Properties of Random Direction Models

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***Abstract*— A number of mobility models have been proposed for the purpose of either analyzing or simulating the movement of users in a mobile wireless network. Two of the more popular are the random waypoint and the random direction models. The random waypoint model is physically appealing but difficult to understand. Although the random direction model is less appealing physically, it is much easier to understand. User speeds are easily calculated, unlike for the waypoint model, and, as we will observe, user positions and directions are uniformly distributed. The contribution of this paper is to establish this last property for a rich class of random direction models that allow future movements to depend on past movements. To this end, we consider finite one- and two-dimensional spaces. We consider two variations, the random direction model with wrap around and with reflection. We establish a simple relationship between these two models and, for both, show that positions and directions are uniformly distributed for a class of Markov movement models regardless of initial position. In addition, we establish a sample path property for both models, namely that any piecewise linear movement applied to a user preserves the uniform distribution of position and direction provided that users were initially uniformly throughout the space with equal likelihood of being pointed in any direction.**

I. Introduction

Researchers have proposed a number of different mobility models. Two of the more popular are the random waypoint model, [6], and the random direction model, [3]. Both usually operate in a finite two dimensional plane, typically a square. Under both models, users traverse piecewise linear segments where speeds vary from segment to segment but are in general maintained constant on a segment. The two models differ in one critical manner, namely how users choose the next segment to traverse. Under the random waypoint model a user chooses a point within the space with equal probability and a speed from some given distribution. On the other hand, under the random direction model a user chooses a direction to travel in, a speed at which to travel, and a time duration for this travel. The first model is appealing due to its physical interpretation. However, it introduced significant issues, [12], regarding its stationary behavior, i.e., distribution of nodes in the space and distribution of speeds, that were only recently resolved in [8], [9]. A different problem arises with the random direction model, namely what to do when a mobile hits a boundary. Several variations exist including random direction with wrap around, [5], and with reflection, [2]. For some problems, these are less physically appealing. However, these models exhibit some nice properties, especially useful in theoretical studies, [2], namely that users are uniformly distributed within the space and that the distributions of speeds are easily calculated and understood with respect to the model inputs.

The focus of our paper is to derive the above mentioned properties of the random direction model with either wrap around or reflection. More specifically, we establish the following for the movement of a fixed population of users either on the interval $[0, 1)$ or the two-dimensional square $[0, 1)^2$:

- given that at time $t = 0$ the position and orientation of users are independent and uniform, they remain uniformly distributed for all times $t > 0$ provided the users move independently of each other;
- given that the movements of users are described by statistically independent aperiodic and recurrent Markovian mobility models, then over time they will be come uniformly distributed over their movement space (either $[0, 1)$ or $[0, 1)^2$);
- we establish a simple relationship between the wrap around and reflection models that allows one to map results for one into results for the other.

The remainder of the paper is organized as follows. Section II introduces the random direction mobility model with its two variants, wrap around and reflection for the one dimensional space. It includes some preliminary results, namely statements and proofs of the result that initial uniform placement of mobiles is preserved over time and the relationship between the wrap around and reflection models. For the case of a one dimensional space, Section III addresses the problem of when an arbitrary initial configuration converges to a uniform spatial distribution. Section IV extends the previous results to a two dimensional square. Section VI describes what ought to be straightforward extension of our results to other mobility models. Last, the paper is summarized in Section VII.

A word on the notation in use: throughout $\{Z(t)\}_t$ and $\{a_j\}_j$ stand for $\{Z(t), t \geq 0\}$ and $\{a_j, j = 1, 2, \ldots\}$, respectively. For any topological set $\mathbf{X}$, $\beta(\mathbf{X})$ denotes the smallest σ-field generated by all subsets of $\mathbf{X}$. Last, $\lfloor x \rfloor$ is the largest integer smaller than or equal to x.

II. Mobility models on $[0,1)$

We are interested in properties regarding a population of N mobiles moving on $[0,1)$. However, as we will soon observe, it suffices to focus on a single mobile. Consider a mobile moving on $[0,1)$ according to the following mobility pattern. At some random times $0 \leq T_1 < T_2 < \cdots$ a new *speed* and a new *direction* (also called *orientation*) are selected. The selection of the speed and direction at time T_j ($j \geq 1$) initializes the j-th movement of the mobile. Let $\tau_j = T_{j+1} - T_j$, $j \geq 0$, be the duration of the j-th movement (with $T_0 = 0$ by convention).

During the interval $[T_j, T_{j+1})$ the mobile travels at constant speed $s_j \in \mathbf{S}$, where $\mathbf{S}$ is any topological subset of $(0,\infty)$. Typically, $\mathbf{S} = (\sigma_1, \sigma_2)$ with $0 < \sigma_1 < \sigma_2$ or $\mathbf{S} = \{\sigma_1, \ldots, \sigma_k\}$ with $\sigma_i > 0$ for $i = 1, \ldots, k$.

Let $\theta_j \in \{-1, +1\}$ be the new direction selected at time T_j, $j \geq 1$, with $\theta_j = +1$ (resp. $\theta_j = -1$) if the mobile is oriented to the right (resp. left) at time T_j. We denote by θ_0 the orientation of the mobile at time $t = 0$. Define $\theta(t)$ as the direction of the mobile at time $t \geq 0$. We assume that $\theta(t)$ is right-continuous, so that $\theta(T_j) = \theta_j$ for $j \geq 0$. The selection of the new direction at time T_j, $j \geq 1$, is done as follows. Let $\gamma_1, \gamma_2, \ldots$ be $\{-1, +1\}$-valued rvs; then

$$\theta_j = \theta(T_j^-)\gamma_j, \quad j = 1, 2 \ldots . \tag{1}$$

The rv γ_j is called the *relative direction* of the mobile at time T_j, $j \geq 1$. The relative direction will remain constant between two consecutive movements (see Remark 3.1).

We consider two models, the *wrap around* model and the *reflection* model that we describe next.

A. The wrap around model

In the wrap around model, when the mobile hits the boundary 0 or 1 it reappears instantaneously at the other boundary. As a result, the direction in which the mobile is moving remains unchanged between two consecutive movements, namely,

$$\theta(t) = \sum_{j \geq 0} \theta_j \, \mathbf{1}(T_j \leq t < T_{j+1}) \tag{2}$$

with (see (1))

$$\theta_j = \theta_{j-1}\gamma_j, \quad j = 1, 2 \ldots . \tag{3}$$

The location $X(t)$ of the mobile at time t satisfies

$$X(t) = X(T_j) + \theta_j s_j (t - T_j) - \lfloor X(T_j) + \theta_j s_j (t - T_j) \rfloor \tag{4}$$

for $T_j \leq t \leq T_{j+1}$, $j \geq 0$. Note that $0 \leq X(t) < 1$ if $0 \leq X(0) < 1$.

In particular,

$$x_{j+1} = x_j + \theta_j s_j \tau_j - \lfloor x_j + \theta_j s_j \tau_j \rfloor, \quad j \geq 0, \tag{5}$$

with $x_j := X(T_j)$ the location of the mobile at time T_j.

We see from (3) and (4) that, given $X(0)$, $\theta(0)$ and the initial speed s_0, the process $\{(X(t), \theta(t)\}_t$ is entirely determined by the *movement pattern* $\{(T_j, s_j, \gamma_j)\}_j$.

B. The model with reflections

The model is identical to the wrap around model, the only difference being that the direction of the mobile is reversed after hitting a boundary. If the mobile hits a boundary at time t, then $\theta^r(t)$, its direction at time t, is such that $\theta^r(t) = -\theta^r(t-)$.

Similar to the wrap around model, we assume that the behavior of the mobile is governed by the movement pattern $\{(T_j, s_j, \gamma_j)\}_j$. We represent the state of the mobile in the model with reflections by the vector $(X^r(t), \theta^r(t))$, with $X^r(t)$ being the location of the mobile at time $t \geq 0$.

However, unlike the wrap around model, there does not exist a simple expression like (2) for $\theta^r(t)$. The approach we will follow to compute the stationary distribution of $(X^r(t), \theta^r(t))$ will consist, first, in establishing a relationship between the wrap around model and the model with reflections (see Lemma 2.1), and then will use the results obtained for the former model (see Section III-B) to develop results for the latter model (see Section III-C).

C. Link between wrap around and reflection models

The result below establishes a simple pathwise relationship between the wrap around and reflection models.

Lemma 2.1 (Link between wrap around & reflection): Consider a reflection model $\{(X^r(t), \theta^r(t))\}_t$ with the movement pattern $\{(T_j^r, s_j^r, \gamma_j^r)\}_j$ and the initial speed s_0^r.

We construct a wrap around model $\{(X^w(t), \theta^w(t))\}_t$ with the movement pattern $\{(T_j^w, s_j^w, \gamma_j^w)\}_j$ and the initial speed s_0^w such that pathwise $\tau_j^w = \tau_j^r$, $s_j^w = s_j^r/2$, $\gamma_j^w = \gamma_j^r$ for all $j \geq 1$ and $s_0^w = s_0^r/2$. In other words, the movement patterns $\{(T_j^w, s_j^w, \gamma_j^w)\}_j$ and $\{(T_j^r, s_j^r, \gamma_j^r)\}_j$ are pathwise identical except that the speed of the mobile in the reflection model is always twice the speed of the mobile in the wrap around model.

If the relations

$$X^r(t) = \begin{cases} 2X^w(t), & 0 \leq X^w(t) < 1/2 \\ 2(1 - X^w(t)), & 1/2 \leq X^w(t) < 1 \end{cases} \tag{6}$$

$$\theta^r(t) = \begin{cases} \theta^w(t), & 0 \leq X^w(t) < 1/2 \\ -\theta^w(t), & 1/2 \leq X^w(t) < 1 \end{cases} \tag{7}$$

hold at $t = 0$ then they hold for all $t > 0$.

◇

Proof of Lemma 2.1: Equations (6) and (7) trivially hold for all $t > 0$ if they hold at $t = 0$ from the definition of each model and from the assumption that the mobile in the reflection model always moves twice as fast as the mobile in the wrap around model. ∎

D. A sample path property

We focus on the following question: under what conditions in either the wrap around or reflection model is the mobile equally likely to be at any position in $[0,1)$? The following lemma states that if the mobile is equally likely to be anywhere in $[0,1)$ at $t = 0$, then *any* movement pattern $\{T_j, s_j, \gamma_j\}_j$ preserves this at $t > 0$.

Lemma 2.2 (Preservation of uniform distribution):

Assume that the initial speed s_0 of the mobile is fixed and the movement pattern $\{T_j, s_j, \gamma_j\}_j$ is deterministic.

If $P(X(0) < x, \theta(0) = \theta) = x/2$ for all $x \in (0, 1]$ and $\theta \in \{-1, 1\}$, then

$$P(X(t) < x, \theta(t) = \theta) = \frac{x}{2}$$

for all $x \in (0, 1]$, $\theta \in \{-1, 1\}$ and $t > 0$.

◇

The proof of Lemma 2.2 relies on the following technical lemma, that we will use several times in the following. Its proof is given in the Appendix.

Lemma 2.3 (Property of the floor function):

For all $x \in [0, 1]$ and $a \in (-\infty, \infty)$,

$$\int_0^1 \mathbf{1}\{u + a - \lfloor u + a \rfloor < x\} \, du = x. \tag{8}$$

◇

Proof of Lemma 2.2: We prove it for the wrap around model. The proof for the reflection model then follows through an application of Lemma 2.1. By conditioning on the position $X(0)$ and orientation $\theta(0)$ of the mobile at time $t = 0$, and by using the assumption that $(X(0), \theta(0))$ is uniformly distributed over $[0, 1) \times \{-1, +1\}$, we find from (4) and Lemma 2.3 that

$$P(X(t) < x, \theta(t) = \theta) = \frac{1}{2} \int_0^1 \mathbf{1}\{u + \theta s_0 t - \lfloor u + \theta s_0 t \rfloor < x\} \, du = \frac{x}{2} \tag{9}$$

for $0 \leq t < T_1$ Consider now the distribution of $(X(T_1), \theta(T_1))$. Since $\theta(T_1) = \theta(0)\gamma_1$ by definition of the wrap around model, we see from (4) that $X(T_1) = X(0) + \theta(0)\gamma_1 s_1 T_1 - \lfloor X(0) + \theta(0)\gamma_1 s_1 T_1 \rfloor$. Conditioning on $(X(0), \theta(0))$, and using again the assumption that this pair of rvs is uniformly distributed over $[0, 1) \times \{-1, +1\}$, we find, similar to the derivation of (9), that $P(X(T_1) < x, \theta(T_1) = \theta) = x/2$.

This shows that $(X(t), \theta(t))$ is uniformly distributed over $[0, 1) \times \{-1, +1\}$ for all $0 \leq t \leq T_1$. The proof is concluded through the following induction argument: assume that the uniform distribution holds for $0 \leq t \leq T_i$ and let us show that it still holds for $0 \leq t \leq T_{i+1}$. Since $(X(T_i), \theta(T_i))$ is uniformly distributed over $[0, 1) \times \{-1, +1\}$ from the induction hypothesis and since $\theta(T_{i+1}) = \theta(T_i)\gamma_{i+1}$ by definition of the wrap around model, we can reproduce the analysis in (9) to show that $(X(t), \theta(t))$ is uniformly distributed over $[0, 1) \times \{-1, +1\}$ for all $t \in [T_i, T_{i+1}]$. Therefore, the uniform distribution holds for all $t \in [0, T_{i+1}]$, which completes the proof. ∎

We have the following corollary:

Corollary 2.1: If N mobiles are uniformly distributed on the unit interval $[0, 1)$ with equally likely orientations at $t = 0$, and if they move independently of each other, then they are uniformly distributed on the unit interval with equally likely orientations for all $t > 0$. ◇

In the next section we provide conditions under which the joint distribution of position and direction of a mobile is uniform over $[0, 1)$ and $\{-1, +1\}$.

III. The stationary distribution of location and direction

Now we address the case where the initial placement of the mobile is not uniform in $[0, 1)$ and determine the conditions under which the distribution of the mobile position converges to the uniform distribution. We will show this for the case that the mobile's movement is Markovian.

By convention we will assume until throughout this section that $\theta_1 = \gamma_1$.

A. Assumptions and examples

Before introducing this Markovian setting, we introduce the rvs $\{\xi_j\}_j$ that take values in the finite set $\mathbf{M} := \{1, 2, \ldots, M\}$. These rvs will allow us to represent the state of some underlying Markovian environment (as illustrated in Examples 3.1-3.2), so as to further enrich the model.

We now present the set of probabilistic assumptions placed on the *movement vector* $\{y_j := (\tau_j, s_j, \gamma_j, \xi_j)\}_j$, $y_j \in \mathbf{Y} := [0, \infty) \times \mathbf{S} \times \{-1, +1\} \times \mathbf{M}$. We recall that $\tau_j = T_{j+1} - T_j$, $j \geq 0$, is the duration of the j-th movement (with T_0 by convention).

Set of assumptions A1:

The movement vector $\{y_j\}_j$ is an aperiodic, ϕ-irreducible [10, Prop. 4.2.1, p. 87] and Harris recurrent [10, Prop. 4.2.1, p. 87] discrete-time Markov chain on the state space $\mathbf{Y}$, with probability transition kernel

$$Q(y; C) = P(y_{j+1} \in C \,|\, y_j = y) \tag{10}$$

for all $y = (\tau, s, \gamma, m) \in \mathbf{Y}$, $C = B \times S \times \{\gamma'\} \times \{m'\}$ with $(B, S) \in \beta([0, \infty) \times \mathbf{S})$, $\gamma' \in \{-1, +1\}$, $m' \in \mathbf{M}$. We further assume that $\{y_j\}_j$ has a unique invariant probability measure[1] q, namely q is the unique solution of the equations

$$q(C) = \int_{\mathbf{Y}} q(dy) \, Q(y; C), \; C \in \beta(\mathbf{Y}), \quad \int_{\mathbf{Y}} q(dy) = 1. \tag{11}$$

◇

Below, we determine $Q(y, C)$ for Markov Modulated Travel Times (MMTTs).

Example 3.1 (MMTTs):

Consider a movement vector $\{y_j\}_j$ where the consecutive travel times $\tau_1, \tau_2 \ldots$ form a Markov modulated sequence: when $\xi_j = m \in \mathbf{M}$ the j-th travel time τ_j is taken from an iid sequence $\{\tau_j(m)\}_j$ with probability distribution $G_m(\cdot)$, namely $\tau_j = \sum_{m=1}^{M} \tau_j(m) \mathbf{1}(\xi_j = m)$. We assume that $\{\tau_j(m)\}_j$, $j = 1, \ldots, M$, are mutually independent sequences, independent of the rvs $\{s_j, \gamma_j, \xi_j\}_j$.

[1] Under the assumptions placed on $\{y_j\}_j$ we already know that it has a unique invariant measure (up to a multiplicative constant). A sufficient condition for this measure to be finite – and therefore for an invariant probability measure to exist – is that there exists a petite set [10, p. 121] such that the expected return to this petite set is uniformly bounded [10, Theorem 10.0.1, p. 230].

We further assume that the sequences $\{\xi_j\}_j$ and $\{s_j, \gamma_j\}_j$ are mutually independent. The latter assumption, together with assumptions **A1**, implies that $\{\xi_j\}_j$ and $\{s_j, \gamma_j\}_j$ are both Markov chains. Let $R(m; m') = P(\xi_{j+1} = m' \,|\, \xi_j = m)$, $m, m' \in \mathbf{M}$ be the one-step probability transition of the Markov chain $\{\xi_j\}_j$ and $K(s, \gamma; S \times \{\gamma'\}) = P(s_{j+1} \in S, \gamma_{j+1} = \gamma' \,|\, s_j = s, \gamma_j = \gamma)$, $S \in \beta(\mathbf{S})$, $s \in \mathbf{S}$, $\gamma, \gamma' \in \{-1, +1\}$, the probability transition kernel of the Markov chain $\{s_j, \gamma_j\}_j$.

Then, the probability transition kernel $Q(y, C)$ of the Markov chain $\{y_j\}_j$ writes (see (10))

$$Q(y; C) = G_{m'}(B)\, K(s, \gamma; S \times \{\gamma'\})\, R(m; m') \tag{12}$$

for all $y = (\tau, s, \gamma, m) \in \mathbf{Y}$, $C = B \times S \times \{\gamma'\} \times \{m'\}$ with $(B, S) \in \beta([0, \infty) \times \mathbf{S})$, $\gamma' \in \{-1, +1\}$.

In the particular case when $M = 1$ the travel times $\{\tau_j\}_j$ are iid rvs.

$\diamond$

Still in the context of MMTTs, we now introduce two models where assumptions **A1** are satisfied.

Example 3.2 (MMTTs – Continued):

Consider the model in Example 3.1. Let us place additional conditions on the transition kernel Q in (12) so that assumptions in **A1** are met.

(i) Consider first the situation where the set of available speeds is countable. Assume that the sequences of speeds $\{s_j\}_j$ and relative directions $\{\gamma_j\}_j$ are mutually independent Markov chains, with probability transition kernels $K_{sp}(s; s')$ and $K_{rd}(\gamma; \gamma')$, respectively. Therefore

$$Q(y; C) = G_{m'}(B)\, K_{sp}(s; s')\, K_{rd}(\gamma; \gamma')\, R(m; m')$$

for $y = (\tau, s, \gamma, m) \in \mathbf{Y}$ and $C = B \times \{s'\} \times \{\gamma'\} \times \{m'\} \in \beta(\mathbf{Y})$.

Assume further that travel times have a density and finite expectation (i.e. $\int_0^\infty (1 - G_m(t))\, dt < \infty$ for every $m \in \mathbf{M}$), and that the (mutually independent) finite-state space Markov chains $\{s_j\}_j$, $\{\gamma_j\}_j$ and $\{\xi_j\}_j$ are all irreducible and aperiodic. Therefore, each of them admits a unique invariant distribution, denoted by π_s, π_r and π_e, respectively.

Under these assumptions, the Markov chain $\{y_j\}_j$ is aperiodic, ϕ-irreducible and Harris recurrent, with the unique invariant probability measure q given by

$$q(B \times \{\sigma\} \times \{\gamma\} \times \{m\}) = G_m(B)\, \pi_s(\sigma)\, \pi_r(\gamma)\, \pi_e(m)$$

for all $B \in \beta([0, \infty))$, $s \in \mathbf{S}$, $\gamma \in \{-1, +1\}$, $m \in \mathbf{M}$.

(ii) Consider now the situation where the set of speeds $\mathbf{S}$ is non-countable. Assume that $\{s_j\}_j$ is an iid sequence of rvs, with common probability distribution H and finite expectation in case the set $\mathbf{S}$ is infinite. We place the same assumptions on the travel time, relative direction and environment sequences as in (i) above.

Then, the Markov chain $\{y_j\}_j$ is aperiodic, ϕ-irreducible and Harris recurrent, and has a unique invariant probability measure q given by

$$q(B \times S \times \{\gamma\} \times \{m\}) = G(B)\, H(S)\, \pi_r(\gamma)\, \pi_e(m)$$

for all $B \in \beta([0, \infty))$, $s \in \beta(\mathbf{S})$, $\gamma \in \{-1, +1\}$, $m \in \mathbf{M}$. More general scenarios can easily be constructed.

$\diamond$

We now proceed with the wrap around model and end the section with extensions to the reflection model.

B. Wrap around model

We introduce some more notation. Let

- $R(t) := \sum_{j \ge 0} (T_{j+1} - t) \mathbf{1}(T_j \le t < T_{j+1})$
- $S(t) := \sum_{j \ge 0} s_j \mathbf{1}(T_j \le t < T_{j+1})$
- $\gamma(t) := \sum_{j \ge 0} \gamma_j \mathbf{1}(T_j \le t < T_{j+1})$
- $\xi(t) := \sum_{j \ge 0} \xi_j \mathbf{1}(T_j \le t < T_{j+1})$

be the remaining travel time, the mobile's speed, the relative direction and the state of the environment, respectively, at time t, where by convention $\gamma_0 = 1$ and $\xi_0 = 1$.

The state of the system at time t is represented by the vector

$$Z(t) := (X(t), \theta(t), Y(t))$$

taking values in the set $\mathbf{Z} := [0, 1) \times \{-1, +1\} \times \mathbf{Y}$, where $Y(t) := (R(t), S(t), \gamma(t), \xi(t))$. Recall that $X(t)$ is the position of the mobile at time t (given in (4)) and $\theta(t)$ is the orientation of the mobile at time t (see (2)). Observe that $\{Z(t)\}_t$ is a Markov process.

Define $z_j := Z(T_j)$ the state of the system at time T_j, namely, $z_j = (x_j, \theta_j, y_j)$. The next result shows that the process $\{z_j\}_j$ inherits the Markovian structure of $\{y_j\}_j$.

Lemma 3.1 (Probability transition kernel of $\{z_j\}_j$):

Under assumptions **A1** $\{z_j\}_j$ is Markov chain on $\mathbf{Z}$, with probability transition kernel $P(z; A)$, $z \in \mathbf{Z}$, $A \in \beta(\mathbf{Z})$, given by

$$P(z; A) = \mathbf{1}\left\{x + \theta s \tau - \lfloor x + \theta s \tau \rfloor \in U, \theta' = \theta \gamma'\right\} Q(y; C) \tag{13}$$

for all $z = (x, \theta, y) \in \mathbf{Z}$ with $y = (\tau, s, \gamma, m)$, $A = U \times \{\theta'\} \times B \times S \times \{\gamma'\} \times \{m'\}$ with $(U, B, S) \in \beta([0, 1) \times [0, \infty) \times \mathbf{S})$, $\gamma' \in \{-1, +1\}$, $m' \in \mathbf{M}$, and $C = B \times S \times \{\gamma'\} \times \{m'\}$.

$\diamond$

The proof of Lemma 3.1 is given in the Appendix.

The rest of this section is devoted to the computation of the limiting distribution of $\{Z(t)\}_t$. Below is the main result of this section:

Proposition 3.1 (Limiting distribution of $\{Z(t)\}_t$):
Assume that (i) assumptions **A1** hold, (ii) the Markov chain $\{z_j\}_j$ is aperiodic and ϕ-irreducible, and (iii) the expected travel time

$$\overline{\tau} := \int_0^\infty (1 - q([0, t) \times \mathbf{S} \times \{-1, +1\} \times \mathbf{M}))\, dt \tag{14}$$

is finite.

Then, the limiting distribution of the process $\{Z(t)\}_t$ exists, is independent of the initial state, and is given by

$$\lim_{t \to \infty} P(Z(t) \in A) = \frac{u}{2\overline{\tau}} \int_0^{\tau'} (1 - q([0, t) \times S \times \{\gamma'\} \times \{m'\}))\, dt \tag{15}$$

for all $A = [0, u) \times \{\theta'\} \times [0, \tau') \times S \times \{\gamma'\} \times \{m'\}$, with $\theta', \gamma' \in \{-1, +1\}$, $m' \in \mathbf{M}$, $u \in (0, 1]$, $\tau' > 0$, $S \in \beta(\mathbf{S})$.

$\diamond$

The proof of Proposition 3.1 can be found after the proof of Proposition 3.3.

A direct consequence of Proposition 3.1 is that in steady-state the mobile is equally likely to be anywhere in $[0, 1)$, with equally likely orientations. More precisely:

Proposition 3.2 (Unif. distr. of location & direction): Assume that (i) assumptions **A1** hold, (ii) the Markov chain $\{z_j\}_j$ is aperiodic and ϕ-irreducible, and (iii) the expected travel time $\overline{\tau}$ (given in (14)) is finite.

Then,

$$\lim_{t\to\infty} P(X(t) < u, \theta(t) = \theta) = \frac{u}{2} \tag{16}$$

for all $u \in (0, 1]$ and $\theta \in \{-1, +1\}$, for any initial position and direction.

$\diamond$

Proof of Proposition 3.2: For all $u \in (0, 1]$, $\theta \in \{-1, +1\}$, we have from (15)

$$\begin{aligned}
\lim_{t\to\infty} P(X(t) < u, \theta(t) = \theta) &= \frac{u}{2\overline{\tau}} \sum_{\substack{\gamma\in\{-1,+1\}\\ m\in\mathbf{M}}} \\
&\int_0^\infty (1 - q([0, t) \times \mathbf{S} \times \{\gamma\} \times \{m\}))\, dt \\
= &\ \frac{u}{2\,\overline{\tau}} \int_0^\infty (1 - q([0, t) \times \mathbf{S} \times \{-1, +1\} \times \mathbf{M}))\, dt \\
= &\ \frac{u}{2}.
\end{aligned}$$

■

Propositions 3.1 and 3.2 hold under a number of assumptions. We have already given two examples where assumptions **A1** hold (see Example 3.2). The Markov chain $\{z_j\}_j$ will be aperiodic and ϕ-irreducible if travel times *or* speeds have a density (which implies that $\tau_j s_j$, the distance to travel during the interval $[T_j, T_{j+1})$, can take a continuum of values), which covers most cases of practical interest. These assumptions will also hold if $Q(y, C) > 0$ for all $y \in \mathbf{Y}$ and $C \in \beta(\mathbf{Y})$.

The next result addresses the invariant distribution of the Markov chain $\{z_j\}_j$. It will be used to prove Proposition 3.1

Proposition 3.3 (Invariant distribution of $\{z_j\}_j$):

Assume that assumptions **A1** hold and that the Markov chain $\{z_j\}_j$ is ϕ-irreducible. Then, $\{z_j\}_j$ admits a unique invariant probability measure p, given by

$$p(A) = \frac{u}{2}\, q(C) \tag{17}$$

for all $A = [0, u) \times \{\theta'\} \times B \times S \times \{\gamma'\} \times \{m'\}$, with $(B, S) \in \beta([0, \infty) \times \mathbf{S})$, $\theta', \gamma' \in \{-1, +1\}$, $m' \in \mathbf{M}$, $u \in (0, 1)$.

$\diamond$

Proof of Proposition 3.3: We first show that p is an invariant probability measure. Take $A = [0, u) \times \{\theta'\} \times B \times S \times \{\gamma'\} \times \{m'\}$. With (13) we find

$$\begin{aligned}
&\int_{\mathbf{Z}} p(dz)\, P(z; A) \\
&= \frac{1}{2} \sum_{\theta\in\{-1,+1\}} \mathbf{1}\{\theta' = \theta\gamma'\} \int_{y\in\mathbf{Y}} Q(y, C) \\
&\quad \times \left(\int_0^1 \mathbf{1}\{x + \theta s\tau - \lfloor x + \theta s\tau \rfloor < u\}\, dx \right) q(dy) \\
&= \frac{u}{2} \int_{\mathbf{Y}} q(dy)\, Q(y; C) = \frac{u}{2}\, q(C) = p(A),
\end{aligned} \tag{18}$$

where the last three equalities follow from Lemma 2.3, (11) and (17), respectively. Moreover

$$\int_{\mathbf{Z}} p(dz) = \sum_{\theta\in\{-1,+1\}} \int_0^1 \frac{dx}{2} \int_{\mathbf{Y}} q(dy) = 1$$

from (11), which shows with (18) that p is an invariant probability measure.

The uniqueness of the invariant probability measure is a consequence of the assumption that the Markov chain $\{z_j\}$ is ϕ-irreducible and of the fact that its admits an invariant probability measure. This implies (by definition, see [10, p. 230]) that it is positive, and therefore Harris recurrent [10, Theorem 10.1.1, p. 231]. The uniqueness result now follows from the fact that a Harris recurrent Markov chain admits a unique invariant measure (up to a multiplicative constant) [10, Theorem 10.0.1, p. 230]. This shows that p in (17) is necessarily the unique invariant probability measure of the Markov chain $\{z_j\}_j$. ■

We are now in position to prove Proposition 3.1.

Proof of Proposition 3.1: Consider the stationary version of the Markov chain $\{y_j\}_j$ (which exists under assumptions **A1**). In particular, the sequence of travel times $\{\tau_j\}_j$ is stationary and ergodic, the latter property being a consequence of the assumption that the expected travel time $\overline{\tau}$ is finite.

We may therefore apply the Palm formula to the (Markov) process $\{Z(t)\}_t$ [1, Formula 4.3.2], which yields

$$\lim_{t\to\infty} P(Z(t) \in A) = \frac{1}{E^0[T_2]} E^0 \left[\int_0^{T_2} \mathbf{1}\{Z(t) \in A\}\, dt \right] \tag{19}$$

for all $A \in \beta(\mathbf{Z})$, independent of the initial condition $Z(0)$. In (19), the symbol E^0 denotes the expectation operator under the Palm measure w.r.t. the sequence $\{T_j\}_j$ (i.e. the expectation operator conditioned on the event $\{T_1 = 0\}$). Hence $E^0[T_2]$ is equal to the stationarty expected travel times, i.e.

$$E^0[T_2] = \overline{\tau}. \tag{20}$$

As already discussed in the proof of Proposition 3.3 we know that the Markov chain $\{z_j\}_j$ is Harris recurrent, in addition (by assumption) to being aperiodic and ϕ-irreducible. We have also shown in Proposition 3.3 that $\{z_j\}_j$ admits a unique invariant probability measure on $\mathbf{Z}$. We may then conclude from the aperiodic ergodic theorem [10, Theorem

13.0.1-(ii), p. 309] that the stationary distribution of $\{z_j\}_j$ coincides with its invariant probability measure, i.e.

$$\lim_{j\to\infty} P(z_j \in A \,|\, z_1 = z) = p(A), \quad A \in \beta(\mathbf{Z}), \tag{21}$$

for every initial condition $z \in \mathbf{Z}$ (see e.g. [10, Theorem 13.0.1, p. 309]).

Take $A \in \beta(\mathbf{Z})$ as in the statement of the proposition. By conditioning in the r.h.s. of (19) on the state of the *stationary* version of the Markov chain $\{z_j\}_j$ and by using (21) and (17), we find that

$$\begin{aligned} & E^0\left[\int_0^{T_2} \mathbf{1}\{Z(t) \in A\}\,dt\right] \\ &= \frac{u}{2}\int_0^{\tau'} (1 - q([0,t) \times S \times \{\gamma'\} \times \{m'\}))\,dt. \end{aligned} \tag{22}$$

The proof of (22) is given in the Appendix. The proof of Proposition 3.1 is completed by combining (19), (20) and (22). ∎

C. Reflection model

Consider a reflection model with the movement vector $\{y_j^r = (\tau_j^r, s_j^r, \gamma_j^r, \xi_j^r)\}_j$ with state space $\mathbf{Y}^r := [0,\infty) \times \mathbf{S} \times \{-1,+1\} \times \mathbf{M}$. We assume that $\{y_j^r\}_j$ satisfies assumptions **A1**. Let $Z^r(t) = (X^r(t), \theta^r(t), R^r(T), S^r(t), \gamma^r(t), \xi^r(t)) \in [0,1) \times \{-1,+1\} \times \mathbf{Y}^r$ be the state of the mobile at time t, where the definition of $Z^r(t)$ is similar to that of $Z(t)$ in the wrap around model.

Following Lemma 2.1, we construct another movement vector $\{y_j^w = (\tau_j^w, s_j^w, \gamma_j^w, \xi_j^w)\}_j$ with state space $\mathbf{Y}^w := [0,\infty) \times \mathbf{S}/2 \times \{-1,+1\} \times \mathbf{M}$ which satisfies assumptions **A1** and such that pathwise $\tau_j^w = \tau_j^r$, $s_j^w = s_j^r/2$, $\gamma_j^w = \gamma_j^r$ and $\xi_j^w = \xi_j^r$ for all $j \geq 1$ (such a construction is always possible). Let q^w be the the invariant probability measure of the Markov chain $\{y_j^w\}_j$. Let

$$\overline{\tau^w} = \int_0^\infty (1 - q^w([0,t) \times (1/2)\mathbf{S} \times \{-1,+1\} \times \mathbf{M}))dt \tag{23}$$

be the expected travel time associated with the movement vector $\{y_j^w\}_j$. With the movement vector $\{y_j^w\}_j$ we construct a wrap around model $\{Z^w(t)\}_t \in [0,1) \times \{-1,+1\} \times \mathbf{Y}^w$ as in Section III-B. Let z_j^w be the state of the mobile at time T_j in this wrap around model.

The following result holds:

Proposition 3.4 (Stationary mobile's behavior): Assume that the Markov chain $\{z_j^w\}_j$ is aperiodic and ϕ-irreducible, and the expected travel time $\overline{\tau^w}$ in (23) is finite.

Then,

$$\begin{aligned} & \lim_{t\to\infty} P(Z^r(t) \in A) = \frac{u}{2\,\overline{\tau^w}} \\ & \times \int_0^\tau (1 - q^w([0,t) \times (1/2)S \times \{\gamma\} \times \{m\}))\,dt \end{aligned} \tag{24}$$

for all $A = [0,u) \times \{\theta\} \times [0,\tau) \times S \times \{\gamma\} \times \{m\}$ with $u \in (0,1]$, $\theta, \gamma \in \{-1,+1\}$, $\tau > 0$, $S \in \beta(\mathbf{S})$ and $m \in \mathbf{M}$.

In particular,

$$\lim_{t\to\infty} P(X^r(t) < u, \theta^r(t) = \theta) = \frac{u}{2} \tag{25}$$

for all $u \in (0,1]$ and $\theta \in \{-1,+1\}$.

◇

Proof of Proposition 3.4: Take the set A as in the statement of the proposition. With (6)-(7) we find

$$\begin{aligned} P(Z^r(t) \in A) &= P(Z^r(t) \in A, 0 < X^w(t) < 1/2) \\ &\quad + P(Z^r(t) \in A, 1/2 \leq X^w(t) < 1) \\ &= P(X^w(t) < u/2, \theta^w(t) = \theta', R^w(t) < \tau, \\ &\quad S^w(t) \in (1/2)S, \gamma^w(t) = \gamma, \xi^w(t) = m') \\ &\quad + P(X^w(t) > 1 - u/2, \theta^w(t) = -\theta, R^w(t) < \tau, \\ &\quad S^w(t) \in (1/2)S, \gamma^w(t) = \gamma, \xi^w(t) = m). \end{aligned}$$

By letting $t \to \infty$ in the above equation and then using (15) (with $Z(t) = Z^w(t)$, $q = q^w$, $\overline{\tau} = \overline{\tau^w}$) we find (24).

To derive (25) sum up the r.h.s. of (15) over all values of $m' \in \mathbf{M}$ and $\theta' \in \{-1,+1\}$, set $S = \mathbf{S}$, let $\tau' \to \infty$ and use the definition of $\overline{\tau^w}$. ∎

In particular, Proposition 3.4 shows that (see (25)), like the wrap around model, the location and the orientation of the mobile in the reflection model are uniformly distributed in steady-state on $[0,1)$ and $\{-1,+1\}$, respectively.

Remark 3.1: All the results in Sections II and III also hold if the relative direction is additive, namely, if $\theta_j = (\theta_{j-1} + \gamma_j) \bmod 2$ with $\gamma_j \in \{0,1\}$, where $\gamma_j = 0$ (resp. $\gamma_j = 1$) if the direction at time T_j is not modified (resp. is reversed). ◇

IV. Mobility models on $[0,1)^2$

In this section we extend the analysis of Sections II and III to dimension 2 (2D). More precisely, we will assume that the mobile evolves in the square $[0,1)^2$. We begin with the wrap around model.

A. The wrap around model in 2D

The movement vector $\{y_j = (\tau_j, s_j, \gamma_j, \xi_j)\}_j$ in $[0,1)^2$ is similar to the movement vector for the mobility models on [0,1) (referred to as 1D) except for the fact that the relative direction γ_j is additive (see (26)) and takes values in $[0, 2\pi)$. Hence, the state space of y_j is now the set $\mathbf{Y}_\star := [0,\infty) \times \mathbf{S} \times [0,2\pi) \times \mathbf{M}$ (we use the subscript $\star$ to distinguish between some sets in 1D and in 2D; other than that we use the same notation as in the 1D case – for the movement vector, state of the system, etc.– as no confusion is possible between 1D and 2D models).

At the beginning of the j-th movement (i.e. at time T_j), the state of the system is represented by the vector $z_j = (x_j, \theta_j, y_j)$ taking values in the set $\mathbf{Z}_\star := [0,1)^2 \times [0,2\pi) \times \mathbf{Y}_\star$. During the time-interval $[T_j, T_{j+1})$ the mobile travels at constant speed s_j and in direction

$$\theta_j = \theta_{j-1} + \gamma_j - 2\pi\lfloor(\theta_{j-1} + \gamma_j)/(2\pi)\rfloor. \tag{26}$$

The direction $\theta(t) \in [0, 2\pi)$ of the mobile at time t is given by (with $T_0 = 0$ by convention)

$$\theta(t) = \sum_{j \geq 0} \theta_j \, \mathbf{1}\{T_j \leq t < T_{j+1}\}. \tag{27}$$

When the mobile hits a boundary in some dimension, it wraps around and reappears instantaneously at the other boundary in that dimension (see Fig. 1: the mobile starts in position A and moves in the direction θ until it reaches the boundary at point B; then, it wraps around to instantaneously reappear in C, and keeps moving in the direction θ until it reaches D. A new movement begins in direction ϕ, that leads the mobile to E, where it wraps around again and reappears in F with the same orientation ϕ. This movement ends in G).

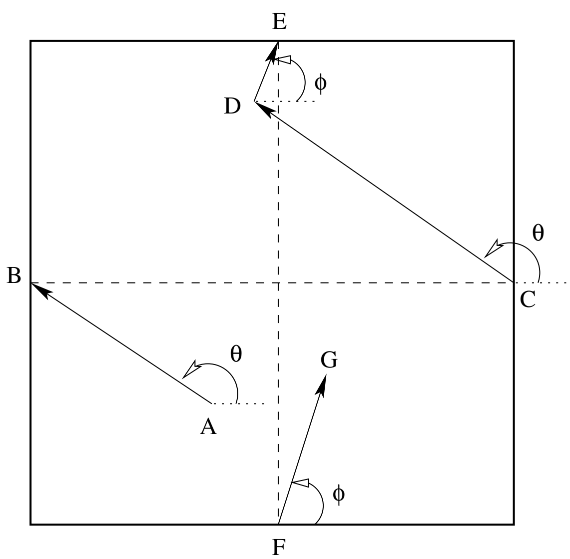

Fig. 1. Wrap around model in 2D

Let $X(t) = (X_1(t), X_2(t)) \in [0, 1)^2$ be the location of the mobile at time t. We have

$$\begin{aligned} X(t) &= X(T_j) + s_j(t - T_j)(\cos\theta_j, \sin\theta_j) \\ &\quad - \lfloor X(T_j) + s_j(t - T_j)(\cos\theta_j, \sin\theta_j) \rfloor \end{aligned} \tag{28}$$

for $T_j \leq t < T_{j+1}$, $j \geq 0$. Here $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$. In particular,

$$\begin{aligned} x_{j+1} &= x_j + s_j \tau_j (\cos\theta_j, \sin\theta_j) \\ &\quad - \lfloor x_j + s_j \tau_j (\cos\theta_j, \sin\theta_j) \rfloor, \quad j \geq 0, \end{aligned} \tag{29}$$

where $x_j := X(T_j)$.

We start the analysis with the analog of Lemma 2.2 in 2D. The lemma shows that, like in 1D, if the position and direction of the mobile are uniformly distributed on $[0, 1)^2 \times [0, 2\pi)$ at time $t = 0$, then this property is preserved at any time.

Lemma 4.1 (Preservation of uniform distr. in 2D): Assume that s_0, the initial speed of the mobile, is fixed and assume that the movement pattern $\{T_j, s_j, \gamma_j\}_j$ is deterministic.

If $P(X_1(0) < u_1, X_2(0) < u_2, \theta(0) < \theta) = u_1 u_2 \theta / 2\pi$ for all $u_1, u_2 \in (0, 1]$ and $\theta \in (0, 2\pi]$, then

$$P(X_1(t) < u_1, X_2(t) < u_2, \theta(t) < \theta) = \frac{u_1 u_2 \theta}{2\pi} \tag{30}$$

for all $u_1, u_2 \in (0, 1]$, $\theta \in (0, 2\pi]$ and $t > 0$.

◇

Proof of Lemma 4.1: By conditioning on the position $X(0)$ and direction $\theta(0)$ of the mobile at time $t = 0$, and by using the assumption that $(X(0), \theta(0))$ is uniformly distributed on $[0, 1)^2 \times [0, 2\pi)$, we find from (29) that for $0 \leq t < T_1$

$$\begin{aligned} &P(X_1(t) < u_1, X_2(t) < u_2, \theta(t) < \theta) \\ &\quad = \frac{1}{2\pi} \int_{\phi=0}^{\theta} \prod_{i=1}^{2} \left(\int_{x_i=0}^{1} \mathbf{1}\{x_i + s_0 t \cos(\phi) - \right. \\ &\qquad \left. \lfloor x_i + s_0 t \cos(\phi) \rfloor < u_i\} dx_i \right) d\phi \\ &\quad = \frac{u_1 u_2 \theta}{2\pi} \end{aligned}$$

where the latter equality follows from Lemma 2.3.

When $t = T_1$, we have

$$\begin{aligned} &P(X_1(T_1) < u_1, X_2(T_1) < u_2, \theta(T_1) < \theta) \\ &\quad = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \prod_{i=1}^{2} \left(\int_{x_i=0}^{1} \mathbf{1}\{x_i + s_0 t \cos(\theta(\phi)) - \right. \\ &\qquad \left. \lfloor x_i + s_0 t \cos(\theta(\phi)) \rfloor < u_i\} dx_i \right) \mathbf{1}\{\theta(\phi) < \theta\} d\phi \\ &\quad = u_1 u_2 \frac{1}{2\pi} \int_0^{2\pi} \mathbf{1}\{\theta(\phi) < \theta\} d\phi \end{aligned} \tag{31}$$

from Lemma 2.3, where $\theta(\phi) := \phi + \gamma_1 - 2\pi \lfloor (\phi + \gamma_1)/(2\pi) \rfloor$ is the direction at time T_1 given that $\theta(0) = \phi$.

Letting $t = \phi / 2\pi$ in the integral in the r.h.s. of (31) gives

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \mathbf{1}\{\theta(\phi) < \theta\} d\phi \\ &\quad = \int_0^1 \mathbf{1}\left\{ \gamma_1/2\pi + t - \lfloor \gamma_1/2\pi + t \rfloor < \frac{\theta}{2\pi} \right\} dt \\ &\quad = \frac{\theta}{2\pi} \end{aligned} \tag{32}$$

by using again Lemma 2.3. Hence, we have shown that (30) holds for all $t \in (0, T_1]$ if it holds at $t = 0$. The proof is concluded by using the same induction argument as in the proof of Lemma 2.2. ∎

We have the following corollary:

Corollary 4.1: If N mobiles are uniformly distributed on $[0, 1)^2$ with equally likely orientations at $t = 0$, and if they move independently of each other, then they are uniformly distributed on $[0, 1)^2$ with equally likely orientations for all $t > 0$. ◇

Set of assumptions A2:

The movement vector $\{y_j\}_j$ is an aperiodic, ϕ-irreducible and Harris recurrent Markov chain on $\mathbf{Y}_\star$, with probability transition kernel $Q(y; C)$, for all $y = (\tau, s, \gamma, m) \in \mathbf{Y}_\star$, $C = B \times S \times \Gamma \times \{m'\}$ with $(B, S, \Gamma) \in \beta([0, \infty) \times \mathbf{S} \times [0, 2\pi))$, $m' \in \mathbf{M}$. We further assume that $\{y_j\}_j$ has a unique invariant probability measure q.

$\diamond$

Observe that the set of assumptions **A2** is identical to the set of assumptions **A1**, except that the relative directions now take values in $[0, 2\pi)$.

Lemma 4.2 (Probability transition $\{z_j\}_j$):
Under assumptions **A2** $\{z_j\}_j$ is a Markov chain on $\mathbf{Z}_\star$, with probability transition kernel $P(z; A)$, $z \in \mathbf{Z}_\star$, $A \in \beta(\mathbf{Z}_\star)$, given by

$$\begin{aligned} P(z; A) = \mathbf{1}\{&x + s\tau(\cos\theta, \sin\theta) \\ &- \lfloor x + s\tau(\cos\theta, \sin\theta)\rfloor \in U\} \\ &\times \int_\Gamma \mathbf{1}\{\gamma + \theta - 2\pi\lfloor(\gamma + \theta)/(2\pi)\rfloor \in V\} \\ &\times Q(y; B \times S \times d\gamma \times \{m'\}) \end{aligned} \tag{33}$$

for all $z = (x, \theta, y) \in \mathbf{Z}_\star$ with $y = (\tau, s, \gamma, m)$, $A = U \times V \times B \times S \times \Gamma \times \{m'\}$, with $(U, V, B, S) \in \beta([0, 1)^2 \times [0, 2\pi) \times [0, \infty) \times \mathbf{S})$ and $m' \in \{1, 2, \ldots, M\}$.

$\diamond$

The proof is identical to the proof of Lemma 3.1 and is therefore omitted.

Similar to the 1D case we represent the state of the system at time $t$by the vector $Z(t) = (X(t), \theta(t), R(t), S(t), \gamma(t), \xi(t)) \in \mathbf{Z}_\star$, where components $R(t)$, $S(t)$, $\gamma(t)$ and $\xi(t)$ are defined like in 1D (see beginning of Section III-B). Below is the main result of this section.

Proposition 4.1 (Uniform distributions in 2D):
Assume that (i) assumptions **A2** hold, (ii) the Markov chain $\{z_j\}_j$ is aperiodic and ϕ-irreducible, and (iii) the expected travel time

$$\overline{\tau}_\star := \int_0^\infty (1 - q([0, t) \times \mathbf{S} \times [0, 2\pi) \times \mathbf{M}))\, dt \tag{34}$$

is finite.

Then,

$$\begin{aligned} \lim_{t\to\infty} P(Z(t) \in A) = \frac{u_1 u_2 v}{2\pi\, \overline{\tau}_\star} \\ \times \int_0^{\tau'} (1 - q([0, t) \times S \times \Gamma \times \{m'\}))\, dt \end{aligned} \tag{35}$$

for all $A = [0, u_1) \times [0, u_2) \times [0, v) \times [0, \tau') \times S \times \Gamma \times \{m'\}$ with $u_1, u_2 \in (0, 1]$, $v \in (0, 2\pi]$, $\tau' > 0$, $(S, \Gamma) \in \beta(\mathbf{S} \times [0, 2\pi))$, and $m' \in \mathbf{M}$.

In particular,

$$\lim_{t\to\infty} P(X_1(t) < u_1, X_2(t) < u_2, \theta(t) < v) = \frac{u_1 u_2 v}{2\pi} \tag{36}$$

for all $u_1, u_2 \in (0, 1]$, $v \in (0, 2\pi]$, and for any initial position and direction.

$\diamond$

The proof of Proposition 4.1 (similar to the proof of Proposition 3.1 in 1D) is sketched in the Appendix.

B. Reflection model in 2D

The model is identical to the wrap around model in 2D, the only difference being that the mobile is reflected when it hits a boundary. If the mobile hits a boundary at time t, then its direction $\theta_r(t)$ at time t is such that $\theta_r(t) = \theta_r(t-) + \pi/2 \bmod 2\pi$ (i.e. the incidence angle is equal to the reflection angle – see Fig. 2).

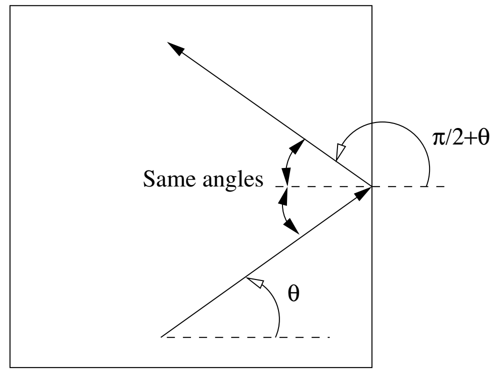

Fig. 2. Reflections in 2D

Similar to the 1D case, there exist simple relations between wrap around and reflection models, as shown in the lemma below.

Lemma 4.3 (Wrap around vs. reflection in 2D):
Consider a reflection model $\{(X_r(t), \theta_r(t))\}_t$, with $X_r(t) = (X_1^r(t), X_2^r(t))$, in 2D with movement vector $\{y_j^r = (\tau_j^r, s_j^r, \gamma_j^r, \xi_j^r)\}_j$.

We construct a wrap around model $\{(X_w(t), \theta_w(t))\}_t$, $X_w(t) = (X_1^w(t), X_2^w(t))$, with movement vector $\{y_j^w = (\tau_j^w, s_j^w, s_j^w, \xi_j^w)\}_j$ such that pathwise $\tau_j^w = \tau_j^r$, $s_j^w = s_j^r/2$, $\gamma_j^w = \gamma_j^r$ and $\xi_j^w = \xi_j^r$ for all $j \geq 1$. In other words, y_j^w and y_j^r are pathwise identical except that the speed of the mobile in the reflection model is always twice the speed of the mobile in the wrap around model.

If the relations $(i = 1, 2)$

$$\begin{aligned} X_i^r(t) &= \begin{cases} 2X_i^w(t), & 0 \leq X_i^w(t) < 1/2; \\ 2(1 - X_i^w(t)), & 1/2 \leq X_i^w(t) < 1, \end{cases} \\ \theta_r(t) &= \begin{cases} \theta_w(t), & X_w(t) \in [0, 1/2)^2; \\ 2\pi - \theta_w(t), & X_w(t) \in [0, 1/2) \times [1/2, 1); \\ 3\pi - \theta_w(t) - 2\pi\lfloor 3/2 - \theta_w(t)/2\pi\rfloor, & X_w(t) \in [1/2, 1) \times [0, 1/2); \\ \pi + \theta_w(t) - 2\pi\lfloor 1/2 + \theta_w(t)/2\pi\rfloor, & X_w(t) \in [1/2, 1)^2, \end{cases} \end{aligned}$$

hold at $t = 0$, then they hold for all $t > 0$.

$\diamond$

The proof of Lemma 4.3 (omitted) follows from the definition of the wrap around and reflection models.

Consider a reflection model $\{(X_r(t), \theta_r(t))\}_t$ in 2D with movement vector $\{y_j\}_j$, $y_j \in \mathbf{Y}_\star := [0, \infty) \times \mathbf{S} \times [0, 2\pi) \times$

$\mathbf{M}$, such that assumptions **A2** hold. Following Lemma 4.3, we construct a wrap around model $\{(X_w(t), \theta_w(t))\}_t$ with movement vector $\{y_j^w\}_j$, $y_j^w \in \mathbf{Y}_\star^w := [0,\infty)\times\mathbf{S}/2\times[0,2\pi)\times\mathbf{M}$, such that assumptions **A2** hold (such a construction is always possible). Let Q^w and q^w be the probability transition kernel and invariant distribution measure, respectively, of the Markov chain $\{y_j^w\}_j$ and let

$$\overline{\tau}_\star^w := \int_0^\infty (1 - q^w([0,t) \times (1/2)\mathbf{S} \times [0,2\pi) \times \mathbf{M}))\, dt \quad (37)$$

be the expected travel time associated with the movement vector $\{y_j^w\}_j$. Last, we denote by $z_j^w := (X_w(T_j), \theta_w(T_j), y_j^w)$ the state of the wrap around model at time T_j.

The next result shows (in particular) that the model with reflection yields a uniform distribution of the position and of direction of the mobile in steady-state.

Proposition 4.2 (Stationary mobile's behavior):
Assume that the Markov chain $\{z_j^w\}_j$ defined above is aperiodic and ϕ-irreducible and $\overline{\tau}_\star^w < \infty$.

Then,

$$\lim_{t\to\infty} P(Z^r(t) \in A) = \frac{u_1 u_2 v}{2\pi\,\overline{\tau}_\star^w} \times \int_0^\tau (1 - q^w([0,t) \times (1/2)S \times \Gamma \times \{m\}))\, dt$$

for all $A = [0,u_1)\times[0,u_2)\times[0,v)\times[0,\tau)\times S\times\Gamma\times\{m\}$ with $u_1, u_2 \in (0,1]$, $v \in (0,2\pi]$, $\tau > 0$, $(S,\Gamma) \in \beta(\mathbf{S}\times[0,2\pi))$, and $m \in \mathbf{M}$.

In particular,

$$\lim_{t\to\infty} P(X_1^r(t) < u_1, X_1^r(t) < u_2, \theta_r(t) < v) = \frac{u_1 u_2 v}{2\pi}$$

for all $u_1, u_2 \in (0,1]$, $v \in (0,2\pi]$, and for any initial position and direction.

◇

The proof is analogous to the proof of Proposition 3.4 [Hint: condition on the values of $X_w(t)$ and use Proposition 4.1] and is omitted.

We conclude this section by briefly discussing the assumptions under which Propositions 4.1 and 4.2 hold. Assumptions **A2** will hold (in particular) in the models described in Example 3.2 if we replace the assumptions made on the relative directions $\{\gamma_j\}_j$ with values in $\{-1,+1\}$ by the assumptions that $\{\gamma_j\}_j$ is an aperiodic, ϕ-irreducible, Harris recurrent Markov chain on $[0,2\pi)$, with probability transition kernel $K_{rd}(\gamma;\Gamma)$ and invariant probability distribution π_{rd} (the latter assumptions will hold if (in particular) $\{\gamma_j\}_j$ is a renewal sequence, with a common probability distribution that has a density w.r.t. the Lebesgue measure). The assumptions related to the aperiodicity and ϕ-irreducibility of the Markov chain $\{z_j\}_j$ in Proposition 4.1 (resp. $\{z_j^w\}_j$ in Proposition 3.4) will hold, for instance, if we travel times or speeds have a density, or if the transition kernel Q (resp. Q^w) of the Markov chain $\{y_j\}_j$ (resp. $\{y_j^w\}_j$) is such that $Q(y,C) > 0$ (resp. $Q^w(y,C) > 0$) for all y and C.

V. Application to simulation

In the spirit of [9], we describe how to start the random direction model with wrap around in the stationary regime under the assumptions that $\{\tau_j\}_j$ are modulated by a finite state Markov process, the speeds $\{s_j\}_j$ and relative directions $\{\gamma_j\}_j$ are independent iid sequences independent of $\{\tau_j\}_j$. Let $R = [R(i;j)]$ denote the transition probability matrix associated with the modulating process with stationary distribution π_e. Let $G_m(x) = P(\tau_j < x \,|\, \xi_j = m)$ denote the conditional distribution function of τ_j given the modulating process is in state $m \in \mathbf{M}$. Let $H(x) = P(s_j < x)$ and $D(x) = P(\gamma_j < x)$, $0 \le x < 2\pi$. In the case of N users, each one is initialized independently of the other. Apply the following procedure to each user where $\{u_i\}_{i=1}^7$ is a set of independent uniformly distributed random variables each in $[0,1)$:

Position and direction at $t = 0$: $(X_1(0), X_2(0), \theta(0)) = (u_1, u_2, 2\pi u_3)$.

Speed and relative direction at $t = 0$: $(S(0), \gamma(0)) = (H^{-1}(u_4), D^{-1}(u_5))$. Here $H^{-1}(\cdot)$ and $D^{-1}(\cdot)$ are the inverses of $H(\cdot)$ and $D(\cdot)$, respectively.

State of modulating process and remaining time until movement change: Label states of modulating process $1, 2, \ldots, M$. Then $\xi(0)$ is given by

$$\xi(0) = \arg\min\{l : u_6 < \frac{\sum_{i=1}^l \pi_e(l)\bar{\tau}^{l)}}{\sum_n \pi_e(n)\bar{\tau}^{n)}}, l = 1, \ldots, M\}$$

where $\bar{\tau}^{(m)} = E[\tau_j \,|\, \xi_j = m]$ for $m = 1, \ldots, M$. The remaining time $R(0)$ is given as $R(0) = F_{\xi(0)}^{-1}(u_7)$ where $F_m^{-1}(\cdot)$ is the inverse of $F_m(x) = \frac{1}{\bar{\tau}_m}\int_0^x (1 - G_m(y))dy$.

An *ns-2* module implementing this is available from the authors.

VI. Extension of the results

The results presented in Sections II - IV extend in a number of different directions.

- users can have non-identical mobility models,
- pause times are easily accounted for,
- the space can be $[0, d_1) \times [0, d_2)$,
- the results apply to other spaces including d-dimensional hypercubes, the most interesting of which is a 3-dimensional cube, d-dimensional hyperspheres, the surface of a d-hypersphere, including the surface of a sphere.

VII. Summary

In this paper we derived properties of the random direction mobility model, which is commonly used in studies concerning mobile ad hoc networks. In particular, we derived a simple relationship between the wrap around and reflection variants of the random direction models. We then showed that if users are uniformly distributed in their movement space, they remain so for arbitrary movement patterns. Furthermore, we showed, for a class of Markovian movement patterns that users converge to a uniform spatial distribution and are equally likely to be pointed in any direction regardless of their initial positions. These results were established for the one and two-dimensional spaces $[0,1)$ and $[0,1)^2$.

APPENDIX

Proof of Lemma 2.3: With the the change of variable $t = x + a$ in the l.h.s. of (8) we find

$$\int_0^1 \mathbf{1}\{x+a-\lfloor x+a\rfloor < u\}dx = \int_a^{1+a} \mathbf{1}\{t-\lfloor t\rfloor < u\}dt.$$

Assume first that $a = n$ is an integer. Then $\lfloor t\rfloor = n$ for $t \in [a, 1+a)$ and $\int_a^{1+a} \mathbf{1}\{t-\lfloor t\rfloor < u\}dt = u = \int_n^{1+n} \mathbf{1}\{t < u+n\}dt = u$.

Assume now that a is not an integer. Therefore, there exists an integer $n = \lfloor a\rfloor$ and $0 < \epsilon < 1$ such that $a = n + \epsilon$. Since $\lfloor t\rfloor = n$ for $t \in [n+\epsilon, n+1)$ and $\lfloor t\rfloor = n+1$ for $t \in [n+1, n+1+\epsilon)$, we have

$$\begin{aligned}&\int_a^{1+a} \mathbf{1}\{t-\lfloor t\rfloor < u\}\,dt\\ &= \int_{n+\epsilon}^{n+1} \mathbf{1}\{t < u+n\}\,dt\\ &\quad + \int_{n+1}^{n+1+\epsilon} \mathbf{1}\{t < u+n+1\}\,dt\\ &= (u-\epsilon)\,\mathbf{1}\{u \geq \epsilon\} + \min(u,\epsilon) = u.\end{aligned}$$

∎

Proof of Lemma of 3.1: For $z = (x, \theta, y)$ and A as defined in the statement of the lemma, we have by using (3) and (5)

$$\begin{aligned}&P(z;A) = P(x+\theta s\tau - \lfloor x+\theta s\tau\rfloor \in U,\\ &\qquad \theta_{j+1} = \theta\gamma', y_{j+1} \in C \,|\, x_j = x, \theta_j = \theta, y_j = y)\\ &= \mathbf{1}\left\{x+\theta s\tau - \lfloor x+\theta s\tau\rfloor \in U, \theta' = \theta\gamma'\right\}\\ &\quad \times P(y_{j+1} \in C \,|\, x_j = x, \theta_j = \theta, y_j = y). \qquad (38)\end{aligned}$$

Also note that the event $\{\theta_j = \theta, \gamma_j = \gamma\}$ in the r.h.s. of (38) can be replaced by the event $\{\theta_{j-1} = \theta/\gamma, \gamma_j = \gamma\}$ since $\theta_j = \theta_{j-1}\gamma_j$. This gives

$$\begin{aligned}P(z;A) &= \mathbf{1}\left\{x+\theta s\tau - \lfloor x+\theta s\tau\rfloor \in U, \theta' = \theta\gamma'\right\}\\ &\times P(y_{j+1} \in C | x_j = x, \theta_{j-1} = \frac{\theta}{\gamma}, y_j = y). \qquad (39)\end{aligned}$$

The assumption that $\{y_j\}_j$ is a Markov chain, added to the fact that x_j and θ_{j-1} are both measurable w.r.t. $\{y_k\}_{k=1}^{j-1}$, implies that we can remove the conditioning on (x_j, θ_{j-1}) in the r.h.s. of (39). Hence, from (10),

$$\begin{aligned}&P(z;A) =\\ &\quad \mathbf{1}\left\{x+\theta s\tau - \lfloor x+\theta s\tau\rfloor \in U, \theta' = \theta\gamma'\right\} Q(y;C).\end{aligned}$$

∎

Proof of (22):

$$\begin{aligned}&E^0\left[\int_0^{T_2} \mathbf{1}\{Z(t) \in A\}\,dt\right] = \frac{1}{2}\int_{x=0}^1 \int_{\tau=0}^\infty \int_{s\in\mathbf{S}} \int_{t=0}^\tau\\ &\quad \mathbf{1}\{x+\theta' st - \lfloor x+\theta' st\rfloor < u\}\,\mathbf{1}\{R(t) < \tau'\}\\ &\quad \times dt\, q(d\tau \times ds \times \{\gamma'\} \times \{m'\}).\end{aligned}$$

Since $\int_0^1 \mathbf{1}\{x+\theta' st - \lfloor x+\theta' st\rfloor < u\}dx = u$ by Lemma 2.3, we find

$$\begin{aligned}&E^0\left[\int_0^{T_2} \mathbf{1}\{Z(t) \in A\}\,dt\right] = \frac{u}{2}\int_{\tau=0}^\infty \int_{s\in\mathbf{S}} \int_{t=0}^\tau\\ &\quad \mathbf{1}\{R(t) < \tau'\}dt\, q(d\tau \times ds \times \{\gamma'\} \times \{m'\})\\ &= \frac{u}{2}\int_{t=0}^\infty \int_{\tau=0}^\infty \mathbf{1}\{t < \tau < t+\tau'\}\\ &\quad \times q(d\tau \times \mathbf{S} \times \{\gamma'\} \times \{m'\})\,dt\\ &= \frac{u}{2}\int_0^{\tau'} (1 - q([0,t) \times \mathbf{S} \times \{\gamma'\} \times \{m'\}))\,dt,\end{aligned}$$

which completes the proof. ∎

Proof of Proposition 4.1: We only sketch the proof of (35) as it is similar to the proof of the corresponding result in 1D (see Proposition 3.1).

The first step is to show that $p(A) := \frac{u_1 u_2 v}{2\pi} q(C)$, $A = [0,u_1) \times [0,u_2) \times [0,v) \times C$, $C \in \beta(\mathbf{Y}_\star)$, is an invariant measure of the Markov chain $\{z_j\}_j$. By using Lemma 4.2 we find

$$\begin{aligned}&\int_{\mathbf{Z}_\star} p(dz)P(z;A) = \frac{1}{2\pi}\int_{y=(\tau,s,\gamma,m)\in\mathbf{Y}_\star} q(dy)\\ &\quad \times\Bigg(\int_{\phi\in\Gamma} Q^*(y, B \times S \times d\phi \times \{m'\})\\ &\quad \times \int_{\theta=0}^{2\pi}\Bigg[\mathbf{1}\{\theta+\phi-2\pi\lfloor(\theta+\phi)/2\pi < v\rfloor\}\\ &\quad \times \prod_{i=1}^2 \int_{x_i=0}^1 \mathbf{1}\{x_i s\tau\cos\theta - \lfloor x_i s\tau\cos\theta\rfloor < u_i\}dx_i\\ &\quad \times d\theta\Bigg]\Bigg) \qquad (40)\end{aligned}$$

for $A = [0,u_1) \times [0,u_2) \times [0,v) \times B \times S \times \Gamma \times \{m'\} \in \beta(\mathbf{Z}_\star)$. By Lemma 2.3 the product $\prod_{i=1}^2 \ldots$ is equal to $u_1 u_2$; therefore, the integral $\int_{\theta=0}^1 \ldots$ is equal to $v/2\pi$ [Hint: same argument as the one used to derive (32)] and from this it is easily seen that the remaining terms are equal to $q(C)$ [Hint: by **A2** $\int_{\mathbf{Y}_\star} q^*(dy)Q(y,C) = q^*(C)$, for all $C \in \beta(\mathbf{Y}_\star)$]. This shows that, as announced, the l.h.s. of (40) is equal to $p(A) = \frac{u_1 u_2 v}{2\pi} q(C)$.

The second step is to use (similarly to the derivation of (21)) the aperiodic ergodic theorem to conclude that, under the assumptions of the proposition, the limiting distribution of $\{z_j\}_j$ coincides with its (unique) invariant measure p, that is

$$\lim_{j\to\infty} P(z_j \in A \,|\, z_1 = z) = \frac{u_1 u_2 v}{2\pi}\, q(C) \qquad (41)$$

for all $z \in \mathbf{Z}_\star$, $A = [0,u_1) \times [0,u_2) \times [0,v) \times C$, with $C \in \beta(\mathbf{Y}_\star)$.

The third step is to apply the Palm formula to the process $\{Z(t)\}_t$, which gives

$$\lim_{t\to\infty} P(Z(t) \in A) = \frac{1}{E^0[T_2]} E^0\left[\int_0^{T_2} \mathbf{1}\{Z(t) \in A\}dt\right]. \qquad (42)$$

The use of the Palm formula is justified under the assumptions of the proposition (see the proof of Proposition 3.1 where the same argument has been used).

The fourth step of the proof of (35) consists in conditioning on the stationary distribution of $\{z_j\}_j$ (given in (41)) in the r.h.s. of (42), in direct analogy with the proof of (22). Easy algebra then yield (35).

The proof of (36) is routinely obtained from (35) (see the proof of Proposition 3.2). ∎

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