i) For each $z \in z(A)$, $\deg \phi(z) \geq 2$.

(ii) If $\deg \phi(x) > 1$ *for at least one* $x \in x(A)$ *or* $\deg \phi(y) > 1$ *for at least one* $y \in y(A)$, *then* $\deg \phi(z) > 2$ *for some* $z \in z(A)$.

We define by $\{\mathcal{F}_j A\}$ the first new $\mathbb{N}$-filtration of A with degrees on the generators of

$$\deg x = \deg y = 1 \quad \text{for all } \ x \in x(A), y \in y(A). \quad (6.2)$$

Proof of Lemma 6.1.2. To prove part (i) we first note that a series of calculations verifies that the normal elements of $A_{n_i}^{E_i,B_i}(T)$ in the first degree of the filtration, $\mathcal{F}_1 A_{n_i}^{E_i,B_i}(T)$ are those of the form $T.1$ and they are simultaneously central elements. As ϕ is an automorphism, it follows that $\phi(z)$ remains a normal element in A for every $z \in z(A)$. But $\phi(z)$ cannot be central, and thus $\phi(z) \notin \mathcal{F}_1 A$.

To verify part (ii) we begin with $\phi \in \mathrm{Aut}_T(A)$ such that $\deg \phi(x) > 1$ or $\deg \phi(y) > 1$ for at least one $x \in x(A)$ or $y \in y(A)$. In this event we have $i \in [1, l]$ and $j \in [1, n_i]$ so that the x- and y-generators, $x_1, y_1, \dots, x_{n_i}, y_{n_i}$, of $A_{n_i}^{E_i,B_i}(T)$, satisfy

$$\deg \phi(x_k) = \deg \phi(y_k) = 1 \ \text{ for } \ k < j \quad \text{and} \quad \deg \phi(x_j) > 1 \ \text{ or } \ \deg \phi(y_j) > 1.$$

Then as $z_j := 1 + (\epsilon_1 - 1)y_1x_1 + \dots + (\epsilon_j - 1)y_jx_j$ it follows that, $\deg \phi(x_j)\phi(y_j) > 2$ and

$$\deg(1 + (\epsilon_1 - 1)\phi(y_1)\phi(x_1) + \dots + (\epsilon_{j-1} - 1)\phi(y_{j-1})\phi(x_{j-1})) \leq 2,$$

and hence $\deg \phi(z_j) > 2$. $\square$

Remark 6.1.3. It is worth noting that the discriminants considered in [9, 10, 11] have unique leading terms in certain generating sets as a general property, which is to say they are linear combinations of monomials, where the powers of the trailing monomials are componentwise less than those of a leading one. This implies the (global) dominance by those discriminants following [9, Lemma 2.2 (1)]. Our discriminants described in Theorem 1.2.2 (ii) do not possess this property in general and the proof of their local dominance is thus more involved.

To demonstrate the effectiveness of discriminants of PI quantized Weyl algebras we now consider our second new filtration on A. Using the trivial filtration on A, where $F_0A := A$ we then let R be any "testing" filtered PI T-algebra. Choosing elements $\theta(x), \theta(y) \in R$ corresponding to each $x \in x(A)$, $y \in y(A)$ ensuring that there exists an x or y, where $\theta(x) \notin F_0R$ or $\theta(y) \notin F_0R$. Again, there exist $i \in [1, l]$ and $j \in [1, n_i]$ such that the x- and y-generators $x_1, y_1, \ldots, x_{n_i}, y_{n_i}$ of $A_{n_i}^{E_i,B_i}(T)$ satisfy

$$\deg \theta(x_k) = \deg \theta(y_k) = 0 \text{ for } k < j \quad \text{and} \quad \deg \theta(x_j) > 0 \text{ or } \deg \theta(y_j) > 0.$$

And thus,

$$\deg(1+(\epsilon_1-1)\theta(y_1)\theta(x_1)+\cdots+(\epsilon_{j-1}-1)\theta(y_{j-1})\theta(x_{j-1})) \in F_0R, \ \ \theta(x_j)\theta(y_j) \notin F_0R.$$

Again ensuring that

$$1 + (\epsilon_1 - 1)\theta(y_1)\theta(x_1) + \cdots + (\epsilon_j - 1)\theta(y_j)\theta(x_j) \notin F_0R.$$

The discriminant $d(A/\mathcal{Z}(A))$ is then effective by (6.1) and Theorem 1.2.2 (ii).

6.2 Classifying $\mathrm{Aut}_T(\bigotimes_{i=1}^{l}(A_{n_i}^{E_i,B_i}))$

As above and in Theorem 1.2.3 we let A be a tensor product of quantized Weyl algebras. For this section we will require the additional notation:

$$E(A) = \{\epsilon \in \mathcal{T}^\times \mid \epsilon^{\pm 1} \in E_1 \cup \cdots \cup E_l\};$$

$$\mathcal{L}_\epsilon(r) = \{r' \in \mathcal{F}_1 A \mid rr' = \epsilon r' r\} \quad \text{for } r \in A,\ \epsilon \in E(A) \cup \{1\},$$

$$\mathcal{L}(r) = \bigoplus_{\epsilon \in E(A)\cup\{1\}} \mathcal{L}_\epsilon(r),\ \ \mathcal{L}^*(r) = \bigoplus_{\epsilon \in E(A)} \mathcal{L}_\epsilon(r).$$

Having established this, we proceed directly to the proof of Theorem 1.2.3.

Proof of Theorem 1.2.3. To prove part (i) we again let $\phi \in \mathrm{Aut}_T(A)$, and now let $\mathbb{K}$ be the field of fractions of T. Extending ϕ to a $\mathbb{K}$-linear automorphism of $A_{\mathbb{K}} := A \otimes_T \mathbb{K}$, we use ϕ to refer to both with context for clarification. ϕ then induces a $\mathbb{K}$-linear automorphism on the polynomial algebra $\mathcal{Z}(A_{\mathbb{K}})$ (established in Theorem 1.2.1 and naturally extended to the finite tensor). Likewise, we have that $\phi(d(A_{\mathbb{K}}/\mathcal{Z}(A_{\mathbb{K}})) =_{\mathbb{K}^\times} d(A_{\mathbb{K}}/\mathcal{Z}(A_{\mathbb{K}}))$. As before the prime divisors of $d(A_{\mathbb{K}}/\mathcal{Z}(A_{\mathbb{K}})) \in \mathcal{Z}(A_{\mathbb{K}})$ are $Z(A)$ and thus, for every $Z \in Z(A)$ there exists $\alpha_0 \in \mathbb{K}^\times$ such that $\alpha_0^{-1}\phi(Z) \in Z(A)$.

Given $z \in z(A)$, there exists $\phi(z)^k \in Z(A)$ for some $k \in \mathbb{Z}_+$. It then follows from the previous conclusion that given any $z \in Z(A)$, there exist $z' \in Z(A)$, as well as $k, k' \in \mathbb{Z}_+$ and $\alpha_0 \in \mathbb{K}^\times$, such that

$$\phi(z)^k = \alpha_0 (z')^{k'}.$$

Using again the filtration (6.2) of A. We have that $d(A_{\mathbb{K}}/\mathcal{Z}(A_{\mathbb{K}}))$ is locally dominating, and by Theorem 2.3.4, $\phi(\mathcal{F}_1 A_{\mathbb{K}}) = \mathcal{F}_1 A_{\mathbb{K}}$ so, $\phi(z) \in \mathcal{F}_2 A_{\mathbb{K}}$, but by Lemma 6.1.2, $\phi(z) \notin \mathcal{F}_1 A_{\mathbb{K}}$. Similarly, we find that $z' \in \mathcal{F}_2 A_{\mathbb{K}}$ and $z' \notin \mathcal{F}_1 A_{\mathbb{K}}$. Therefore, $k' = k$ by equivalence of degree in the filtration. As both $z', \phi(z) \in \mathcal{F}_2 A$ are normal

elements, (3.1) imply that z' and $\phi(z)$ commute. Hence $\phi(z)^k = \alpha_0(z')^k$ requires that $z' = \alpha\phi(z)$ for some $\alpha \in \overline{\mathbb{K}}$, necessarily in $\mathbb{K}$, as $z' \in A_{\mathbb{K}}$. Thus

$$\text{for every } z \in z(A) \text{ there exists } \alpha \in \mathbb{K}^\times \text{ such that } \alpha^{-1}\phi(z) \in z(A). \tag{6.3}$$

Now, for $i \in [1, l]$ we have $z \in z(A)$ where $\mathcal{L}^*(z) = A_{n_i}^{E_i,B_i}$ and for each $z_0 \in z(A) \cap A_{n_i}^{E_i,B_i}$, we have $\mathcal{L}^*(z_0) \subseteq A_{n_i}^{E_i,B_i}$. Then Theorem 1.2.3 part (i) will follow from $\phi(\mathcal{F}_1 A) = \mathcal{F}_1 A$ by applying Theorem 2.3.4 and using the fact that $d(A_{\mathbb{K}}/\mathcal{Z}(A_{\mathbb{K}}))$ is locally dominating.

To do so we begin by letting $z_1, \ldots, z_{n_i}$ and $z'_1, \ldots, z'_{n_i}$ denote the sequences of normal elements in $A_{n_i}^{E_i,B_i}$ and $A_{n_{\sigma(i)}}^{E_{\sigma(i)},B_{\sigma(i)}}$ belonging to $z(A)$. Choose $z_0 = 1$. Then by (3.1), we have $\mathcal{L}^*(z_{j-1}) \subsetneq \mathcal{L}^*(z_j)$, $\forall j \in [1, n_i]$. In conjunction with the facts (6.3) and that $\phi(\mathcal{F}_1 A) = \mathcal{F}_1 A$, there must exist a corresponding sequence $\alpha_1, \ldots, \alpha_n \in \mathbb{K}^\times$ such that

$$\phi(z_j) = \alpha_j z'_j \quad \text{for } \ j \in [1, n]. \tag{6.4}$$

By (3.1) we have that for all $\epsilon \neq 1$

$$\mathcal{L}_\epsilon(z_j) \cap \mathcal{L}_1(z_{j-1}) \neq 0 \ \text{ if and only } \ \epsilon = \epsilon_j^{\pm 1}$$

and

$$\mathcal{L}_{\epsilon_j^{-1}}(z_j) \cap \mathcal{L}_1(z_{j-1}) = Tx_j, \quad \mathcal{L}_{\epsilon_j}(z_j) \cap \mathcal{L}_1(z_{j-1}) = Ty_j.$$

Then equation (6.4) and the fact that $\phi(\mathcal{F}_1 A) = \mathcal{F}_1 A$ implies that either

$$\phi(x_j) = \mu_j x_j, \qquad \phi(y_j) = \nu_j y_j \quad \text{or} \tag{6.5}$$

$$\phi(x_j) = \mu_j y_j, \qquad \phi(y_j) = \nu_j x_j \tag{6.6}$$

for some $\mu_j, \nu_j \in T$. Considering the same setup for ϕ^{-1} gives $\mu_j, \nu_j \in T^\times$ then $\tau_j = 1$ describes the first case and $\tau_j = -1$ the second case.

Recalling that $[x_j, y_j] = z_j$ yields $\alpha_j = \tau_j \mu_j \nu_j$, i.e.

$$\phi(z_j) = \tau_j \mu_j \nu_j z'_j. \tag{6.7}$$

The two equalities in (1.8) then follow by applying ϕ to our identity $x_j y_j - \epsilon_j y_j x_j = z_{j-1}$ and using (6.7) given (6.5) or (6.6). Likewise (1.9) follows from applying ϕ to the homogeneous defining relations of $A_n^{E,B}(T)$ and using (6.5) or (6.6), which concludes part(i).

By contrast, part (ii) can be shown directly, while part (iii) combines Theorem 1.2.2 (iii) and 2.3.4 (ii) with the fact that the quantized Weyl algebras have finite GK-dimension. □

6.3 Special Cases of Theorem 1.2.3 (i)-(ii)

It is worth covering two important special cases of Theorem 1.2.3 (i)–(ii) in more detail. The following two corollaries classify the automorphisms and isomorphisms of PI quantized Weyl algebras.

Corollary 6.3.1. *Let $A_n^{E,B}(T)$ and $A_{n'}^{E',B'}(T)$ be two quantized Weyl algebras over an integral domain T satisfying the conditions in Theorem 1.2.1 (ii), where $E = (\epsilon_1, \ldots, \epsilon_n)$, $E' = (\epsilon'_1, \ldots, \epsilon'_{n'})$, $B = (\beta_{jk})$ and $B' = (\beta'_{jk})$. Then the algebras $A_n^{E,B}(T)$ and $A_{n'}^{E',B'}(T)$ are isomorphic if and only if $n' = n$ and there exists a sequence $(\tau_1, \ldots, \tau_n) \in \{\pm 1\}^n$ such that*

$$\epsilon'_j = \epsilon_j^{\tau_j}, \ \ \forall j \quad \textit{and} \quad \beta'_{jk} = \begin{cases} \beta_{jk}^{\tau_j}, & \textit{if } \tau_k = 1, \\ (\epsilon_j \beta_{jk})^{-\tau_j}, & \textit{if } \tau_k = -1, \end{cases} \qquad \forall j < k.$$

This theorem for the non-PI case was found in [22], while the theorem for the case where $n = 1$ was obtained in [15, 11]. All homogenized PI quantized Weyl algebras were discussed by [16] building on the results of [4] for the isomorphism problem on $\mathbb{N}$ graded algebras. But this last result does not apply to quantized Weyl algebras as they lack a nontrivial $\mathbb{N}$-grading.

Corollary 6.3.2. *Let $A_n^{E,B}(T)$ be a quantized Weyl algebra over an integral domain T satisfying the conditions in Theorem 1.2.1 (ii).*

(i) For all scalars $\mu_1, \nu_1, \ldots, \mu_n, \nu_n \in T^\times$ such that $\mu_j\nu_j = 1$, $\forall j$,

$$\phi(x_j) = \mu_j x_j, \quad \phi(y_j) = \nu_j y_j$$

defines a T-linear automorphism of $A_n^{E,B}(T)$.

(ii) Assume that for some $k \in [1, n]$, $\epsilon_k = -1$, $\beta_{jk}^2 = \epsilon_j$ for $j < k$, $\beta_{jk}^2 = 1$ for $j > k$. For all scalars $\mu_1, \nu_1, \ldots, \mu_n, \nu_n \in T^\times$ such that $\mu_j\nu_j = 1$ for $j \leq k$ and $\mu_j\nu_j = -1$ for $j > k$,

$$\phi(x_j) = \mu_j x_j, \quad \phi(y_j) = \nu_j y_j, \ \text{ for all } \ j \neq k,$$
$$\phi(x_k) = \mu_k y_k, \quad \phi(y_k) = \nu_k x_k$$

defines a T-linear automorphism of $A_n^{E,B}(T)$.

All elements of $\mathrm{Aut}_T(A_n^{E,B}(T))$ have one of the above two forms.

In particular, $\mathrm{Aut}_T(A_n^{E,B}(T)) \cong (T^\times)^n \ltimes \mathbb{Z}_2$ if the pair (E, B) satisfies the condition in (2) for some $k \in [1, n]$ and $\mathrm{Aut}_T(A_n^{E,B}(T)) \cong (T^\times)^n$ otherwise.

Importantly, when Corollary 6.3.2 (ii) is satisfied, then in Theorem 1.2.1 (ii) when it requires that $d_j | d_k$ for $j < k$, this implies that $\epsilon_j = -1$ for all $j < k$.

The theorem in the non-PI version was obtained in [33], while the case of the theorem for $n = 1$ was found by [11].

References

[1] M. Artin, W. Schelter, and J. Tate, *Quantum deformations of* GL_n, Commun. Pure Appl. Math. **44** (1991), 879–895.

[2] E. Backelin, *Endomorphisms of quantized Weyl algebras*, Lett. Math. Phys. **97** (2011), 317–338.

[3] V. V. Bavula and D. A. Jordan, *Isomorphism problems and groups of automorphisms for generalized Weyl algebras*, Trans. Amer. Math. Soc. **353** (2001), 769–794.

[4] J. Bell and J. J. Zhang, *An isomorphism lemma for graded rings*, preprint arXiv:1509.08812.

[5] J. Bell and J. J. Zhang, *Zariski cancellation problem for noncommutative algebras*, preprint arXiv:1601.04625.

[6] G. Benkart, *Multiparameter Weyl algebras*, preprint arXiv:1306.0485.

[7] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005), 405–455.

[8] K. A. Brown and I. Gordon, *Poisson orders, symplectic reflection algebras and representation theory*, J. Reine Angew. Math. **559** (2003), 193–216.

[9] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant controls automorphism groups of noncommutative algebras*, Adv. Math. **269** (2015), 551–584.

[10] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang, *The discriminant criterion and automorphism groups of quantized algebras*, Adv. Math. **286** (2016), 754–801.

[11] K. Chan, A. A. Young, and J. J. Zhang, *Discriminant formulas and applications*, preprint arXiv:1503.06327, to appear in Algebra Number Theory.

[12] C. De Concini, V. Kac, and C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. **5** (1992), 151–189.

[13] S. Fomin and A. Zelevinsky, *Cluster algebras, I. Foundations,* J. Amer. Math. Soc. **15** (2002), 497–529.

[14] H. Fujita, E. Kirkman, and J. Kuzmanovich, *Global and Krull dimensions of quantum Weyl algebras*, J. Algebra **216** (1999), 405–416.

[15] J. Gaddis, *Isomorphisms of some quantum spaces*, In: Ring theory and its applications, Eds.: D. Van Huynh et al., pp. 107–116, Contemp. Math., 609, Amer. Math. Soc., Providence, RI, 2014.

[16] J. Gaddis, *The isomorphism problem for quantum affine spaces, homogenized quantized Weyl algebras, and quantum matrix algebras*, preprint arXiv:1605.08711.

[17] M. Gekhman, M. Shapiro, A. Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, American Mathematical Society Providence, Rhode Island, 2010

[18] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Mod. Birkhäuser Class., Birkhäuser Boston, 2008.

[19] A. Giaquinto and J. J. Zhang, *Quantum Weyl algebras*, J. Algebra **176** (1995), 861–881.

[20] K. R. Goodearl, *Prime ideals in skew polynomial rings and quantized Weyl algebras*, J. Algebra **150** (1992), 324–377.

[21] K. R. Goodearl and E. S. Letzter, *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. **352** (2000), 1381–1403.

[22] K. R. Goodearl and J. T. Hartwig, *The isomorphism problem or multiparameter quantized Weyl algebras*, São Paulo J. Math. Sci. **9** (2015), 53–61.

[23] K. R. Goodearl and M. T. Yakimov, *From quantum Ore extensions to quantum tori via noncommutative UFDs*, preprint arXiv:1208.6267, to appear in Adv. Math.

[24] K. R. Goodearl and M. T. Yakimov, *Quantum cluster algebra structures on quantum nilpotent algebras*, preprint arXiv:1309.7869, to appear in Mem. Amer. Math. Soc.

[25] T. Hayashi, *Sugawara operators and Kac–Kazhdan conjecture*, Invent. Math. **94** (1988), 13–52.

[26] D. A. Jordan, *A simple localization of the quantized Weyl algebra*, J. Algebra **174** (1995), 267–281.

[27] A. P. Kitchin and S. Launois, *Endomorphisms of quantum generalized Weyl algebras*, Lett. Math. Phys. **104** (2014), 837–848.

[28] L. Makar-Limanov, *Locally nilpotent derivations, a new ring invariant and applications*, preprint, http://www.math.wayne.edu/~lml/lmlnotes.pdf.

[29] G. Maltsiniotis, *Groupes quantiques et structures différentielles*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 831–834.

[30] B. Nguyen, K. Trampel, and M. Yakimov, *Noncommutative discriminants via Poisson primes*, preprint arXiv:1603.02585.

[31] I. Reiner, *Maximal Orders*, London Math. Soc. Monogr. New Ser., vol. **28**, The Clarendon Press, Oxford Univ Press, Oxford, 2003.

[32] L. Richard and A. Solotar, *Isomorphisms between quantum generalized Weyl algebras*, J. Algebra Appl. **5** (2006), 271–285.

[33] L. Rigal, *Spectre de l'algèbre de Weyl quantique*, Beiträge Algebra Geom. **37** (1996), 119–148.

[34] W. A. Stein, *Algebraic Number Theory: A Computational Approach*, book preprint, http://wstein.org/books/ant/.

Appendix: The Generalized Discriminant Formula

This appendix contains the proof for a general formula for the discriminants of PI quantized Weyl algebras over polynomial central subalgebras generated by powers of pairs of the standard generators of the Weyl algebra. The proof comes from an extension the approach from Sect. 2.4, which used quantum cluster algebra techniques then combined with ideas from field theory.

A.1 Setting Up the Generalized Formula

In this section we will work with algebras that are slightly more general than the quantized Weyl algebras $A_n^{E,B}(T)$ to enable the use of inductive arguments. Working over the commutative integral domain T, we define $A_n^{E,B,c}(T)$ over the indeterminate c to be the $T[c]$-algebra with generators $x_1, y_1, \ldots, x_n, y_n$ and relations (1.1), but with the final relation redefined to be

$$x_j y_j - \epsilon_j y_j x_j = c + \sum_{k=1}^{j-1} (\epsilon_k - 1) y_k x_k, \quad \forall k.$$

The T-algebra $A_n^{E,B}(T)$ is then the result of specialization:

$$A_n^{E,B}(T) \cong A_n^{E,B,c}(T)/(c-1)A_n^{E,B,c}(T).$$

Assuming again (1.3), for $j < k$, let

$$\frac{m_j}{d_j} + \frac{m_{jk}}{d_{jk}} = \frac{m'_{jk}}{d'_{jk}}$$

where $m'_{jk} \in \mathbb{N}$, $d'_{jk} \in \mathbb{Z}_+$ such that $\gcd(m'_{jk}, d'_{jk}) = 1$ and define $d'_{kj} := d'_{jk}$. Fortunately, Theorem 1.2.1 and 3.1.1 remain true in this slightly more general situation. In fact Theorem 1.2.1 (i) directly implies the following lemma.

Lemma A.1.1. *For $l \in \mathbb{Z}_+$ the following hold:*

(i) $x_j^l \in \mathcal{Z}(A_n^{E,B,c}(T))$ *if and only if* $x_j^l \in \mathcal{Z}(A_n^{E,B}(T))$ *if and only if*

$$\operatorname{lcm}(d_j, d'_{jk}, 1 \leq k \leq n, k \neq j) | l.$$

(ii) $y_j^l \in \mathcal{Z}(A_n^{E,B,c}(T))$ *if and only if* $y_j^l \in \mathcal{Z}(A_n^{E,B}(T))$ *if and only if*

$$\operatorname{lcm}(d_j, d_{jk}, 1 \leq k \leq n, k \neq j) | l.$$

Now, every polynomial central subalgebra of $A_n^{E,B,c}(T)$ of the form

$$\mathcal{C} \coloneqq T[c, x_1^{L_1}, y_1^{L_1}, \ldots, x_n^{L_n}, y_n^{L_n}], \tag{A.8}$$

has $A_n^{E,B,c}(T)$ as a free $\mathcal{C}$-module. We define $\operatorname{tr}\colon A_n^{E,B,c}(T) \to \mathcal{C}$ to be the internal trace function associated to the embedding of $A_n^{E,B,c}(T) \hookrightarrow M_\Lambda(\mathcal{C})$ over $\mathcal{C}$-bases of $A_n^{E,B}(T)$, where

$$\Lambda \coloneqq L_1^2 \ldots L_n^2.$$

For $A_n^{E,B,c}(T)$ satisfying the conditions of Theorem 1.2.1 (ii) with $L_j = d_j$, this becomes the trace map of §5.1 under the specialization $c = 1$.

Continuing in the general case as in the case for $c = 1$, we show that

$$z_j \coloneqq c + (\epsilon_1 - 1)y_1x_1 + \cdots + (\epsilon_j - 1)y_jx_j = [x_j, y_j]$$

are normal elements of $A_n^{E,B,c}(T)$ satisfying (3.1). Beginning with $z_0 = c$, we find that

$$z_j = (\epsilon_j - 1)y_jx_j + z_{j-1} \quad \text{and} \quad z_j^{d_j} = -(1 - \epsilon_j)^{d_j} y_j^{d_j} x_j^{d_j} + z_{j-1}^{d_j} \tag{A.9}$$

for $j \in [1, n]$, with the second identity is verified in the same manner as its twin in Proposition 4.1.2. Again we have that $x_j^{d_j}$ and $y_j^{d_j}$ commute.

Setting up our induction, let $E^{\check{}} \coloneqq (\epsilon_2, \ldots, \epsilon_n)$ and $B^{\check{}}$ be the $(n-1) \times (n-1)$ submatrix of B obtained by deleting the first row and column.

Then by Lemma A.1.1, given $x_1^{L_1}, y_1^{L_1}, \ldots, x_n^{L_n}, y_n^{L_n} \in \mathcal{Z}(A_n^{E,B,c}(T))$, then

$$d_j | L_k \quad \text{for } j \leq k. \tag{A.10}$$

Theorem A.1.2. *Let $A_n^{E,B,c}(T)$ be an arbitrary PI quantized Weyl algebra over an integral domain $T[c]$ satisfying* (1.2)*. For a choice of central elements*

$$x_1^{L_1}, y_1^{L_1}, \ldots, x_n^{L_n}, y_n^{L_n} \in \mathcal{Z}(A_n^{E,B}(T)),$$

denote $\mathcal{A}_n := A_n^{E,B,c}(T)$, $\mathcal{C}_n := T[c, x_1^{L_1}, y_1^{L_1}, \ldots, x_n^{L_n}, y_n^{L_n}]$ and $\mathcal{A}_{n-1} := A_{n-1}^{E^{\check{}},B^{\check{}},c'}(T)$, $\mathcal{C}_{n-1} := T[c', x_2^{L_2}, y_2^{L_2}, \ldots, x_n^{L_n}, y_n^{L_n}]$ for $n > 1$, $\mathcal{A}_0 = \mathcal{C}_0 = T[c']$ for $n = 1$.

(i) The discriminant $d(\mathcal{A}_n/\mathcal{C}_n)$ is a polynomial in $c^{\gcd(L_1,\ldots,L_n)}$.

(ii) By part (i) *and* (A.10) *the discriminant $d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})$ is a polynomial in $(c')^{d_1}$, which will be denoted by $d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})((c')^{d_1})$. We have,*

$$\begin{aligned} d(\mathcal{A}_n/\mathcal{C}_n) =_{T^\times} \theta x_1^{(L_1-d_1)\Lambda} y_1^{(L_1-d_1)\Lambda} (c^{L_1} - (1-\epsilon_1)^{L_1} y_1^{L_1} x_1^{L_1})^{(d_1-1)\Lambda/L_1} \\ \times \prod_{i=0}^{L_1/d_1-1} \left[d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})(c^{d_1} - \zeta^i (1-\epsilon_1)^{d_1} y_1^{d_1} x_1^{d_1}) \right]^{d_1 L_1}, \end{aligned}$$

where $\Lambda = L_1^2 \ldots L_n^2$, $\theta = L_1^\Lambda (L_1 (1-\epsilon_1)^{-d_1+1})^\Lambda$, and ζ is a primitive L_1/d_1-st root of unity.

Here we have that $\mathcal{C}_n^\times = \mathcal{C}_{n-1}^\times = T[c]^\times = T^\times$, usefully connecting back to the fact from (5.3), that $d_1(1-\epsilon_1)^{-d_1+1} \in T$. Before continuing our proof let us consider two examples for small n.

Example A.1.3. (i) Let $n = 1$. Here the quantized Weyl algebra $A_1^{\epsilon_1,c}(T)$ is defined given $E = (\epsilon_1)$, with $B = (1)$ as B is multiplicatively skewsymmetric. Then the discriminant formula is

$$d(A_1^{\epsilon_1,c}(T)/T[c, x_1^{L_1}, y_1^{L_1}]) =_{T^\times} \theta x_1^{(L_1-d_1)L_1^2} y_1^{(L_1-d_1)L_1^2} (c^{L_1} - (1-\epsilon_1)^{L_1} y_1^{L_1} x_1^{L_1})^{(d_1-1)L_1}$$

for $\theta = L_1^{L_1^2}(L_1(1-\epsilon_1)^{-d_1+1})^{L_1^2}$.

(ii) Now letting $n = 2$, the discriminant formula becomes

$$\begin{aligned} &d(A_2^{E,B,c}(T)/T[c, x_1^{L_1}, y_1^{L_1}, x_2^{L_2}, y_2^{L_2}]) \\ &=_{T^\times} \theta x_1^{(L_1-d_1)\Lambda} y_1^{(L_1-d_1)\Lambda} x_2^{(L_2-d_2)\Lambda} y_2^{(L_2-d_2)\Lambda} \big(c^{L_1} - (1-\epsilon_1)^{L_1} y_1^{L_1} x_1^{L_1}\big)^{(d_1-1)\Lambda/L_1} \\ &\times \prod_{i=0}^{L_1/d_1-1} \Big[\big(c^{d_1} - \zeta^i(1-\epsilon_1)^{d_1} y_1^{d_1} x_1^{d_1}\big)^{L_2/d_1} - (1-\epsilon_2)^{L_2} y_2^{L_2} x_2^{L_2}\Big]^{(d_2-1)d_1 L_1 L_2}, \end{aligned}$$

now with $\Lambda = L_1^2 L_2^2$, $\theta = \Lambda^{\Lambda/2} \prod_{i=1}^2 (L_i(1-\epsilon_i)^{-d_i+1})^\Lambda$, and ζ a primitive (L_1/d_1)-st root of 1. Here, the final product in discriminant expression is a polynomial in $c^{\gcd(L_1,L_2)}$.

A.2 Proof of Theorem A.1.2

We set up this proof with the following notations. Given a field extension $\mathbb{K}'/\mathbb{K}$, let

$$\mathrm{tr}_{\mathbb{K}'/\mathbb{K}}, \mathcal{N}_{\mathbb{K}'/\mathbb{K}} \colon \mathbb{K}' \to \mathbb{K}$$

indicate the standard trace and norm functions respectively. For $\mathbb{K}(\alpha)/\mathbb{K}$, a finite separable extension, let $f(t) \in \mathbb{K}[t]$ be the minimal polynomial of α over $\mathbb{K}$. Let $\mu(\alpha)$ be the $\mathbb{K}$-linear endomorphism of $\mathbb{K}(\alpha)$ defined by multiplication by α. When $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_k$ are the roots of $f(t)$ over its splitting field, then

$$\text{the characteristic polynomial of } \mu(\alpha) \text{ is } \ (t-\alpha_1)\ldots(t-\alpha_k) \in \mathbb{K}[t],$$

see e.g. [33, p. 67, Ex. 14]. In particular,

$$\mathrm{tr}\, \mu(\alpha)^j = \sum_{i=1}^k \alpha_i^j, \quad \mathcal{N}_{\mathbb{K}(\alpha)/\mathbb{K}}(g(\alpha)) = \prod_{i=1}^k g(\alpha_i) \ \ \forall g(t) \in \mathbb{K}[t]. \tag{A.11}$$

We also set

$$\begin{aligned} \Delta \coloneqq_{T^\times} &\ \theta x_1^{(L_1-d_1)\Lambda} y_1^{(L_1-d_1)\Lambda} (c^{L_1} - (1-\epsilon_1)^{L_1} y_1^{L_1} x_1^{L_1})^{(d_1-1)\Lambda/L_1} \\ &\times \prod_{i=0}^{L_1/d_1-1} \big[d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})(c^{d_1} - \zeta^i(1-\epsilon_1)^{d_1} y_1^{d_1} x_1^{d_1})\big]^{d_1 L_1}, \end{aligned} \tag{A.12}$$

Proof of Theorem A.1.2 (ii). Instead of using full quantum clusters as described in Sect. 2.4, we will use a part of a cluster consisting of the cluster variables x_1 and z_1. Localizing by x_1, we work inductively by relating the discriminant of $\mathcal{A}_n$ to that of $\mathcal{A}_{n-1}$.

To start we reduce the statement of Theorem A.1.2 (ii) to a form that has to do with the localization in question. Theorem A.1.2 (ii) follows once we show that

$$d(\mathcal{A}_n[x_1^{-L_1}]/\mathcal{C}_n[x_1^{-L_1}]) =_{T[x_1^{L_1}]^\times} \Delta. \tag{A.13}$$

In fact, when this holds we also have that

$$d(\mathcal{A}_n/\mathcal{C}_n) =_{T^\times} x_1^{kL_1}\Delta \tag{A.14}$$

for some $k \in \mathbb{Z}$. Recalling again the filtration from §1.1, then by [10, Proposition 4.10],

$$\operatorname{gr} d(\mathcal{A}_n/\mathcal{C}_n) =_{T^\times} d(\operatorname{gr}\mathcal{A}_n/\operatorname{gr}\mathcal{C}_n),$$

with this second discriminant computed with respect to the trace on $\operatorname{gr}\mathcal{A}$ derived from its freeness over $\operatorname{gr}\mathcal{C}_n$. As $\operatorname{gr} A_n^{E,B,c}(T)$ defines a localization of a skew polynomial algebra over a power of one of its generators $\overline{x}_1$, [10, Proposition 2.8] applies, yielding

$$d(\operatorname{gr}\mathcal{A}_n/\operatorname{gr}\mathcal{C}_n) =_{T^\times} \Lambda^\Lambda(\overline{x}_1\overline{y}_1)^{\Lambda(L_1-1)}\dots(\overline{x}_n\overline{y}_n)^{\Lambda(L_n-1)} =_{T^\times} \operatorname{gr}\Delta. \tag{A.15}$$

Thus, in (A.14), $k = 0$, and (A.13) will imply Theorem A.1.2 (ii).

By a similar argument used in §5.2, it will suffice to prove the theorem for the case where $\epsilon_j - 1 \in T^\times$ for all $j \in [1, n]$.

Again, we will assume that

$$\epsilon_1 - 1 \in T^\times \tag{A.16}$$

for the remainder of this proof and we will then prove (A.13).

Working with the associated graded in (A.14) and using (A.15) yields

$$\overline{x}^{kL_1} \operatorname{gr}\Delta =_{T^\times} \operatorname{gr}(\mathcal{A}_n/\mathcal{C}_n) =_{T^\times} \operatorname{gr}\Delta.$$

Hence $k = 0$ again and so (A.13) implies the statement in Theorem A.1.2 (ii).

Now, z_1 commutes with all x_j, y_j for $j > 1$. Thus, we find that $\mathcal{A}_{n-1}$ is isomorphic to the T-subalgebra of $\mathcal{A}_n$ generated by z_1 and x_j, y_j for $j > 1$ when $c' = z_1$. We will also let this algebra be $\mathcal{A}_{n-1}$ and with central subalgebra $T[z_1, x_j^{L_j}, y_j^{L_j}, 2 \le j \le n]$ being $\mathcal{C}_{n-1}$ in standard abuses of notation.

We will define the prospective bases

$$\begin{aligned}
\mathcal{B}'' &\coloneqq \{x_2^{l_2'} y_2^{l_2} \dots x_n^{l_n'} y_n^{l_n} \mid l_j \in [0, L_j - 1]\},\\
\mathcal{B}' &\coloneqq \{1, z_1, \dots, z_1^{L_1-1}\}\mathcal{B}'',\\
\mathcal{B} &\coloneqq \{1, x_1, \dots, x_1^{L_1-1}\}\mathcal{B}', \quad \text{and}\\
\mathcal{B}^{\mathrm{op}} &\coloneqq \mathcal{B}'\{1, x_1, \dots, x_1^{L_1-1}\}.
\end{aligned}$$

Showing first from (A.9) that

$$y_1^{L_1} = \frac{(c^{d_1} - z_1^{d_1})^{L_1/d_1}}{(1-\epsilon_1)^{L_1} x_1^{L_1}}.$$

And thus that,

$$\mathcal{C}_n[x_1^{-L_1}] = T[c, x_1^{\pm L_1}, (c^{d_1} - z_1^{d_1})^{L_1/d_1}, x_j^{L_j}, y_j^{L_j}, 2 \le j \le n].$$

Then, since x_1 normalizes z_1 and x_j, y_j for $j > 1$, and as z_1, x_j, y_j, $j > 1$ generate our $T[z_1]$-algebra $\mathcal{A}_{n-1}$, we find that $\mathcal{B}$ is a $\mathcal{C}_n[x_1^{-L_1}]$-basis of $\mathcal{A}_n[x_1^{-L_1}]$ and $\mathcal{B}'$ is a $\widetilde{\mathcal{C}}_{n-1}$-basis of $\mathcal{A}_{n-1}[c]$, where

$$\widetilde{\mathcal{C}}_{n-1} = T[c, (c^{d_1} - z_1^{d_1})^{L_1/d_1}, x_j^{L_j}, y_j^{L_j}, 2 \le j \le n].$$

Now let $\mathrm{tr}' \colon \mathcal{A}_{n-1}[c] \to \widetilde{\mathcal{C}}_{n-1}$ be the $T[c]$-linear trace function from this latter basis.

With this preamble complete we now prove (A.13), and thus Theorem A.1.2 (ii), in two steps.

Step I. We first connect $d(\mathcal{A}_n[x_1^{-L_1}]/\mathcal{C}_n[x_1^{-L_1}])$ to $d(\mathcal{A}_{n-1}[c]/\widetilde{\mathcal{C}}_{n-1})$. For all $b_1', b_2' \in \mathcal{B}'$ and $i, k \in [0, L_1 - 1]$,

$$\operatorname{tr}(b_1' x_1^i \cdot x_1^k b_2') = \operatorname{tr}(b_2' b_1' x_1^{i+k}) = L_1 x_1^{i+k} \operatorname{tr}'(b_2' b_1') = L_1 x_1^{i+j} \operatorname{tr}'(b_1' b_2')$$

if $i + j = 0$ or L_1 and $\operatorname{tr}(b_1 x_1^i \cdot x_1^k b_2) = 0$ otherwise. Then the determinant of a Kronecker product of matrices calculated by its standard formula gives

$$\begin{aligned} d(\mathcal{A}_n[x_1^{-L_1}]/\mathcal{C}_n[x_1^{-L_1}]) &=_{T[x_1^{-L_1}]^\times} L_1^\Lambda x_1^{(L_1-1)\Lambda} \det([\operatorname{tr}(b_1 b_2)]_{b_1 \in \mathcal{B}^{\mathrm{op}}, b_2 \in \mathcal{B}})^{L_1} \\ &=_{T[x_1^{L_1}]^\times} L_1^\Lambda d(\mathcal{A}_{n-1}[c]/\widetilde{\mathcal{C}}_{n-1})^{L_1}. \end{aligned}$$

Step II. Now, we may relate $d(\mathcal{A}_{n-1}[c]/\widetilde{\mathcal{C}}_{n-1})$ to $d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})$. The set $\mathcal{B}''$ forms a $\mathcal{C}_{n-1}$-basis of $\mathcal{A}_{n-1}$, as $\mathcal{A}_{n-1}$ and $\mathcal{C}_{n-1}$ can both be viewed as $T[z_1]$-algebras. This defines $\operatorname{tr}'' \colon \mathcal{A}_{n-1} \to \mathcal{C}_{n-1}$, the associated $T[z_1]$-linear trace of this basis, which extends to a map $\operatorname{tr}'' \colon \mathcal{A}_{n-1}[c] \to \mathcal{C}_{n-1}[c]$ by c-linearity.

Let $\mathbb{K}$ be the fraction field of $T[x_1^{L_1} y_1^{L_1}]$ and

$$f(t) \coloneqq (c^{d_1} - t^{d_1})^{L_1/d_1} - (1 - \epsilon_1)^{L_1} x_1^{L_1} y_1^{L_1} \in \mathbb{K}[t].$$

This polynomial is irreducible, separable and z_1 is a root of it, with irreducibility coming from the fact that $x_1^{L_1} y_1^{L_1} \in \mathbb{K}$ but $x_1^{d_1} y_1^{d_1} \notin \mathbb{K}$. Given the field extension $\mathbb{K}(z_1)/\mathbb{K}$. We can relate the traces tr' and tr'' by

$$\operatorname{tr}' = \operatorname{tr}_{\mathbb{K}(z_1)/\mathbb{K}} \circ \operatorname{tr}'' \tag{A.17}$$

with $\operatorname{tr}_{\mathbb{K}(z_1)/\mathbb{K}}$ extended to $\mathbb{K}[c, z_1, x_j^{L_j}, y_j^{L_j}, 2 \le j \le n]$ by linearity on c, $x_j^{L_j}$, and $y_j^{L_j}$, for $j > 1$. In proving (A.17) we will use that z_1 is central in $\mathcal{A}_{n-1}[c]$.

Let $\alpha_1 = z_1, \alpha_2, \ldots, \alpha_{L_1}$ be the roots of $f(t)$ in its splitting field. These are given by

$$\xi^k \big(c^{d_1} - \zeta^i (1 - \epsilon_1)^{d_1} y_1^{d_1} x_1^{d_1}\big)^{1/d_1} \quad \text{for } \ k \in [0, d_1 - 1], i \in [0, L_1/d_1 - 1], \tag{A.18}$$

for ζ a primitive (L_1/d_1)-st root of unity, as stated in the theorem, with ξ a primitive d_1-st root of unity.

For $a \in \mathcal{A}_{n-1}$ we denote by $\mathrm{tr}''(a)(z_1)$ the polynomial dependance of $\mathrm{tr}''(a)$ on z_1. By (A.11) and (A.17) it follows that

$$\mathrm{tr}'(a) = \sum_{j=1}^{L_1} \mathrm{tr}''(a)(\alpha_j).$$

Thus, considering the bases $\mathcal{B}$ and $\mathcal{B}'$ we see that

$$d(\mathcal{A}_{n-1}[c]/\widetilde{\mathcal{C}}_{n-1}) = \det\left[\mathrm{tr}'(z_1^{i-1}b_1'' \cdot z_1^{k-1}b_2'')\right]_{i,k;b_1'',b_2''} = \det\left[\sum_j \alpha_j^{i+k-2}\,\mathrm{tr}'(b_1''b_2'')(\alpha_j)\right]_{i,k;b_1'',b_2''}$$

where in each matrix $i, j, k \in [1, L_1]$, and $b_1'', b_2'' \in \mathcal{B}''$. The final matrix is factored as the product of block matrices, with square matrix blocks of size $\Lambda/L_1^2 = |\mathcal{B}''|$ by:

$$[\alpha_j^{i-1}I]_{i,j} \cdot [\alpha_j^{i-1}I]_{i,j} \cdot \mathrm{diag}(Q(\alpha_1), \ldots, Q(\alpha_{L_1})) \tag{A.19}$$

with $i, j \in [1, L_1]$, and where I indicates the identity matrix of size Λ/L_1^2 and

$$Q(z_1) := [\mathrm{tr}''(b_1''b_2'')(z_1)]_{b_1'',b_2'' \in \mathcal{B}''}.$$

That $\det Q(z_1) =_{T\times} d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})(z_1^{d_1})$ follows directly from it's definition. It then follows from the fact that the roots $\alpha_1, \ldots, \alpha_{L_1}$ are given by (A.18), and from Theorem A.1.2 (i), that the determinant of the third matrix in (A.19) above gives the product appearing in the second line of (A.12) where Δ is defined.

By (A.18) we then find that for all $k \in [1, L_1]$,

$$f'(\alpha_j) = L_1\alpha_j^{d_1-1}\frac{(1-\epsilon_1)^{L_1}x_1^{L_1}y_1^{L_1}}{(c^{d_1}-\alpha_j^{d_1})} = L_1\zeta^{-i}\alpha_j^{d_1-1}(1-\epsilon_1)^{L_1-d_1}x_1^{L_1-d_1}y_1^{L_1-d_1}$$

for some $i \in [0, L_1/d_1 - 1]$. Revisiting (A.19), the determinant of the product of the first two matrices is

$$\begin{aligned}
&\prod_{1\le i<j\le n}(\alpha_i-\alpha_j)^{2\Lambda/L_1^2} = \pm\left[\mathcal{N}_{\mathbb{K}(z_1)/\mathbb{K}}(f'(z_1))\right]^{\Lambda/L_1^2} = \pm\prod_i f'(\alpha_i)^{\Lambda/L_1^2}\\
&= \pm L_1^{\Lambda/L_1}(1-\epsilon_1)^{(L_1-d_1)\frac{\Lambda}{L_1}}x_1^{(L_1-d_1)\frac{\Lambda}{L_1}}y_1^{(L_1-d_1)\frac{\Lambda}{L_1}}\left[\mathcal{N}_{\mathbb{K}(z_1)/\mathbb{K}}(z_1^{d_1}-1)\right]^{\Lambda/L_1^2}\\
&= \pm L_1^{\frac{\Lambda}{L_1}}(1-\epsilon_1)^{(L_1-d_1)\frac{\Lambda}{L_1}}x_1^{(L_1-d_1)\frac{\Lambda}{L_1}}y_1^{(L_1-d_1)\frac{\Lambda}{L_1}}\left(c^{L_1}-(1-\epsilon_1)^{L_1}x_1^{L_1}y_1^{L_1}\right)^{(d_1-1)\Lambda/L_1^2},
\end{aligned}$$

which follows from the standard expression for discriminants of finite separable field extensions as a product of norms [31, pp. 66-67, Ex. 14]. Replacing the determinants of the matrices in (A.19) in our expression for $d(\mathcal{A}_{n-1}[c]/\widetilde{\mathcal{C}}_{n-1})$, in conjunction with Step I and (A.16), proves (A.13), which completes the proof of Theorem A.1.2 (ii). □

Proof of Theorem A.1.2 (i). We proceed by induction on n. First assume that $d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})$ is a polynomial in z_1^L for $L = \gcd(L_2, \ldots, L_n)$. We will show that $d(\mathcal{A}_n/\mathcal{C}_n)$ is a polynomial in $c^{\gcd(L_1,\ldots,L_n)}$. Let

$$d(\mathcal{A}_{n-1}/\mathcal{C}_{n-1})(z_1^L) = \prod_s (z_1^L - a_s)$$

for some a_s in the algebraic closure of the fraction field of $T[x_j^{L_j}, y_j^{L_j}, 2 \le j \le n]$. Then the product

$$\prod_{i=0}^{L_1/d_1-1}\left((c^{d_1}-\zeta^i(1-\epsilon_1)^{d_1}y_1^{d_1}x_1^{d_1})^{L/d_1}-a_s\right)$$

defines a polynomial in $c^{\gcd(L_1,L)}$, implying that the product given in the formula for $d(\mathcal{A}_n/\mathcal{C}_n)$ on the second line in part (ii) of the theorem is a polynomial in $c^{\gcd(L_1,L)}$. As the product from the first line said formula is a polynomial in c^{L_1}, $d(\mathcal{A}_n/\mathcal{C}_n)$ is thus a polynomial in

$$c^{\gcd(L_1,L)} = c^{\gcd(L_1,\ldots,L_n)}.$$

This proves the first part of the theorem. □

Vita

Jesse S. F. Levitt was born in 1986, in Boston, Massachusetts. He finished his undergraduate studies at Williams College in June 2008. In August 2010 he came to Louisiana State University to pursue graduate studies in mathematics. He earned a master of science degree in mathematics from Louisiana State University in May 2012. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2016.