Journal of Applied Mathematics and Stochastic Analysis
6, Number 3, Fall 1993, 217-236

PROPERTIES OF SOLUTION SET OF STOCHASTIC INCLUSIONS[1]

MICHAŁ KISIELEWICZ

Institute of Mathematics
Higher College of Engineering
Podgórna 50, 65-246 Zielona Góra, POLAND

ABSTRACT

The properties of the solution set of stochastic inclusions $x_t - x_s \in cl_{L^2}(\int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\widetilde{\nu}(d\tau, dz))$ are investigated. They are equivalent to properties of fixed points sets of appropriately defined set-valued mappings.

Key words: Stochastic inclusions, existence solutions, solution set, weak compactness.

AMS (MOS) subject classifications: 93E03, 93C30.

1. INTRODUCTION

There is a large number of papers (see for example [1], [4] and [5]) dealing with the existence of optimal controls of stochastic dynamical systems described by integral stochastic equations. Such problems can be described (see [10]) by stochastic inclusions $(SI(F,G,H))$ of the form

$$x_t - x_s \in cl_{L^2}\left(\int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\widetilde{\nu}(d\tau, dz)\right),$$

where the stochastic integrals are defined by Aumann's procedure (see [7], [9]).

The results of the paper are concerned with properties of the set of all solutions to $SI(F,G,H)$. To begin with, we recall the basic definitions dealing with set-valued stochastic integrals and stochastic inclusions presented in [10]. We assume, as given, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where a family $(\mathcal{F}_t)_{t \geq 0}$, of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ is assumed to be increasing: $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. We set $\mathbb{R}_+ = [0, \infty)$, and $\mathcal{B}_+$ will denote the Borel σ-algebra on

[1]Received: April, 1993. Revised: July, 1993.

$\mathbb{R}_+$. We consider set-valued stochastic processes $(F_t)_{t\geq 0}, (\mathcal{G}_t)_{t\geq 0}$ and $(\mathcal{R}_{t,z})_{t\geq 0, z\in\mathbb{R}^n}$, taking on values from the space $Comp(\mathbb{R}^n)$ of all nonempty compact subsets of n-dimensional Euclidean space $\mathbb{R}^n$. They are assumed to be predictable and such that $E\int_0^\infty \| \mathcal{F}_t \|^p dt < \infty$, $p \geq 1$, $E\int_0^\infty \| \mathcal{G}_t \|^2 dt < \infty$ and $E\int_0^\infty \int_{\mathbb{R}^n} \| \mathcal{R}_{t,z} \|^2 dt q(dz) < \infty$, where q is a measure on the Borel σ-algebra $\mathcal{B}^n$ of $\mathbb{R}^n$ and $\| A \| := sup\{ |a| : a \in A\}$, $A \in Comp(\mathbb{R}^n)$. The space $Comp(\mathbb{R}^n)$ is considered with the Hausdorff metric h defined in the usual way, i.e., $h(A,B) = max\{\bar{h}(A,B), \bar{h}(B,A)\}$, for $A, B \in Comp(\mathbb{R}^n)$, where $\bar{h}(A,B) = \{dist(a,B): a \in A\}$ and $\bar{h}(B,A) = \{dist(b,A): b \in B\}$. Although the classical theory of stochastic integrals (see [3], [8], [12]) usually deals with measurable and $\mathcal{F}_t$-adapted processes, it can be finally reduced (see [4], pp. 60-62) to predictable ones.

2. BASIC DEFINITIONS AND NOTATIONS

Throughout the paper we shall assume that a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfies the following usual hypotheses: (i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, (ii) $\mathcal{F} = \vee_{t\geq 0}\mathcal{F}_t$ and (iii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, for all $t, 0 \leq t < \infty$. As usual, we consider a set $\mathbb{R}_+ \times \Omega$ as a measurable space with the product σ-algebra $\mathcal{B}_+ \otimes \mathcal{F}$. Moreover, we introduce on $\mathbb{R}_+ \times \Omega$ the predictable σ-algebra $\mathcal{P}$ generated by a semiring $\mathcal{K}$ of all predictable rectangles in $\mathbb{R}_+ \times \Omega$ of the form $\{0\} \times A_0$ and $(s,t] \times A_s$, where $A_0 \in \mathcal{F}_0$ and $\mathcal{A}_s \in \mathcal{F}_s$ for $s < t$ in $\mathbb{R}_+$. Similarly, besides the usual product σ-algebra on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, we also introduce the predictable σ-algebra $\mathcal{P}^n$ generated by a semiring $\mathcal{K}^n$ of all sets of the form $\{0\} \times A_0 \times D$ and $(s,t] \times A_s \times D$, with $A_0 \in \mathcal{F}_0$, $A_s \in \mathcal{F}_s$ for $s < t$ in $\mathbb{R}_+$ and $D \in \mathcal{B}_0^n$, where $\mathcal{B}_0^n$ consists of all Borel sets $D \subset \mathbb{R}^n$ such that their closure does not contain the point 0.

An n-dimensional stochastic process x, understood as a function $x: \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ with $\mathcal{F}$-measurable sections x_t, each $t \geq 0$, is denoted by $(x_t)_{t\geq 0}$. It is measurable (predictable) if x is $\mathcal{B}_+ \otimes \mathcal{F}$ ($\mathcal{P}$, resp.)-measurable. The process $(x_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted if x_t is $\mathcal{F}_t$-measurable for $t \geq 0$. It is clear (see [3], [8], [11]) that every predictable process is measurable and $\mathcal{F}_t$-adapted. In what follows the Banach space $L^p(\mathbb{R}_+ \times \Omega, \mathcal{P}, dt \times P, \mathbb{R}^n)$, $p \geq 1$, with the norm $\| \cdot \|_{\mathcal{L}_n^p}$ defined in the usual way, will be denoted by $\mathcal{L}_n^p$. Similarly, the Banach spaces

$L^p(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ and $L^p(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ with the usual norm $\|\cdot\|_{L^p_n}$ are denoted by $L^p_n(\mathcal{F}_t)$ and $L^p_n(\mathcal{F})$, respectively.

Throughout the paper, by $(w_t)_{t\geq 0}$, we mean a one-dimensional $\mathcal{F}_t$-Brownian motion starting at 0, i.e., such that $P(w_0 = 0) = 1$. By $\nu(t,A)$ we denote a $\mathcal{F}_t$-Poisson measure on $\mathbb{R}_+ \times \mathcal{B}^n$, and then define a $\mathcal{F}_t$-centered Poisson measure $\widetilde{\nu}(t,A)$, $t \geq 0$, $A \in \mathcal{B}^n$, by taking $\widetilde{\nu}(t,A) = \nu(t,A) - tq(A)$, $t \geq 0$, $A \in \mathcal{B}^n$, where q is a measure on $\mathcal{B}^n$ such that $E\nu(t,B) = tq(B)$ and $q(B) < \infty$ for $B \in \mathcal{B}^n_0$.

For a given $\mathcal{F}_t$-centered Poisson measure $\widetilde{\nu}(t,A)$, $t \geq 0$, $A \in \mathcal{B}^n$, $\mathcal{W}^2_n$ denotes the space $L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^n,\ \mathcal{P}^n, dt \times P \times q)$, with the norm $\|\cdot\|_{\mathcal{W}^2_n}$ defined in the usual way. We shall also consider the Banach spaces $L^p(\mathbb{R}_+, \mathcal{B}_+, dt, \mathbb{R}_+)$, $p \geq 1$ and $L^2(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_+ \otimes \mathcal{B}^n,\ dt \times q, \mathbb{R}_+)$, with the usual norms by $|\cdot|_p$ and $\|\cdot\|_2$, respectively. They will be denoted by $L^p(\mathcal{B}_+)$ and $L^2(\mathcal{B}_+ \times \mathcal{B}^n)$, respectively. Finally, by $\mathcal{M}^p_n(\mathcal{P})$, $p \geq 1$ and $\mathcal{M}^2_n(\mathcal{P}^n, q)$ we shall denote the families of all $\mathcal{P}$-measurable and $\mathcal{P}^n$-measurable functions $f: \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ and $h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, respectively, such that $\int_0^\infty |f_t|^p dt < \infty$ and $\int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dt q(dz) < \infty$, *a.s.* Elements of $\mathcal{M}^p_n(\mathcal{P})$, $p \geq 1$ and $\mathcal{M}^2_n(\mathcal{P}^n, q)$ will be denoted by $f = (f_t)_{t \geq 0}$ and $h = (h_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, respectively. We have

$$\mathcal{L}^p_n = \{f \in \mathcal{M}^p(\mathcal{P}): E\int_0^\infty |f_t|^p dt < \infty\},\ p \geq 1,$$

and

$$\mathcal{W}^2_n = \{h \in \mathcal{M}^2(\mathcal{P}^n, q): E\int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dt q(dz) < \infty\}.$$

Given $g \in \mathcal{M}^2(\mathcal{P})$ and $h \in \mathcal{M}^2(\mathcal{P}^n, q)$, by $(\int_0^t g_\tau dw_\tau)_{t \geq 0}$ and $(\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \widetilde{\nu}(d\tau, dz))_{t \geq 0}$, we denote their stochastic integrals with respect to a $\mathcal{F}_t$-Brownian motion $(w_t)_{t \geq 0}$ and a $\mathcal{F}_t$-centered Poisson measure $\widetilde{\nu}(t,A)$, $t \geq 0$, $A \in \mathcal{B}^n$, respectively. These integrals, understood as n-dimensional stochastic processes, have quite similar properties (see [6]).

Let us denote by D the family of all n-dimensional $\mathcal{F}_t$-adapted cádlág processes $(x_t)_{t \geq 0}$ such that

$$E sup_{t \geq 0} |x_t|^2 < \infty$$

and

$$lim_{\delta \to 0} sup_{t \geq 0} sup_{t \leq s \leq t+\delta} E|x_t - x_s|^2 = 0.$$

Recall that an n-dimensional stochastic process is said to be a cádlág process if it has almost all sample paths right continuous with finite left limits. The space D is considered as a normed space with the norm $\|\cdot\|_{\ell}$ defined by

$\| \xi \|_{\ell} = \| sup_{t \geq 0} | \xi_t | \|_{L_1^2}$ for $\xi = (\xi_t)_{t \geq 0} \in D$. It can be verified that $(D, \| \cdot \|_{\ell})$ is a Banach space.

Given $0 \leq \alpha < \beta < \infty$ and $(x_t)_{t \geq 0} \in D$ let $x^{\alpha,\beta} = (x_t^{\alpha,\beta})_{t \geq 0}$ be defined by $x_t^{\alpha,\beta} = x_\alpha$ and $x_t^{\alpha,\beta} = x_\beta$ for $0 \leq t \leq \alpha$ and $t \geq \beta$, respectively, and $x_t^{\alpha,\beta} = x_t$ for $\alpha \leq t \leq \beta$. It is clear that $D^{\alpha,\beta}: = \{x^{\alpha,\beta}: x \in D\}$ is a linear subspace of D, closed in the $\| \cdot \|_{\ell}$-norm topology. Then $(D^{\alpha,\beta}, \| \cdot \|_{\ell})$ is also a Banach space. Finally, as usual, by $\sigma(D, D^*)$ we shall denote a weak topology on D.

In what follows we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping $\mathcal{R}$ with nonempty values in a topological space $(Y, \mathcal{T}_Y)$ is said to be upper (lower) semicontinuous [u.s.c. (l.s.c.)] on a topological space $(X, \mathcal{T}_X)$ if $\mathcal{R}^-(C): = \{x \in X: \mathcal{R}(x) \cap C \neq \emptyset\}$ $(\mathcal{R}_-(C): = \{x \in X: \mathcal{R}(x) \subset C\})$ is a closed subset of X for every closed set $C \subset Y$. In particular, for $\mathcal{R}$ defined on a metric space $(\mathcal{S}, d)$ with values in $Comp(\mathbb{R}^n)$, it is equivalent (see [9]) to $\lim_{n \to \infty} \bar{h}(\mathcal{R}(x_n), \mathcal{R}(x)) = 0$ $(\lim_{n \to \infty} \bar{h}(\mathcal{R}(x), \mathcal{R}(x_n)) = 0)$ for every $x \in \mathcal{S}$ and every sequence (x_n) of $\mathcal{S}$ converging to x. If, moreover, $\mathcal{R}$ takes convex values then it is equivalent to upper (lower) semicontinuity of a real-valued function $s(p, \mathcal{R}(\cdot))$ on $\mathbb{R}^n$ for every $p \in \mathbb{R}^n$, where $s(\cdot, A)$ denotes a support function of a set $A \in Comp(\mathbb{R}^n)$. In what follows, we shall need the follow well-known (see [9]) fixed point and continuous selection theorems.

Theorem (Schauder, Tikhonov): *Let $(X, \mathcal{T}_X)$ be a locally convex topological Hausdorff space, $\mathcal{K}$ a nonempty compact convex subset of X and f a continuous mapping of $\mathcal{K}$ into itself. Then f has a fixed point in $\mathcal{K}$.*

Theorem (Covitz, Nadler): *Let $(\mathcal{S}, d)$ be a complete metric space and $\mathcal{R}: \mathcal{S} \to Cl(\mathcal{S})$ a set-valued contraction mapping, i.e., such that $H(\mathcal{R}(x), \mathcal{R}(y)) \leq \lambda d(x, y)$ for $x, y \in \mathcal{S}$ with $\lambda \in [0, 1)$, where H is the Hausdorff metric induced by the metric d on the space $Cl(\mathcal{S})$ of all nonempty closed bounded subsets of $\mathcal{S}$. Then there exists $x \in \mathcal{S}$ such that $x \in \mathcal{R}(x)$.*

Theorem (Kakutani, Fan): *Let $(X, \mathcal{T}_X)$ be a locally convex topological Hausdorff space, $\mathcal{K}$ a nonempty compact convex subset of X and $CCl(\mathcal{K})$ a family of all nonempty closed convex subsets of $\mathcal{K}$. If $\mathcal{R}: \mathcal{K} \to CCl(\mathcal{K})$ is u.s.c. on $\mathcal{K}$ then there exists $x \in \mathcal{K}$ such that $x \in \mathcal{R}(x)$.*

Theorem (Michael): *Let $(X, \mathcal{T}_X)$ be a paracompact space and let $\mathcal{R}$ be a set-valued mapping from X to a Banach space $(Y, \|\cdot\|)$ whose values are closed and convex. Suppose, further $\mathcal{R}$ is l.s.c. on X. Then there is a continuous function $f: X \to Y$ such that $f(x) \in \mathcal{R}(x)$, for each $x \in X$.*

3. SET-VALUED STOCHASTIC INTEGRALS

Let $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ be a set-valued stochastic process with values in $Comp(\mathbb{R}^n)$, i.e. a family of $\mathcal{F}$-measurable set-valued mappings $\mathcal{G}_t: \Omega \to Comp(\mathbb{R}^n)$, $t \geq 0$. We call $\mathcal{G}$ measurable (predictable) if it is $\mathcal{B}_+ \otimes \mathcal{F}$ ($\mathcal{P}$, resp.)-measurable. Similarly, $\mathcal{G}$ is said to be $\mathcal{F}_t$-adapted if $\mathcal{G}_t$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. It is clear that every predictable set-valued stochastic process is measurable and $\mathcal{F}_t$-adapted. It follows from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [9]) that every measurable (predictable) set-valued process with nonempty compact values possesses a measurable (predictable) selector. We shall also consider $\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n$ and $\mathcal{P}^n$-measurable set-valued mappings $\mathcal{R}: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$. They will be denoted as families $(\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ and called measurable and predictable, respectively set-valued stochastic processes depending on a parameter $z \in \mathbb{R}^n$. The process $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ is said to be $\mathcal{F}_t$-adapted if $\mathcal{R}_{t,z}$ is $\mathcal{F}_t$-measurable for each $t \geq 0$ and $z \in \mathbb{R}^n$.

Denote by $\mathcal{M}^p_{s-v}(\mathcal{P})$, $p \geq 1$, and $\mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$ the families of all set-valued predictable processes $F = (F_t)_{t \geq 0}$ and $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, respectively, such that $E\int_0^\infty \| F_t \|^p dt < \infty$ and $E\int_0^\infty \int_{\mathbb{R}^n} \| \mathcal{R}_{t,z} \|^2 dt q(z) < \infty$. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem it follows that for every $F \in \mathcal{M}^p_{s-v}(\mathcal{P})$, $p \geq 1$, and $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$ the sets

$$\mathcal{S}^p(F) := \{f \in \mathcal{L}^p_n : f_t(\omega) \in F_t(\omega),\ dt \times P \text{ - } a.e.\}$$

and

$$\mathcal{S}^2_q(\mathcal{R}) := \{h \in \mathcal{W}^2_n : h_{t,z}(\omega) \in \mathcal{R}_{t,z}(\omega), dt \times P \times q \text{ - } a.e.\}$$

are nonempty.

Given set-valued processes $F = (F_t)_{t \geq 0} \in \mathcal{M}^p_{s-v}(\mathcal{P})$, $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \in \mathcal{M}^2_{s-v}(\mathcal{P})$ and $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$ by their stochastic integrals $\mathcal{I}F$, $\mathcal{J}\mathcal{G}$ and $\mathcal{T}\mathcal{R}$ we mean families $\mathcal{I}F = (\mathcal{I}_t F)_{t \geq 0}$, $\mathcal{J}\mathcal{G} = (\mathcal{J}_t \mathcal{G})_{t \geq 0}$, and $\mathcal{T}\mathcal{R} = (\mathcal{T}_t \mathcal{R})_{t \geq 0}$ subsets of $L^p_n(\mathcal{F}_t)$, $p \geq 1$ and $L^2_n(\mathcal{F}_t)$, respectively, defined by

$\mathfrak{I}_tF = \{\mathfrak{I}_tf: f \in \mathcal{S}^p(F)\}$, $\mathfrak{J}_t\mathcal{G} = \{\mathfrak{J}_tg: g \in \mathcal{S}^2(\mathcal{G})\}$ and $\mathcal{T}_t\mathcal{R} = \{\mathcal{T}_th: h \in \mathcal{S}^2_q(\mathcal{R})\}$, where $\mathfrak{I}_tf = \int_0^t f_sds$, $\mathfrak{J}_tg = \int_0^t g_sdw_s$ and $\mathcal{T}_th = \int_0^t\int_{\mathbb{R}^n} h_{s,z}\tilde{\nu}(ds,dz)$. Given $0 \leq \alpha < \beta < \infty$, we also define $\int_\alpha^\beta F_sds = \{\int_\alpha^\beta f_sds: f \in \mathcal{S}^p(F)\}$, $\int_\alpha^\beta \mathcal{G}_sdw_s = \{\int_\alpha^\beta g_sdw_s: g \in \mathcal{S}^2(\mathcal{G})\}$ and $\int_\alpha^\beta\int_{\mathbb{R}^n} \mathcal{R}_{s,z}\tilde{\nu}(ds,dz) = \{\int_\alpha^\beta\int_{\mathbb{R}^n} h_{s,z}\tilde{\nu}(ds,dz): h \in \mathcal{S}^2(\mathcal{R})\}$. The following properties of set-valued stochastic integrals are given in [10].

Proposition 1: *Let* $F \in \mathcal{M}^p_{s-v}(\mathcal{P})$, $p \geq 1$, $\mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{P})$ *and* $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$. *Then*

(i) $\mathfrak{I}_t\mathcal{G}$ *and* $\mathcal{T}_t\mathcal{R}$ *are closed subsets of* $L^2_n(\mathcal{F}_t)$ *for each* $t \geq 0$.

(ii) *If, moreover,* $F, \mathcal{G}$ *and* $\mathcal{R}$ *take on convex values then* $\mathfrak{I}_tF$, $\mathfrak{J}_t\mathcal{G}$ *and* $\mathcal{T}_t\mathcal{R}$ *are convex and weakly compact in* $L^p_n(\mathcal{F}_t)$ *and* $L^2_n(\mathcal{F}_t)$, *respectively, for each* $t \geq 0$.

Proposition 2: *Let* $F \in \mathcal{M}^2_{s-v}(\mathcal{P})$, $\mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{P})$ *and* $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$. *Assume* $(x_t)_{t \geq 0} \in D$ *is such that*

$$x_t - x_s \in cl_{L^2}\left(\int_s^t F_\tau d\tau + \int_s^t \mathcal{G}_\tau dw_\tau + \int_s^t\int_{\mathbb{R}^n} \mathcal{R}_{\tau,z}\tilde{\nu}(d\tau,dz)\right)$$

for every $0 \leq s < t < \infty$. *Then for every* $\epsilon > 0$ *there are* $f^\epsilon \in \mathcal{S}^p(F)$, $g^\epsilon \in \mathcal{S}^2(\mathcal{G})$ *and* $h^\epsilon \in \mathcal{S}^2_q(\mathcal{R})$ *such that*

$$\sup_{t \geq 0} \| \, |(x_t - x_0) - \left(\int_0^t f^\epsilon_\tau d\tau + \int_0^t g^\epsilon_\tau dw_\tau + \int_0^t\int_{\mathbb{R}^n} h^\epsilon_{\tau,z}\tilde{\nu}(d\tau,dz)\right)| \, \|_{L^2} \leq \epsilon.$$

Proposition 3: *Assume* $F \in \mathcal{M}^2_{s-v}(\mathcal{P})$, $\mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{P})$ *and* $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$ *take on convex values and let* $(x_t)_{t \geq 0} \in D$. *Then*

$$x_t - x_s \in \int_s^t F_\tau d\tau + \int_s^t \mathcal{G}_\tau dw_\tau + \int_s^t\int_{\mathbb{R}^n} \mathcal{R}_{\tau,z}\tilde{\nu}(d\tau,dz)$$

for $0 \leq s < t < \infty$ *if and only if there are* $f \in \mathcal{S}^2(F)$, $g \in \mathcal{S}^2(\mathcal{G})$ *and* $h \in \mathcal{S}^2_q(\mathcal{R})$ *such that*

$$x_t = x_0 + \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t\int_{\mathbb{R}^n} h_{\tau,z}\tilde{\nu}(d\tau,dz), \text{ a.s. for each } t \geq 0.$$

4. STOCHASTIC INCLUSIONS

Let $F = \{(F_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$. Assume F, G and H are such that $(F_t(x))_{t \geq 0} \in \mathcal{M}^p_{s-v}(\mathcal{P})$, $(G_t(x))_{t \geq 0} \in \mathcal{M}^2_{s-v}(\mathcal{P})$ and $(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$ for each $x \in \mathbb{R}^n$.

By a stochastic inclusion, denoted by $SI(F,G,H)$, corresponding to F,G and H given above, we mean the relation

$$x_t - x_s \in cl_{L^2}\left(\int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\widetilde{\nu}(d\tau, dz) \right)$$

that is to be satisfied for every $0 \leq s < t < \infty$ by a stochastic process $x = (x_t)_{t \geq 0} \in D$ such that $F \circ mx \in \mathcal{M}^p_{s-v}(\mathcal{P})$, $G \circ mx \in \mathcal{M}^2_{s-v}(\mathcal{P})$ and $H \circ mx \in \mathcal{M}^2_{s-v}(\mathcal{P}^n, q)$, where $F \circ mx = (F_t(x_t))_{t \geq 0}$, $G \circ mx = (G_t(x_t))_{t \geq 0}$ and $H \circ mx = (H_{t,z}(x_t))_{t \geq 0, z \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \geq 0} \in D$, satisfying the conditions mentioned above, is said to be global solution to $SI(F,G,H)$.

Corollary 1: *If F,G and H take on convex values then $SI(F,G,H)$ has a form*

$$x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\widetilde{\nu}(d\tau, dz)$$

and $(x_t)_{t \geq 0} \in D$ is a global solution to $SI(F,G,H)$ if and only if there are $f \in \mathcal{S}^2(F \circ mx)$, $g \in \mathcal{S}^2(G \circ mx)$ and $h \in \mathcal{S}^2_q(H \circ mx)$ such that

$$x_t = x_0 + \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,z}\widetilde{\nu}(d\tau, dz), \text{a.s. for each } t \geq 0.$$

Given $0 \leq \alpha < \beta < \infty$, a stochastic process $(x_t)_{t \geq 0} \in D$ is said to be a local solution to $SI(F,G,H)$ on $[\alpha, \beta]$ if

$$x_t - x_s \in cl_{L^2}\left(\int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\widetilde{\nu}(d\tau, dz) \right)$$

for $\alpha \leq s < t \leq \beta$.

Corollary 2: *A stochastic process $(x_t)_{t \geq 0} \in D$ is a local solution to $SI(F,G,H)$ on $[\alpha, \beta]$ if and only if $x^{\alpha,\beta}$ is a global solution to $SI(F^{\alpha\beta}, G^{\alpha\beta}, H^{\alpha\beta})$, where $F^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta]}F$, $G^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta]}G$ and $H^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta]}H$.*

A stochastic process $(x_t)_{t\geq 0}\in D$ is called a global (local on $[\alpha,\beta]$, resp.) solution to an initial value problem for stochastic inclusion $SI(F,G,H)$ with an initial condition $y\in L^2(\Omega,\mathcal{F}_0,\mathbb{R}^n)$ ($y\in L^2(\Omega,\mathcal{F}_\alpha,\mathbb{R}^n)$, resp.) if $(x_t)_{t\geq 0}$ is a global (local on $[\alpha,\beta]$, resp.) solution to $SI(F,G,H)$ and $x_0=y$ ($x_\alpha=y$, resp.). An initial-value problem for $SI(F,G,H)$ mentioned above will be denoted by $SI_y(F,G,H)$ ($SI_y^{\alpha,\beta}(F,G,H)$, resp.). In what follows, we denote a set of all global (local on $[\alpha,\beta]$, resp.) solutions to $SI_y(F,G,H)$ by $\Lambda_y(F,G,H)$ ($\Lambda_y^{\alpha,\beta}(F,G,H)$, resp.).

Suppose F,G and H satisfy the following conditions $(\mathcal{A}_1)$:

(*i*) $F=\{(F_t(x))_{t\geq 0}: x\in\mathbb{R}^n\}$, $G=\{(G_t(x))_{t\geq 0}: x\in\mathbb{R}^n\}$ and $H=\{(H_{t,z}(x))_{t\geq 0, z\in\mathbb{R}^n}: x\in\mathbb{R}^n\}$ are such that mappings $\mathbb{R}^+\times\Omega\times\mathbb{R}^n\ni(t,\omega,x)\to F_t(x)(\omega)\in Cl(\mathbb{R}^n)$, $\mathbb{R}_t\times\Omega\times\mathbb{R}^n\ni(t,\omega,x)\to G_t(x)(\omega)\in Cl(\mathbb{R}^n)$ and $\mathbb{R}_t\times\Omega\times\mathbb{R}^n\times\mathbb{R}^n\ni(t,\omega,z,x)\to H_{t,z}(x)(\omega)\in Cl(\mathbb{R}^n)$ are $\mathcal{P}\otimes\mathcal{B}^n$ and $\mathcal{P}^n\otimes\mathcal{B}^n$-measurable, respectively.

(*ii*) $(F_t(x))_{t\geq 0}$, $(G_t(x))_{t\geq 0}$, $(H_{x,z}(x))_{t\geq 0, z\in\mathbb{R}^n}$ are uniformly p- and square-integrable bounded, respectively, i.e.,

$$(sup_{x\in\mathbb{R}^n}\| F_t(x)\|)_{t\geq 0}\in\mathcal{L}_1^p,\ (sup_{x\in\mathbb{R}^n}\| G_t(x)\|)_{t\geq 0}\in\mathcal{L}_1^2 \quad\text{and}\quad (sup_{x\in\mathbb{R}^n}\| H_{t,z}(x)\|)_{t\geq 0, z\in\mathbb{R}^n}\in\mathcal{W}_1^2.$$

Corollary 3: *For every $(x_t)_{t\geq 0}\in D$ and F,G,H satisfying $(\mathcal{A}_1)$ one has $F\circ mx\in\mathcal{M}_{x-v}^p(\mathcal{P})$, $G\circ mx\in\mathcal{M}_{s-v}^2(\mathcal{P})$ and $H\circ mx\in\mathcal{M}_{s-v}^2(\mathcal{P}^n,q)$.*

Now define a linear continuous mapping Φ on $\mathcal{L}_n^p\times\mathcal{L}_n^2\times\mathcal{W}_n^2$ by taking $\Phi(f,g,h)=(\mathcal{I}_tf+\mathcal{J}_tg+\mathcal{T}_th)_{t\geq 0}$ to each $(f,g,h)\in\mathcal{L}_n^p\times\mathcal{L}_n^2\times\mathcal{W}_n^2$. It is clear that Φ maps $\mathcal{L}_n^p\times\mathcal{L}_n^2\times\mathcal{W}_n^2$ into D. Given above F,G and H satisfying $(\mathcal{A}_1)$, define a set-valued mapping $\mathcal{H}$ on D by setting

$$\mathcal{H}(x)=cl_{\ell}(\Phi(\mathcal{S}^p(F\circ mx)\times\mathcal{S}^2(G\circ mx)\times\mathcal{S}_q^2(H\circ mx))) \tag{1}$$

for $x=(x_t)_{t\geq 0}\in D$, where the closure is taken in the norm topology in $(D,\|\cdot\|_{\ell})$. Similarly, for given $0\leq\alpha<\beta<\infty$, we define a set-valued mapping $\mathcal{H}^{\alpha,\beta}$ on D by taking

$$\mathcal{H}^{\alpha,\beta}(x)=cl_{\ell}(\Phi(\mathcal{S}^p(F^{\alpha\beta}\circ mx)\times\mathcal{S}^2(G^{\alpha\beta}\circ mx)\times\mathcal{S}_q^2(H^{\alpha\beta}\circ mx)) \tag{2}$$

where $F^{\alpha\beta}$, $G^{\alpha\beta}$ and $H^{\alpha\beta}$ are as above.

Corollary 4: *For every F,G and H taking on convex values and* *satisfying* $(\mathcal{A}_1)$, *one has* $\mathcal{H}(x) = \Phi(\mathcal{S}^p(F \circ mx) \times \mathcal{S}^2(G \circ mx) \times \mathcal{S}_q^2(H \circ mx))$ *and* $\mathcal{H}^{\alpha,\beta}(y) = \Phi(\mathcal{S}^p(F^{\alpha\beta} \circ mx) \times \mathcal{S}^2(G^{\alpha\beta} \circ mx) \times \mathcal{S}_q^2(H^{\alpha\beta} \circ mx))$ *for* $x \in D$.

Let $S(F,G,H)$ and $S^{\alpha,\beta}(F,G,H)$ denote the set of all fixed points of $\mathcal{H}$ and $\mathcal{H}^{\alpha,\beta}$, respectively. It will be shown below that $S^{\alpha,\beta}(F,G,H) \subset D^{\alpha,\beta}$. Immediately from Proposition 2 (see [10]) the following result follows.

Proposition 4: *Assume* F, G *and* H *satisfy* $(\mathcal{A}_1)$ *and take on convex values. Then* $\Lambda_0(F,G,H) = S(F,G,H)$ *and* $\Lambda_0^{\alpha,\beta}(F,G,H) = S^{\alpha,\beta}(F,G,H)$ *for every* $0 \leq \alpha < \beta < \infty$, *respectively.*

Proposition 5: *Assume* F, G *and* H *satisfy* $(\mathcal{A}_1)$ *and let* $0 \leq \alpha < \beta < \infty$. *Then* $x \in S^{\alpha,b}(F,G,H)$ *if and only if*

(*i*) $x_t = 0$ *a.s. for* $t \in [0,\alpha]$,

(*ii*) $x_t = x_\beta$ *a.s. for* $t \geq \beta$,

(*iii*) *for every* $\epsilon > 0$ *there is* $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{S}^p(F^{\alpha\beta} \circ mx) \times \mathcal{S}^2(G^{\alpha\beta} \circ mx) \times \mathcal{S}_q^2(H^{\alpha\beta} \circ mx))$ *such that* $\| sup_{\alpha \leq t \leq \beta} | x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) | \|_{L_1^2} < \epsilon$.

Proof: $(\Rightarrow)$ Let $x \in S^{\alpha,\beta}(F,G,H)$. By the definition of $\mathcal{H}^{\alpha,\beta}$, for every $\epsilon > 0$, there is $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{S}^p(F^{\alpha\beta} \circ mx) \times \mathcal{S}^2(G^{\alpha\beta} \circ mx) \times \mathcal{S}_q^2(H^{\alpha\beta} \circ mx))$ such that $\| x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon) \|_{\varrho} < \epsilon$. We have of course $\Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) = 0$ and $\Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) = \Phi_\beta(f^\epsilon, g^\epsilon, h^\epsilon)$, *a.s.* for $0 \leq t \leq \alpha$ and $t \geq \beta$, respectively. Then

$$\| \sup_{0 \leq t \leq \alpha} | x_t | \|_{L_1^2} = \| \sup_{0 \leq t \leq \alpha} | x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) | \|_{L_1^2}$$

$$\leq \| x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon) \|_{\varrho} < \epsilon.$$

and

$$\| \sup_{t \geq \beta} | x_t - x_\beta | \|_{L_1^2} = \| \sup_{t \geq \beta} | x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) | \|_{L_1^2}$$

$$+ \| \sup_{t \geq \beta} | x_\beta - \Phi_\beta(f^\epsilon, g^\epsilon, h^\epsilon) | \|_{L_1^2} < 2\epsilon.$$

Therefore, $sup_{0 \leq t \leq \alpha} | x_t | = 0$ and $sup_{t \geq \beta} | x_t - x_\beta | = 0$ *a.s.*

By the properties of $\Phi(f^\epsilon, g^\epsilon, h^\epsilon)$, (*i*) and (*ii*), (*iii*) easily follow.

$(\Leftarrow)$ Conditions (*i*) – (*iii*) imply

$$\| x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon) \|_{\varrho} = \| \sup_{\alpha \leq t \leq \beta} | x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) | \|_{L_1^2} < \epsilon.$$

Therefore, $x \in cl_{\varrho}\Phi(\mathcal{S}^p(F^{\alpha\beta} \circ mx) \times \mathcal{S}^2(G^{\alpha\beta} \circ mx) \times \mathcal{S}^2_q(H^{\alpha\beta} \circ mx))$. □

Proposition 6: *Assume F, G and H satisfy $(\mathcal{A}_1)$ and let $(\tau_n)_{n=1}^{\infty}$ be a sequence of positive numbers increasing to $+\infty$. If $x^1 \in S^{0,\tau_1}(F,G,H)$ and $x^{n+1} \in x^n_{\tau_n} + S^{\tau_n,\tau_{n+1}}(F,G,H)$ for $n = 1,2,\ldots$, then $x = \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_{n-1},\tau_n)}(x^n - x^{n-1}_{\tau_{n-1}})$ belongs to $S(F,G,H)$, where $x^0_0 = 0$.*

Proof: For every $n = 1,2,\ldots$ one has $x^n - x^{n-1}_{\tau_{n-1}} \in S^{\tau_{n-1},\tau_n}(F,G,H)$. Then, by Proposition 5, for every $n = 1,2,\ldots$ and $\epsilon > 0$ there is $(f^n, g^n, h^n) \in \mathcal{S}^p(F^{\tau_{n-1}\tau_n} \circ mx^n) \times \mathcal{S}^2(G^{\tau_{n-1}\tau_n} \circ mx^n) \times \mathcal{S}^2_q(H^{\tau_{n-1}\tau_n} \circ mx^n)$ such that

$$\| \sup_{\tau_{n-1} \le t \le \tau_n} | (x^n_t - x^{n-1}_{\tau_{n-1}}) - \Phi_t(f^n, g^n, h^n) | \|_{L^2_1} < \epsilon/2^n.$$

Put $f^\epsilon = \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_{n-1},\tau_n)} f^n$, $g^\epsilon = \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_{n-1},\tau_n)} g^n$ and $h^\epsilon = \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_{n-1},\tau_n)} h^n$. By the decomposability (see [9], [10]) of $\mathcal{S}^2(F \circ mx)$, $\mathcal{S}^2(G \circ mx)$ and $\mathcal{S}^2_q(h \circ mx)$, we get $f^\epsilon \in \mathcal{S}^2(F \circ mx)$, $g^\epsilon \in \mathcal{S}^2(G \circ mx)$ and $h^\epsilon \in \mathcal{S}^2_q(H \circ mx)$. Moreover

$$\| x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon) \|_{\varrho}$$

$$\le \| \sum_{n=1}^{\infty} \sup_{\tau_{n-1} \le t \le \tau_n} | (x^n_t - x^{n-1}_{\tau_{n-1}} - \Phi_t(f^n, g^n, h^n) | \|_{L^2_1}$$

$$\le \sum_{n=1}^{\infty} \| \sup_{\tau_{n-1} \le t \le \tau_n} | (x^n_t - x^{n-1}_{\tau_{n-1}} - \Phi_t(f^n, g^n, h^n) | \|_{L^2_1} < \epsilon.$$

Therefore, $x \in cl_{\varrho}\Phi(\mathcal{S}^2(F \circ mx) \times \mathcal{S}^2(G \circ mx) \times \mathcal{S}^2_q(H \circ mx)$. □

In what follows we shall deal with $F = \{(F_t(x))_{t \ge 0}: x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \ge 0}: x \in \mathbb{R}^n\}$ and $H = \{H_{t,z}(x))_{t \ge 0, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$ satisfying conditions $(\mathcal{A}_1)$ and any one of the following conditions.

$(\mathcal{A}_2)$ *F, G and H are such that set-valued functions $D \ni x \to (F \circ mx)_t(\omega) \subset \mathbb{R}^n$, $D \ni x \to (G \circ mx)_t(\omega) \subset \mathbb{R}^n$ and $D \ni x \to (H \circ mx)_{t,z}(\omega) \subset \mathbb{R}^n$ are w.-w.s.u.s.c. on D, i.e., for every $x \in D$ and every sequence (x_n) of $(D, \|\cdot\|_{\varrho})$ converging weakly to x, one has $\bar{h}(\int \int_A (F \circ mx_n)_t \, dtdP, \int \int_A (F \circ mx)_t dtdP) \to 0$, $\bar{h}(\int \int_A (G \circ mx_n)_t dtdP, \int \int_A (G \circ mx)_t dtdP) \to 0$ and $\bar{h}(\int \int \int_B (H \circ mx_n)_{t,z} \, dtq(dz)dP, \int \int \int_B (H \circ mx)_{t,z} dtq(dz) \, dP) \to 0$.*

($\mathcal{A}_3$) *F, G and H are such that set-valued functions $D \ni x \to (F \circ mx)_t(\omega) \subset \mathbb{R}^n$, $D \ni x \to (G \circ mx)_t(\omega) \subset \mathbb{R}^n$ and $D \ni x \to (H \circ mx)_{t,z}(\omega) \subset \mathbb{R}^n$ are s. – w.s.l.s.c. on D, i.e., for every $x \in D$ and every sequence (x_n) of $(D, \| \cdot \|_{\ell})$ converging weakly to x, one has $\bar{h}((F \circ mx)_t(\omega), (F \circ mx^n)_t(\omega)) \to 0$, $\bar{h}((G \circ mx)_t(\omega), (G \circ mx^n)_t(\omega)) \to 0$ and $\bar{h}((H \circ mx)_{t,z}(\omega), (H \circ mx^n)_{t,z}(\omega)) \to 0$ a.e.*

($\mathcal{A}_4$): *There are $k, \ell \in \mathcal{L}_1^2$ and $m \in \mathcal{W}_1^2$ such that $\| \int_0^\infty h[(F \circ mx)_t, (F \circ my)_t] dt \|_{L_1^2} \leq E \int_0^\infty k_t \mid x_t - y_t \mid dt$, $\| h(G \circ mx, G \circ my) \|_{\mathcal{L}_1^2} \leq E \int_0^\infty \ell_t \mid x_t - y_t \mid dt$ and $\| h(H \circ mx, H \circ my) \|_{\mathcal{W}_1^2} \leq E \int_0^\infty \int_{\mathbb{R}^n} m_{t,z} \mid x_t - y_t \mid dt q(dz)$ for $x, y \in D$.*

($\mathcal{A}_4'$) *There are $k, \ell \in L^2(\mathcal{B}_+)$ and $m \in L^2(\mathcal{B}_+ \times \mathcal{B}^n)$ such that $h(F_t(x_2)(\omega), F_t(x_1)(\omega)) \leq k(t) \mid x_1 - x_2 \mid$, $h(G_t(x_2)(\omega), G_5(x_1)(\omega)) \leq \ell(t) \mid x_1 - x_2 \mid$ and $h(H_{t,z}(x_2)(\omega), H_{t,z}(x_1)(\omega)) \leq m(t,z) \mid x_1 - x_2 \mid$ a.e., each $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.*

It is clear that the upper (lower) semicontinuity of F, G and H does not imply their weak (strong) - weak sequential upper (lower) semicontinuity presented above. We shall show that in some special cases, i.e., for concave (convex, resp.), set-valued mappings such implication holds true. Recall a set-valued mapping $\mathcal{R}$, defined on a locally convex topological space $(X, \mathcal{T}_X)$ with values in a normed space is said to be concave (convex) if $\mathcal{R}(\alpha x_1 + \beta x_2) \subset \alpha \mathcal{R}(x_1) + \beta \mathcal{R}(x_2)$ $(\alpha \mathcal{R}(x_1) + \beta \mathcal{R}(x_2) \subset \mathcal{R}(\alpha x_1 + \beta x_2))$, for every $x_1, x_2 \in X$ and $\alpha, \beta \in [0,1]$ satisfying $\alpha + \beta = 1$.

Lemma 1: *Suppose F, G and H satisfy ($\mathcal{A}_1$) with $p = 1$, take on convex values and are concave (convex) with respect to $x \in \mathbb{R}^n$. If moreover F, G and H are u.s.c. (l.s.c.) with respect to $x \in \mathbb{R}^n$ then they are w.-w.s.u.s.c. (s.-w.s.l.s.c.).*

Proof: Let $x \in D$ be fixed and let (x^n) be a sequence of D weakly converging to x. Denote $K_p(t, \omega, y) := -s(p, F_t(y_t)(\omega))$ for $p \in \mathbb{R}^n$, $y \in D$, $t \geq 0$ and $\omega \in \Omega$. We shall show that for every $A \in \mathcal{P}$ and every $p \in \mathbb{R}^n$ one has

$$\int_A \int K_p(t, \omega, x) dt dP \leq \liminf_{n \to \infty} \int_A \int K_p(t, \omega, x^n) dt dP,$$

which is equivalent to the weak-weak sequential upper semicontinuity of F at $x \in D$ in the sense defined in ($\mathcal{A}_2$). Similarly, the weak-weak sequential upper semicontinuity of G and H can be verified.

Let $A \in \mathcal{P}$, $p \in \mathbb{R}^n$ be given. Denote $j_n = \int\int_A K_p(t,\omega,x^n)dtdP$ for $n = 1,2,\ldots$ and put $i: = liminf_{n\to\infty}\int\int_A K_p(t,\omega,x^n)dtdP$. By taking a suitable subsequence, say (n_k) of (n) we may well assume that $j_{n_k}\to i$ as $k\to\infty$. By the Banach and Mazur theorem (see [2]) for every $s = 1,2,\ldots$ there are numbers $\alpha_k^s \geq 0$ with $k = 1,2,\ldots,N$ and $N = 1,2,\ldots$ satisfying $\sum_{k=1}^N \alpha_k^s = 1$ and such that $\| z_N^s - x \|_{\mathcal{L}}\to 0$ as $N\to 0$, where $z_N^s(t,\omega) = \sum_{k=1}^N \alpha_k^s x_t^{n_k+s}(\omega)$. By the definition of the norm $\| \cdot \|_{\mathcal{L}}$ there is a subsequence, say again (z_N^s), of (z_N) such that $sup_{t\geq 0} | z_N^s(t,\omega) - x_t(\omega) | \to 0$ *a.s.* for $s = 1,2,\ldots$. Put

$$\eta_N^s: = \sum_{k=1}^N \alpha_k^s K_p(\cdot,\cdot,x^{n_k+s}),$$
$$j_k^s = \int\int_A K_p(t,\omega,x^{n_k+s})dtdP$$

and let $\delta_s = max_{N\geq s+1} max_{1\leq k\leq N} | j_k^s - i |$ for $s = 1,2,\ldots$. We have $\delta_s \to 0$ as $s\to\infty$. By the uniform square boundedness of F there is $m_F \in \mathcal{L}_1^2$ such that $\eta_N^s \geq -m_F$ *a.e.* for N, $s = 1,2,\ldots$. Therefore, $liminf_{N\to\infty}\eta_N^s \geq -m_F$ *a.e.* for $s = 1,2,\ldots$. Then by Fatou's lemma one obtains

$$\int\int_A \underset{N\to\infty}{liminf}\,\eta_N^s dtdP \leq \underset{N\to\infty}{liminf}\int\int_A \eta_N^s dtdP \leq i + \delta_s$$

for $s = 1,2,\ldots$, because for every $s = 1,2,\ldots$, we have $i - \delta_s \leq \int\int_A \eta_N^s dtdP \leq i + \delta_s$. Taking $\eta = liminf_{s\to\infty}[liminf_{N\to\infty}\eta_N^s]$ *a.e.*, we get $\eta \geq -m_F$ *a.e.* and $\int\int_A \eta dtdP \leq i$. We shall verify that we also have $K(t,\omega,x) \leq \eta(t,\omega)$ for *a.e.* $(t,\omega) \in \mathbb{R}_+ \times \Omega$. Indeed, by upper semicontinuity of F with respect to $x \in \mathbb{R}^n$, a real valued function $x\to -s(p,F_t(x))$ is lower semicontinuous on $\mathbb{R}^n$, *a.s.* for every $t \geq 0$ and $p \in \mathbb{R}^n$. Therefore for every $m,s = 1,2,\ldots$ there is $M \geq 1$ such that

$$-s(p,F_t(x_t)) - \frac{1}{m} < -s(p,F_t(\sum_{k=1}^N \alpha_k^s x_t^{n_k+s}))$$

a.s. for every $t \geq 0$ and $N \geq M$. Hence, by the properties of F, it follows

$$-s(p,F_t(x_t)) - \frac{1}{m} < \sum_{k=1}^N \alpha_k^s[-s(p,F_t(x_t^{n_k+s}))] = :\eta_N^s(t,\cdot)$$

a.s. for $t \geq 0$, $s,m = 1,2\ldots$ and $N \geq M$. Therefore, for $m = 1,2,\ldots$ almost everywhere, one gets

$$K_p(\cdot,\cdot,x) - \frac{1}{m} \le \liminf_{s\to\infty} f[\liminf_{N\to\infty} f \eta_N^s] = \eta.$$

Finally, we get

$$\int_A\int K_p(t,\omega,x)dtdP \le \int_A\int \eta(t,\omega)dtdP \le i. \qquad \square$$

5. PROPERTIES OF SOLUTION SET

We shall prove here the existence theorems for $SI(F,G,H)$. We show first that conditions $(\mathcal{A}_1)$ and anyone of conditions $(\mathcal{A}_2)$-$(\mathcal{A}_4)$ or $(\mathcal{A}_4')$ imply the existence of fixed points for the set-valued mappings $\mathcal{H}$ and $\mathcal{H}^{\alpha,\beta}$ defined above. Hence, by Propositions 4 and 5, the existence theorems for $SI(F,G,H)$ will follow. We begin with the following lemmas.

Lemma 2: *Assume F,G and H take on convex values, satisfy $(\mathcal{A}_1)$ with $p=2$ and $(\mathcal{A}_2)$. Then a set-valued mapping $\mathcal{H}$ is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D,\sigma(D,D^*))$ with nonempty values in $(D,\sigma(D,D^*))$.*

Proof: Let C be a nonempty weakly closed subset of D and select a sequence (x^n) of $\mathcal{H}^-(C)$ weakly converging to $x\in D$. There is a sequence (y^n) of C such that $y^n\in\mathcal{H}(x^n)$ for $n=1,2,\ldots$. By the uniform square-integrable boundedness of F,G and H, there is a convex weakly compact subset $\mathcal{B}\subset\mathcal{L}_n^2\times\mathcal{L}_n^2\times\mathcal{W}_n^2$ such that $\mathcal{H}(x^n)\subset\Phi(\mathcal{B})$. Therefore, $y^n\in\Phi(\mathcal{B})$, for $n=1,2,\ldots$ which, by the weak compactness of $\Phi(\mathcal{B})$, implies the existence of a subsequence, say for simplicity (y^k), of (y^n) weakly converging to $y\in\Phi(\mathcal{B})$. We have $y^k\in\mathcal{H}(x^k)$ for $k=1,2,\ldots$. Let $(f^k,g^k,h^k)\in\mathcal{S}^2(F\circ mx^k)\times\mathcal{S}^2(G\circ mx^k)\times\mathcal{S}_q^2(H\circ mx^k)$ be such that $\Phi(f^k,g^k,h^k)=y^k$, for each $k=1,2,\ldots$. We have of course $(f^k,g^k,h^k)\in\mathcal{B}$. Therefore, there is a subsequence, say again $\{(f^k,g^k,h^k)\}$ of $\{(f^k,g^k,h^k)\}$ weakly converging in $\mathcal{L}_n^2\times\mathcal{L}_n^2\times\mathcal{W}_n^2$ to $(f,g,h)\in\mathcal{B}$. Now, for every $A\in\mathcal{P}$ one obtains

$$dist\left(\int_A\int f_t dtdP,\ \int_A\int F_t(x)dtdP\right)\le$$

$$\le\left|\int_A\int[f_t-f_t^k]dtdP\right|+dist\left(\int_A\int f_t^k dtdP, \int_A\int F_t(x_k)dtdP\right)$$

$$+\bar{h}\left(\int\int_A F_t(x^k)dtdP, \int\int_A F_t(x)dtdP\right).$$

Therefore (see [8], Lemma 4.4) $f \in \mathcal{S}^2(F \circ mx)$. Quite similarly, we also get $t \in \mathcal{S}^2(G \circ mx)$ and $h \in \mathcal{S}^2_q(H \circ mx)$. Thus, $\Phi(f, g, h) \in \mathcal{H}(x)$, which implies $y \in \mathcal{H}(x)$. On the other hand we also have $y \in C$, because C is weakly closed. Therefore, $x \in \mathcal{H}^-(C)$. Now the result follows immediately from Eberlein and Šmulian's theorem. □

Lemma 3: *Assume F, G and H take on convex values, satisfy $(\mathcal{A}_1)$ with $p = 2$ and $(\mathcal{A}_3)$. Then a set-valued mapping $\mathcal{H}$ is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D, \sigma(D, D^*))$ with nonempty values in $(D, \sigma(D, D^*))$.*

Proof: Let C be a nonempty weakly closed subset of D and (x^n) a sequence of $\mathcal{H}_-(C)$ weakly converging to $x \in D$. Select arbitrarily $y \in \mathcal{H}(x)$ and suppose $(f, g, h) \in \mathcal{S}^2(F \circ mx) \times \mathcal{S}^2(G \circ mx) \times \mathcal{S}^2_q(H \circ mx)$ is such that $y = \Phi(f, g, h)$. Let $(f^n, g^n, h^n) \in \mathcal{S}^2(F \circ mx^n) \times \mathcal{S}^2(G \circ mx^n) \times \mathcal{S}^2_q(H \circ mx^n)$ be such that

$$|f_t(\omega) - f^n_t(\omega)| = dist(f_t(\omega), (F \circ mx^n)_t(\omega)),$$

$$|g_t(\omega) - g^n_t(\omega)| = dist(g_t(\omega), (G \circ mx^n)_t(\omega))$$

and $|h_{t,z}(\omega) - g^n_{t,z}(\omega)| = dist(h_{t,z}(\omega), (H \circ mx^n)_{t,z}(\omega))$ on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, respectively, for each $n = 1, 2, \ldots$. By virtue of $(\mathcal{A}_3)$ one gets $|f_t(\omega) - f^n_t(\omega)| \to 0$, $|g_t(\omega) - g^n_t(\omega)| \to 0$ and $|h_{t,z}(\omega) - h^n_{t,z}(\omega)| \to 0$ *a.e.*, as $n \to \infty$. Hence, by $(\mathcal{A}_1)$ we can easily see that a sequence (y_n), defined by $y^n = \Phi(f^n, g^n, h^n)$, weakly converges to y. But $y^n \in \mathcal{H}(x^n) \subset C$ for $n = 1, 2, \ldots$ and C is weakly closed. Then $y \in C$ which implies $\mathcal{H}(x) \subset C$. Thus $x \in \mathcal{H}_-(C)$. □

Lemma 4. *Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$. Then $H(\mathcal{H}(x), \mathcal{H}(y)) \leq L \| x - y \|_{\ell}$ or $H(\mathcal{H}(x), \mathcal{H}(y)) \leq L' \| x - y \|_{\ell}$, respectively, for every $x, y \in D$, where H is the Hausdorff metric induced by the norm $\| \cdot \|_{\ell}$, $L = \| \int_0^\infty k_t dt \|_{L^2_1} + 2 \| \int_0^\infty \ell_t dt \|_{L^2_1} + 2 \| \int_0^\infty \int_{\mathbb{R}^n} m_{\tau, z} d\tau q(dz) \|_{L^2_1}$ and $L' = |k|_1 + 2 |\ell|_2 + 2 \| m \|_2$.*

Proof: Let $x, y \in D$ be given and let $u \in \mathcal{H}(x)$. For every $\epsilon > 0$, there is $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{S}^2(F \circ mx) \times \mathcal{S}^2(G \circ mx) \times \mathcal{S}^2_q(H \circ mx)$ such that $\| u - \Phi(f^\epsilon, g^\epsilon, h^\epsilon) \|_{\ell} < \epsilon$. Select now $(\tilde{f}^\epsilon, \tilde{g}^\epsilon, \tilde{h}^\epsilon) \in \mathcal{S}^2(F \circ my) \times \mathcal{S}^2(G \circ my) \times \mathcal{S}^2_q(H \circ my)$ such that

$$| f_t^\epsilon(\omega) - \widetilde{f}_t^\epsilon(\omega) | = dist(f_t^\epsilon(\omega), (F \circ my)_t(\omega)),$$

$$| g_t^\epsilon(\omega) - \widetilde{g}_t^\epsilon(\omega) | = dist(g_t^\epsilon(\omega), (G \circ my)_t(\omega)) \qquad \text{and}$$

$| h_{t,z}^\epsilon(\omega) - \widetilde{h}_{t,z}^\epsilon(\omega) | = dist(h_{t,z}^\epsilon(\omega), (H \circ my)_{t,z}(\omega))$ on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, respectively. Now, by $(\mathcal{A}_4)$ it follows

$$E\left[\sup_{t \geq 0} \left| \int_0^t (f_\tau^\epsilon - \widetilde{f}_\tau^\epsilon) d\tau \right| \right]^2 \leq E \left[\int_0^\infty | f_\tau^\epsilon - \widetilde{f}_\tau^\epsilon | d\tau \right]^2$$

$$\leq E \left[\int_0^t \bar{h}((F \circ mx)_\tau, (F \circ my)_\tau) d\tau \right]^2 \leq \left(E \int_0^t k_\tau | x_\tau - y_\tau | d\tau \right)^2$$

$$\leq \left[E\left(\sup_{t \geq 0} | x_t - y_t | \cdot \int_0^\infty k_\tau d\tau \right) \right]^2 \leq E \left(\int_0^\infty k_\tau d\tau \right)^2 \cdot \| x - y \|_{\ell}^2.$$

Similarly, by Doob's inequality, we obtain

$$E \left[\sup_{t \geq 0} \left| \int_0^t (g_\tau^\epsilon - \widetilde{g}_\tau^\epsilon) dw_\tau \right| \right]^2 \leq 4E \int_0^\infty | g_\tau^\epsilon - \widetilde{g}_\tau^\epsilon |^2 d\tau$$

$$\leq 4E \int_0^\infty [\bar{h}((G \circ mx)_\tau, (G \circ my)_\tau)]^2 d\tau \leq 4\left(E \int_0^\infty \ell_\tau | x_\tau - y_\tau | d\tau \right)^2$$

$$\leq 4\left[E \left(\sup_{t \geq 0} | x_t - y_t | \cdot \int_0^\infty \ell_\tau d\tau \right) \right]^2 \leq 4E\left(\int_0^\infty \ell_\tau d\tau \right)^2 \| x - y \|_{\ell}^2.$$

Quite similarly, we also get

$$E \left[\sup_{t \geq 0} \left| \int_0^t \int_{\mathbb{R}^n} h_\tau^\epsilon - \widetilde{h}_{\tau,z}^\epsilon) \widetilde{\nu}(d\tau, dz) \right| \right]^2$$

$$\leq 4E\left(\int_0^\infty \int_{\mathbb{R}^n} m_{\tau,z} d\tau q(dz) \right)^2 \cdot \| x_\tau - y_\tau \|_{\ell}^2.$$

Therefore

$$\| u - \Phi(\widetilde{f}^\epsilon, \widetilde{g}^\epsilon, \widetilde{h}^\epsilon \|_{\ell}$$

$$\leq \| u-\Phi(f^{\epsilon},g^{\epsilon},h^{\epsilon}) \|_{\ell} + \| \Phi(f^{\epsilon},g^{\epsilon},h^{\epsilon})-\Phi(\tilde{f}^{\epsilon},\tilde{g}^{\epsilon},\tilde{h}^{\epsilon}) \|_{\ell} \leq \epsilon + L \| x-y \|_{\ell},$$

where L is such as above. This implies $\bar{H}(\mathcal{H}(x),\mathcal{H}(y)) \leq L \| x-y \|_{\ell.}$ Quite similarly we also get $\bar{H}(\mathcal{H}(y),\mathcal{H}(x)) \leq L \| x-y \|_{\ell}$. Therefore $H(\mathcal{H}(x),\mathcal{H}(y)) \leq L | x-y |_{\ell}$. Using conditions $(\mathcal{A}_4')$ instead of $(\mathcal{A}_4)$ we also get $H(\mathcal{H}(y),\mathcal{H}(x)) \leq L' \| x-y \|_{\ell}$. □

Lemma 5: *Suppose F,G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}_4')$. Then for every $0 \leq \alpha < \beta < \infty$ one has $H(\mathcal{H}^{\alpha,\beta}(x),\mathcal{H}^{\alpha,\beta}(y)) \leq L_{\alpha\beta} \| x-y \|_{\ell}$ or $H(\mathcal{H}^{\alpha,\beta}(x),\mathcal{H}^{\alpha,\beta}(y)) \leq L'_{\alpha\beta} | x-y |_{\ell}$, respectively, for every $x,y \in D^{\alpha,\beta}$, where H is a Hausdorff metric induced by the norm $\| \cdot \|_{\ell}$, $L_{\alpha,\beta} = \| \int_0^\infty \mathbb{1}_{[\alpha,\beta]}(t) k_t dt \|_{L_1^2} + 2 \| \int_0^\infty \mathbb{1}_{[\alpha,\beta]}(t) \ell_t dt \|_{L_1^2} + 2 \| \int_0^\infty \int_{\mathbb{R}^n} \mathbb{1}_{[\alpha,\beta]}(t)\, m_{t,z} dt q(dz) \|_{L_1^2}$ and $L'_{\alpha,\beta} = | \mathbb{1}_{[\alpha,\beta]} k |_1 + 2 | \mathbb{1}_{[\alpha,\beta]} \ell |_2 + 2 \| \mathbb{1}_{[\alpha,\beta]} m \|_2$.*

Proof: The proof follows immediately from Lemma 4 applied to $F^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta]}F$, $G^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta[}G$ and $H^{\alpha\beta} = \mathbb{1}_{[\alpha,\beta]}H$. □

Immediately from Lemma 2 and the Kakutani and Fan fixed point theorem the following result follows.

Lemma 6: *If F,G and H take on convex values and satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then $S(F,G,H) \neq \emptyset$.*

Proof: Let $\mathcal{B} = \{(f,g,h) \in \mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2 \colon | f_t(\omega) | \leq \| F_t(\omega) \|$, $| g_t(\omega) | \leq \| G_t(\omega) \|$, $| h_{t,z}(\omega) | \leq \| H_{t,z}(\omega) \|$ and put $\mathcal{K} = \Phi(\mathcal{B})$. It is clear that $\mathcal{K}$ is a nonempty convex weakly compact subset of D such that $\mathcal{H}(x) \subset \mathcal{K}$ for $x \in D$. By (ii) of Proposition 1, $\mathcal{H}(x)$ is a convex and weakly compact subset of D, for each $x \in D$. By Lemma 2, $\mathcal{H}$ is *u.s.c.* on a locally convex topological Hausdorff space $(D, \sigma(D,D^*))$. Therefore, by the Kakutani and Fan fixed point theorem, we get $S(F,G,H) \neq \emptyset$. □

Lemma 7. *If F,G and H take on convex values and satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$, then $S(F,G,H) \neq \emptyset$.*

Proof. Let $\mathcal{K}$ be as in Lemma 6. By virtue of Lemma 3, $\mathcal{H}$ is *l.s.c.* as a set-valued mapping from a paracompact space $\mathcal{K}$ considered with its relative topology induced by a weak topology $\sigma(D,D^*)$ on D into a Banach space $(D, \| \cdot \|_{\ell})$. By (ii) of Proposition 1, $\mathcal{H}(x)$ is a closed and convex subset of D, for each $x \in \mathcal{K}$. Therefore, by Michael's theorem, there is a continuous selection $f: \mathcal{K} \to D$ for $\mathcal{H}$. But $\mathcal{H}(\mathcal{K}) \subset \mathcal{K}$. Then f maps $\mathcal{K}$ into itself and is continuous with respect to the relative topology on $\mathcal{K}$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $x \in \mathcal{K}$ such that $x = f(x) \in \mathcal{H}(x)$. □

Lemma 8. *If F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}_4')$ then $S(F,G,H) \neq \emptyset$.*

Proof. Let $(\tau_n)_{n=1}^{\infty}$ be a sequence of positive numbers increasing to $+\infty$. Select a positive number σ such that $L_{k\sigma,(k+1)\sigma} < 1$ or $L'_{k\sigma,(k+1)\sigma} < 1$, respectively, for $k = 0, 1, \ldots$, where $L_{k\sigma,(k+1)\sigma}$ and $L'_{k\sigma,(k+1)\sigma}$ are as in Lemma 5. Suppose a positive integer n_1 is such that $n_1\sigma < \tau_1 \leq (n_1+1)\sigma$. By virtue of Lemma 5, $\mathcal{H}^{k\sigma,(k+1)\sigma}$ is a set-valued contraction for every $k = 0, 1, \ldots$. Therefore, by the Covitz and Nadler fixed point theorem, there is $z^1 \in S^{0,\sigma}(F,G,H)$. By the same argument, there is $z^2 \in z^1_\sigma + S^{\sigma,2\sigma}(F,G,H)$, because $z^1_\sigma + \mathcal{H}^{\sigma,2\sigma}$ is again a set-valued contraction mapping. Continuing the above procedure we can finally find a $z^{n_1+1} \in z^{n_1}_{n_1\sigma} + S^{n_1\sigma,\tau_1}(F,G,H)$. Put

$$x^1 = \sum_{k=0}^{n_1-1} \mathbb{I}_{[k\sigma,(k+1)\sigma)}(z^{k+1} - z^k_{k\sigma})$$

$$+ \mathbb{I}_{[n_1\sigma,,\tau_1]}(z^{n_1+1} - z^{n_1}_{n_1\sigma}) + \mathbb{I}_{(\tau_1,\infty)}(z^{n_1+1}_{\tau_1} - z^{n_1}_{n_1\sigma}),$$

where $z^0_0 = 0$. Similarly, as in the proof of Proposition 6, we can easily verify that $x^1 \in S^{0,\tau_1}(F,G,H)$. Repeating the above procedure to the interval $[\tau_1, \tau_2]$, we can find $x^2 \in x^1_{\tau_1} + S^{\tau_1,\tau_2}(F,G,H)$. Continuing this process we can define a sequence (x^n) of D satisfying the conditions of Proposition 6. Therefore $S(f,G,H) \neq \emptyset$. □

Now as a corollary of Proposition 4 and Lemmas 6-8, the following results follow.

Theorem 1. *Suppose F, G and H take on convex values, satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ or $(\mathcal{A}_3)$. Then $\Lambda_0(F,G,H) \neq \emptyset$.*

Theorem 2. *Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}_4')$ and* *take on convex values. Then* $\Lambda_0(F,G,H) \neq \emptyset$.

From the stochastic optimal control theory point of view (see [6]), it is important to know whether the set $\Lambda_0(F,G,H)$ is at least weakly compact in $(D, \|\cdot\|_\ell)$. We have the following result dealing with this topic.

Theorem 3. *Suppose* F,G *and* H *take on convex values and satisfy* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$. *Then* $\Lambda_0(F,G,H)$ *is a nonempty weakly compact subset of* $(D, \|\cdot\|_\ell)$.

Proof. Nonemptiness of $\Lambda_0(F,G,H)$ follows immediately from Theorem 1. By virtue of Proposition 4 and the Eberein and Šmulian theorem for the weak compactness of $\Lambda_0(F,G,H)$, it suffices only to verify that $S(F,G,H)$ is sequentially weakly compact. But $S(F,G,H) \subset \Phi(\mathcal{B})$, where $\mathcal{B}$ is a weakly compact subset of $\mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ defined in Lemma 6. Hence, by the properties of the linear mapping Φ, the relative sequential weak compactness of $S(F,G,H)$ follows. Suppose (x^n) is a sequence of $S(F,G,H)$ weakly converging to $x \in \Phi(\mathcal{B})$, and let $(f^n, g^n, h^n) \in \mathcal{S}^2(F \circ mx^n) \times \mathcal{S}^2(G \circ mx^n) \times \mathcal{S}_q^2(H \circ mx^n)$ be such that $x^n = \Phi(f^n, g^n, h^n)$, for $n = 1,2,\ldots$. By the weak compactness of $\mathcal{B}$, there is a subsequence, denoted again by $\{(f^n, g^n, h^n)\}$, of $\{(f^n, g^n, h^n)\}$ weakly converging to $(f,g,h) \in \mathcal{B}$. Similarly, as in the proof of Lemma 2, we can verify that $(f,g,h) \in \mathcal{S}^2(F \circ mx) \times \mathcal{S}^2(G \circ mx) \times \mathcal{S}_q^2(H \circ mx)$. This and the weak convergence of $\{\Phi(f^n, g^n, h^n)\}$ to $\Phi(f,g,h)$ imply that $x = \Phi(f,g,h) \in \mathcal{H}(x)$. Thus $x \in S(F,G,H)$□

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