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PROPERTIES OF SPECTRAL EXPANSIONS CORRESPONDING TO NON-SELF-ADJOINT DIFFERENTIAL OPERATORS

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Introduction

This paper is a survey of results in the spectral theory of differential operators generated by ordinary differential expressions and also by partial differential expressions of elliptic type. Our main focus is on the non-self-adjoint case.

In contrast to the theory of self-adjoint differential operators, in which a firm foundation of functional analysis methods was laid due to the efforts of many mathematicians, in many respects, a universal conception of approaches to studying the problems arisen was created, and, finally much experience was accumulated in scientific publications for more than a century, the spectral theory of non-self-adjoint operators contains at present fairly many open problems. This does not mean at all that little has been done in this field: a list of all publications devoted to this theme, if it has ever been composed, would look at least like that in the theory of self-adjoint problems. All this is explained by the fact that often to study a new class of non-self-adjoint problems, we need to elaborate new methods using a "fine adjustment" of the functional analysis technique.

Of course, the present survey does not claim to be an exhaustive presentation of scientific results and methods of the theory of non-self-adjoint differential operators. Here, we pay considerable attention to the studies carried out at the Chair of General Mathematics of the Department of Computational Mathematics and Cybernetics of the M. V. Lomonosov Moscow State University over a period of more than 30 years. They mainly concern one aspect or another of convergence of spectral expansions related to non-self-adjoint differential operators. The methodology elaborated there turns out to be a fairly effective tool for solving many new problems in this field. The authors try to make the reader familiar with the main results obtained up to now and give an idea of the methods elaborated for their proof. This specific character of the survey explains a certain "narrow specialization" and subjectiveness of the list of literature cited.

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Chapter 1

ORDINARY DIFFERENTIAL OPERATORS

In the first part of the survey, we speak about the properties of spectral expansions related to ordinary differential operators of an arbitrary order on finite intervals of the real axis.

1. One-Dimensional Schrödinger Operators on a Finite Interval

We start our study of a number of problems arising in studying spectral expansions related to differential operators from the study of the following object, which is the simplest from the technical

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point of view; namely, we study the operator on a finite interval $G = (a, b)$ of the real axis $\mathbb{R}$ which is generated by a differential expression of the form

$$\mathcal{L}\,u = -u'' + q(x)u, \tag{1.1}$$

in which the coefficient $q(x)$, called the potential of the operator $\mathcal{L}$ in what follows, is an arbitrary, for now, complex-valued function on the interval G.

1.1. Samarskii–Ionkin nonlocal problem. We demonstrate a specific character of non-self-adjoint problems for the differential expression (1.1) examining the following example of the boundary-value problem arising in studying the heat propagation in a homogeneous rod [75].

Let $G = (0, 1)$. For expression (1.1) with the coefficient $q(x) \equiv 0$, let us consider the eigenvalue problem[1]

$$\begin{cases} \mathcal{L}\,u(x) \equiv -u''(x) = \lambda u(x), & 0 < x < 1, \\ u(0) = 0, \quad u'(0) = u'(1). \end{cases} \tag{1.2}$$

The second boundary condition here is nonlocal, since it connects the values of the function $u(x)$ at distinct points of the interval. Such a nonlocality in the boundary condition leads to the violation of the self-adjointness. Indeed, an elementary integration by parts shows that the problem adjoint to (1.2) is given by other boundary conditions[2]:

$$\begin{cases} \mathcal{L}^*v(x) \equiv -v''(x) = \lambda v(x), & 0 < x < 1, \\ v'(1) = 0, \quad v(0) = v(1). \end{cases} \tag{1.3}$$

The eigenvalues of problem (1.2) are equal to $\lambda_k = (2\pi k)^2$, $k = 0, 1, 2, \dots$, and exactly one eigenfunction corresponds to each eigenvalue λ_k: $\overset{0}{u}_0(x) = x$ for $k = 0$ and $\overset{0}{u}_k(x) = \sin(2\pi kx)$ for $k \geq 1$. We cannot restrict ourselves to the eigenfunctions only, since, to obtain a complete trigonometric system on the interval $(0, 1)$, we need infinitely many additional functions (namely, the functions $\cos(2\pi kx)$, $k \geq 1$). We complete the set of eigenfunctions by the so-called associated functions, i.e., the solutions of the problems

$$\begin{cases} \mathcal{L}\,u(x) = \lambda_k u(x) - \overset{0}{u}_k(x), & 0 < x < 1, \\ u(0) = 0, \quad u'(0) = u'(1). \end{cases} \tag{1.4}$$

Here, the functions $\overset{1}{u}_k(x) = (4\pi k)^{-1} x \cdot \cos(2\pi kx)$, $k \geq 1$, can be taken as such functions (for $k = 0$, problem (1.4) has no solutions). By the Keldysh theorem [86], the system of eigenfunctions $\{\overset{0}{u}_k(x),\, k \geq 0\}$, together with the associated functions $\{\overset{1}{u}_k(x),\, k \geq 1\}$, is now complete in $L_2(0, 1)$, and any function from this space can be approximated (with an arbitrary accuracy in the metric of $L_2(0, 1)$) by a linear combination of functions from this system. However, owing to the fact that the functions of the constructed system are not orthogonal to each other (for example, $(\overset{0}{u}_k, \overset{1}{u}_k) = -(32\pi^2k^2)^{-1} \neq 0$), the completeness of such a system in the space $L_2(0, 1)$ does not ensure its basis property in this space, i.e., the possibility of uniquely expanding any function from $L_2(0, 1)$ into a series in functions of this system that converges in the metric of $L_2(0, 1)$.

To construct such an expansion in eigenfunctions and associated functions of problem (1.2), we take eigenfunctions and associated functions of the adjoint problem (1.3), since precisely these functions, together with the eigenfunctions and associated functions of the direct problem, form a biorthogonal pair.

The eigenvalues $\lambda = \lambda_k^*$ of the adjoint problem (1.3) coincide with λ_k. With each eigenvalue, one associates a unique eigenfunction: $\overset{0}{v}_k(x) = 1$ for $k = 0$ and $\overset{0}{v}_k(x) = \cos(2\pi kx)$ for $k \geq 1$, and only eigenfunctions with serial numbers $k \geq 1$ have the associated functions $\overset{1}{v}_k(x) = (4\pi k)^{-1}(x - 1)\sin(2\pi kx)$ as in the direct problem.

[1] In the literature, this problem is conventionally called the Ionkin–Samarskii problem.

[2] Only under such a choice of the boundary conditions does the relation $(\mathcal{L}\,u, v) = (u, \mathcal{L}^*v)$ hold; here, $(\cdot, \cdot)$ stands for the inner product in the space $L_2(0, 1)$.

The biorthogonality of the chosen pair of function systems is implied, for example, from the following argument.[3]

For a pair of $\overset{0}{u}_k$ and $\overset{1}{v}_k$ corresponding to the eigenvalues with the same serial number k, we have $\lambda_k(\overset{0}{u}_k, \overset{1}{v}_k) = (\mathcal{L}\,\overset{0}{u}_k, \overset{1}{v}_k) = (\overset{0}{u}_k, \mathcal{L}^*\overset{1}{v}_k) = \lambda_k(\overset{0}{u}_k, \overset{1}{v}_k) - (\overset{0}{u}_k, \overset{0}{v}_k)$; this yields $(\overset{0}{u}_k, \overset{0}{v}_k) = 0$. If $l \neq k$, then, on one hand, it follows from the relation $\lambda_k(\overset{0}{u}_k, \overset{0}{v}_l) = (\mathcal{L}\,\overset{0}{u}_k, \overset{0}{v}_l) = (\overset{0}{u}_k, \mathcal{L}^*\overset{0}{v}_l) = \lambda_l(\overset{0}{u}_k, \overset{0}{v}_l)$ that $(\overset{0}{u}_k, \overset{0}{v}_l) = 0$, and, on the other hand, the relation $\lambda_k(\overset{0}{u}_k, \overset{1}{v}_l) = (\mathcal{L}\,\overset{0}{u}_k, \overset{1}{v}_l) = (\overset{0}{u}_k, \mathcal{L}^*\overset{1}{v}_l) = \lambda_l(\overset{0}{u}_k, \overset{1}{v}_l) - (\overset{0}{u}_k, \overset{0}{v}_l) = \lambda_l(\overset{0}{u}_k, \overset{1}{v}_l)$ implies $(\overset{0}{u}_k, \overset{1}{v}_l) = 0$. Therefore, for $k \geq 1$, the eigenfunction $\overset{0}{u}_k(x)$ of problem (1.2) is orthogonal to all eigenfunctions and associated functions of the adjoint problem (1.3), except for $\overset{1}{v}_k(x)$.

Using similar arguments for the eigenfunction $\overset{0}{v}_k(x)$, we obtain its orthogonality to all eigenfunctions and associated functions of the direct problem (1.2), except for the associated function $\overset{1}{u}_k(x)$.

Taking into account that $(\overset{0}{u}_0, \overset{0}{v}_0) = 1/2$ and $(\overset{0}{u}_k, \overset{1}{v}_k) = -(\overset{1}{u}_k, \overset{0}{v}_k) = -(16\pi k)^{-1}$, we renumber and normalize the eigenfunctions and the associated functions of problems (1.2) and (1.3) so that they satisfy the relations[4] $(u_k, v_l) = \delta_{kl}$:

$$\begin{gathered} u_0(x) = x,\ u_{2k-1}(x) = \sin(2\pi kx),\ u_{2k}(x) = \frac{x}{4\pi k}\cos(2\pi kx), \\ v_0(x) = 2,\ v_{2k-1}(x) = 4(1-x)\sin(2\pi kx),\ v_{2k}(x) = 16\pi k\cos(2\pi kx),\ k \in \mathbb{N}. \end{gathered} \tag{1.5}$$

Since the biorthogonal adjoint system was already constructed for the system $\{u_k(x)\}$, this system is minimal in $L_2(0,1)$, and the spectral expansion of an arbitrary function $f \in L_2(0,1)$ can be written by using the biorthogonal series $\sum_{k=0}^{\infty}(f, v_k)u_k$.

It remains to answer the question whether or not this biorthogonal expansion converges in the metric of $L_2(0,1)$ to the function being expanded, or, in other words, whether or not the system $\{u_k(x)\}$ forms a basis in the space $L_2(0,1)$. In the system considered, we have infinitely many associated functions, and in such a case, the results of [96, 145, 162] and also the results of the theory of spectral operators [33] are no longer applicable, since the boundary conditions covered by them ensure the simplicity of eigenvalues starting from a certain serial number and hence the existence of only finitely many associated functions in the system considered.

We indicate one more specific peculiarity of Samarskii–Ionkin-type problems. Let us change all the associated functions of problem (1.2), adding the corresponding eigenfunctions to them. We obtain the following biorthogonal pair of eigenfunctions and associated functions of problems (1.2) and (1.3):

$$\begin{gathered} \widetilde{u}_0(x) = x,\ \widetilde{u}_{2k-1}(x) = \sin(2\pi kx), \\ \widetilde{u}_{2k}(x) = \frac{x}{4\pi k}\cos(2\pi kx) + A_k\sin(2\pi kx), \\ \widetilde{v}_0(x) = 2,\ \widetilde{v}_{2k-1}(x) = 4(1-x)\sin(2\pi kx) - 16A_k\pi k\cos(2\pi kx), \\ \widetilde{v}_{2k}(x) = 16\pi k\cos(2\pi kx),\ k \in \mathbb{N}, \end{gathered} \tag{1.6}$$

where A_k are numbers that are arbitrary for now.

On one hand, after such a change, each of the systems remains complete and minimal in $L_2(0,1)$. On the other hand, we can choose the constants A_k so that the necessary base condition in $L_2(0,1)$ will be violated:

$$\sup_{l\geq 0}\left(\|\widetilde{u}_l\|_{L_2(0,1)} \cdot \|\widetilde{v}_l\|_{L_2(0,1)}\right) < \infty. \tag{1.7}$$

[3] We note that this argument in a slightly different form can be applied to any pair of self-adjoint operators, even if they have associated functions of a higher order.

[4] As a result of this, the associated functions of the adjoint problem satisfy the equation $\mathcal{L}^* v_{2k-1} = \lambda_k v_{2k-1} + v_{2k}$.

Indeed, $\|\widetilde{u}_{2k-1}\|^2_{L_2(0,1)} = 1/2$, $\|\widetilde{v}_{2k-1}\|^2_{L_2(0,1)} = 16(8A_k^2\pi^2k^2 + A_k) + 8/3 - (\pi^2k^2)^{-1}$, and if A_k are chosen so that $A_k \geq A_0 k^{\varepsilon-1}$, where $\varepsilon, A_0 > 0$ are arbitrary, then condition (1.7) is violated. Therefore, none of the systems from the biorthogonal pair (1.6) forms a basis in $L_2(0,1)$.

However, as will be shown below, each of systems (1.5) forms a basis in $L_2(0,1)$, moreover, even an unconditional basis.

The "sensitivity" of the basis property to the choice of associated functions demonstrated by this example shows that theorems on the convergence of spectral expansions for such problems cannot be formulated in terms of belonging of boundary conditions to one type or another. The conditions of such a theorem should include the conditions tracing a concrete form of the root functions or, for example, only their asymptotics.

Therefore, in this sense, the class of so-called strengthened regular boundary conditions[5] for an ordinary differential operator is the only case where one can immediately obtain a positive answer to the question on the basis property of eigenfunctions and associated functions for a whole class of operators generated by a certain type of boundary conditions [96, 145].

To study the properties of spectral expansions for a wider range of problems, a new treatment of the concept of root (i.e., eigen- and associated) functions of differential operators was proposed in [46]; it consists in the fact that we do not specify the boundary conditions in a certain concrete form but consider each root function as only a regular solution of the corresponding differential equation with a spectral parameter.

1.2. Generalization of the concept of root function. Let the potential $q(x)$ in (1.1) be an arbitrary locally Lebesgue integrable complex-valued function on G.

A *regular solution on* G of the equation

$$\mathcal{L}\, u = \lambda u + f, \tag{1.8}$$

where $\lambda \in \mathbb{C}$ and $f \in L_1(G)$, is an arbitrary function $u = u(x)$ that, together with its first derivative, is absolutely continuous on any compact set in G and satisfies Eq. (1.8) almost everywhere on G.

By an *eigenfunction* of the operator $\mathcal{L}$, which is given by only the differential expression (1.1), we mean any nontrivial regular solution $u_0(x,\lambda)$ of the equation

$$\mathcal{L}\, u_0(x,\lambda) = \lambda u_0(x,\lambda) \tag{1.9}$$

that belongs to the space $L_2(G)$. The number λ in Eq. (1.9) is called an *eigenvalue* of the operator $\mathcal{L}$. We also say that the eigenfunction $u_0(x,\lambda)$ is an associated function of zero order.

If the associated function $u_{k-1}(x,\lambda)$ of order $k-1 \geq 0$ is already defined, then the *associated function of order* k corresponding to the eigenfunction $u_0(x,\lambda)$ and the eigenvalue λ is any regular solution $u_k(x,\lambda)$ of the equation[6]

$$\mathcal{L}\, u_k(x,\lambda) = \lambda u_k(x,\lambda) - \widetilde{\mu}\, u_{k-1}(x,\lambda) \tag{1.10}$$

from the space $L_2(G)$. Here, $\widetilde{\mu} = 1$ for $|\lambda| \leq 1$ and $\widetilde{\mu} = \mu \equiv \sqrt{\lambda}$ for $|\lambda| > 1$, where the value of the square root of a complex number $\lambda = \rho \exp(i\phi)$ $(-\pi < \phi \leq \pi)$ everywhere means the number $\sqrt{\lambda} = \sqrt{\rho}\exp(i\phi/2)$.

Let the eigenvalues of the operator $\mathcal{L}$ form a certain countable set Λ on the complex plane. With each eigenvalue $\lambda \in \Lambda$, the definition of eigenfunctions and associated functions (root functions for brevity) associates a chain of functions $u_0(x,\lambda), u_1(x,\lambda), u_2(x,\lambda), \dots$. We consider only those systems of root functions which, along with each associated function $u_k(x,\lambda)$, contain all the associated functions $u_l(x,\lambda)$ of lower orders $l < k$ corresponding to the same eigenvalue λ and the same eigenfunction $u_0(x,\lambda)$. The maximum order of the associated function in the chain corresponding to the eigenvalue $\lambda \in \Lambda$ is denoted by $m(\lambda)$, and the number $m(\lambda)+1$ is called the *rank* of the eigenfunction $u_0(x,\lambda)$. If the chain is infinite, we set $m(\lambda) = \infty$.

[5]In the Samarskii–Ionkin problem, the boundary condition are regular but not strengthened regular.

[6]The appearance of the normalizing factor $\widetilde{\mu}$ in the definition of an associated function becomes clear from what follows.

Such a treatment of the concept of root function allows us to include into consideration not only systems of root functions of various boundary-value problems with a point spectrum but also function systems consisting of only solutions of differential equations with a parameter not satisfying any boundary conditions at all (for example, the system of generalized exponentials) and also the systems obtained by uniting subsets of root functions of two distinct boundary-value problems.

To make more precise the potential $q(x)$ at the ends of the interval G, which ensures the belonging of the root functions to the class $L_2(G)$, we consider the main integral representations of regular solutions of Eq. (1.8).

1.3. Integral representations. By a direct integration, it is easy to make sure that the following "shift" formula [164] holds,[7] which expresses the value of a solution of Eq. (1.8) at a point $x \pm t \in G$ through the values of the solution and its derivative at the point $x \in G$:

$$u(x \pm t) = u(x)\cos\mu t \pm \frac{u'(x)}{\mu}\sin\mu t + \frac{1}{\mu}\int_0^t u(x \pm \tau)q(x \pm \tau)\sin\mu(t-\tau)\,d\tau$$
$$-\frac{1}{\mu}\int_0^t f(x \pm \tau)\sin\mu(t-\tau)\,d\tau. \tag{1.11}$$

Adding term-by-term relations (1.11) with plus and minus signs, we obtain the so-called mean-value formula [163, p. 20]:

$$\frac{u(x+t)+u(x-t)}{2} = u(x)\cos\mu t + \frac{1}{2\mu}\int_0^t [u(x+\tau)q(x+\tau) + u(x-\tau)q(x-\tau)]\sin\mu(t-\tau)\,d\tau$$
$$-\frac{1}{2\mu}\int_0^t [f(x+\tau)+f(x-\tau)]\sin\mu(t-\tau)\,d\tau. \tag{1.12}$$

For a fixed x, representation (1.11) is, in essence, an integral equation with respect to the unknown function $u(x \pm t)$ of the variable t. If we solve it using the successive approximation method, then we obtain the "explicit" variant of "shift" (1.11):

$$u(x \pm t) = u(x)F_0^{\pm}(t,x;\lambda) \pm \mu^{-1}u'(x)\Phi_0^{\pm}(t,x;\lambda) + F(t,x;\lambda), \tag{1.13}$$

where the following notation was used:

$$F_0^{\pm}(t,x;\lambda) = (E-T_\pm)^{-1}\cos\mu t, \quad \Phi_0^{\pm}(t,x;\lambda) = (E-T_\pm)^{-1}\sin\mu t,$$
$$F(t,x;\lambda) = \mu^{-1}(E-T_\pm)^{-1}\int_0^t f(x \pm \tau)\sin\mu(t-\tau)\,d\tau, \tag{1.14}$$

in which E is the identity operator, $T_\pm$ are integral operators acting on a function $\chi(t)$ of the variable t according to the rule

$$T_\pm\chi(t) = \mu^{-1}\int_0^t \chi(\tau)q(x \pm \tau)\sin\mu(t-\tau)\,d\tau, \tag{1.15}$$

and the operator $(E-T_\pm)^{-1}$ is equal to $E + \sum_{l=1}^{\infty} T_\pm^l$. Namely, the convergence of the series in (1.14) implies the correctness of the application of the successive approximation method in obtaining formula (1.13).

[7]Here, as above, $\mu = \sqrt{\lambda}$, and the expression $\mu^{-1}\sin\mu\xi$ for $\mu = 0$ is assumed to be equal to ξ.

Definition (1.15) of the operator $T_\pm$ obviously implies that if the potential $q(x)$ is locally Lebesgue integrable on G and the integral $\int_a^b (\xi - a)(b - \xi)|q(\xi)|\,d\xi$ converges, then for all $\mu \in \mathbb{C}$, $x \in G$, and $t > 0$ such that $x \pm t \in \overline{G}$, the following inequality holds:

$$|T_\pm \chi(t)| \le 2\omega(t)(b-a)^{-1} \cdot \sup_{0 \le \tau \le t} |\chi(\tau) \cosh \operatorname{Im} \mu(t - \tau)|,$$

where $\omega(t) \equiv \sup\limits_{e \subset G, \operatorname{mes} e \le t} \int_e (\xi - a)(b - \xi)|q(\xi)|\,d\xi$ tends to zero as $t \to 0 + 0$. Thus, series (1.14) for $F_0^\pm(t, x; \lambda)$, $\Phi_0^\pm(t, x; \lambda)$, and for $F(t, x; \lambda)$ if $f \in L_1(G)$ converge under the condition that t is sufficiently small.

These arguments and also the analysis of the "shift" formula (1.11), being differentiated in the variable t, allow us to reveal the smoothness of a regular solution of Eq. (1.8) up to the ends of the interval G.

Theorem 1.1. *Let a function $f(x)$ be Lebesgue integrable on G.*

(1) *If the potential $q(x)$ of the operator $\mathcal{L}$ is such that*

$$\int_a^b (\xi - a)(b - \xi)|q(\xi)|\,d\xi < \infty, \tag{1.16}$$

then any regular solution of Eq. (1.8) *is absolutely continuous on $\overline{G}$.*

(2) *If the potential $q(x)$ in the operator $\mathcal{L}$ is Lebesgue integrable on G, then any regular solution of Eq.* (1.8), *together with its derivative, is absolutely continuous on $\overline{G}$.*

In particular, this theorem implies that any nontrivial regular solution of Eqs. (1.9) and (1.10) can serve as a root function of the operator $\mathcal{L}$ with the potential satisfying (1.16).

The representations of regular solutions of Eq. (1.8) presented in this subsection are the main tool for analyzing the functional properties of systems of root functions of the operator $\mathcal{L}$.

1.4. Bases in Banach spaces. Let us present briefly the main definitions and facts which will be used in what follows.

Let $\mathfrak{B}$ be a Banach space with norm $\|\cdot\|_{\mathfrak{B}}$, and let $\mathfrak{B}^*$ be its dual with norm $\|\cdot\|_{\mathfrak{B}^*}$.

A system of elements $\{e_k\}_{k=1}^\infty$ is said to be *closed* in $\mathfrak{B}$ if the linear span of this system is everywhere dense in $\mathfrak{B}$, i.e., any element of the space $\mathfrak{B}$ can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space $\mathfrak{B}$.

A system $\{e_k\}_{k=1}^\infty$ is said to be *minimal* in $\mathfrak{B}$ if none of its elements belongs to the closure of the linear span of the other elements of this system.

Theorem 1.2 ([168, p. 65])**.** *A system $\{e_k\}_{k=1}^\infty$ is minimal iff there exists a biorthogonal system dual to it, i.e., a system of linear functionals $\{g_k\}_{k=1}^\infty$ from $\mathfrak{B}^*$ such that $(e_k, g_l) = \delta_{kl}$ for all $k, l \in \mathbb{N}$. Moreover, if the initial system is simultaneously closed and minimal in $\mathfrak{B}$, then the system biorthogonally dual to it is uniquely defined.*

We say that a system $\{e_k\}_{k=1}^\infty$ is *uniformly minimal* in $\mathfrak{B}$ if there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$,

$$\operatorname{dist}(e_k, E_{(k)}) > \gamma \|e_k\|_{\mathfrak{B}}, \tag{1.17}$$

where $E_{(k)}$ is the closure of the linear span of all elements e_l with serial numbers $l \ne k$.

Theorem 1.3 ([168, p. 66])**.** *A closed and minimal system $\{e_k\}_{k=1}^\infty$ is uniformly minimal in $\mathfrak{B}$ iff*

$$\sup_{k \ge 1} (\|e_k\|_{\mathfrak{B}} \cdot \|g_k\|_{\mathfrak{B}^*}) < \infty. \tag{1.18}$$

A system $\{e_k\}_{k=1}^{\infty}$ forms a *basis* of the space $\mathfrak{B}$ if, for any element $f \in \mathfrak{B}$, there exists a unique expansion of it in the elements of the system, i.e., the series $\sum_{k=1}^{\infty} c_k e_k$ convergent to f in the norm of the space $\mathfrak{B}$. Any basis is a closed and minimal system in $\mathfrak{B}$, and, therefore, we can uniquely find its biorthogonally dual systems $\{g_k\}_{k=1}^{\infty}$, and hence the expansion of any element of f with respect to the basis $\{e_k\}_{k=1}^{\infty}$ coincides with its biorthogonal expansion, i.e., $c_k = (f, g_k)$ for all $k \in \mathbb{N}$.

Any basis in $\mathfrak{B}$ is a uniformly minimal system, and, therefore, (1.18) holds. However, it is well known that a closed and uniformly minimal system may not form a basis in $\mathfrak{B}$.

A system biorthogonally dual to a basis in a reflexive Banach space $\mathfrak{B}$ itself forms a basis in $\mathfrak{B}^*$.

A basis $\{e_k\}_{k=1}^{\infty}$ in the space $\mathfrak{B}$ is said to be *unconditional* if it remains a basis for any permutation of its elements.

In a Hilbert space $\mathfrak{H}$, along with the concept of an unconditional basis, we have the close concept of a Riesz basis. A system $\{e_k\}_{k=1}^{\infty}$ is called a *Riesz basis* of the space $\mathfrak{H}$ if there exists a bounded invertible operator U such that the system $\{Ue_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $\mathfrak{H}$.

Theorem 1.4 ([188]). *A system $\{e_k\}_{k=1}^{\infty}$ forms a Riesz basis of the space $\mathfrak{H}$ iff it is an unconditional basis almost normalized in $\mathfrak{H}$, i.e.,*

$$0 < \inf \|e_k\|_{\mathfrak{H}} \le \sup \|e_k\|_{\mathfrak{H}} < \infty. \tag{1.19}$$

Any Riesz basis in $\mathfrak{H}$ can also be characterized in terms of the behavior of coefficients (f, e_k) of the biorthogonal basis in the dual system.

A system $\{e_k\}_{k=1}^{\infty}$ is said to be *Bessel* in $\mathfrak{H}$ if there exists a constant $M > 0$ such that for any $f \in \mathfrak{H}$, the following Bessel-type inequality holds:

$$\sum_{k=1}^{\infty} |(f, e_k)|^2 \le M \|f\|_{\mathfrak{H}}^2; \tag{1.20}$$

it is *Hilbert* in $\mathfrak{H}$ if there exists a constant $m > 0$ such that for any $f \in \mathfrak{H}$, the following Hilbert-type inequality holds:

$$m\|f\|_{\mathfrak{H}}^2 \le \sum_{k=1}^{\infty} |(f, e_k)|^2. \tag{1.21}$$

The Bessel and Hilbert properties of systems in a biorthogonal pair are dual to one another: if one of the systems is a Bessel system in $\mathfrak{H}$, then the other is a Hilbert system in $\mathfrak{H}$, and vice versa [5].

Theorem 1.5 ([5]). *A system $\{e_k\}_{k=1}^{\infty}$ closed and minimal in $\mathfrak{H}$ forms a Riesz basis iff it is simultaneously a Bessel and Hilbert system in $\mathfrak{H}$.*

We note that the Hilbert property of a system in $\mathfrak{H}$ implies that if $(f, e_k) = 0$ for all $k \in \mathbb{N}$, then $f = 0$. This property is called the *completeness* property of the system $\{e_k\}_{k=1}^{\infty}$ in $\mathfrak{H}$. In a Hilbert space, the properties of completeness and closedness of a system are equivalent.

It should be noted that inequalities (1.20) and (1.21) are a key characteristic of Riesz bases consisting of root functions of differential operators and are the base of many proofs.

The behavior of coefficients (f, e_k) can be characterized not only for Riesz bases in $\mathfrak{H}$ but for systems of elements $\{e_k\}_{k=1}^{\infty}$ of a uniformly convex and uniformly smooth Banach space $\mathfrak{B}$ that form an almost normalized basis in $\mathfrak{B}$.

Theorem 1.6 ([32]). *For an almost normalized basis $\{e_k\}_{k=1}^{\infty}$ of a uniformly convex and uniformly smooth Banach space $\mathfrak{B}$, we can find positive constants A_1 and A_2 and numbers s_1 and s_2 connected by the inequalities $1 < s_2 \le 2 \le s_1 < \infty$ such that for any $f \in \mathfrak{B}^*$, the following inequalities hold:*

$$A_1 \left(\sum_{k=1}^{\infty} |(f, e_k)|^{s_1} \right)^{1/s_1} \le \|f\|_{\mathfrak{B}^*} \le A_2 \left(\sum_{k=1}^{\infty} |(f, e_k)|^{s_2} \right)^{1/s_2}. \tag{1.22}$$

The left inequality in (1.22) generalizes the Bessel-type inequality (1.20), and the right one generalizes the Hilbert-type inequality (1.21).

Also, we mention a result concerning the stability of Riesz bases. We say that two systems $\{e_k\}_{k=1}^{\infty}$ and $\{\widetilde{e}_k\}_{k=1}^{\infty}$ of the space $\mathfrak{H}$ are *quadratically close* if $\sum_{k=1}^{\infty} \|e_k - \widetilde{e}_k\|_{\mathfrak{H}}^2 < \infty$. In [5], it was proved that any minimal system that is quadratically close to a Riesz basis in $\mathfrak{H}$ is also a Riesz basis in $\mathfrak{H}$.

Finally, we note that in this survey, as the Banach space $\mathfrak{B}$, we take spaces of Lebesgue integrable functions[8] $L_p(G)$ and $L_p(K)$, $1 \le p \le \infty$, where G is the domain of the differential expression considered and K is any compact set in G. The norms in these spaces are denoted for brevity by $\|\cdot\|_p$ and $\|\cdot\|_{p,K}$, respectively.

1.5. Unconditional basis property of a system of root functions. We now consider the main aspects of the approach elaborated for studying the convergence of spectral expansions in root functions of differential operators by examining the solution of the problem on the unconditional basis property in $L_2(G)$ of the system of root functions for the Schrödinger operator (1.1) with an arbitrary Lebesgue integrable potential.

We assume that all chains of root functions entering the system are finite, i.e., $m(\lambda) < \infty$. Denote by $\mathfrak{U}$ the system of root functions enumerated in a certain way. We write relations (1.9) and (1.10), which define the root functions of the operator $\mathcal{L}$, in the unified form

$$\mathcal{L}\, u_k(x;\lambda) = \lambda u_k(x;\lambda) - \operatorname{sign} k\, \widetilde{\mu}\, u_{k-1}(x;\lambda). \tag{1.23}$$

We require the fulfillment of the following

Condition A:

(A1) the system of root functions $\mathfrak{U}$ of the operator $\mathcal{L}$ is complete and minimal in $L_2(G)$;

(A2) the ranks of eigenfunctions are uniformly bounded:

$$\sup_{\lambda\in\Lambda} m(\lambda) < \infty; \tag{1.24}$$

(A3) the following estimate holds uniformly in $t \ge 0$:

$$\sum_{\lambda\in\Lambda:|\operatorname{Re}\sqrt{\lambda}-t|\le 1} 1 \le B_1, \tag{1.25}$$

which will be called the *"sum of units"* in what follows;

(A4) the set of eigenvalues Λ lies inside a certain parabola, i.e., the following estimate holds uniformly in $\lambda \in \Lambda$:

$$|\operatorname{Im}\sqrt{\lambda}| \le B_2. \tag{1.26}$$

The latter condition on the spectrum of the operator will be called the *Carleman condition.*

By Theorem 1.2, condition (A1) ensures the existence of a unique system $\mathfrak{V}$ biorthogonally dual to $\mathfrak{U}$. Also, we assume that

(A5) the biorthogonal dual system $\mathfrak{V}$ consists of root functions of the formally adjoint operator $\mathcal{L}^* v \equiv -v'' + \overline{q(x)}v$ that are understood in the generalized sense, i.e., for all $\lambda \in \Lambda$ and $k = \overline{0, m(\lambda)}$, the functions of the system $\mathfrak{V}$ are solutions of the following equation that are regular on G:

$$\mathcal{L}^* v_k(x;\overline{\lambda}) = \overline{\lambda} v_k(x;\overline{\lambda}) - \operatorname{sign} k\, \overline{\widetilde{\mu}}\, v_{k-1}(x;\overline{\lambda}). \tag{1.27}$$

We note that, as in the Samarskii–Ionkin problem, the relation $(u_k(\cdot,\lambda_1), v_l(\cdot,\overline{\lambda}_2)) = 1$ holds only if $\lambda_1 = \lambda_2$ and $l = m(\lambda) - k$.

[8] For $1 < p < \infty$, these spaces are reflexive, uniformly convex, and uniformly smooth Banach spaces.

Theorem 1.7 ([49]). *Let the potential $q(x)$ of the operator $\mathcal{L}$ be Lebesgue integrable on G, and let conditions* (A1)–(A5) *hold. Then each of the systems $\mathfrak{U}$ and $\mathfrak{V}$ forms an unconditional basis in $L_2(G)$ iff the following estimate of the product of norms holds uniformly in $\lambda \in \Lambda$ and $k = \overline{0, m(\lambda)}$*:

$$\|u_k(\cdot;\lambda)\|_2 \cdot \|v_{m(\lambda)-k}(\cdot;\overline{\lambda})\|_2 \le C. \tag{1.28}$$

We note that for the system of root functions of a concrete boundary-value problem, all the conditions of Theorem 1.7 are easily verified. Indeed, the completeness of the system $\mathfrak{U}$ is proved by using the well-known abstract theorems (see [86, 87, 150]). The minimality is implied by the completeness in $L_2(G)$ of the biorthogonally dual system $\mathfrak{V}$. Conditions (A2), (A3), and (A4) are verified by using the leading term of the asymptotics of eigenvalues, and estimate (1.28) is verified by using the leading term of the asymptotics of root functions.

Let us present a brief scheme for proving Theorem 1.7. Since the necessity of estimate (1.28) is obviously implied by Theorem 1.3, we prove its sufficiency only.

The base of the proof of the sufficiency in Theorem 1.7 consists in the justification of the Bessel property for the systems $\mathfrak{U}$ and $\mathfrak{V}$ normalized in $L_2(G)$. It follows from the definition of an unconditional basis that each of the systems $\mathfrak{U}$ and $\mathfrak{V}$ forms an unconditional basis in $L_2(G)$ iff each of the systems $\mathfrak{U}' = \{\gamma_k(\lambda)u_k(x;\lambda)\}$ and $\mathfrak{V}' = \{\gamma_k^{-1}(\lambda)v_{m(\lambda)-k}(x;\overline{\lambda})\}$, where $\gamma_k(\lambda) \equiv \|u_k(\cdot;\lambda)\|_2^{-1}$, forms an unconditional basis in $L_2(G)$. By Theorem 1.5 and the remark before it, it suffices to prove that each of the systems $\mathfrak{U}'$ and $\mathfrak{V}'$ is a Bessel system in $L_2(G)$. It follows from condition (1.28) that the Bessel property of the system $\mathfrak{V}'$ will be implied by the Bessel property of the normalized system $\mathfrak{V}'' = \{\gamma_k^*(\lambda)v_{m(\lambda)-k}(x;\overline{\lambda})\}$, where $\gamma_k^*(\lambda) \equiv \|v_{m(\lambda)-k}(\cdot;\lambda)\|_2^{-1}$. Among two systems $\mathfrak{U}'$ and $\mathfrak{V}''$ normalized in $L_2(G)$, we choose the first one and restrict ourselves to the verification of the Bessel property for it, because this can be done in a similar way for the second one by condition (A5).

Let us consider an arbitrary function $f(x)$ of the space $L_2(G)$. We justify the estimate

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)} |(\gamma_k(\lambda)u_k(\cdot;\lambda), f(\cdot))|^2 \le M\|f\|_2^2 \tag{1.29}$$

with the constant $M > 0$ independent of f.

By (1.24) and (1.25), no more than countably many summands in the left-hand side of (1.29), each of which does not exceed $\|f\|_2^2$, correspond to the values of $\lambda : |\lambda| \le 1$. Therefore, without loss of generality, we assume that all $\lambda \in \Lambda$ satisfy the condition $|\lambda| > 1$.

To transform the coefficients of the series in the left-hand side of (1.29), we use the "shift" formula (1.11) being applied to solutions of Eq. (1.23). We set $x = (a+b)/2$ and $R = (b-a)/2$. Then $\int_a^b u_k(\xi)\overline{f(\xi)}\,d\xi = \int_0^R u_k(x+t;\lambda)\overline{f(x+t)}\,dt + \int_0^R u_k(x-t;\lambda)\overline{f(x-t)}\,dt$, and after the application of the "shift" formula to $u_k(x+t;\lambda)$ and $u_k(x-t;\lambda)$, it becomes clear that to justify (1.29), it suffices to estimate the following series:

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)} |\gamma_k(\lambda)u_k(x;\lambda)|^2 \left|\int_0^R \overline{f(x\pm t)}\cos\mu t\,dt\right|^2, \tag{1.30}$$

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)} \left|\gamma_k(\lambda)\mu^{-1}u_k'(x;\lambda)\right|^2 \left|\int_0^R \overline{f(x\pm t)}\sin\mu t\,dt\right|^2, \tag{1.31}$$

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)}\left|\frac{\gamma_k(\lambda)}{\mu}\right|^2\left|\int_0^R\overline{f(x\pm t)}\int_0^t u_k(x\pm\tau;\lambda)q(x\pm\tau)\sin\mu(t-\tau)\,d\tau dt\right|^2, \tag{1.32}$$

$$\sum_{\lambda\in\Lambda}\sum_{k=1}^{m(\lambda)}|\gamma_k(\lambda)|^2\left|\int_0^R\overline{f(x\pm t)}\int_0^t u_{k-1}(x\pm\tau;\lambda)\sin\mu(t-\tau)\,d\tau dt\right|^2. \tag{1.33}$$

We note that if we obtain the estimates

$$\sup_{x\in G}|u_k(x;\lambda)| = O(1)\|u_k(\cdot;\lambda)\|_2, \tag{1.34}$$

$$\sup_{x\in G}|u_k'(x;\lambda)| = O\left(1+|\sqrt{\lambda}|\right)\|u_k(\cdot;\lambda)\|_2, \tag{1.35}$$

uniform in $\lambda\in\Lambda$ and $k=\overline{0,m(\lambda)}$, then to estimate series (1.30) and (1.31) it suffices to estimate each of the following series:

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)}\left|\int_0^R\overline{f(x\pm t)}\cos\mu t\,dt\right|^2, \quad \sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)}\left|\int_0^R\overline{f(x\pm t)}\sin\mu t\,dt\right|^2. \tag{1.36}$$

Since the square of the module of the integral in (1.32) does not exceed

$$\sup_{x\in G}|u_k(x;\lambda)|^2\cdot R\exp(2B_2R)\|q\|_1^2\,\|f\|_2^2,$$

where B_2 is the constant from condition (1.26), it follows from (1.34) that to estimate series (1.32), it suffices to estimate the series

$$\sum_{\lambda\in\Lambda}\sum_{k=0}^{m(\lambda)}|\mu|^{-2}\|f\|_2^2. \tag{1.37}$$

Finally, in each of the summands of series (1.33), we change the order of integration:

$$\int_0^R\overline{f(x\pm t)}\int_0^t u_{k-1}(x\pm\tau;\lambda)\sin\mu(t-\tau)\,d\tau dt$$

$$=\int_0^R u_{k-1}(x\pm\tau;\lambda)\int_\tau^R\overline{f(x\pm t)}\sin\mu(t-\tau)\,dtd\tau;$$

therefore, the square of the module of the integral in (1.33) does not exceed

$$\sup_{x\in G}|u_{k-1}(x;\lambda)|^2\,2e^{2B_2R}R\int_0^R\left[\left|\int_\tau^R\overline{f(x\pm t)}\sin\mu t\,dt\right|^2+\left|\int_\tau^R\overline{f(x\pm t)}\cos\mu t\,dt\right|^2\right].$$

This implies that if we obtain the estimate

$$\sup_{x\in G}|u_{k-1}(x;\lambda)| = O(1)\|u_k(\cdot;\lambda)\|_2, \tag{1.38}$$

uniform in $\lambda\in\Lambda$ and $k=\overline{1,m(\lambda)}$, then to estimate series (1.33) it suffices to estimate the series

$$\sum_{\lambda\in\Lambda}\sum_{k=1}^{m(\lambda)}\left|\int_\tau^R\overline{f(x\pm t)}\cos\mu t\,dt\right|^2, \quad \sum_{\lambda\in\Lambda}\sum_{k=1}^{m(\lambda)}\left|\int_\tau^R\overline{f(x\pm t)}\sin\mu t\,dt\right|^2 \tag{1.39}$$

uniformly in $\tau \in [0, R]$.

We now divide the summation in each of the series (1.36), (1.37), and (1.39) into blocks (or "batches"). The summands corresponding to the eigenvalues λ for which $2\pi l/R \le \operatorname{Re}\sqrt{\lambda} < 2\pi(l+1)/R$ and all $k = \overline{0, m(\lambda)}$ enter the lth block ($l = 0, 1, 2, \dots$). For $\mu = \sqrt{\lambda}$ entering the lth block, the representation $\mu = 2\pi l/R + \delta_l$ holds, and, moreover, by (1.26), $|\delta_l| = O(1)$ uniformly in l. Thus, for example, the first of the series (1.36) can be written as follows:

$$\sum_{l=0}^{\infty} \sum_{\lambda \in \Lambda_l} \sum_{k=0}^{m(\lambda)} \left| \int_0^R \overline{f(x \pm t)} \cos \mu t \, dt \right|^2, \tag{1.40}$$

where $\Lambda_l = \Lambda \cap \{2\pi l/R \le \operatorname{Re}\sqrt{\lambda} < 2\pi(l+1)/R\}$. In each of the integrals, we take into account the representation for the parameter μ and perform integration by parts:

$$\int_0^R \overline{f(x \pm t)} \cos \mu t \, dt = \int_0^R \overline{f(x \pm \tau)} \cos(2\pi l\tau/R) \, d\tau$$

$$- \int_0^R \delta_l \sin \delta_l t \int_t^R \overline{f(x \pm \tau)} \cos(2\pi l\tau/R) \, d\tau \, dt - \int_0^R \delta_l \cos \delta_l t \int_t^R \overline{f(x \pm \tau)} \sin(2\pi l\tau/R) \, d\tau \, dt. \tag{1.41}$$

Each of the integrals

$$\int_0^R \overline{f(x \pm \tau)} \cos(2\pi l\tau/R) \, d\tau, \quad \int_t^R \overline{f(x \pm \tau)} \cos(2\pi l\tau/R) \, d\tau, \text{ and } \int_t^R \overline{f(x \pm \tau)} \sin(2\pi l\tau/R) \, d\tau$$

is a Fourier coefficient with respect to the almost normalized orthogonal trigonometric system of the function $f(x \pm \tau)$ or its restriction to the interval (t, R). By the classical Bessel inequality, the series in squares of modules of these coefficients does not exceed $\|f\|_2^2$. Thus, it follows from relation (1.41) and the estimate for δ_l that the series (1.40) is

$$O(1)\|f\|_2^2 \sup_{l \ge 0} \sum_{\lambda \in \Lambda_l} (1 + m(\lambda)) = O(1)\|f\|_2^2$$

by conditions (A2) and (A3) of the theorem.

In a similar way, we estimate the second of the series (1.36) and each of the series (1.39). Following this line of reasoning, we obtain that series (1.37) is $O(1)\|f\|_2^2 \sum_{l=1}^{\infty} l^{-2} = O(1)\|f\|_2^2$.

Therefore, we may assume that the required estimate (1.29) is proved whenever we prove that estimates (1.34), (1.35), and (1.38) hold for the root functions of the operator $\mathcal{L}$.

1.6. Estimates of root functions. For operator (1.1) with the Lebesgue integrable potential $q(x)$, the estimates connecting the L_p norms ($1 \le p \le \infty$) of the root functions and their first derivatives and also the estimates for the norm of the associated function of order $k-1$ through the norm of the associated function of order k of the same chain were obtained in a more general situation than that required in Theorem 1.7.

Theorem 1.8 ([109, 111, 164, 165, 172, 175, 181]). (1) *The following estimates hold uniformly in* $\lambda \in \mathbb{C}$:

$$\|u_k(\cdot;\lambda)\|_p = O(1)\left(1 + |\operatorname{Im}\sqrt{\lambda}|\right)^{1/s - 1/p} \|u_k(\cdot;\lambda)\|_s, \tag{1.42}$$

$$\|u_k'(\cdot;\lambda)\|_p = O(1)\left(1 + |\sqrt{\lambda}|\right) \|u_k(\cdot;\lambda)\|_p, \tag{1.43}$$

$$\|u_{k-1}(\cdot;\lambda)\|_p = O(1)\left(1 + |\operatorname{Im}\sqrt{\lambda}|\right) \|u_k(\cdot;\lambda)\|_p, \tag{1.44}$$

where $1 \le p, s \le \infty$.

(2) *There exists* $\lambda_0 > 0$ *such that the following estimate holds uniformly in* $\lambda \in \mathbb{C}$, $|\lambda| \geq \lambda_0$, *and for all* $p \in [1, \infty]$:

$$\left(1 + |\sqrt{\lambda}|\right) \|u_k(\cdot;\lambda)\|_p = O(1)\|u_k'(\cdot;\lambda)\|_p. \tag{1.45}$$

(3) *Let* K *be a certain compact set in* $\overline{G}$, *and let a compact set* $K_h \subset K$ *satisfy the condition* $\operatorname{dist}(K_h, \partial K) = h > 0$. *Then the following estimates hold uniformly in* $\lambda \in \mathbb{C}$ *for all* $p \in [1, \infty]$:

$$\|u_k(\cdot;\lambda)\|_{p,K} = O(1) \exp\left(h|\operatorname{Im}\sqrt{\lambda}|\right) \|u_k(\cdot;\lambda)\|_{p,K_h}, \tag{1.46}$$

$$\|u_k(\cdot;\lambda)\|_{p,K_h} = O(1) \exp\left(-h|\operatorname{Im}\sqrt{\lambda}|\right) \left(1 + |\operatorname{Im}\sqrt{\lambda}|\right)^k \|u_k(\cdot;\lambda)\|_{p,K}. \tag{1.47}$$

All the constants in $O(1)$ *of estimates* (1.42)–(1.47) *depend only on the order of the associated function* k *and the potential* $q(x)$.

We stress that all the estimates of Theorem 1.8 are sharp in λ; this is easily verified by taking the system of generalized exponentials as the system of root functions for the operator $\mathcal{L}$ with the potential $q(x) \equiv 0$.

Estimate (1.44) of the norm of the preceding associated function through the norm of the subsequent associated function is, in fact, the estimate of the norm of the right-hand side of an equation of a special form $\mathcal{L}\, u_k(x;\lambda) - \lambda u_k(x;\lambda) = -\widetilde{\mu} u_{k-1}(x;\lambda)$ through its solution. Therefore, such estimates are conventionally called *anti-a priori* bounds. For the first time an anti-a priori bound for the root function of a differential operator was proved by V. A. Il'in in [46].

1.7. Necessity of the conditions of the theorem on the unconditional basis property. In this subsection, we discuss the necessity of the first four conditions A of Theorem 1.7.[9]

The condition of uniform boundedness of the ranks of eigenfunctions is rather natural for the basis property of systems of root functions connected with ordinary differential operators. So, for example, this condition is necessary for the uniform minimality in $L_2(G)$ of the system of generalized exponentials $\{t^s \exp(i\mu_k t),\ s = 0, 1, \ldots, m_k\}_{k=1}^{\infty}$ without any structural restrictions on the set $\{\mu_k\} \subset \mathbb{C}$ [103]. The fact that there are infinitely many eigenfunctions of infinite rank in the basis of root functions $\mathfrak{U}$ is probably also not possible, as well as (by the Müntz theorem [3, p. 53]) the basis property in $L_2(0,1)$ of the system of powers $\{t^k\}_{k=0}^{\infty}$ is not possible (see [94]).

Let us present conditions that ensure the fulfillment of estimate (1.24) for the ranks of eigenfunctions of the Schrödinger operator (1.1).

Theorem 1.9. *Each of the following three conditions ensures the uniform boundedness of the ranks of eigenfunctions for the operator* $\mathcal{L}$:

(1) *the system* $\mathfrak{U}$ *is almost normalized and Bessel in* $L_2(G)$ [97];

(2) *the system* $\mathfrak{U}$ *forms an almost normalized basis of the space* $L_p(G)$, $1 < p < \infty$ [98];

(3) *the system* $\mathfrak{U}$ *is uniformly minimal in* $L_p(G)$, $1 < p < \infty$, *and the refined anti-a priori bound* (1.44) *holds for the root functions, namely, there exists a constant* $C_0 > 0$ *such that the estimate*

$$\|u_{k-1}(\cdot;\lambda)\|_p \leq C_0 k^{1/2-\varepsilon} \|u_k(\cdot;\lambda)\|_p \tag{1.48}$$

with certain $\varepsilon > 0$ [94, 95] *holds uniformly in* $\lambda \in \Lambda$ *and* $k = \overline{1, m(\lambda)}$.

The "sum of units" condition (1.25) characterizes the density of the distribution of eigenvalues on the complex plane, and, for the first time, it was proved in [60] for an arbitrary nonnegative extension of operator (1.1) having a complete system of eigenfunctions orthonormal in $L_2(G)$ (the potential $q(x)$ should belong to a certain space $L_p(G)$, $p > 1$). Later on, in [52] it was proved that condition (1.25) is necessary for the unconditional basis property in $L_2(G)$ of the system of root functions for operator

[9] In all the theorems of this subsection, it is assumed that the potential $q(x)$ of the operator $\mathcal{L}$ is Lebesgue integrable on G.

(1.1) with an arbitrary Lebesgue integrable potential $q(x)$ (under the Carleman condition (1.26) and the condition of uniform boundedness of the ranks (1.24)).

We note that if the Carleman condition (1.26) holds, then the "sum-of-units" condition is equivalent to the following estimate uniform in $z \in \mathbb{C}$:

$$\sum_{\lambda\in\Lambda:\,|\sqrt{\lambda}-z|\le 1} 1 \le \widetilde{B}_1. \tag{1.49}$$

First of all, for this estimate to be valid, it is necessary that there be no finite accumulation points for the set Λ.

Theorem 1.10. *Each of the following two conditions ensures the absence of finite accumulation points for the set of eigenvalues of the operator $\mathcal{L}$:*

(1) *the system $\mathfrak{U}$ forms a basis in $L_p(G)$, $1 < p < \infty$, and either the following estimate uniform in $\lambda \in \Lambda$ and $k = \overline{1, m(\lambda)}$ holds*:

$$\|u_{k-1}(\cdot;\lambda)\|_p \le C_0 \|u_k(\cdot;\lambda)\|_p \tag{1.50}$$

or the system $\mathfrak{U}$ is almost normalized in $L_p(G)$ [94, 98];

(2) *the system $\mathfrak{U}$ is uniformly minimal in $L_p(G)$, $1 < p < \infty$, and either the anti-a priori bound* (1.48) *holds or the ranks of the eigenvalues are uniformly bounded* [94, 95].

Obviously, condition (1.49) holds if the set $\{\mu = \sqrt{\lambda}, \lambda \in \Lambda\}$ is Hausdorff, i.e., $\inf\{|\mu' - \mu''| : (\mu')^2, (\mu'')^2 \in \Lambda, \mu' \ne \mu''\} > 0$. The Hausdorff condition arises in a natural way in studying the systems of exponentials [161]; however, for the systems of root functions corresponding to second-order differential operators and operators of higher order, it is not necessary to require the Hausdorff property of the set of eigenvalues.

For example, on $[0, 1]$, let us consider the system of functions

$$\mathfrak{U} = \{1\} \cup \{\cos 2\pi kx, \sin(2\pi k + \delta_k)x\}_{k=1}^{\infty},$$

in which $\delta_k > 0$ are numbers satisfying the condition $\sum_{k=1}^{\infty} \delta_k^2 < 3$. Then $\mathfrak{U}$ is minimal and quadratically close to a complete orthonormal system and hence forms a Riesz basis in $L_2(0, 1)$ (see [5]). However, the set $\{0\} \cup \{2\pi k, 2\pi k + \delta_k\}_{k=1}^{\infty}$ corresponding to $\mathfrak{U}$ in this case turns out to be Hausdorff, although all the eigenvalues are simple.

The following assertion holds for operator (1.1) with a Lebesgue integrable potential $q(x)$.

Theorem 1.11 ([104]). *If the system $\mathfrak{U}$ is uniformly minimal in $L_p(G)$, $1 \le p < \infty$, then the multiple Hausdorff property holds, i.e., there exists a number $\delta_0 > 0$ such that for any $z \in \mathbb{C}$,*

$$\sum_{\lambda\in\Lambda:\,|\sqrt{\lambda}-z|\le\delta_0} 1 \le 2. \tag{1.51}$$

Such a multiple Hausdorff property obviously ensures the fulfillment of the uniform estimate (1.49).

We note that the "sum of units" condition (1.25) is sharp in Theorem 1.7 in the following sense.

Theorem 1.12 ([99]). *If the system $\mathfrak{U}$ forms an almost orthonormal basis in $L_p(G)$, $1 < p < \infty$, then there exist constants $M_0, \nu_0 > 0$, such that in each rectangle $\{|\operatorname{Re}\sqrt{\lambda} - t| \le M_0, |\operatorname{Im}\sqrt{\lambda}| \le \nu_0\}$, $t \ge 0$, there is the number $\sqrt{\lambda}$ for at least one of the eigenvalues $\lambda \in \Lambda$.*

For the first time, condition (1.26) of the belonging of eigenvalues to a certain parabola appeared in the work [171] of Carleman in connection with the study of the spectral asymptotics of elliptic operators. The attempts undertaken to justify the necessity of this condition [10, 27] led to the following final result.

Theorem 1.13 ([91–93]). *Let the ranks of eigenvalues be uniformly bounded, and moreover, let the system dual to $\mathfrak{U}$ satisfy condition* (A5) *of Theorem* 1.7. *Then the condition of uniform minimality of the system $\mathfrak{U}$ in $L_p(G)$, $1 < p < \infty$, is sufficient for the fulfillment of the Carleman condition* (1.26).

Taking into account all the results presented above, we can give the following final form to the theorem on the unconditional basis property.

Theorem 1.14 ([93]). *Let the potential $q(x)$ be Lebesgue integrable on G. Let conditions* (A1) *and* (A5) *and condition* (A2) *of the uniform boundedness of the rank hold. Then each of the systems $\mathfrak{U}$ and $\mathfrak{V}$ forms an unconditional basis in $L_2(G)$ iff the uniform estimate* (1.28) *of the product of norms holds.*

Indeed, under the conditions of this theorem, the "sum-of-units" estimate (1.25) and the Carleman condition (1.26) (by Theorems 1.11 and 1.13) are consequences of the uniform minimality of the system $\mathfrak{U}$ in $L_2(G)$, which is equivalent to estimate (1.20) by Theorem 1.3.

The following assertion of principal character is also implied by the theorems of this subsection.

Theorem 1.15 ([93, 98]). *Let the potential $q(x)$ be Lebesgue integrable on G, and let condition* (A5) *hold. Moreover, let one of the following three conditions hold*:

(a) *the system $\mathfrak{U}$ is almost orthonormal in $L_2(G)$*;

(b) *the uniform a priori bound* (1.48) *holds*;

(c) *the ranks of eigenfunctions are uniformly bounded.*

Then if the system $\mathfrak{U}$ forms a basis of the space $L_2(G)$, then this basis is unconditional.

Theorem 1.15 shows that among systems of functions connected with second-order differential operators, it is not possible to construct examples of conditional bases in $L_2(G)$ that are analogs of those in [1, 4, 28].

Also, we note that by [29] any system normalized in $L_p(G)$ for $p > 1$, $p \neq 2$, and uniformly bounded is not an unconditional basis in $L_p(G)$. But by estimate (1.42) and the results of Theorem 1.13, any system of root functions $\mathfrak{U}$ almost normalized in $L_p(G)$ is uniformly bounded. Thus, if all the conditions of Theorem 1.15 hold, then the system $\mathfrak{U}$ can form only conditional bases in the spaces $L_p(G)$ for $p > 1$, $p \neq 2$.

1.8. Uniform equiconvergence of spectral expansions with the trigonometric Fourier series. The Schrödinger operator (1.1) with a Lebesgue integrable potential can be considered as a perturbation of operator (1.1) with $q(x) \equiv 0$ for which the classical trigonometric system is its system of eigenfunctions satisfying the periodic boundary conditions. Therefore, it is natural to compare the biorthogonal expansion in the system of root functions of the general operator (1.1) with the expansion into the trigonometric Fourier series from the viewpoint of their convergence.

Without loss of generality, we consider operator (1.1) on the interval $G = (0, 1)$ and assume that $q(x) \in L_1(0, 1)$. We claim that the system $\mathfrak{U}$ of root functions of the operator $\mathcal{L}$, being understood in the generalized sense, satisfies the following conditions[10] A_p:

(1) the system $\mathfrak{U}$ is closed and minimal in the space $L_p(G)$ for a certain $p \geq 1$;

(2) the ranks of the eigenfunctions are uniformly bounded (condition (1.24)) and the "sum-of-units" condition (1.25) and the Carleman condition (1.26) hold.

The second of these conditions allows us to enumerate the root functions of the system $\mathfrak{U}$ in nondecreasing order of $|\sqrt{\lambda}|$, where λ are eigenvalues from the set Λ considered, and the root functions in each of the chains so that after each associated function, we have the associated function of order that is one greater from the same chain or the eigenfunction of the next chain. For the root functions of the system enumerated in such a way, we use the notation $u_k(x)$, $k = 1, 2, \dots$, in which, in contrast to the notation $u_k(x; \lambda)$ used above, the subscript k indicates not the order of a given associated function but its serial number in the system $\mathfrak{U}$. The fact that two functions $u_k(x)$ and $u_j(x)$ belong to one and the same chain is denoted by the symbol $u_k \sim u_j$.

The first of the conditions A_p guarantees (by Theorem 1.2) the existence of a unique closed and minimal biorthogonal dual system $\mathfrak{V} = \{v_l(x)\}$ each of whose elements belongs to the space $L_s(G)$ for

[10]For $p = 2$, these conditions coincide with the first four conditions A of the theorem on the unconditional basis property.

$s = p/(p-1)$. In contrast to the conditions of Theorem 1.7 on the unconditional basis property, it is not assumed here that the functions $v_l(x)$ satisfy any differential equation (condition (A5)).

For an arbitrary complex-valued function $f \in L_p(G)$, we compose the nth partial sum of its biorthogonal series in the system $\mathfrak{U}$,

$$\sigma_n(x, f) = \sum_{k=1}^{n} (f, v_k) u_k(x), \tag{1.52}$$

and compare this sum with the following modified sum of the trigonometric Fourier series of the same function $f(x)$[11]:

$$S_\tau(x, f) = \frac{1}{\pi} \int_0^1 \frac{\sin \tau(x-y)}{x-y} f(y)\, dy, \tag{1.53}$$

of order $\tau = |\mu_n| = |\sqrt{\lambda_n}|$, where λ_n is the eigenvalue corresponding to the root function $u_n(x)$.

One says that the expansions of a function $f(x)$ into the biorthogonal series in the system $\mathfrak{U}$ and into the trigonometric Fourier series are *equiconvergent uniformly in any compact subset* of the interval G if

$$\lim_{n\to\infty} |\sigma_n(x, f) - S_{|\mu_n|}(x, f)| = 0 \tag{1.54}$$

uniformly in x on each compact subset K of the interval G.

The property of the uniform equiconvergence of the spectral expansion in the system of root functions $\mathfrak{U}$ with the trigonometric Fourier series on any compact subset means that the biorthogonal series behaves itself inside G as the usual trigonometric Fourier series; this allows us to apply many fairly fine results on the convergence of the usual Fourier series to the biorthogonal expansions considered.

Theorem 1.16 ([56])**.** *Let the potential $q(x)$ be Lebesgue integrable on G, and let two conditions A_p hold for a certain $p \geq 1$. Then the expansions of an arbitrary function $f \in L_p(G)$ into the biorthogonal series in the system $\mathfrak{U}$ and into the trigonometric Fourier series are equiconvergent on any compact subset of the interval G iff, for each compact subset $K \subset G$, there exists a constant $C(K) > 0$ such that the following inequality holds uniformly in $k \in \mathbb{N}$:*

$$\|u_k\|_{p,K} \cdot \|v_k\|_s \leq C(K), \tag{1.55}$$

where $s = p/(p-1)$.

In particular, all the conditions of Theorem 1.16 hold whenever the potential $q(x)$ is a Lebesgue integrable real-valued function on G and $\mathcal{L}$ is an arbitrary self-adjoint extension of operator (1.1). As follows from [25], this expansion is semibounded, and, therefore, the Carleman condition holds. In the self-adjoint case, $v_k(x) = \|u_k\|_2^{-1} u_k(x)$, and hence inequality (1.55) follows from estimates (1.42). The rank of any eigenfunction is equal to 1, and the necessity of the "sum-of-units" condition for an arbitrary biorthogonal system of eigenfunctions of operator (1.1) follows from [60] and [52]. Thus, the result of Theorem 1.16 completely exhausts the problem of equiconvergence for the self-adjoint Schrödinger operator (1.1) with a Lebesgue integrable potential.

The property of equiconvergence of the biorthogonal expansion with the trigonometric Fourier series is closely related to the so-called local basis property in L_p.

We say that the system $\mathfrak{U}$ has the *basis property in L_p on any compact set* if, for each function $f \in L_p(G)$ and each compact subset $K \subset G$, the following relation holds:

$$\lim_{n\to\infty} \|\sigma_n(\cdot, f) - f(\cdot)\|_{p,K} = 0. \tag{1.56}$$

[11]It is known [39, p. 94] that sum (1.53) differs from the usual partial sum of the trigonometric Fourier series of order $[2\pi\tau]$ by a summand that tends to zero as $\tau \to \infty$ uniformly in x on an arbitrary compact subset of the interval $(0, 1)$.

Relation (1.54) and the fact that the complete trigonometric system forms a basis in $L_p(G)$ for any $p > 1$ [6, p. 593] imply that the equiconvergence of the biorthogonal series with the trigonometric Fourier series uniform on any compact set for any function from the space $L_p(G)$ for $p > 1$ ensures the local basis property of the system $\mathfrak{U}$ in L_p.

Let us sketch a scheme for proving Theorem 1.16.

We prove that condition (1.55) is necessary for the system $\mathfrak{U}$ to have the basis property in L_p on any compact set if $p > 1$. By what was said above, this implies the necessity of condition (1.55) in Theorem 1.16 for $p > 1$.[12]

Assume the contrary: let there exist a compact set $K \subset G$ such that

$$\overline{\lim_{k\to\infty}} (\|u_k\|_{p,K} \cdot \|v_k\|_s) = \infty.$$

Then, by the resonant-type theorem [76, p. 104], there exists a function $f \in L_s(G)$ with support $\operatorname{supp} f \subseteq K$ such that

$$\overline{\lim_{k\to\infty}} (|(f, u_k)| \cdot \|v_k\|_s) = \infty. \tag{1.57}$$

On the other hand, for this function f and for any function $g \in L_p(G)$, by the identity $(\sum_{i=1}^k (f, u_i)v_i - f, g) = (f, \sum_{i=1}^k (g, v_i)u_i - g)$ and relation (1.56), we have $\lim\limits_{k\to\infty} (\sum_{i=1}^k (f, u_i)v_i - f, g) = 0$, which means the weak convergence in $L_s(G)$ of the sequence $\sum_{i=1}^k (f, u_i)v_i$ to f. Therefore, it is bounded on $L_s(G)$: $\| \sum_{i=1}^k (f, u_i)v_i\|_s \le C_0$, and hence $\|(f, u_k)v_k\|_s \le 2C_0$, which contradicts (1.57).

The proof of sufficiency of condition (1.55) in Theorem 1.16 develops the ideas founded in the monograph [55]; a detailed presentation of it is contained in [56] (see also [46, 159]). We only dwell on the principal aspects of the technique used for it.

First of all, we note that to prove equiconvergence (1.54), it suffices to prove that for any compact set $K \subset G$, there exists a constant $C_0(K) > 0$ such that for all functions $f \in L_p(G)$ and all $x \in K$,

$$|\sigma_n(x, f) - S_{|\mu_n|}(x, f)| \le C_0(K)\|f\|_p. \tag{1.58}$$

Indeed, the closedness of the system $\mathfrak{U}$ in $L_p(G)$ is ensured by the fact that for any $\varepsilon > 0$ given in advance, there exist constants α_i, $i = \overline{1, N_0}$, such that the function $T(x) = \sum_{i=1}^{N_0} \alpha_i u_i(x)$ absolutely continuously differentiable on G satisfies the inequality

$$\|f - T\|_p < \varepsilon. \tag{1.59}$$

We note that $\sigma_n(x, T) = T(x)$ for $n \ge N_0$, and by the linearity of the partial sums, we have

$$\sigma_n(x, f) - S_{|\mu_n|}(x, f) = \left[\sigma_n(x, f - T) - S_{|\mu_n|}(x, f - T)\right] - \left[S_{|\mu_n|}(x, T) - T(x)\right].$$

Owing to the choice of a sufficiently large serial number n, the module of the second square bracket can be made less than ε for $x \in K$, and the module of the first square bracket (by (1.58)) does not exceed $C_0(K)\|f - T\|_p < C_0(K)\varepsilon$. Thus, for a sufficiently large serial number n, the inequality

$$|\sigma_n(x, f) - S_{|\mu_n|}(x, f)| < (1 + C_0(K))\varepsilon$$

holds uniformly in $x \in K$; this implies relation (1.54).

In turn, estimate (1.58) is implied by the following inequality, which holds uniformly in $x \in K$:

$$\left\| \sum_{k=1}^n u_k(x)\overline{v_k(y)} - [\pi(x - y)]^{-1} \sin[|\mu_n|(x - y)] \right\|_{L_s(y\in G)} \le C_0(K). \tag{1.60}$$

Inequality (1.60) yields an estimate for the sum $\sum_{k=1}^n u_k(x)\overline{v_k(y)}$, which is the spectral function of the operator $\mathcal{L}$ by (1.52).

[12] The necessity of condition (1.55) for $p = 1$ is proved in [51].

Let us fix an arbitrary compact set $K \subset G$ and an arbitrary $R_0 < \operatorname{dist}(K, \partial G)/2$. Consider an arbitrary number $R \in [R_0/2, R_0]$ and denote by $\theta(r, |\mu_n|, R)$ the function which is equal to $(\pi r)^{-1} \sin|\mu_n| r$ for $r < R$ and vanishes for $r \geq R$. Let $S_{R_0}\varphi(R)$ denote the operation of averaging a function $\varphi(R)$ on the closed interval $R_0/2 \leq R \leq R_0$, which is carried out by the formula $S_{R_0}\varphi(R) = (3R_0^2/8)^{-1} \int_{R_0/2}^{R_0} R\varphi(R)\, dR$.

We set $\widehat{\theta}(x-y, |\mu_n|) = S_{R_0}\theta(x-y, |\mu_n|, R)$. Obviously, for all $x \in K$ and $y \in G$, the following inequality holds:

$$\left|\widehat{\theta}(x-y, |\mu_n|) - [\pi(x-y)]^{-1} \sin[|\mu_n|(x-y)]\right| \leq C_1(K).$$

Therefore, to prove (1.60), it suffices to justify the following estimate uniform in $x \in K$:

$$\left\| \sum_{k=1}^{n} u_k(x)\overline{v_k(y)} - \widehat{\theta}(x-y, |\mu_n|) \right\|_{L_s(y \in G)} \leq C_2(K). \tag{1.61}$$

Let us consider the expression under the sign of norm in this estimate as a function $\Phi(y)$ of the variable y. Then the coefficient Φ_k of the function $\Phi(y)$ in its biorthogonal expansion in the system $\overline{\mathfrak{V}}$ has the form

$$\Phi_k = \int_G \Phi(y) u_k(y)\, dy = \begin{cases} -\widehat{\theta}_k(x, |\mu_n|) + u_k(x) & \text{if } k \leq n, \\ -\widehat{\theta}_k(x, |\mu_n|) & \text{if } k > n, \end{cases} \tag{1.62}$$

where $\widehat{\theta}_k(x, |\mu_n|) = \int_G \widehat{\theta}(x-y, |\mu_n|) u_k(y)\, dy$. Obviously, $\widehat{\theta}_k(x, |\mu_n|) = S_{R_0}\theta_k(x, |\mu_n|, R)$, where $\theta_k(x, |\mu_n|, R)$ is the coefficient of $\theta(x-y, |\mu_n|, R)$ as a function of the variable y in its biorthogonal expansion in the system $\mathfrak{V}$. We transform this coefficient into

$$\theta_k(x, |\mu_n|, R) = \pi^{-1} \int_0^R r^{-1} \sin(|\mu_n| r)[u_k(x+r) + u_k(x-r)]\, dr,$$

and for the expression in the square brackets under the integral sign, we use the mean-value formula (1.12). As a result, we obtain the relation

$$\theta_k(x, |\mu_n|, R) = 2\pi^{-1} u_k(x) \int_0^R \frac{\sin|\mu_n| r \, \cos \mu_k r}{r}\, dr + I_k(x, |\mu_n|, R),$$

in which $I_k(x, |\mu_n|, R)$ is obviously connected with the integral summand on the right-hand side of (1.12). We have from this relation that

$$\widehat{\theta}_k(x, |\mu_n|) = u_k(x)\, S_{R_0} \left\{ \frac{2}{\pi} \int_0^R \frac{\sin|\mu_n| r \, \cos \mu_k r}{r}\, dr \right\} + S_{R_0} I_k(x, |\mu_n|, R). \tag{1.63}$$

For averaging in the first summand of the right-hand side of this relation, we have proved the representation

$$S_{R_0} \left\{ \frac{2}{\pi} \int_0^R \frac{\sin|\mu_n| r \, \cos \mu_k r}{r}\, dr \right\} = \delta_k(|\mu_n|) + \overset{0}{I}_k(|\mu_n|), \tag{1.64}$$

in which the so-called discontinuous Dirichlet multiplier $\delta_k(|\mu_n|)$ is calculated by the formulas

$$\delta_k(|\mu_n|) = \begin{cases} 1 & \text{for } |\mu_k| < |\mu_n|, \\ 1/2 & \text{for } |\mu_k| = |\mu_n|, \\ 0 & \text{for } |\mu_k| > |\mu_n|, \end{cases} \tag{1.65}$$

and the summand $\overset{0}{I}_k(|\mu_n|)$ satisfies the estimate

$$\overset{0}{I}_k(|\mu_n|) = O\left(\min\left\{1, |\,|\mu_k| - |\mu_n|\,|^{-1}\right\}\right). \tag{1.66}$$

Taking (1.63)–(1.65) into account in the representation of the coefficient Φ_k in (1.62), we obtain

$$\Phi_k = \begin{cases} \widehat{I}_k(x, |\mu_n|) & \text{for } |\mu_k| < |\mu_n|, \\ 0.5u_k(x) + \widehat{I}_k(x, |\mu_n|) & \text{for } |\mu_k| = |\mu_n|, \\ \widehat{I}_k(x, |\mu_n|) & \text{for } |\mu_k| > |\mu_n|, \end{cases} \tag{1.67}$$

where $\widehat{I}_k(x, |\mu_n|) = -\overset{0}{I}_k(|\mu_n|) + S_{R_0} I_k(x, |\mu_n|, R)$.

We now consider the biorthogonal series $\sum_{k=1}^{\infty} \Phi_k \overline{v_k(y)}$ of the function $\Phi(y)$ in the system $\mathfrak{V}$. If we prove that this series converges in the metric of $L_s(G)$, then by the closedness of the system $\mathfrak{U}$ in $L_p(G)$, which is equivalent to the completeness of the system $\mathfrak{V}$ in $L_s(G)$, this series converges precisely to the function $\Phi(y)$. In fact, we can prove a more general property: the following estimate holds uniformly in $x \in K$:

$$\sum_{k=1}^{\infty} |\Phi_k| \cdot \|v_k\|_s = O(1). \tag{1.68}$$

In particular, this estimate implies that the relation $\|\Phi(y)\|_s = O(1)$ holds uniformly in $x \in K$, which is equivalent to the required estimate (1.61).

It follows from relations (1.67) that

$$\sum_{k=1}^{\infty} \Phi_k \overline{v_k(y)} = 0.5 \sum_{k\colon |\mu_k| = |\mu_n|} u_k(x)\overline{v_k(y)} + \sum_{k=1}^{\infty} \widehat{I}_k(x, |\mu_n|)\overline{v_k(y)}.$$

The estimate

$$\sum_{k\colon |\mu_k| = |\mu_n|} |u_k(x)| \|v_k\|_s = O(1)$$

uniform in $x \in K$ is directly implied by estimate (1.42) and the second condition of A_p. The proof of the estimate

$$\sum_{k=1}^{\infty} \left|\widehat{I}_k(x, |\mu_n|)\right| \, \|v_k\|_s = O(1)$$

uniform on the compact set K requires the refinement of the integral summands in the mean-value formula (1.12) (see [71]) and a considerable analytical calculation (see [56] for more details).

Remark 1. It is necessary to call attention to the fact that in contrast to Theorems 1.7 and 1.14 on the unconditional basis property in $L_2(G)$, in Theorem 1.16 we do not assume that the functions composing the biorthogonally dual system $\mathfrak{V}$ satisfy the differential equations related to the differential operator $\mathcal{L}^*$. Therefore, as the system $\mathfrak{U}$ in Theorem 1.16, we can consider the system of generalized exponentials $\{x^l \exp(i\mu_k x),\ l = \overline{0, m_k}, k \in \mathbb{N}\}$ or nonorthogonal systems of sines and cosines. All these systems are composed of the regular solutions of Eqs. (1.23) in the case where $q(x) \equiv 0$ [48]. In particular, the system $\mathfrak{U}$ can be composed of infinite subsets of root functions for distinct boundary-value problems [72] (see also [30, 31]).

Remark 2. In [26, 173, 179], a technique allowing one to prove the equiconvergence of the spectral expansion with the trigonometric Fourier series uniform in any compact set was elaborated; it does not require the Carleman condition among the conditions A_p. However, as follows from Theorem 1.13, the Carleman condition (1.26) is necessary for the uniform minimality of the system $\mathfrak{U}$ in $L_p(G)$, $1 < p < \infty$. Since it was additionally required in [173, 179] that the system $\mathfrak{U}$ form a Riesz basis in $L_2(G)$, the results of these works are merely contained in Theorem 1.16. Moreover, condition (1.55) is close to condition

(1.18), which means the uniform minimality of $\mathfrak{U}$ in $L_p(G)$. Therefore, the Carleman condition is probably necessary for the equiconvergence considered, and, therefore, the technique elaborated in the mentioned works does not allow us to obtain slightly more than that in Theorem 1.16.

1.9. Essential non-self-adjointness. It follows from Theorems 1.7 and 1.16 presented above that in proving the basis property, as well as the equiconvergence of the spectral expansion in the root system of the Schrödinger operator with the trigonometric Fourier series, a key role is played by the upper bound for the product of norms of the corresponding functions of the direct and dual systems: this is estimate (1.28) for the unconditional basis property in $L_2(G)$, and estimate (1.55) for the equiconvergence on any compact set and the basis property in L_p on any compact set.

If the systems $\mathfrak{U}$ and $\mathfrak{V}$ contain only finitely many associated functions (i.e., eigenfunctions such that at least one associated function corresponds to them or there are no associated functions at all or there are finitely many such functions), then the verification of the boundedness of the product of norms of the corresponding root functions reduces to the verification of the product of norms of only eigenfunctions starting from a certain moment. In this case, an answer to the question whether or not these products are bounded in totality is independent of the choice of associated functions. Namely such systems of root functions arise if we add the so-called strengthened regular boundary conditions (see [151, p. 71] and further [96, 145]) to the differential expression (1.1) or if we require the spectrality of the operator [33].

If, as in the example of Subsection 1.1, the total number of associated functions is infinite, then each chain of root functions $\overset{0}{u}, \overset{1}{u}, \dots, \overset{m}{u}$ can be replaced by a new chain

$$\overset{k}{\widetilde{u}} = \overset{k}{u} + \sum_{j=1}^{k} A_j \overset{k-j}{u}, \quad k = \overline{0, m},$$

where $A_1, \dots, A_k$ are perfectly arbitrary complex constants. Since under such a transformation, infinitely many functions are changed in the system, this can lead to a principal change in the functional properties of the system, as takes place in the Samarskii–Ionkin problem.

Indeed, by Theorems 1.7 and 1.16, the system $\{u_k(x)\}_{k=0}^{\infty}$ of root functions of this problem constructed in Subsection 1.1 forms an unconditional basis in $L_2(0, 1)$ and has the basis property in L_p $(p \geq 1)$ on any compact subset of the interval $(0, 1)$, and the biorthogonal expansion of any function from the class $L_p(0, 1)$ $(p \geq 1)$ in this system is equiconvergent with the trigonometric Fourier series uniformly on any compact set. At the same time, the modified system $\{\widetilde{u}_k(x)\}_{k=0}^{\infty}$ has none of the listed properties; moreover, the latter holds for any $p \geq 1$.

Therefore, in such problems, it is not possible to reveal uniquely the indicated properties of the system of root functions by using only the concrete form of boundary conditions. Precisely owing to this, in the case of the regular but not strengthened regular boundary conditions, without revealing the asymptotic behavior of the eigenfunctions and associated functions being chosen, one succeeded only in proving results on the Riesz basis property with brackets [170] or on the equiconvergence (1.54) but by using only a certain sequence of subscripts n [155, 162, 169, 189, 190].

To stress the specific character of operators for which the total number of associated functions is infinite, we call them *essentially non-self-adjoint.*

In choosing associated functions for essentially non-self-adjoint operators, the so-called reduced system is one of the orienting factors [41].

For an arbitrary eigenvalue λ_k, let us consider the corresponding chain of eigenfunctions and associated functions $\overset{0}{u}_k, \overset{1}{u}_k, \dots, \overset{m_k}{u}_k$. If the whole system of root functions is complete and minimal in $L_2(G)$, then there exists a unique biorthogonal system dual to it. Denote by $\overset{0}{v}_k, \overset{1}{v}_k, \dots, \overset{m_k}{v}_k$ the part of this system that corresponds to the chain considered, i.e., $(\overset{i}{u}_k, \overset{j}{v}_k) = \delta_{ij}$, $i, j = \overline{0, m_k}$. We orthonormalize these functions via the Hilbert–Schmidt method starting from the function $\overset{m_k}{v}_k$, and then denote by $\overset{0}{V}_k, \overset{1}{V}_k, \dots, \overset{m_k}{V}_k$ the obtained system. Then, correspondingly, the chain of root functions biorthogonal to it changes: $\overset{0}{U}_k, \overset{1}{U}_k, \ldots, \overset{m_k}{U}_k$. We note that new root functions remain in the same space, but, in contrast to the initial chain, they do not form now a new chain (in the sense of relations (1.9) and (1.10)) in general.

A system in which the root functions of each chain were changed in such a way is called a *reduced* system. Obviously, any reduced system remains complete and minimal in $L_2(G)$.

The following assertion[13] holds.

Theorem 1.17 ([41, 56]). *Let the potential $q(x)$ be Lebesgue integrable on G, and let condition A_2 hold. Then the system of root functions of the Schrödinger operator $\mathcal{L}$ has the basis property in L_2 on any compact set for at least one choice of associated functions iff the reduced system has this property, i.e., it is necessary and sufficient that for any compact subset $K \subset G$, the following inequality hold uniformly in $i = \overline{0, m_k}$ and $k \in \mathbb{N}$:*

$$\|\overset{i}{U}_k\|_{2,K} \|\overset{i}{V}_k\|_2 \le C(K).$$

We indicate one more peculiarity of essentially non-self-adjoint operators.

Let us consider the Samarskii–Ionkin problem (1.2) for operator (1.1) with an arbitrary absolutely continuous potential $q(x)$. As follows from [137], this problem has the following properties:

(1) if $q(x) \equiv 0$, then the rank of each eigenfunction is equal to 2, and the system of root functions can be chosen so that it forms an unconditional basis in $L_2(0,1)$;

(2) if $q(0) \ne q(1)$, then all the eigenvalues, probably except for finitely many of them, are simple, and the product of the L_2 norms of the eigenfunctions grows as the module of the square root of the eigenvalue; therefore, the system of root function does not form a basis in $L_2(0,1)$ for any choice of the associated functions. We note that the condition $q(0) \ne q(1)$ is satisfied, for example, by the potential $q(x) = \varepsilon x^n/n!$, and, moreover, $\max\limits_{0\le x\le 1} |q^{(k)}(x)| \le \varepsilon$ for all $k = \overline{0, n}$.

This example shows that for essentially non-self-adjoint problems, even arbitrarily small changes of the coefficients of the operator in the metric of the space $C^{(n)}[0,1]$ (under preservation of the form of the boundary conditions) can lead to a change in basis properties of the system of root functions for the operator considered.[14]

An analogous instability of properties of the system of root functions is observed with respect to boundary conditions of the problem.

Consider the problem

$$\begin{cases} -u'' = \lambda u, \quad 0 < x < 1, \\ u(0) = 0, \qquad u(1) = -2u(1/2) + \varepsilon \int\limits_0^1 u(x)\, dx. \end{cases} \tag{1.69}$$

If $\varepsilon = 0$, then the numbers $\lambda_n = (2\pi n)^2$, $n \in \mathbb{N}$, are its eigenvalues. Each eigenvalue λ_{2k} is simple, and one eigenfunction and two associated functions

$$\overset{0}{u}_{2k+1}(x) = \sin(4k+2)\pi x, \quad \overset{1}{u}_{2k+1}(x) = -0.5x\cos(4k+2)\pi x,$$
$$\overset{2}{u}_{2k+1}(x) = -0.5x^2 \sin(4k+2)\pi x - \frac{x\cos(4k+2)\pi x}{2(4k+2)\pi} \tag{1.70}$$

correspond to the eigenvalues λ_{2k+1}.

This system of root functions satisfies all the conditions of Theorem 1.7 (on the unconditional basis property) and Theorem 1.16 (on the equiconvergence).

[13] In [41], the concept of reduced system and Theorem 1.17 are considered for ordinary differential operators of arbitrary order with sufficiently smooth coefficients.

[14] For the first time, this phenomenon was discovered by V. A. Il'in in [58], and the corresponding example was constructed for a second-order differential operator of general form.

If ε is any arbitrarily small number different from zero, then the picture changes. The eigenvalues $\lambda_{2k} = (4\pi k)^2$ are also simple (the eigenfunction $\overset{0}{u}_{2k+1}(x) = \sin 4k\pi x$). Now one eigenfunction and one associated function (the functions $\overset{0}{u}_{2k+1}(x)$ and $\overset{1}{u}_{2k+1}(x)$ from relations (1.70)) correspond to the eigenvalues $\lambda'_{2k+1} = ((4k+2)\pi)^2$. Also, one more series of eigenvalues $\lambda''_{2k+1} = [(4k+2)\pi + O(k^{-1})]^2$ is added; one eigenfunction

$$\overset{0}{u}{}^{*}_{2k+1}(x) = \sin(4k+2)\pi x + O(k^{-1}) \tag{1.71}$$

corresponds to each of them.

It follows from (1.70) and (1.71) that

$$\left\| \|\overset{0}{u}{}^{*}_{2k+1}\|_2^{-1} \overset{0}{u}{}^{*}_{2k+1}(x) - \|\overset{0}{u}_{2k+1}\|_2^{-1} \overset{0}{u}_{2k+1}(x) \right\|_2 \to 0$$

for $k \to \infty$; this shows the absence of the uniform minimality for the system of root functions in $L_2(0,1)$ (cf. (1.17)), and hence the absence of its basis property in $L_2(0,1)$.

2. Ordinary Differential Operators of General Form

The methodology for studying biorthogonal expansions elaborated for the Schrödinger operator (1.1), turns out to be also appropriate in many respects for the case where the differential expression generating a given operator has the general form ($n \geq 2$)

$$\mathcal{L}\, u = u^{(n)} + p_1(x)u^{(n-1)} + p_2(x)u^{(n-2)} + \cdots + p_n(x)u, \quad x \in G = (a,b). \tag{2.1}$$

The results obtained in this case showed that in this case, the first step of the study of spectral expansions consists in obtaining integral representations of solutions of differential equations with spectral parameters that are analogous to the mean-value formula (1.12) and "shift" formulas (1.11) and (1.13), and that here, one of the most important stages consists in proving estimates for root functions of the operator in various L_p spaces. To prove the unconditional basis property in $L_2(G)$ for the system of root functions, first of all, it is necessary to reveal the Bessel condition for this system, and to prove the equiconvergence with the trigonometric Fourier series uniform in any compact set, it is necessary to obtain an estimate of the spectral function of the operator analogous to estimate (1.60).

Of course, the arbitrary order n of operator (2.1) and the existence of the whole set of coefficients $p_m(x)$ give their own specificity to the results obtained; however, these results can undoubtedly be qualified as generalizations of the theorems presented in Sec. 1. The peculiarities that arise here are first of all related to the existence of the coefficient $p_1(x) \not\equiv 0$ of the $(n-1)$th derivative in (2.1) and the order of the operator $\mathcal{L}$, which is greater than 2.

2.1. Generalized concept of root functions. As for the Schrödinger operator, we introduce into consideration the root functions of the operator $\mathcal{L}$ as only regular solutions of a differential equation with a spectral parameter.

A *regular solution on* G of the equation

$$\mathcal{L}\, u = \lambda u + f, \tag{2.2}$$

where $\lambda \in \mathbb{C}$ and $f \in L_1(G)$, is an arbitrary function $u = u(x)$ such that it, together with its derivatives up to the order $n-1$ inclusively, is absolute continuous on any compact set in G and satisfies Eq. (2.2) almost everywhere on G.

An *eigenfunction* of the operator $\mathcal{L}$ given by the differential expression (2.1) is any nontrivial regular solution $u_0(x;\lambda)$ of the equation

$$\mathcal{L}\, u_0(x;\lambda) = \lambda u_0(x;\lambda) \tag{2.3}$$

that belongs to the space $L_2(G)$. Also, an eigenfunction $u_0(x;\lambda)$ will be called an associated function of zero order. The complex number λ in (2.3) is called an *eigenvalue* of the operator $\mathcal{L}$. Along with λ, in what follows, we will use the spectral parameter $\mu = \mu(\lambda)$ defined by the relation

$$\mu = \begin{cases} \left[(-1)^{n/2}\lambda\right]^{1/n} & \text{if } n \text{ is even,} \\ (i\lambda)^{1/n} & \text{if } n \text{ is odd and } \operatorname{Im}\lambda \geq 0, \\ (-i\lambda)^{1/n} & \text{if } n \text{ is odd and } \operatorname{Im}\lambda < 0, \end{cases}$$

where $[r\exp(i\phi)]^{1/n} = r^{1/n}\exp(i\phi/n)$ for $-\pi < \phi \leq \pi$.

If the associated function $u_{k-1}(x;\lambda)$ of order $k-1 \geq 0$ was already defined, then by the *associated function of order k* corresponding to the eigenfunction $u_0(x;\lambda)$ and the eigenvalue λ, we mean any regular solution $u_k(x;\lambda)$ of the equation

$$\mathcal{L}\,u_k(x;\lambda) = \lambda u_k(x;\lambda) + \widetilde{\mu}\,u_{k-1}(x;\lambda) \tag{2.4}$$

belonging to the space $L_2(G)$. Here,[15] $\widetilde{\mu} = 1$ for $|\lambda| \leq 1$ and $\widetilde{\mu} = \mu^{n-1}$ for $|\lambda| > 1$.

Let the eigenvalues of the operator $\mathcal{L}$ form a certain countable set Λ. We will consider only those systems of root functions (i.e., eigenfunctions and associated functions) of the operator $\mathcal{L}$ which, for each $\lambda \in \Lambda$, along with each associated function $u_k(x;\lambda)$ of kth order, contain the whole chain of root functions $u_0(x;\lambda)$, $u_1(x;\lambda), \ldots,$ $u_{k-1}(x;\lambda)$ of lesser orders. The maximum order of an associated function in the chain corresponding to an eigenvalue λ is denoted by $m(\lambda)$; the *rank* of the eigenfunction $u_0(x;\lambda)$ is the number $m(\lambda)+1$. If a chain is infinite, we set $m(\lambda) = \infty$.

2.2. Integral representations. The regularity of a solution of Eq. (2.2) is closely connected with the smoothness of the coefficients $p_j(x)$, $j = \overline{1,n}$, of expression (2.1) defining the operator $\mathcal{L}$.

In relation (2.2), we set $\lambda = z^n$, $z \in \mathbb{C}$, and introduce linearly independent solutions of the equation $u^{(n)} - u = 0$ by the relations

$$s_m(x) = \frac{1}{n}\sum_{k=1}^{n} \eta_k^m \exp(\eta_k x), \quad m = \overline{0, n-1}, \tag{2.5}$$

where $\eta_k = \exp(2\pi ik/n)$, $k = \overline{1,n}$. Then for any regular solution $u = u(x)$ of Eq. (2.2), using a direct integration by parts, we easily make sure that the following relation analogous to the "shift" formula (1.11) holds:

$$u(x \pm t) = \sum_{m=0}^{n-1} (\pm 1)^m \frac{u^{(m)}(x)}{z^m} s_m(zt)$$
$$-z^{1-n}\int_0^t \sum_{m=0}^{n-1} p_m(x\pm\tau)u^{(m)}(x\pm\tau)s_{n-1}(z(t-\tau))\,d\tau + z^{1-n}\int_0^t f(x\pm t)s_{n-1}(z(t-\tau))\,d\tau. \tag{2.6}$$

Here, $x \in G$ and $t > 0$ are such that $x \pm t \in G$. Solving (2.6) as an integro-differential equation in the unknown function $u(x \pm t)$ of the variable t via the successive approximation method, we obtain (for a sufficiently small t) the following "explicit" analog of formula (2.6):[16]

$$u(x \pm t) = \sum_{m=0}^{n-1} (\pm 1)^m \frac{u^{(m)}(x)}{z^m}\left[s_m(zt) + \widetilde{s}_m(t,x,z)\right] + F(t,x,z), \tag{2.7}$$

where $\widetilde{s}_m(t,x,z)$ and $F(t,x,z)$ depend on t, x, and z, and the coefficients of the operator $\mathcal{L}$ and $F(t,x,z)$ depend additionally on the function f in the right-hand side of (2.2).

[15]In some works, in Eq. (2.4) defining an associated function, there is no multiplier $\widetilde{\mu}$. As for the Schrödinger operator, this leads in essence to only a change in anti-a priori bounds of the root functions.

[16]See [101] for more details.

If all the coefficients $p_j(x)$, $j = \overline{1,n}$, are Lebesgue integrable on any compact subset of the interval G (i.e., are locally Lebesgue integrable on G), then we can show that the solution given by formula (2.7) is absolutely continuous strictly inside G, together with its derivatives up to the $(n-1)$th order inclusively (see, e.g., [89, 112, 178, 185]).

To ensure the belonging of all regular solutions of Eq. (2.2) to the class $L_2(G)$ and, therefore, to give the possibility of taking any regular solution of Eqs. (2.3) and (2.4) as a root function of the operator $\mathcal{L}$, it is necessary to refine the behavior of the coefficients of the operator at the ends of the interval G. For example, the following conditions allow us to consider a wide class of operators whose coefficients admit Lebesgue nonintegrable singularities at the points a and b.

Theorem 2.1 ([101])**.** *In expression* (2.1), *let each of the functions* $p_m(x)$, $m = \overline{1,n}$, *be complex-valued and satisfy the condition*

$$\int_a^b |p_m(x)|(x-a)^m(b-x)^m\,dx < \infty. \tag{2.8}$$

Then any regular solution of Eq. (2.2) *belongs to the class* $L_\infty(G)$.

Also, we note that if the coefficient $p_1(x)$ is sufficiently smooth ($p_1(x) \in W^{n-1}_{1,\mathrm{loc}}(G)$) and other coefficients are locally Lebesgue integrable on G, then passing to a new function

$$\widetilde{u}(x) = u(x)\exp\left(n^{-1}\int_0^x p_1(\xi)\,d\xi\right),$$

we obtain

$$\mathcal{L}\,u(x) = [\widetilde{\mathcal{L}}\,\widetilde{u}(x)]\exp\left(-n^{-1}\int_0^x p_1(\xi)\,d\xi\right),$$

and, moreover, the coefficients $\widetilde{p}_m(x)$ of the new operator $\widetilde{\mathcal{L}}$ are locally Lebesgue integrable on G and $\widetilde{p}_1(x) \equiv 0$.

Representations (2.6) and (2.7) are not convenient for studying biorthogonal expansions in root functions of the operator $\mathcal{L}$, since the coefficients of $u^{(m)}(x)$, $m = \overline{0, n-1}$, in these representations written for solutions of Eqs. (2.3) and (2.4) grow exponentially as $\operatorname{Re}\mu \to +\infty$ even under the Carleman condition

$$\sup_{\lambda\in\Lambda} |\operatorname{Im}\mu| < \infty. \tag{2.9}$$

The first modification of representation (2.6) that allows one to reject such a growth condition was proposed by E. I. Moiseev in [147]. Here, we present one of the one-sided analogs of the Moiseev mean-value formula obtained for the operator $\mathcal{L}$ of any even order n by V. D. Budaev.

Theorem 2.2 ([13, 17])**.** *Let the coefficients of operator* (2.1) *satisfy the conditions* $p_m(x) \in W_1^{n-m}(G)$, $m = \overline{1,n}$. *Moreover, let the Carleman condition* (2.9) *hold. Then we can find constants* $\rho_0 > 0$ *and* $C_0 > 0$ *such that for all* $x \in \overline{G}$, *for all* $\mu : \operatorname{Re}\mu \geq \rho_0$, *and for any* $r \in (0, C_0R_0)$, *where* $R_0 > 0$ *is such* *that $x \pm R_0 \in \overline{G}$, the following formula holds*:

$$u_k(x \pm r; \lambda) = \cos \mu r \left\{ D^{\pm}_{k1}(x) + \sum_{2 \le 2s \le k} A_{2s} r^{2s} D^{\pm}_{k-2s,1}(x) \right.$$
$$\left. + \sum_{2 \le 2s \le k+1} A_{2s-1} r^{2s-1} D^{\pm}_{k+1-2s,2}(x) + \sum_{s=1}^{k} \sum_{l=1}^{(n-2)/2} O(|\mu|^{-1}) D^{\pm}_{k-s,3,l}(x) \right\}$$
$$+ \sin \mu r \left\{ D^{\pm}_{k2}(x) + \sum_{2 \le 2s \le k} B_{2s} r^{2s} D^{\pm}_{k-2s,2}(x) \right.$$
$$\left. + \sum_{2 \le 2s \le k+1} B_{2s-1} r^{2s-1} D^{\pm}_{k+1-2s,1}(x) + \sum_{s=1}^{k} \sum_{l=1}^{(n-2)/2} O(|\mu|^{-1}) D^{\pm}_{k-s,3,l}(x) \right\}$$
$$+ \sum_{l=1}^{(n-2)/2} \exp\{i\mu\eta_l r\} \left\{ D^{\pm}_{k,3,l}(x) + \sum_{s=1}^{k} (A_{sl} r^s + O(|\mu|^{-1})) D^{\pm}_{k-s,3,l}(x) \right\}$$
$$+ O(|\mu|^{-1}) \left(\|u_k(\cdot; \lambda)\|_2 + \sum_{s=1}^{k} \|u_{k-s}(\cdot; \lambda)\|_2 \right), \tag{2.10}$$

in which the constants η_l are defined after relations (2.5), *A_k and B_k are nonzero constants, and $D^{\pm}_{k,j}$ ($j = 1, 2$) and $D^{\pm}_{k,3,l}$ ($l = 1, (n-2)/2$) are*[17] *linear combinations of values of the function $u_k(x; \lambda)$ and its derivatives up to order $n - 1$ inclusively at the point x.*

We note that the one-sided analog of the Moiseev mean-value formula for μ satisfying the condition $|\operatorname{Im} \mu| \ge \nu_0$, where $\nu_0 > 0$ is sufficiently large, is presented in [18].

In a recent work [123], it was shown that the Moiseev mean-value formula remains valid if the order n of the operator is even and all the coefficients $p_m(x)$, $m = \overline{1, n}$, are only Lebesgue integrable on G.

Another method for adopting "shift" formulas was proposed in [174,176,178]. Let us fix an arbitrary $x \in G$ and choose any positive $r \le \operatorname{dist}(x, \partial G)/n$. Further, we write, for example, formula (2.6) for the root function $u_k(x; \lambda)$ with the displacement $t = r, 2r, \dots, nr$, and express $u_k(x; \lambda)$ from these relations. Thus, we obtain a representation of the root functions of the form

$$u_k(x; \lambda) = \sum_{m=1}^{n} u_k(x \pm mr; \lambda) f^{\pm}_m(\mu r) + I(r, x; \mu), \tag{2.11}$$

where the coefficients $f^{\pm}_m(\mu r)$ are calculated via the functions $s_k(m\mu r)$ and the integral expression $I(r, x; \mu)$ depends on r, x, and μ, the coefficients of the operator $\mathcal{L}$, and the values of the functions $u_k(x; \lambda)$ and $u_{k-1}(x; \lambda)$ (for $k \ge 1$) on the closed interval between the points x and $x \pm nr$.

Other variants of representation (2.11) are presented in [101,106,177].

A slightly different approach to the obtaining of integral representations of regular solutions of Eq. (2.2) was proposed by N. B. Kerimov [90,94].

2.3. Estimates of root functions. In the most general case, the estimates of root functions of the operator $\mathcal{L}$ and their derivatives up to the order $n-1$ inclusively were obtained by N. B. Kerimov [94,95].

Theorem 2.3. *Let all the coefficients of the operator $\mathcal{L}$ be Lebesgue integrable on G. Then for all $1 \le p, s \le \infty$, and $j = \overline{0, n-1}$, we have*:

[17] All quantities entering formula (2.10) can be explicitly written.

(1) *the following estimates hold uniformly in* $\lambda \in \mathbb{C}$:

$$\|u_k^{(j)}(\cdot;\lambda)\|_\infty = O(1)(1+|\mu|)^{j+(1/p)}\|u_k(\cdot;\lambda)\|_p, \tag{2.12}$$

$$\|u_k^{(j)}(\cdot;\lambda)\|_p = O(1)(1+|\mu|)^{j}\|u_k(\cdot;\lambda)\|_p, \tag{2.13}$$

where the constants bounding $O(1)$ *depend on the order* k *of the root function considered*;

(2) *the following estimates hold uniformly in* $\lambda \in \mathbb{C}$ *and* $k \in \mathbb{N}$:

$$\begin{aligned}\|u_k^{(j)}(\cdot;\lambda)\|_\infty = O(1)(1+|\mu|)^{j+(1/p)}\Big\{&\|u_k(\cdot;\lambda)\|_p \\ &+ (1+|\mu|)^{-1+(1/s)-(1/p)}\|u_{k-1}(\cdot;\lambda)\|_s\Big\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\|u_k^{(j)}(\cdot;\lambda)\|_p = O(1)(1+|\mu|)^{j}\Big\{&\|u_k(\cdot;\lambda)\|_p \\ &+ (1+|\mu|)^{-1+(1/s)-(1/p)}\|u_{k-1}(\cdot;\lambda)\|_s\Big\}.\end{aligned} \tag{2.15}$$

Remark 1. For $n = 2$, estimates (2.12), (2.14), and (2.15) (as their comparison with estimates (1.42) and (1.44) of Theorem 1.8 shows) require some refinement. In this case, $(1+|\mu|)^{j+(1/p)}$ in (2.12) and (2.14) should be replaced by $(1+|\mu|)^j(1+|\operatorname{Im}\mu|)^{1/p}$, and the multiplier $(1+|\mu|)^{-1+(1/s)-(1/p)}$ in (2.14) and (2.15) should be replaced by $(1+|\operatorname{Im}\mu|)^{-1+(1/s)-(1/p)}$.

Remark 2. In contrast to the case $n = 2$, estimate (2.12) cannot be completed by a lower bound of the L_∞ norm of a root function through its L_p norm with preservation of the same order with respect to μ. As is shown in [174], even for eigenfunctions of the operator $\mathcal{L}\,u = u^{(n)}$, we have the following estimate for $n \geq 3$:

$$\|u_0(\cdot;\lambda)\|_\infty \geq C\left(1+\min_k|\operatorname{Re}(\mu\eta_k)|\right)^{1/p}\|u_0(\cdot;\lambda)\|_p, \tag{2.16}$$

which is sharp on the linear subspace of all eigenfunctions corresponding to a given λ (see also [183]).

Under the Carleman condition (2.9), estimates (2.12) and (2.13) were obtained in [89] (see also [182, 184]). The methodology for obtaining estimates (2.14) and (2.15) by using the preparatory estimate of the value $u_k^{(j)}(x;\lambda)$ through L_p norms of the root functions $u_k(\xi;\lambda)$ and $u_{k-1}(\xi;\lambda)$ on an arbitrary compact set containing x of length no more than $|\mu|^{-1}$ was elaborated by B. D. Budaev in [11] for the operator $\mathcal{L}$ of even order with smooth coefficients $p_m(x) \in W_1^{n-m}(G)$, $m = \overline{1,n}$.

Theorem 2.4. *Let all coefficients of the operator* $\mathcal{L}$ *be Lebesgue integrable on* G. *Then for all* $1 \leq p \leq \infty$, $j = \overline{0, n-1}$, *and* $k \geq 1$, *the following assertions hold*:

(1) *the following estimate holds uniformly in* $\lambda \in \mathbb{C}$ *for* $n \geq 3$:

$$\|u_{k-1}(\cdot;\lambda)\|_p = O(1)(1+|\mu|)\|u_k(\cdot;\lambda)\|_p, \tag{2.17}$$

and for $n = 2$, *we have the estimate*

$$\|u_{k-1}(\cdot;\lambda)\|_p = O(1)(1+|\operatorname{Im}\mu|)\|u_k(\cdot;\lambda)\|_p; \tag{2.17a}$$

(2) *for any compact sets* K_1 *and* K_2 *of the interval* G *satisfying the condition* $K_1 \subset \operatorname{int} K_2$, *the following estimates hold uniformly in* $\lambda \in \mathbb{C} : |\operatorname{Im}\mu| \leq c_0$:

$$\|u_{k-1}^{(j)}(\cdot;\lambda)\|_{p,K_1} = O(1)(1+|\mu|)^{j}\|u_k(\cdot;\lambda)\|_{p,K_2}, \tag{2.18}$$

$$\|u_{k-1}^{(j)}(\cdot;\lambda)\|_{\infty,K_1} = O(1)(1+|\mu|)^{j}\|u_k(\cdot;\lambda)\|_{p,K_2}, \tag{2.19}$$

$$\|u_{k}^{(j)}(\cdot;\lambda)\|_{\infty,K_1} = O(1)(1+|\mu|)^{j}\|u_k(\cdot;\lambda)\|_{p,K_2}. \tag{2.20}$$

As follows from this theorem, bounds of anti-a priori type depend on over which sets we take the norms in the left- and right-hand sides of these bounds. The comparison of estimates (2.17) and (2.18) shows that the second bound is better than the first one by $(1+|\mu|)$.

For the first time, the sharp-in-order estimates (2.18)–(2.20) were obtained in [46] for the operator $\mathcal{L}$ with smooth coefficients $p_m(x) \in C^{(n-m+1)}(G)$. The estimates of the same theorem for the operator $\mathcal{L}$ with only Lebesgue integrable coefficients (uniform in all $\lambda \in \mathbb{C}$ satisfying the Carleman condition) were entirely obtained by N. B. Kerimov [89] (see also [101, 112, 184, 185]). Some generalizations of estimates (2.18)–(2.20) to the case where the Carleman condition is violated are contained in [18].

2.4. Uniform equiconvergence of spectral expansions with the trigonometric Fourier series. Let us indicate conditions that ensure the equiconvergence of the biorthogonal expansion in root functions of the operator $\mathcal{L}$ of arbitrary order $n \geq 2$ and general form uniform on any compact set and also present a result on the basis property of the system of root functions in L_p on any compact set in G.

Let the coefficients of the operator $\mathcal{L}$ be smooth functions $p_m(x) \in C^{(n+1-m)}(G)$. We require that the system $\mathfrak{U}$ of root functions of the operator $\mathcal{L}$, being understood in the generalized sense, satisfy the following four conditions A_p:

(1) the system $\mathfrak{U}$ is closed and minimal in the space $L_p(G)$ for a certain $p \geq 1$;

(2) the ranks of the eigenfunctions of the system $\mathfrak{U}$ are uniformly bounded:

$$\sup_{\lambda \in \Lambda} m(\lambda) < \infty; \tag{2.21}$$

(3) the "sum-of-units" condition holds, i.e., the following estimate holds uniformly in $t \geq 0$:

$$\sum_{\lambda \in \Lambda:\, |\operatorname{Re} \sqrt[n]{\lambda} - t| \leq 1} 1 \leq B_1; \tag{2.22}$$

(4) the Carleman condition holds, i.e., the following inequality holds uniformly in $\lambda \in \Lambda$:

$$|\operatorname{Im} \sqrt[n]{\lambda}| \leq B_2. \tag{2.23}$$

Let us enumerate the root functions of the system $\mathfrak{U}$ in nondecreasing order of $|\sqrt[n]{\lambda}|$, $\lambda \in \Lambda$, and in each chain, do this in increasing order of orders of associated functions. For the functions of the system $\mathfrak{U}$ enumerated in such a way, we use the notation $\mathfrak{U} = \{u_k(x)\}_{k=1}^{\infty}$.[18]

The first of the conditions A_p guarantees the existence of a unique closed and minimal biorthogonal dual system $\mathfrak{V} = \{v_k(x)\}_{k=1}^{\infty}$ each of whose elements belongs to the class $L_s(G)$, $s = p/(p-1)$. The functions of the system $\mathfrak{V}$ may not satisfy any differential equation with a spectral parameter.

For an arbitrary complex-valued function $f \in L_p(G)$, we compose a partial sum of the biorthogonal series in the system $\mathfrak{U}$:

$$\sigma_t(x,f) = \sum_{1 \leq k \leq t} (f, v_k) u_k(x) \tag{2.24}$$

and a modified partial sum $S_t(x,f)$ of the trigonometric Fourier series by formula (1.53).

If, as $t \to \infty$,

$$\sigma_t(x,f) - \exp\left(-\frac{1}{n}\int_a^x p_1(\xi)\,d\xi\right) S_{|\mu_{[t]}|}\left(x, \exp\left(\frac{1}{n}\int_a^x p_1(\xi)\,d\xi\right) f(x)\right) \to 0 \tag{2.25}$$

uniformly in x on an arbitrary compact set K in G, then we say that the expansions of the function $f(x)$ into the biorthogonal series and into the trigonometric Fourier series are *equiconvergent on any compact set* in G.

[18] Here, as in the case of the Schrödinger operator (Subsection 1.8), the subscript k denotes not the order of an associated function but its serial number in the system $\mathfrak{U}$.

If, for each function $f \in L_p(G)$ and any compact set $K \subset G$,

$$\lim_{t\to\infty} \|\sigma_t(\cdot, f) - f(\cdot)\|_{p,K} = 0, \tag{2.26}$$

then, as in Sec. 1, we say that the system $\mathfrak{U}$ has the *basis property in* L_p *on any compact set* in G.

The following assertions hold.

Theorem 2.5 ([46,48])**.** *Let* $1 \le p < \infty$*, and let all conditions* A_p *hold. Then the expansions of an arbitrary function* $f \in L_p(G)$ *into the biorthogonal series in the system* $\mathfrak{U}$ *of root functions of the operator* $\mathcal{L}$ *and into the trigonometric Fourier series are equiconvergent uniformly on any compact set in* G *iff the uniform bound* (1.55) *of the product of norms holds.*

Theorem 2.6 ([46,48])**.** *Let* $1 < p < \infty$*, and let all conditions* A_p *hold. Then the system* $\mathfrak{U}$ *has the basis property in* L_p *on any compact set in* G *iff the uniform estimate* (1.55) *holds.*

If we do not take into account the technical conditions related to the application of the general mean-value formula of E. I. Moiseev, then the method for proving these theorems is completely analogous to that described in Sec. 1 for the Schrödinger operator. We note that for the operator $\mathcal{L}$ of general form, the central role of the proof is also the proof of estimate (1.60) for the spectral function of the operator $\mathcal{L}$.

Remark 1. The proof of Theorems 2.5 and 2.6 in [46] is based on the application of the Moiseev mean-value formula, and the restrictions on the smoothness of the coefficients of the operator are in essence related only to the conditions under which this formula was obtained in [147]. As follows from the result of [123], this formula also remains valid in the case of Lebesgue integrable coefficients of the operator. Thus, Theorems 2.5 and 2.6 remain valid for operators (2.1) of any even order n with coefficients $p_1(x) \in W_1^{n-1}(G)$, $p_m(x) \in L_1(G)$, $m = \overline{2, n}$.

Remark 2. The property of the equiconvergence (uniform on any compact set in G) of the biorthogonal expansion of an arbitrary function from the class $L_2(G)$ in the system $\mathfrak{U}$ of root functions of an operator of an arbitrary order $n \ge 2$ whose coefficients $p_m(x) \in W_{2,\text{loc}}^{n-m}(G)$, $m = \overline{2, n}$, and $p_1(x) \equiv 0$, with the expansion of the same function into the trigonometric Fourier series was proved in [180, 186] under the condition that $\mathfrak{U}$ forms a Riesz basis in $L_2(G)$ and the set Λ of eigenvalues in general satisfies no additional conditions.

2.5. Rate of equiconvergence. The problem of estimating the rate of equiconvergence of the spectral expansion in the system of root functions of the differential operator with expansion into the trigonometric Fourier series was studied in [62] for the first time, where for the difference $\sigma_m(x, f) - S_{|\mu_m|}(x, f)$, in which $f(x)$ is an absolutely continuous function and $\sigma_m(x, f)$ is a partial sum of the expansion of this function into Fourier series in the complete system of eigenfunctions of an arbitrary nonnegative self-adjoint expansion of the Schrödinger operator (1.1) with the potential $q(x) \in L_p(G)$, $p > 1$, there was obtained the estimate $O(|\mu_m|^{-1})$ uniform in any compact set. Since the latter summands of each of the sums being compared has the decay order $O(|\mu_m|^{-1})$ as $m \to \infty$, this estimate is sharp on the class of all absolutely continuous functions $f(x)$.

Later on, in [156], the mentioned estimate was extended to the non-self-adjoint Schrödinger operator (1.1) with the potential from the same class, and in the case where the potential is only Lebesgue integrable on G, the estimate $O(|\mu_m|^{-1} \ln |\mu_m|)$ was obtained in [152].

Since the root functions of a differential operator may not satisfied the periodicity conditions on G, of course, in the general case, there is no uniform equiconvergence of the biorthogonal expansion with the expansion into the trigonometric Fourier series on the whole G. However, it is interesting to study the equiconvergence on the whole closed interval G in the integral metric. For the self-adjoint Schrödinger operator, this was done in [110], and, moreover, the function being expanded is a function of bounded variation or a Lebesgue integrable function whose Fourier coefficients tend to zero at a certain rate.

Let us present, in more details, the results obtained in this direction for the general second-order operator $\mathcal{L}$,[19] i.e., for the operator

$$\mathcal{L}\, u = u'' + p_1(x)u' + p_2(x)u, \quad x \in G, \tag{2.27}$$

with the complex-valued coefficients $p_1(x)$ and $p_2(x)$.

Let the coefficients of operator (2.27) $p_1(x) \in L_s(G)$, $s \geq 1$, and $p_2(x) \in L_1(G)$. For a certain $r \geq 1$, let the system $\mathfrak{U}$ of root functions of this operator satisfy conditions A_r, and, moreover, let the following estimate hold uniformly in $k \in \mathbb{N}$:

$$\|u_k\|_r \cdot \|v_k\|_{r'} \leq C, \tag{2.28}$$

where $r' = r/(r-1)$ and $\mathfrak{V} = \{v_k\}$ is the system biorthogonally dual to $\mathfrak{U}$.

It follows from Theorems 2.5 and 2.6 that under these conditions, if the coefficient $p_1(x)$ is absolutely continuous on G, then the system $\mathfrak{U}$ has the basis property in L_r on any compact set in G and the biorthogonal series $\sum_{k=1}^{\infty}(f, v_k)u_k(x)$ in this system of any function f from the class $L_r(G)$ is equiconvergent with the trigonometric Fourier series uniformly on any compact set in G.

We say that the *coefficient condition* K_ν with a certain $\nu > 0$ holds for a function $f(x)$ if the following estimate holds for $|\lambda_k| \geq 1$:

$$\|v_k\|_{r'}^{-1}(f, v_k) = O(|\lambda_k|^{-\nu}), \tag{2.29}$$

and, moreover, the ordinary Fourier coefficients of the function $f(x)$ satisfy condition (2.29) with the same exponent ν.

The following theorem shows the role played by the coefficient condition (2.29) in studying the convergence of the spectral expansion in the metric $L_p(G)$, $p > 1$.

Theorem 2.7 ([121]). *Let $p > 1$ be arbitrary, and let $f(x)$ be any complex-valued function belonging to $L_r(G) \cap L_p(G)$. If conditions A_r hold, the uniform estimate* (2.28) *holds, and the coefficient condition K_ν, in which*

$$\nu > \nu_* \equiv \left[\min(2, p/(p-1), s)\right]^{-1}, \tag{2.30}$$

holds for $f(x)$, then the biorthogonal expansion of the function $f(x)$ in the system $\mathfrak{U}$ converges to $f(x)$ in the metric of $L_p(G)$.

Remark. Condition (2.30) is sharp; this is justified by the example from [148]. Let $s \geq p/(p-1)$, $p \geq 2$. Then $\nu_* = (p-1)/p$. Consider the sine system $u_k(x) = \sin \pi(k + (2p)^{-1})x$, $k = 0, 1, 2, \ldots$. This system consists of eigenfunctions of the operator $\mathcal{L}\, u = u''$ on the interval $G = (0, 1)$ corresponding to the eigenvalues $\mu_k^2 = [\pi(k + (2p)^{-1})]^2$. We note that $s = \infty$ here.

The indicated system satisfies conditions A_p and estimate (2.28) and, as was proved in [148], is closed and minimal in $L_p(G)$. Using the explicit form of the biorthogonally dual system, we can show that the following relation holds for $f(x) \equiv 1$:

$$0 < c_1 k^{-\nu_*} \leq \|v_k\|_{p/(p-1)}^{-1} |(f, v_k)| \leq c_2 k^{-\nu_*} < \infty,$$

i.e., the coefficient condition K_ν holds with $\nu = \nu_*$. At the same time, it was proved in [148] that the biorthogonal series of this function $f(x)$ converges to a function not belonging to the class $L_p(G)$, i.e., the sine system considered does not form a basis in $L_p(G)$.

In studying the rate of convergence of orthogonal series (for example, in the theory of trigonometric Fourier series), the coefficient condition K_ν is usually formulated in terms of belonging of the function being expanded to one class or another. In the case of nonorthogonal expansions, one should note the multiplier by the inner product in (2.29). A detailed consideration of the example from [148] shows that

[19] Analogous results on the convergence and the equiconvergence for the operators of any even order are contained in [124, 125]. A number of results on the rate of equiconvergence in various metrics on compact sets in the interval G are presented in [106].

even for infinitely differentiable functions $f(x)$, the exponent ν can assume any values on $(1 - r^{-1}, 1]$ depending on the choice of systems of the biorthogonal pair. Therefore, it is not possible to write the condition K_ν in terms of belonging of the function $f(x)$ to a certain class (see [122]).

Now let us present the most general theorem on the estimation of the rate of equiconvergence in the L_p metric on the whole interval G.

Theorem 2.8 ([121])**.** *Let $p \geq 1$ be arbitrary, and let conditions A_r and the uniform estimate* (2.28) *hold. Then for any function $f \in L_r(G)$ for which the coefficient condition K_ν holds and for all sufficiently large serial numbers m, we have the estimate*

$$\|\sigma_m(\cdot, f) - S_{|\mu_m|}(\cdot, f)\|_p = O(1) \begin{cases} |\lambda_m|^{-\min(1, \nu - \nu_*)} & \text{if } \nu \neq 1 + \nu_*, \\ |\lambda_m|^{-1} \ln^{\nu_*} |\lambda_m| & \text{if } \nu = 1 + \nu_*. \end{cases} \tag{2.31}$$

We note that the estimates of the equiconvergence on the whole G contained in Theorem 2.8 coincide with sharp estimates of the convergence of the trigonometric Fourier series (see also [119, 120]).

In the estimates of the rate of equiconvergence on compact sets, it is necessary to note the considerable role of the exponent s of the Lebesgue integrability $p_1(x)$ in the case where the norm is taken not over the whole G but only over a certain compact subset.

Theorem 2.9 ([122])**.** *Let $p \in [1, \infty)$ and $s > 1$ be arbitrary, and let conditions A_r and the uniform estimate* (2.28) *hold. Let $f \in L_r(G)$ be such that the coefficient condition K_ν holds. Then for any compact set $K \subset G$ and all sufficiently large serial numbers m, the following estimate holds*:

$$\|\sigma_m(\cdot, f) - S_{|\mu_m|}(\cdot, f)\|_{p,K} = O(1) \max\left\{|\lambda_m|^{-1}, |\lambda_m|^{-\nu} \ln^2 |\lambda_m|, |\lambda_m|^{-\nu-(1/p)+(1/s)} \ln |\lambda_m|\right\}, \tag{2.32}$$

and for $s = \infty$, the following estimate holds:

$$\|\sigma_m(\cdot, f) - S_{|\mu_m|}(\cdot, f)\|_{p,K} = O(1) \max\{|\lambda_m|^{-1}, |\lambda_m|^{-\nu} \ln |\lambda_m|\}. \tag{2.33}$$

For $s \in [p, \infty]$, in the right-hand side of (2.32), we can reject the third argument of the maximum, and, as in estimate (2.31), the obtained estimate (2.32) is independent of s. For $s < p$, the third argument of the maximum begins to dominate, and the rate of convergence considerably depends on s.

In [122], estimate (2.32) was generalized to the case $p = \infty$. Moreover, it was shown that one can modify estimates of the rate of equiconvergence in Theorem 2.9 if one of the coefficients $p_1(x)$ or $p_2(x)$ in (2.27) is absent or if the total number of associated functions in the system of root functions is finite.

We finally note that the elaborated technique for studying the convergence of spectral expansions can also be applied to the study of convergence of derivatives in x of the partial sums $\sigma_m(x, f)$ [107].

2.6. Unconditional basis property of a system of root functions. Even in studying systems of root functions of a fourth-order operator [88], a specific feature of results on the unconditional basis property of operators of order $n > 2$ was revealed. Along with the five conditions A (Sec. 1) that are usual for a second-order operator, one should require refined anti-a priori bounds for functions of the systems $\mathfrak{U}$ and $\mathfrak{V}$; among the necessary and sufficient conditions for the unconditional basis property, one needs a condition characterizing the behavior of L_∞ norms of root functions.

Let us consider the system $\mathfrak{U}$ of root functions $u_k(x; \lambda)$ understood in the generalized sense for which the eigenvalues λ belong to a certain countable set $\Lambda \subset \mathbb{C}$ and the order k varies from zero to the maximum order $m(\lambda)$ of a root function in the corresponding chain. The functions of the biorthogonally dual system $\mathfrak{V}$ will be denoted similarly by $v_k(x; \lambda)$. We combine relations (2.3) and (2.4) into one relation:

$$\mathcal{L}\, u_k(x; \lambda) = \lambda u_k(x; \lambda) + \operatorname{sign} k\, \widetilde{\mu}\, u_{k-1}(x; \lambda). \tag{2.34}$$

Conditions A for the operator $\mathcal{L}$ of an arbitrary order $n \geq 2$ are as follows:

(A1) the completeness and minimality in $L_2(G)$ of the system of root functions $\mathfrak{U}$ of the operator $\mathcal{L}$;

(A2) the uniform boundedness of ranks of eigenfunctions (2.21);

(A3) the "sum-of-units" condition (2.22);

(A4) the Carleman condition (2.23);

(A5) the fact that the biorthogonally dual system $\mathfrak{V}$ consists of root functions (being understood in the generalized sense) of the formally adjoint operator

$$\mathcal{L}^*v = (-1)^n v^{(n)} + (-1)^{n-1}(\overline{p_1(x)}v)^{(n-1)} + (-1)^{n-2}(\overline{p_2(x)}v)^{(n-2)} + \cdots + \overline{p_n(x)}v. \tag{2.35}$$

We see from the form of expression (2.35) for $\mathcal{L}^*$ that condition (A5) consists of certain smoothness requirements for the coefficients of the operator $\mathcal{L}$.

Theorem 2.10 ([88]). *Let the order n of operator (2.1) be equal to 4, and let the coefficients $p_m(x) \in W_1^{4-m}(G)$. Let the system $\mathfrak{U}$ of root functions of the operator $\mathcal{L}$ satisfy condition A, and, in addition, let the following anti-a priori bounds hold uniformly in $\lambda \in \Lambda$ and $k = \overline{1, m(\lambda)}$:*

$$\begin{aligned} \|u_{k-1}(\cdot;\lambda)\|_2 &= O(1)\|u_k(\cdot;\lambda)\|_2, \\ \|v_{k-1}(\cdot;\lambda)\|_2 &= O(1)\|v_k(\cdot;\lambda)\|_2. \end{aligned} \tag{2.36}$$

Then each of the systems $\mathfrak{U}$ and $\mathfrak{V}$ has the unconditional basis property in $L_2(G)$ if the following two conditions hold: there exists the uniform bound (1.28) of the product of norms and there exists a constant $\tau_0 > 0$ such that the following estimates hold uniformly in $\tau \geq \tau_0$:

$$\sum_{\tau_0 \leq \operatorname{Re}\mu \leq \tau} \sum_{k=0}^{m(\lambda)} \|u_k(\cdot;\lambda)\|_\infty^2 \cdot \|u_k(\cdot;\lambda)\|_2^{-2} = O(\tau), \tag{2.37}$$

$$\sum_{\tau_0 \leq \operatorname{Re}\mu \leq \tau} \sum_{k=0}^{m(\lambda)} \|v_k(\cdot;\lambda)\|_\infty^2 \cdot \|v_k(\cdot;\lambda)\|_2^{-2} = O(\tau). \tag{2.38}$$

We note that for a second-order operator, by estimate (1.44), under the condition of the uniform boundedness of ranks for eigenfunctions and the Carleman condition, the anti-a priori bounds (2.36) hold automatically, and estimate (1.42) implies that estimates (2.37) and (2.38) are consequences of the "sum-of-units" condition and the uniform boundedness of ranks. For operators of higher order, this is no longer true. In the right-hand side of the anti-a priori bound (2.17), there is a factor that grows as $|\mu| \to \infty$, and the "sum-of-units" condition and the uniform boundedness imply estimates of the left-hand sides of (2.37) and (2.38) only through $O(\tau^2)$.

In [13], the result of Theorem 2.10 is extended to the differential operator (2.1) of any even order with coefficients from the classes $p_m(x) \in W_1^{n-m}(G)$, $m = \overline{2, n}$, and with the coefficient $p_1(x) \equiv 0$.

It turns out that it is considerably more difficult to prove that the "sum-of-units" condition is necessary for the basis property of the system $\mathfrak{U}$.

The first result in this direction was obtained by V. D. Budaev [14,17]. For this purpose, he suggested dividing the whole system $\mathfrak{U}$ into three classes $\mathfrak{U}_1$, $\mathfrak{U}_2$, and U_3 depending on the asymptotics of root functions as $|\lambda| \to \infty$ and studying the root functions of each of the classes separately. The first class $\mathfrak{U}_1$ contains all those root functions for which the leading term of the asymptotics oscillates, or, which is the same, there exists a constant $C_1 > 0$ such that the estimate $\|u_k(\cdot;\lambda)\|_\infty \leq C_1\|u_k(\cdot;\lambda)\|_2$ holds. The second class $\mathfrak{U}_2$ contains those root functions which do not belong to the first class and are such that for a certain compact set $K \subset G$, the estimate $\|u_k(\cdot;\lambda)\|_{2,K} \geq C_2\|u_k(\cdot;\lambda)\|_2$ holds with the constant $C_2 > 0$ that is the same for all functions of the class $\mathfrak{U}_2$. Thus, the class $\mathfrak{U}_2$ contains all root functions such that the leading term of their asymptotics contains oscillating as well as exponential summands. And finally, the class $\mathfrak{U}_3$ consists of all other functions of the system $\mathfrak{U}$ (i.e., those root functions whose leading term of the asymptotics contains only exponential summands).

By using the suggested partition of the system $\mathfrak{U}$ into these classes, he has succeeded in proving the criterion for the unconditional basis property in the following form.

Theorem 2.11 ([15,19,21]). *Let the order n of operator* (2.1) *be even, let the coefficients $p_m(x) \in W_1^{n-m}(G)$, $m = \overline{2,n}$, and let $p_1(x) \equiv 0$. Let the system $\mathfrak{U}$ of root functions of the operator $\mathcal{L}$ be complete and minimal in $L_2(G)$, let the Carleman condition* (2.23) *and condition* (A5) *hold, and, moreover, let the functions of the system $\mathfrak{U}$ and the biorthogonally dual system $\mathfrak{V}$ satisfy the uniform anti-a priori bounds* (2.36)*. Then each of the systems $\mathfrak{U}$ and $\mathfrak{V}$ has the unconditional basis property in $L_2(G)$ iff*:

(1) *the "sum-of-units" condition* (2.22) *and the condition of the uniform boundedness of ranks* (2.21) *hold*;

(2) *the uniform estimate* (1.28) *of the product of norms holds*;

(3) *the uniform bounds* (2.37) *and* (2.38) *hold.*

For operators of odd order, Theorem 2.11 was proved in [117].

The role of the anti-a priori bounds (2.36) in this theorem was revealed in [22].

Theorem 2.12. *Let operator* (2.1) *satisfy all the conditions of Theorem* 2.11, *and let the system $\mathfrak{U}$ of its root functions be complete and minimal in $L_2(G)$. Moreover, let the ranks of eigenfunctions be uniformly bounded. Then it is possible to construct a new system $\widetilde{\mathfrak{U}}$ of root functions of the operator $\mathcal{L}$ such that*:

(1) *each function $\widetilde{u}_k(x;\lambda)$, $k = \overline{0, m(\lambda)}$, $\lambda \in \Lambda$, of this system is a linear combination of the root functions $u_0(x;\lambda)$, $u_1(x;\lambda), \dots, u_k(x;\lambda)$ of the system $\mathfrak{U}$*;

(2) *the functions of the system $\widetilde{\mathfrak{U}}$ and the system $\widetilde{\mathfrak{V}}$ biorthogonally dual to it satisfy anti-a priori bounds* (2.36);

(3) *the system $\widetilde{\mathfrak{U}}$ is complete in $L_2(G)$ whenever the system $\mathfrak{U}$ is complete in $L_2(G)$.*

Under the additional condition (1.28) *imposed on the product of norms in $L_2(G)$ of the corresponding functions of the systems $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathfrak{V}}$,*

(4) *the unconditional basis property of the system $\mathfrak{U}$ in $L_2(G)$ implies the unconditional basis property of the system $\widetilde{\mathfrak{U}}$ in $L_2(G)$.*

Therefore, it follows from Theorem 2.12 and the result of [21] that the requirement of the uniform boundedness of the ranks for eigenfunctions and the requirement of fulfillment of the anti-a priori bounds (2.36) are interchangeable in studying the unconditional basis property of the system of root functions of the operator $\mathcal{L}$.

The further study of the necessity of conditions A for the basis property of the system $\mathfrak{U}$ of root functions of the operator $\mathcal{L}$ was carried out by N. B. Kerimov in the case where the order $n \geq 3$ of the operator is arbitrary and the coefficients $p_m(x)$, $m = \overline{1,n}$, are only Lebesgue integrable on G.

Theorem 2.13 ([95]). *The finiteness of the rank $m(\lambda)$ of the eigenfunctions of the system $\mathfrak{U}$ ensures its uniform minimality in $L_p(G)$, $1 \leq p \leq \infty$, under the condition that the following anti-a priori bound holds uniformly in $k = \overline{1, m(\lambda)}$ and $\lambda \in \Lambda$*:

$$\|u_{k-1}(\cdot;\lambda)\|_p \leq C_0(\lambda) k^{(n/2)-\varepsilon} \|u_k(\cdot;\lambda)\|_p, \tag{2.39}$$

where $\varepsilon > 0$ is a certain constant and the constant $C_0(\lambda) > 0$ is independent of k.

Theorem 2.14 ([94,95]). *The uniform boundedness of the rank $m(\lambda)$ of eigenfunctions of the system $\mathfrak{U}$ is ensured by its uniform minimality in $L_p(G)$, $1 \leq p \leq \infty$, under the condition that the anti-a priori bound of the following form holds uniformly in $k = \overline{1, m(\lambda)}$ and $\lambda \in \Lambda$*:

$$\|u_{k-1}(\cdot;\lambda)\|_p \leq C_0 k^{(1/2)-\varepsilon} \|u_k(\cdot;\lambda)\|_p, \tag{2.40}$$

where the positive constants ε and C_0 are independent of k, as well as of λ.

Theorem 2.15 ([94,95]). *The absence of finite accumulation points of the set of eigenvalues Λ is ensured by any of the following two conditions*:

(1) *the system* $\mathfrak{U}$ *forms a basis in* $L_p(G)$, $1 < p < \infty$, *and, moreover, either the anti-a priori bound of the form below holds uniformly in* $\lambda \in \Lambda$ *and* $k = \overline{1, m(\lambda)}$:

$$\|u_{k-1}(\cdot;\lambda)\|_p \le C_0(\lambda)\|u_k(\cdot;\lambda)\|_p, \tag{2.41}$$

where the constant $C_0(\lambda) > 0$ *is independent of* k, *or the rank of the eigenfunctions of the system* $\mathfrak{U}$ *is finite*;

(2) *the system* $\mathfrak{U}$ *is uniformly minimal in* $L_p(G)$, $1 \le p \le \infty$, *and, moreover, the uniform anti-a priori bound* (2.39) *holds.*

The distribution of the eigenvalues on the complex plane in conditions A is characterized by the following two requirements: the "sum-of-units" condition (2.22) and the Carleman condition (2.23). As in the case of a second-order operator, under the Carleman condition, the "sum-of-units" condition is equivalent to the following estimate uniform in $z \in \mathbb{C}$:

$$\sum_{\lambda\in\Lambda:\,|\sqrt[n]{\lambda}-z|\le 1} 1 \le \widetilde{B}_1.$$

Theorem 2.16 ([94, 104])**.** *If the system* $\mathfrak{U}$ *is uniformly minimal in* $L_p(G)$, $1 \le p < \infty$, *then the multiple Hausdorff condition holds, i.e., there exists a number* $\delta_0 > 0$ *such that for any* $z \in \mathbb{C}$,

$$\sum_{\lambda\in\Lambda:\,|\sqrt[n]{\lambda}-z|\le \delta_0} 1 \le n. \tag{2.42}$$

Therefore, under the Carleman condition, the "sum-of-units" condition is a necessary condition not only for the unconditional basis property of the system $\mathfrak{U}$ in $L_2(G)$ but also for its uniform minimality in one of the spaces $L_p(G)$, $1 \le p < \infty$.

It should be noted that the problem on the necessity of the Carleman condition (2.23) for the unconditional basis property of the system of root functions of operator (2.1) of order $n > 2$ remains open.

2.7. Riesz means of spectral expansions. As follows from the theorems on the unconditional basis property (Theorems 1.7 and 2.10), condition (1.28) of the uniform boundedness for the product of L_2 norms of the corresponding functions of the biorthogonally dual systems $\mathfrak{U}$ and $\mathfrak{V}$ is a necessary and sufficient condition for the convergence of the biorthogonal series to the function being expanded in the metric of the space $L_2(G)$.

However, even for a second-order operator, it is possible to find a wide class of boundary-value problems for which the product of norms of the root functions grows when $|\lambda|$ grows.

Let the operator $\mathcal{L}$ be defined by the differential expression (1.1) with the potential $q(x) \in W_1^1(G)$ on the interval $G = (0, 1)$. We consider the general two-point boundary-value problem

$$\begin{cases} \mathcal{L}u = \mu^2 u, \\ A(u'(0)\ u'(1)\ u(0)\ u(1))^T = 0, \end{cases} \tag{2.43}$$

where

$$A = \begin{pmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{pmatrix},$$

with regular but not strengthened regular boundary conditions [151, pp. 71–73]. If we denote by A_{ij} the second-order minor of the matrix A at its ith and jth columns, then this is equivalent to the conditions $A_{12} = 0$ and $0 \ne A_{14} + A_{23} = \mp(A_{13} + A_{24})$. Let $\{u_k(x)\}$ be the system of root functions of problem (2.43). Its completeness and minimality in $L_2(G)$ [170] imply the existence of the system $\{v_k(x)\}$ biorthogonally dual to it.

Theorem 2.17 ([137]). *If the conditions $A_{14} \neq A_{23}$ and $2A_{34}^2 \neq (A_{13}+A_{24})(A_{14}-A_{23})(q(1)-q(0))$ hold, then all the eigenvalues μ_k^2, probably except for finitely many of them, are simple, the Carleman condition* (1.26) *holds, and*

$$c_1(|\mu_k|+1) \leq \|u_k\|_2 \, \|v_k\|_2 \leq c_2(|\mu_k|+1), \tag{2.44}$$

where $c_1, c_2 > 0$.

Now let $q(x) \equiv 0$. We complement the differential equation (1.1) by nonlocal Bitsadze–Samarskii boundary conditions[20]:

$$\begin{cases} -u'' = \mu^2 u, \quad 0 < x < 1, \\ u(0) = 0, \; u'(1) + \varepsilon u(1) = \alpha_0 u'(0) + \sum_{l=1}^m \alpha_l u'(\xi_l), \end{cases} \tag{2.45}$$

where ε, α_l, $l = \overline{0,m}$, are arbitrary complex numbers, $0 < \xi_1 < \cdots < \xi_m < 1$. The system of root functions $\{u_k(x)\}$ of this problem is also complete and minimal in $L_2(G)$ [170], there exists a biorthogonally dual system $\{v_k(x)\}$ for it, and the eigenvalues of problem (2.45) satisfy the Carleman condition.

Theorem 2.18 ([137,138]). (a) *If all the numbers ξ_l, $l = \overline{1,m}$, are rational, then the right inequality in* (2.44) *holds, and, moreover, $|\mu_k|$ cannot be replaced by $|\mu_k|^{1-\varepsilon}$, $\varepsilon > 0$, simultaneously for all boundary-value problems of the form* (2.45).

(b) *If at least one of the points ξ_l is irrational, then the product $\|u_k\|_2 \, \|v_k\|_2$ can grow on a certain sequence of serial numbers k more rapidly than a function of k given in advance.*

In such cases, to "improve" the convergence of biorthogonal series, one uses one or another summation method. It seems to be natural[21] to use for summing multiple trigonometric series the method suggested by S. Bochner, the summation via Riesz means. In [55, Chaps. 2 and 3], the method for studying Riesz means of spectral expansions corresponding to an arbitrary nonnegative self-adjoint expansion of the Laplace operator on $\mathbb{R}^N$, which was suggested by V. A. Il'in, is described. This method does not use the Carleman technique and Tauberian theorems. Its modification allows one to study the convergence of Riesz means of biorthogonal expansions corresponding to a non-self-adjoint operator $\mathcal{L}$ of order n of the general form in [74].

Let the coefficients in (2.1) belong to the class $p_m(x) \in C^{n+1-m}(G)$. Let the root functions of the operator $\mathcal{L}$ which are understood in the generalized sense (see (2.3) and (2.4)) be enumerated in nondecreasing order of $|\mu| = |\sqrt[n]{\lambda}|$, and in each of the chains, in increasing order of the orders of associated functions.

Following the classical definition, we introduce the *Riesz means of order $\alpha \geq 0$* of a partial sum of the biorthogonal series by the relation

$$\sigma_t^\alpha(x,f) = \sum_{1 \leq k \leq t} \left(1 - \frac{\mu_k^2}{|\mu_{[t]}|^2}\right)^\alpha (f, v_k) u_k(x). \tag{2.46}$$

We say that Riesz means *have the basis property in L_2 on any compact set* in G if, for any function $f \in L_2(G)$ and any compact set $K \subset G$, we have the relation

$$\lim_{t \to \infty} \|\sigma_t^\alpha(\cdot, f) - f(\cdot)\|_{2,K} = 0. \tag{2.47}$$

Let us introduce the following modified partial sum of Riesz means of order $\alpha \geq 0$ for the trigonometric Fourier series:

$$S_\tau^\alpha(x,f) = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{2\pi}} \tau^{(1/2)-\alpha} \int\limits_{|x-y| \leq R} |x-y|^{-\alpha-(1/2)} J_{\alpha+(1/2)}(\tau|x-y|) f(y) \, dy, \tag{2.48}$$

[20] The well-known work [9] called attention to such problems for elliptic operators.

[21] See the discussion of the summation methods via Riesz means below in Subsection 5.2.

where $R > 0$ is any number less than $\operatorname{dist}(x, \partial G)$. As for the modified partial sum (1.53) of the trigonometric Fourier series, $S_\tau^\alpha(x, f)$ differs from the ordinary partial Riesz sum of order α for the trigonometric Fourier series by the summand that tends to zero as $\tau \to \infty$ uniformly on any compact set of the interval G.

We say that the spectral expansion of a function $f(x)$ in the system of root functions $\{u_k(x)\}$ and the expansion of the same function into the trigonometric Fourier series *is equisummable via the Riesz method of order* α uniformly on any compact set in G if, for each compact set $K \subset G$,

$$\lim_{t\to\infty} |\sigma_t^\alpha(x, f) - S^\alpha_{|\mu_{[t]}|}(x, f)| = 0 \tag{2.49}$$

uniformly in $x \in K$.

In [74], under the assumption that the total number of associated functions in the system $\{u_k(x)\}$ is finite, the following assertion was proved.

Theorem 2.19. *Let $\alpha < 1$, and let the first four conditions A hold. Then, if for any compact set $K \subset G$, the estimate*

$$\|u_k\|_{2,K}\, \|v_k\|_2 \le C(K)(1 + |\mu_k|)^\alpha \tag{2.50}$$

holds uniformly in $k \in \mathbb{N}$, we have the following:

(a) *the Riesz means* (2.46) *of order α have the basis property in L_2 on any compact set in G*;

(b) *the spectral expansion of an arbitrary function $f \in L_2(G)$ in the system $\{u_k(x)\}$ and the expansion of the same function into the trigonometric Fourier series are equisummable via the Riesz method of order α uniformly on any compact set K in G, and, moreover, for any $\alpha' \in [\alpha, 1)$, the following estimate holds*:

$$\sup_{x\in K} |\sigma_t^{\alpha'}(x, f) - S^{\alpha'}_{|\mu_{[t]}|}(x, f)| = o(1)|\mu_{[t]}|^{\alpha-\alpha'}\|f\|_2. \tag{2.51}$$

In [130], for a one-dimensional Schrödinger operator with potential belonging to the Hölder class $C^\alpha(G)$, A. S. Makin has succeeded in omitting the restriction $\alpha < 1$ and proving the assertion of Theorem 2.19 in this case for any $\alpha \ge 0$.[22]

The omitting of the finiteness requirement of the total number of associated functions became possible only after works [156–158] of A. Sh. Salimov, in which a new definition of Riesz means especially adapted for the essentially non-self-adjoint case was introduced. For a one-dimensional Schrödinger operator, the basis property of Riesz means in L_2 on any compact set and the equisummability of Riesz means uniform on any compact set in the case of infinitely many associated functions were proved in [139].

It was revealed for the operators of higher order that this definition of Riesz means is also not appropriate. Let us present a modification of the classical definition suggested by A. I. Zuev in [40] for operator (2.1) of any order $n \ge 2$.

The *Riesz means of order* $\alpha \ge 0$ of a partial sum of the biorthogonal series for an arbitrary function $f \in L_2(G)$ are

$$\sigma_t^\alpha(x, f) = \sum_{k:\,|\operatorname{Re}\mu_k|\le|\operatorname{Re}\mu_{[t]}|} (f, v_k)\Big\{u_k(x)\Big[\Big(1 - \frac{(\operatorname{Re}\mu_k)^2}{(\operatorname{Re}\mu_{[t]})^2}\Big)^\alpha$$

$$-2i\alpha \operatorname{Im}\mu_k \frac{\operatorname{Re}\mu_k}{\operatorname{Re}\mu_{[t]}}\Big(1 - \frac{(\operatorname{Re}\mu_k)^2}{(\operatorname{Re}\mu_{[t]})^2}\Big)^{\alpha-1}\Big] + \sum_{\substack{1\le l<\alpha+1,\\ u_{k-l}\sim u_k}} \frac{(-\widehat{C})^l u_{k-l}(x)}{l!}\frac{d^l}{d(\operatorname{Re}\mu_k)^l}\Big(1 - \frac{(\operatorname{Re}\mu_k)^2}{(\operatorname{Re}\mu_{[t]})^2}\Big)^\alpha\Big\}, \tag{2.52}$$

where $u_{k-l} \sim u_k$ means that the root functions $u_{k-l}(x)$ and $u_k(x)$ belong to one and the same chain, and the constant $\widehat{C}$ depending on the order of the operator n is taken from the Moiseev mean-value formula [147].[23] We note that for real μ_k and the order n of the operator being equal to two, the expression in curly brackets on the right-hand side of (2.52) coincides with the definition of Riesz means given in [139].

[22]Estimate (2.51) was proved for any $\alpha' \in [\alpha, \alpha + 2)$ if additionally it is known that $q(x) \in C^{\alpha'}(G)$.

[23]For an even n, the constant $\widehat{C}$ is equal to $(-1)^{(n/2)+1}/n$.

For the Riesz means defined by relation (2.52), the following result was proved.

Theorem 2.20 ([40]). *Let all four conditions A hold. Moreover, let estimate* (2.50) *with constant* $\alpha \in [0,1)$ *hold uniformly in* $k \in \mathbb{N}$. *Then*:

(a) *the basis property in* L_2 *on any compact set in* G *holds for the Riesz means of order* α' *with*

$$\alpha' - [\alpha'] > \alpha, \quad \alpha' > M - 1, \tag{2.53}$$

where $M = \sup_{\lambda_k \in \Lambda} m(\lambda_k)$ *is the maximum rank of eigenfunctions in* $\{u_k(x)\}$, *and, moreover, the spectral expansion of an arbitrary function* $f \in L_2(G)$ *in the system* $\{u_k(x)\}$ *and the expansion of* $f(x)$ *into the trigonometric Fourier series are equisummable via the Riesz method of order* α' *uniformly on each compact set in* G; *moreover, estimate* (2.51) *in whose right-hand side we have the value* $o(1)|\mu_{[t]}|^{\alpha+[\alpha']-\alpha'}\|f\|_2$ *holds*;

(b) *if the set* $\{|\operatorname{Re}\mu_k| : \lambda_k \in \Lambda\}$ *is Hausdorff, then among the conditions imposed on* α', *the second condition in* (2.53) *can be omitted completely.*

We note that as in studying the equiconvergence, the central place in proving Theorems 2.19 and 2.20 is occupied by the obtaining of an estimate for Riesz means of the spectral functions of the biorthogonal expansion.

Despite the results mentioned here, the study of Riesz means of the spectral expansion in root functions of an ordinary differential operator of an arbitrary order is still not complete.

2.8. Abel–Poisson means of spectral expansions. In [126], another method for summing spectral expansions was proposed for the case where the condition of uniform boundedness of the product of norms (1.28) does not hold.

For the spectral expansion of a function $f \in L_2(G)$ in root functions of the Schrödinger operator (1.1), in this work, the following *modified Abel–Poisson means* are studied:

$$A_t(x,f) = \sum_{k=1}^{\infty} e^{-\lambda_k t}(f, v_k) \sum_{\substack{l \geq 0 \\ u_{k-l} \sim u_k}} (l!)^{-1} t^l u_{k-l}(x).$$

It was proved that if the potential $q(x)$ of the operator belongs to the class $C^{[2\alpha]+2}(G)$, then under the first four conditions A, when estimate (2.50) holds for any compact set $K \subset G$ and any function $f \in L_2(G)$, we have the relation

$$\lim_{t \to 0+0} \|A_t(x,f) - f(x)\|_{2,K} = 0,$$

i.e., the modified Abel–Poisson means $A_t(x,f)$ have the basis property in L_2 on any compact set in G.

2.9. Singular operators. As was already shown in Subsections 1.2 and 2.2, the coefficients $p_m(x)$ of operator (2.1) can have singularities not Lebesgue integrable at the ends of the interval G; nevertheless, all root functions of such an operator are absolutely continuous on the closed interval $\overline{G}$. Therefore, for such a singular operator $\mathcal{L}$, it is natural, for example, to pose the problem on the unconditional basis property of the system of root functions in $L_2(G)$.

For the first time, such studies were carried out by A. V. Kritskov for the Schrödinger operator (1.1) whose potential satisfies condition (1.16). Using an explicit representation of regular solutions of Eq. (1.8), Theorem 1.7 on the basis property of Riesz means of the system of root functions $\mathfrak{U}$ of such an operator was proved in [99, 100], estimates (1.42) and (1.44) of the root functions were obtained, and an analog of estimate (1.42) in which the constant in $O(1)$ is independent of the order of an associated function was also proved in [98, 100]. The necessity of the conditions implied by conditions A of the theorem on the unconditional basis property was proved.

Theorem 2.21 ([98]). *If the system* $\mathfrak{U}$ *of root functions with potential satisfying condition* (1.16) *forms an almost normed basis in* $L_p(G)$ *for a certain* $p > 1$, *then*:

(a) *the rank of eigenfunctions of the system is uniformly bounded*;

(b) *in the set of eigenvalues* Λ*, we can always find a sequence* $\{\lambda_k\}$ *such that* $c_1 k \le \operatorname{Re}\sqrt{\lambda_k} \le c_2 k$ *and* $|\operatorname{Im}\lambda_k| \le c_3$ *with certain positive constants* c_1*,* c_2*, and* c_3;

(c) *under the Carleman condition* (1.26), *the uniform "sum-of-units" estimate* (1.25) *holds.*

Some results on the properties of root functions of the operator $\mathcal{L}$ of an arbitrary order $n \ge 2$ with coefficients satisfying conditions (2.8) were stated in [101, 102]. The problem on the fulfillment of the theorem on the unconditional basis property for singular differential operators of arbitrary order remains open for now.

The basis property of root functions of a second-order differential operator with a stronger singularity, the Bessel operator $\mathcal{L} = -u'' - x^{-1}u' + \nu^2 x^{-2} u$, $0 < x < 1$, on the weighted space $L_{2,1} = \{u(x) : \int_0^1 u^2(x)x\,dx < \infty\}$ was considered in [38].

2.10. Operators with matrix coefficients. The main results on the unconditional basis property of the system of root functions and those on the equiconvergence of the spectral expansion and the expansion into the trigonometric Fourier series uniform on any compact set can be extended, practically without changes, to the case of operator (2.1) with matrix coefficients.

Let $U(x) = (u_1(x), \dots, u_m(x))^T$ be an m-dimensional vector-valued function of the argument $x \in G = (a, b)$. We consider the differential expression

$$\mathcal{L}U = U^{(n)} + P_1(x)U^{(n-1)} + \dots + P_{n-1}(x)U' + P_n(x)U, \tag{2.54}$$

in which $P_j(x)$, $j = \overline{1, n}$, are $(m \times m)$-matrices all of whose entries are only Lebesgue integrable on G in general.

In [54], operator (2.54) of the second order ($n = 2$) for which $P_1(x) \equiv 0$ was considered. Theorem 1.7 on the unconditional basis property of the system of vector-valued root functions understood in the generalized sense in the space $L_2^m(G)$ with the norm $\|f\| = \left(\sum_{i=1}^m \int_G |f_i(x)|^2\,dx\right)^{1/2}$ was proved; also, the necessity of the "sum-of-units" condition for such a basis property was proved. In [57], for the same operator, the equiconvergence of each component of the biorthogonal expansion

$$f(x) \sim \sum_{k=1}^{\infty} (f, V_k)_m U_k(x) \tag{2.55}$$

of an arbitrary function $f \in L_p^m(G)$, $p \ge 1$, in the system of vector-valued root functions and the corresponding component of $f(x)$ into the trigonometric Fourier series uniform in each compact set was proved.

As a consequence of the last result, in [57], the componentwise localization principle was presented; the essence of this principle can be expressed as follows. The convergence and the divergence of each component of the biorthogonal expansion depend only on the behavior of this component of the function being expanded in a certain neighborhood of a point, despite the fact that, as follows from (2.55), the coefficients $(f, V_k)_m = \sum_{l=1}^m (f_l, \{V_k\}_l)$ of each component contain all the components $f_l(x)$, $l = \overline{1, m}$, of the functions being expanded.

The case of operator (2.54) of an arbitrary order $n \ge 2$ with matrix coefficients $P_m(x)$, $m = \overline{1, n}$, whose entries $p_{mij}(x)$ satisfy the conditions $p_{mij}(x) \in W_1^{n-m}(G)$ was considered in [16, 20, 117]. By using the mean-value formula obtained for such an operator, the properties of the vector-valued root functions were studied, and a theorem on the unconditional basis property of the system formed by them in $L_2^m(G)$ was proved.

The componentwise estimate of the difference of a partial sum of the expansion of a function $f(x)$ in root vector-valued functions of the Schrödinger operator with matrix-valued potential and a partial sum of expansion of this component of the function $f(x)$ into the trigonometric Fourier series was proved in [154] for the vector-valued function $f(x)$ with monotone components.

In [34–37], a more general statement of the eigenvalue problem was studied for operator (2.54). As eigenvalues of this operator, we consider not complex numbers but complex diagonal matrices. In other words, the vector-valued root functions of operator (2.54) satisfy not Eq. (2.34) but the equation

$$\mathcal{L}\, U_k(x;\Lambda) = \Lambda U_k(x;\Lambda) + \operatorname{sign} k\, \widetilde{M} U_{k-1}(x;\Lambda), \tag{2.56}$$

in which $\Lambda = \operatorname{diag}\{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}\}$, $\widetilde{M} = E$ for $|\Lambda|_{\min} \le 1$, and $\widetilde{M} = M^{n-1}$ for $|\Lambda|_{\min} > 1$, where $M = \Lambda^{1/n}$ and $|\Lambda|_{\min} = \min\limits_{1\le l\le m} |\lambda^{(l)}|$.

Let $\{U_k(x)\}$ be a system of chains of vector-valued root functions of operator (2.54) enumerated in a certain way, and let $\{\Lambda_k\}$ be the set of corresponding eigenvalues. Let the following conditions hold:

(1) for a certain $p \ge 1$, the system $\{U_k(x)\}$ is closed and minimal in $L_p^m(G)$;

(2) the sequence of diagonal matrices $\{M_k\}$ corresponding to the sequence of diagonal eigenvalues $\{\Lambda_k\}$ has the following four properties:

(a) $\sup_k |\operatorname{Im} M_k|_{\max} < \infty$, where $|\operatorname{diag}\{\alpha_1, \dots, \alpha_m\}|_{\max} = \max\limits_{1\le l\le m} |\alpha_l|$;

(b) $\sup\limits_{t\ge 0} \sum\limits_{t\le |\operatorname{Re} M_k|_{\min} \le |\operatorname{Re} M_k|_{\max} \le t+1} 1 < \infty$;

(c) condition numbers of the matrices M_k, which are equal to $|M_k|_{\max}/|M_k|_{\min}$, are uniformly bounded;

(d) the elements of the sequence $\{M_k\}$ can be enumerated in an order such that for any $l = \overline{1,m}$, the sequence $\{|\mu_k^{(l)}|\}$ monotonically nondecreases starting from a certain serial number.

For a system of vector-valued root functions $\{U_k(x)\}$ satisfying these conditions, the theorem on the componentwise equiconvergence of the biorthogonal expansion of any vector-valued function from the class $L_p^m(G)$ with its expansion into the trigonometric Fourier series was proved.

2.11. Other results. We briefly mention a number of results characterizing other functional properties of systems of root functions of the ordinary-differential operator (2.1).

If the Bessel-type inequality (1.20) characterizes the coefficients of the biorthogonal expansion of a function $f \in L_2(G)$ as elements of the space l_2, then the left-hand side of the double inequality (1.22) shows that the sequence of these coefficients belongs to other spaces l_s, $s > 1$. In this sense, the well-known theorem of F. Riesz [39, pp. 153–154] on the Hausdorff–Young inequality

$$\left(\sum_{k=1}^{\infty} |(f, e_k)|^{p/(p-1)} \right)^{(p-1)/p} \le A \|f\|_p, \quad 1 < p \le 2, \tag{2.57}$$

for an arbitrary uniformly bounded orthonormal system $\{e_k\}$ shows that the coefficients $\{(f, e_k)\}$ belong to the space $l_{(p-1)/p}$ if the function $f(x)$ being expanded belongs to the space $L_p(G)$, where $1 < p \le 2$.

For systems of root functions of the Schrödinger differential operators (1.1) and an operator of an arbitrary order, the Hausdorff–Young inequality was proved in [83, 84, 105, 117, 120] based on the technique developed for eigenfunctions of the Laplace operator in [61].

Conditions ensuring the fulfillment of the Hilbert-type inequality (1.21) for systems of root functions of ordinary differential operators were considered in [85, 140–142]. The technique of these works develops ideas established in [53].

A slightly different approach to the analysis of the Bessel and Hilbert properties of systems of root functions was suggested in [2, 23, 24].

In a number of works [42–44], the basis properties and the equiconvergence property with the trigonometric Fourier series of expansions in eigenfunctions and associated functions of M. V. Keldysh's pencils of ordinary differential operators of arbitrary order were studied.

The relation between the basis property of systems of root functions and the integrability of nonlinear evolutionary equations admitting P. Laks' $L - A$-representation was studied in [63, 64, 143].

3. Discontinuous Differential Operators

3.1. Nonlocal boundary-value problems: the concept and examples. As was already mentioned above, attention was drawn to the nonlocal problem[24] in the work [9] of A. A. Samarskii and A. V. Bitsadze.

Conditions containing values of the unknown function or its derivatives at interior points of the domain of the differential expression considered are conventionally said to be nonlocal.

For ordinary differential operators, for example, these are the so-called multipoint problems in whose boundary conditions we have values at fixed interior points of the interval considered. Let us present the statements of model problems of such a type.

Let the operator $\mathcal{L}$ be defined by the second-order differential expression (1.1) on the interval $G = (a, b)$ with a Lebesgue integrable potential $q(x)$. Consider an arbitrary set of points $\{\xi_l\}_{l=1}^N$ of the interval G enumerated in increasing order: $a < \xi_1 < \xi_2 < \dots < \xi_N < b$. The problem

$$\begin{cases} \mathcal{L}\,u \equiv -u'' + q(x)u = \lambda u, \quad x \in G, \\ u(a) = 0,\ u(b) = \sum_{l=1}^N \alpha_l u(\xi_l) \end{cases} \tag{3.1}$$

is called a nonlocal boundary-value problem of the first kind [68], and the problem

$$\begin{cases} \mathcal{L}\,u \equiv -u'' + q(x)u = \lambda u, \quad x \in G, \\ u(a) = 0,\ u'(b) + \varepsilon u(b) = \sum_{l=1}^N \alpha_l u'(\xi_l) \end{cases} \tag{3.2}$$

is called a nonlocal boundary-value problem of the second kind [69].[25]

A specific feature of multipoint problems is that the problems adjoint to them are defined on a set of discontinuous functions.

Denote by $y[\xi]$ the jump of a function $y(x)$ at a point ξ, which is equal to $y(\xi+0) - y(\xi-0)$. Taking into account this notation, we see that the problems adjoint to (3.1) and (3.2) have the following form:

$$\begin{cases} \mathcal{L}^*v \equiv -v'' + \overline{q(x)}v = \lambda^* v, \quad x \in G,\, x \neq \xi_l, l = \overline{1, N}, \\ v(a) = v(b) = 0,\ v[\xi_l] = 0,\ v'[\xi_l] = \alpha_l v'(b), l = \overline{1, N}, \end{cases} \tag{3.1*}$$

$$\begin{cases} \mathcal{L}^*v \equiv -v'' + \overline{q(x)}v = \lambda^* v, \quad x \in G,\, x \neq \xi_l, l = \overline{1, N}, \\ v(a) = 0,\ v'(b) + \varepsilon v(b) = 0,\ v[\xi_l] = \alpha_l v(b),\ v'[\xi_l] = 0, l = \overline{1, N}. \end{cases} \tag{3.2*}$$

If spectral properties of multipoint boundary-value problems themselves can be studied on the set of regular solutions of the corresponding differential equations with a spectral parameter, then the following example [114] shows that the direct and adjoint operators can have discontinuous functions in their domains:

$$\begin{cases} \mathcal{L}\,u \equiv -u'' = \lambda u, \quad 0 < x < 1, \\ u'(0) = 0,\, u'(1) = u'(1/2),\, u'[1/2] = 0,\, u[1/2] = u(0), \end{cases} \tag{3.3}$$

$$\begin{cases} \mathcal{L}^*v \equiv -v'' = \lambda^* v, \quad 0 < x < 1, \\ v'(1) = 0,\, v'(0) = -v'(1/2),\, v'[1/2] = 0,\, v[1/2] = v(1). \end{cases} \tag{3.3*}$$

Since we need to consider simultaneously the root functions of the adjoint operator in studying the basis property of root functions of the operator, in the case of nonlocal boundary-value problems it is necessary to omit the requirement of regularity of solutions to the equation with a spectral parameter.

The boundary conditions of multipoint problems are, in essence, linear functionals on the space of functions that are continuous, together with their first derivatives on the closed interval G. If we admit the possibility of considering the vanishing of a linear functional of an arbitrary form as a boundary condition, then by the Riesz theorem, the range of problems under study should include operators whose boundary conditions contain integral summands.

[24] Further references on this subject are presented in [68].

[25] Here, $\varepsilon, \alpha_1, \dots, \alpha_N$ are arbitrary complex numbers.

As an example, we consider the following modification of problem (3.1):

$$\begin{cases} \mathcal{L}u \equiv -u'' + q(x)u = \lambda u, \quad x \in G, \\ u(a) = 0,\ u(b) = \sum_{l=1}^{N} \alpha_l u(\xi_l) + \int\limits_a^b \rho(x)v(x)\,dx, \end{cases} \tag{3.4}$$

where $\rho(x)$ is an arbitrary complex-valued function that is Lebesgue integrable on G. The problem adjoint to (3.3) is the problem in which the differential expression now contains the following additional summand:

$$\begin{cases} \mathcal{L}^*v \equiv -v'' + \overline{q(x)}v + v'(b)\overline{\rho(x)} = \lambda^* v, \quad x \in G,\ x \neq \xi_l, l = \overline{1,N}, \\ v(a) = v(b) = 0,\ v[\xi_l] = 0,\ v'[\xi_l] = \alpha_l v'(b), l = \overline{1,N}. \end{cases} \tag{3.4*}$$

Such differential expressions are conventionally said to be *loaded*.

The pairs of adjoint problems (3.1) and (3.1*), (3.2) and (3.2*), (3.3) and (3.3*), and (3.4) and (3.4*) presented here demonstrate one of the features of nonlocal boundary-value problems with integral conditions of general form,[26] the complicated structure of the adjoint operator. We indicate the Braun result presented in [187] in this connection, where the adjoint operator for a general differential expression of the nth order was constructed in an explicit form.

3.2. Unconditional basis property of the system of root functions. For the first time, the case where the operator $\mathcal{L}$ and its adjoint are defined on the class of discontinuous functions was studied in [52].[27] The theorem on the unconditional basis property for root functions of the discontinuous Schrödinger operator (1.1) obtained there literally repeats Theorem 1.7 presented in the regular case.[28] Only the definition of a root function of the operator $\mathcal{L}$ was changed.

Let points $\{\xi_l\}_{l=1}^N$ give a partition of the interval $G = (a, b)$ considered, and, moreover, let $a \equiv \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_N < b \equiv \xi_{N+1}$. An *eigenfunction* (*associated function of kth order*) is any nontrivial solution of Eq. (1.9) (resp. (1.10)) on $G \setminus \{\xi_l\}_{l=1}^N$ that is absolutely continuous, together with its first derivative, on each partial closed interval $[\xi_{l-1}, \xi_l]$, $l = \overline{1, N+1}$, of the partition.

Such a treatment of the concept of a root function of the operator requires again neither indication of the concrete form of boundary conditions nor correction of "gluing" conditions of the root function and its derivative at the points ξ_l.

For example, as the analysis of problems (3.3) and (3.3*) carried out in [114] and also the study of problem (3.2) in [137, 138] shows, to verify all the conditions ensuring the unconditional basis property of the system of root functions in $L_2(G)$, as in the regular case, it suffices to know only the leading terms of the asymptotics of eigenvalues and root functions of the operator.

In [166], it was shown for the first time that Theorem 1.7 on the unconditional basis property can also be justified for root functions of a second-order operator with a loaded differential expression containing the value of the unknown function with deviated argument of the form $u(\nu(u(x))$, where $\nu(x) : G \to G$ is an arbitrary monotone function whose inverse function is continuously differentiable. However, the loaded differential operators of such a form do not cover boundary-value problems adjoint to boundary-value problems with integral conditions (such as problem (3.4*)). The first steps in studying such operators were made in [113, 167], where representations and estimates of the corresponding root functions were obtained.

We give here a result on the unconditional basis property of root functions of a discontinuous scalar operator generated by a second-order differential expression of the general form and with general integral conditions on G in more detail (see [115, 116]).

[26] Obviously, the second conditions of problems (3.1) and (3.2) can also be written by using appropriate Stieltjes integrals with respect to discrete measures.

[27] A survey of results that appeared before this publication is contained in [115].

[28] In [54], the theorem on the unconditional basis property was also extended to the case of the discontinuous Schrödinger operator with a matrix potential.

On the closed interval $\overline{G} = [a,b]$, let vector singular measures[29] $\nu_j(x) = (\nu_{j1}(x), \nu_{j2}(x))$, $j = 1,2$, of bounded variations that are right-continuous at all points of the interval G be given. We write their representation $\nu_j = \nu_j^{\rm sc} + \nu_j^{\rm sa}$ in the form of the sum of the continuous parts $\nu_j^{\rm sc}$ and the jump functions of the measure $\nu_j^{\rm sa}$. Points of discontinuity of the jump functions are denoted by $\{\xi_p\}_{p=0}^{\infty} \subset \overline{G}$, and, without loss of generality, we assume that they are common for $\nu_1^{\rm sa}$, as well as for $\nu_2^{\rm sa}$.

Denote by $\mathcal{D}$ the set of those functions $u \in L_2(G)$ for which there exists a vector[30] $\varphi = (\varphi_1, \varphi_2)^T \in \mathbb{C}^2$ such that the following conditions hold: (1) the functions $u(x) + \nu_1[a,x]\varphi$ and $-u'(x) + \nu_2[a,x]\varphi$ are absolutely continuous on G; (2) the function $u(x)$ has finite limit values $u(a+0)$ and $u(b-0)$ at the ends of G.

Let $T = \{\tau_p\}_{p=0}^{\infty}$ be an arbitrary partition of $\overline{G}$, and, moreover, let $\tau_0 = a$, $\tau_1 = b$, $\tau_p \neq \tau_j$ for $p \neq j$, $p, j \geq 0$. With each point $\tau_p \in T$, we associate the functions $R_{jp}^{\pm}(x)$, $j = 1,2$, and, moreover, $R_{j1}^{+} = R_{j0}^{-}(x) \equiv 0$.

Let us consider the operator $\mathcal{L}$ generated by the differential expression

$$\mathcal{L}\,u(x) = -u''(x) + q(x)u(x) + \sum_{j=1}^{2} {}^{\pm} \sum_{p=0}^{\infty} R_{jp}^{\pm}(x) u^{(2-j)}(\tau_p^{\pm}) + V(x)\varphi \tag{3.5}$$

on the set $\mathcal{D}$. Here, $\tau_p^{\pm} = \tau_p \pm 0$, $\tau_0^{-} = a$, $\tau_1^{+} = b$, and the sign $\pm$ of the sum means that first, we take the sum $\sum_{j=1}^{2}$ of all summands with a plus sign and then the same sum of summands with a minus sign.

We assume that the functions $q(x)$ and $R_{jp}^{\pm}(x)$ and each of the components of the vector-valued function $V(x)$ are Lebesgue integrable on G, and, moreover, the numerical series

$$\sum_{p=0}^{\infty} \|R_{jp}^{\pm}\|_1 < \infty, \quad j = 1,2, \tag{3.6}$$

converge.

Assume that the adjoint operator $\mathcal{L}^*$ is generated by the following expression analogous to (3.5):

$$\mathcal{L}^* v(x) = -v''(x) + \overline{q(x)} v(x) + \sum_{j=3}^{4} {}^{\pm} \sum_{p=0}^{\infty} R_{jp}^{\pm}(x) v^{(2-j)}(\chi_p^{\pm}) + \widetilde{V}(x)\varphi \tag{3.7}$$

on the set of functions $\mathcal{D}^*$, which is similar to $\mathcal{D}$, but which is introduced according to a more complicated scheme.[31] Let the singular measures ν_3 and ν_4 in the definition of the set $\mathcal{D}^*$ have discontinuities at the same points $\{\xi_p\}_{p=0}^{\infty}$, and let the functions $R_{jp}^{\pm}(x)$ and the components $\widetilde{V}(x)$ be Lebesgue integrable on G and series (3.6) with $j = 3,4$ converge.

We impose the following restrictions on the discontinuous operator considered:

(1) set $\mathcal{N}_V = \bigcap\limits_{x \in \overline{G}} \ker V(x)$, $\mathcal{N}_j^a = \bigcap\limits_{p=2}^{\infty} \ker(\nu_j^{\rm sa}[\xi_p])$, $\mathcal{N}_j^c = \bigcap\limits_{x \in \overline{G}} \ker(\nu_j^{\rm sc}[a,x])$, $j = \overline{1,4}$, and require that these kernels satisfy the inclusions

$$\begin{aligned} &\mathcal{N}_1^a \subseteq \mathcal{N}_1^c, \quad \exists j,l = 1,2 : \mathcal{N}_j^a \subseteq \mathcal{N}_2^c, \mathcal{N}_l^a \subseteq \mathcal{N}_V; \\ &\mathcal{N}_3^a \subseteq \mathcal{N}_3^c, \quad \exists j,l = 3,4 : \mathcal{N}_j^a \subseteq \mathcal{N}_4^c, \mathcal{N}_l^a \subseteq \mathcal{N}_{\widetilde{V}}; \end{aligned} \tag{3.8}$$

(2) if $\mathcal{N}_1^a \not\subseteq \mathcal{N}_V$, then we require that $V \in L_2(G) \times L_2(G)$, and if $\mathcal{N}_3^a \not\subseteq \mathcal{N}_{\widetilde{V}}$, then $\widetilde{V} \in L_2(G) \times L_2(G)$;

[29] As follows from [187], only the singular part of a measure influences the structure of solutions of the adjoint operator.

[30] In most boundary-value problems, the vector φ is expressed through the values of u and u' at the ends of the interval G [116].

[31] The classes $\mathcal{D}$ and $\mathcal{D}^*$ on which the discontinuous operators $\mathcal{L}$ and $\mathcal{L}^*$ act can be introduced in a symmetrical way [118].

(3) the coefficients $R^{\pm}_{jp}(x)$ in the differential expressions (3.5) and (3.7) for $j = 1$ and $j = 3$ are such that

$$\sum_{p=0}^{\infty} |R^{\pm}_{jp}(x)| \in L_2(G). \tag{3.9}$$

By a *root function* $u_k(x;\lambda)$ of the operator $\mathcal{L}$, we mean an arbitrary nontrivial solution of the equation

$$\mathcal{L}\, u_k(x;\lambda) = \lambda u_k(x;\lambda) - \operatorname{sign} k\, \widetilde{\mu}\, u_{k-1}(x;\lambda)$$

that belongs to the class $\mathcal{D}$; here, as in (1.10), $\lambda \in \mathbb{C}$ is an eigenvalue, $k \geq 0$ is the order of the root function, and $\mu = \sqrt{\lambda}$ is an especially chosen value of the square root of λ, $\widetilde{\mu} = \mu$ for $|\lambda| > 1$ and $\widetilde{\mu} = 1$ for $|\lambda| \leq 1$.

We consider the set $\mathfrak{U}$ of chains of root functions of the operator $\mathcal{L}$ corresponding to a certain countable set of eigenvalues $\Lambda \subset \mathbb{C}$ such that $\mathfrak{U} = \{u_k(x;\lambda) \mid k = \overline{0, m(\lambda)}, \lambda \in \Lambda\}$. Let all conditions A of Theorem 1.7 on the unconditional basis property in the regular case hold, i.e.,

(A1) the system $\mathfrak{U}$ is complete and minimal in $L_2(G)$;

(A2) the ranks $m(\lambda)$ of eigenvalues are uniformly bounded in $\lambda \in \Lambda$;

(A3) the uniform "sum-of-units" estimates (1.25) hold;

(A4) the Carleman condition (1.26) holds,

(A5) the system $\mathfrak{V}$ biorthogonally dual to $\mathfrak{U}$ consists of the functions of class $\mathcal{D}^*$ satisfying the equation $\mathcal{L}^* v_k(x;\overline{\lambda}) = \overline{\lambda} v_k(x;\overline{\lambda}) - \operatorname{sign} k\, \overline{\widetilde{\mu}}\, v_{k-1}(x;\overline{\lambda})$.

Theorem 3.1 ([116])**.** *Each of the systems* $\mathfrak{U}$ *and* $\mathfrak{V}$ *forms an unconditional basis in* $L_2(G)$ *iff the uniform estimate* (1.28) *of the product of norms holds.*

Remark. In [116], the necessity of the "sum-of-units" estimate for the unconditional basis property of the system $\mathfrak{U}$ was proved, but this was done only under additional restrictions imposed on the operator $\mathcal{L}$.

The base of the proof of Theorem 3.1, as in the regular case, consists in the justification of the Bessel property for the system $\mathfrak{U}$ normalized in $L_2(G)$. The main tool for this justification consists in the integral representations of root functions, the "shift" formula [115, Theorem 1]. As compared with the similar formula (1.11) of the regular case, it contains additional summands depending on the "load" of the differential expression and the singular measures $\nu_j(x)$.

A principal role of such a proof is the obtaining of estimates for norms of root functions and their derivatives in the space L_p. The corresponding result for the case of operator (3.5) considered above was proved in [115, Theorem 2].

Theorem 3.2. *Let the coefficients in the differential expression* (3.5) *be Lebesgue integrable, and let conditions* (3.6) and (3.8) *and the Carleman condition* (1.26) *hold. Then there exists a compact set* $K \subset G$ *such that the following estimates hold uniformly in* λ:

$$\|u_k^{(\alpha)}(\cdot;\lambda)\|_\infty = O(1)\left(1 + |\sqrt{\lambda}|\right)^{\alpha-\beta} \|u_l^{(\beta)}(\cdot;\lambda)\|_{2,K}, \tag{3.10}$$

where $\alpha, \beta \in \{0, 1\}$, $0 \leq k \leq l$, *and the constant in* $O(1)$ *depends only on the orders* k *and* l *of the root functions.*

Estimates (3.10) in the regular case transform into estimates (1.42)–(1.46) of Theorem 1.8.[32]

We note that the results of Theorems 3.1 and 3.2 extend to the case of expression (3.5) with matrix coefficients without changes [115, 116].

[32]The case where the Carleman condition does not hold and the norms of root functions are taken in the spaces L_p, $1 \leq p \leq \infty$, is not considered in [115].

The basis property of root functions of the discontinuous operator generated by the general loaded nth-order differential expression

$$\mathcal{L}\,u = u^{(n)}(x) + \sum_{m=1}^{n}\left[p_m(x)u^{(n-m)}(x) + \sum_{p=0}^{\infty}{}^{\pm}R^{\pm}_{mp}(x)u^{(n-m)}(\tau_p^{\pm})\right] + V(x)\varphi \tag{3.11}$$

is studied in [118]. The representation obtained in this case and the estimates of root functions generalize the analogous results of the regular case (see Theorems 2.3, 2.4, and 2.10).

In [7, 8], the discontinuous operator (3.5) was studied in the case where its coefficient $q(x)$ admits nonintegrable singularities at the ends of the interval G such that condition (1.16) holds.

3.3. Problems with multipoint boundary conditions: the uniform convergence of spectral expansions. To justify the Fourier method for solving the mixed problem for nonstationary equations with multipoint boundary conditions,[33] it is necessary to note which conditions on the function being expanded ensure the absolute and uniform convergence of the corresponding biorthogonal series on the closed interval $\overline{G}$.

In [160], for the nonlocal boundary-value problem (3.1) of the first kind with $q(x) \equiv 0$, it was shown that the conditions ensuring the absolute and uniform convergence of the spectral expansion are the same as in the self-adjoint case: the function $f(x)$ being expanded should belong to the Sobolev space $W_2^1[0,1]$ and satisfy the boundary conditions.

The general case was studied by I. S. Lomov.[34] He considered the Schrödinger operator (1.1) with an arbitrary potential $q(x)$ Lebesgue integrable on G on the class of functions $u = u(x)$ that are absolutely continuous, together with their first derivatives, on $\overline{G}$ and satisfy the multipoint boundary conditions of an arbitrary form

$$B(u) \equiv \sum_{i=1}^{2}\sum_{p=0}^{\infty} u^{(2-i)}(\xi_p)\alpha_i^p = (0\ 0)^T, \tag{3.12}$$

where the points ξ_p form a partition of the closed interval $\overline{G}$, and, moreover, $\xi_0 = a$, $\xi_1 = b$, and the coefficients $\alpha_i^p = (\alpha_{i1}^p, \alpha_{i2}^p)^T \in \mathbb{C}^2$ are such that $\sum_{i,j=1}^{2}\sum_{p=0}^{\infty}|\alpha_{ij}^p| < \infty$.

Theorem 3.3. *Let the domain of the operator be dense in $L_2(G)$, and let the first four of conditions A of Theorem 1.7 hold.*[35] *If $f \in W_2^1(G)$ and*

$$B_2(f) \equiv \sum_{p=0}^{\infty} f(\xi_p)\alpha_2^p = (0\ 0)^T, \tag{3.13}$$

then the biorthogonal expansion of $f(x)$ in root functions of the operator considered converges absolutely and uniformly on $\overline{G}$.

The requirements on the smoothness of the function being expanded are obviously sharp in terms of Sobolev classes with integer exponents. A further refinement of the exponent of smoothness is possible only in terms of Hölder or Nikol'skii classes.

Remark. We note that a part of formula (3.12) with $i = 1$ is not connected with $f(x)$ at all in the theorem. An analogous situation also takes place for ordinary two-point boundary conditions.

[33] The correctness of the statement of such problems was studied in [68–70].

[34] The corresponding publication is in preparation.

[35] The fifth of conditions A obviously holds automatically.

3.4. Convergence of spectral expansions at a discontinuity point of coefficients. On the interval $G = (a, b)$, we consider the differential operator

$$\mathcal{L}\, u = -(r(x)u')' + q(x)u, \tag{3.14}$$

whose coefficient $r(x)$ is discontinuous at some point x_0 of the interval G. We assume that

$$r(x) \in C^2(a, x_0] \cap C^2[x_0, b), \quad \inf_{x \in G} r(x) > 0,$$

and $q(x)$ is an arbitrary complex-valued function continuous on G.

We define root functions $u_k(x; \lambda)$ of operator (3.14) as solutions of Eq. (1.10) regular on each of the intervals (a, x_0) and (x_0, b) and satisfying the following "gluing" condition at the point x_0:

$$\begin{cases} u_k(x_0 - 0; \lambda) = u_k(x_0 + 0; \lambda), \\ r(x_0 - 0)u_k'(x_0 - 0; \lambda) = r(x_0 + 0)u_k'(x_0 + 0; \lambda) + \beta u_k(x_0 + 0; \lambda), \end{cases} \tag{3.15}$$

where $\beta \in \mathbb{C}$ is arbitrary.

Let the set of eigenvalues Λ and the system $\mathfrak{U}$ of root functions defined in such a way satisfy two conditions in A_p from Subsection 1.8 for a certain $p > 1$, which allows us to enumerate the root functions in nondecreasing order of $|\sqrt{\lambda}|$, $\lambda \in \Lambda$, and in each chain, in increasing order of the orders of associated functions.

It is natural to ask: to what does the biorthogonal series of such a system $\mathfrak{U}$ of discontinuous root functions converge at the point of discontinuity x_0? An answer to this question can be given by comparing a partial sum $\sigma_n(x, f)$ of the biorthogonal series (1.52) with the modified partial sum

$$\widetilde{S}_{|\sqrt{\lambda_n}|}(x, f) = \pi^{-1} \int_{x-\delta}^{x+\delta} \frac{\sin |\sqrt{\lambda_n}|(y - x)}{y - x} f(y)\, dy \tag{3.16}$$

of the trigonometric Fourier series ($\delta < \min\{b - x_0, x_0 - a\}$) at the point x_0.

For this purpose, we consider the function $t = t(x) = \int_{x_0}^{x} [r(\tau)]^{-1/2}\, d\tau$ and its inverse $x = x(t)$. We set $\rho(t) = x(t) - x_0$ and $\gamma(x) = [r(x)/r(x_0 \pm 0)]^{-1/4}$, where the plus sign is chosen for $x > x_0$, and for $x < x_0$, the minus sign is chosen.

Using the function $f \in L_p(G)$ expanded into series, we construct the new function

$$\widetilde{f}(x) = \begin{cases} \dfrac{2\sqrt{r(x_0 + \rho(x - x_0))}}{\sqrt{r(x + 0)} + \sqrt{r(x - 0)}} \gamma(x_0 + \rho(x - x_0)) f(x_0 + \rho(x - x_0)) & \text{if } |x - x_0| < \delta, \\ 0 & \text{if } |x - x_0| \geq \delta. \end{cases} \tag{3.17}$$

Theorem 3.4 ([12])**.** *In addition to conditions A_p, let the uniform estimate* (1.55) *of the product of norms hold. Then*

$$\lim_{n \to \infty} |\sigma_n(x_0, f) - \widetilde{S}_{|\sqrt{\lambda_n}|}(x_0, \widetilde{f})| = 0. \tag{3.18}$$

Remark. Such an "equiconvergence" (3.18) was studied in [45] for the self-adjoint case.

The well-known fact on the convergence of the trigonometric Fourier series at a point of discontinuity of the function being expanded to the half-sum of its right and left limit values at the point x_0 implies that the biorthogonal expansion $\sigma_n(x, f)$ converges to the weighted mean, which is equal to $[\sqrt{r(x_0 + 0)} f(x_0 + 0) + \sqrt{r(x_0 - 0)} f(x_0 - 0)]/(\sqrt{r(x_0 + 0)} + \sqrt{r(x_0 - 0)})$.

Chapter 2

ELLIPTIC DIFFERENTIAL OPERATORS

4. Second-Order Elliptic Differential Operators and Their Root Functions

4.1. Statement of the problem. The main object of the study in this chapter is the general second-order elliptic operator

$$\mathcal{L}\,u = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^{N} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{4.1}$$

which is considered on an arbitrary domain Ω of the space $\mathbb{R}^N$ of dimension $N \geq 2$.

We assume that the coefficients $a_{ij}(x)$ are real-valued functions satisfying the uniform ellipticity condition in Ω, i.e., for all $x = (x_1, \ldots, x_N) \in \Omega$ and all real numbers $\xi_1, \ldots, \xi_N$, the following relations hold:

$$a_{ij}(x) = a_{ji}(x), \quad i, j = \overline{1, N},$$
$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{N} \xi_i^2 \equiv \alpha\|\xi\|^2 \tag{4.2}$$

with a certain positive constant α.[36] Other coefficients in (4.1) can be complex-valued functions on Ω. First, the following minimal assumptions are made on the smoothness of the coefficients of the operator $\mathcal{L}$:

$$a_{ij}(x), \quad \frac{\partial a_{ij}(x)}{\partial x_i}, \quad b_j(x), \quad c(x) \in C(\Omega). \tag{4.3}$$

An important particular case of operator (4.4) is the higher-dimensional Schrödinger operator

$$\mathcal{L}\,u = \Delta u - q(x)u \tag{4.4}$$

with complex-valued potential $q(x)$ and also the simplest second-order elliptic operator, the Laplace operator, i.e., operator (4.4) with $q(x) \equiv 0$.

Along with operator (4.1), we consider its formal adjoint operator

$$\mathcal{L}^*v = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right) - \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[\overline{b_j(x)}v\right] + \overline{c(x)}v, \tag{4.5}$$

which under the condition

$$\frac{\partial b_j(x)}{\partial x_j} \in C(\Omega) \tag{4.6}$$

belongs to the same class of elliptic operators as the operator introduced above.

Since the operators $\mathcal{L}$ and $\mathcal{L}^*$ are distinct in general, on any appropriate domain we deal with the non-self-adjoint case. However, even for the Laplace operator, we can pose the boundary conditions in such a way that the corresponding problem is not self-adjoint.[37]

In this connection, along with eigenfunctions, we need to consider associated functions, and, as in the one-dimensional case, to cover all possible particular statements of problems for operator (4.1), we introduce the concept of a root (i.e., eigen- and associated) function of the operator $\mathcal{L}$ with the only requirement that it be a regular solution of the corresponding differential equation with a spectral parameter.

[36] The uniform ellipticity condition can hold not on the whole domain Ω but on any compact set K in this domain; in this case, the constant α in inequality (4.2) can depend on the compact set considered.

[37] For example, the oblique derivative problem for the Laplace operator in the disk [73] and also various nonlocal boundary-value problems (see, e.g., [149]) are such problems.

An *eigenfunction* $u_0(x;\lambda)$ of operator (4.1) is any function of class $C^2(\Omega)$ not identically equal to zero such that in Ω, it satisfies the equation

$$\mathcal{L}\, u_0(x;\lambda) + \lambda u_0(x;\lambda) = 0 \tag{4.7}$$

for a certain $\lambda \in \mathbb{C}$ and belongs to the class $L_2(\Omega)$. The number λ in (4.7) is called an *eigenvalue* of the operator $\mathcal{L}$. An *associated function* $u_l(x;\lambda)$ *of order* $l \geq 1$ corresponding to this eigenvalue λ and this eigenfunction $u_0(x;\lambda)$ is any function of class $C^2(\Omega)$ such that in Ω, it satisfies the equation

$$\mathcal{L}\, u_l(x;\lambda) + \lambda u_l(x;\lambda) = u_{l-1}(x;\lambda) \tag{4.8}$$

and belongs to the class $L_2(\Omega)$ (here, $u_{l-1}(x;\lambda)$ is either an eigenfunction if $l = 1$ or an associated function of order $l-1$ if $l \geq 2$). The sequence of functions $u_0(x;\lambda)$, $u_1(x;\lambda), \ldots,$ $u_n(x;\lambda)$ is called a chain of root functions.

For simplicity, we consider only those systems $\mathfrak{U}$ of root functions of operator (4.1) which consist of chains of root functions corresponding to eigenvalues from a certain countable set $\Lambda \subset \mathbb{C}$: $\{u_l(x;\lambda) \,|\, l = \overline{0, m(\lambda)}, \lambda \in \mathbb{C}\}$.

It should be noted that not for all problems related to the elliptic operator can the system $\mathfrak{U}$ be divided into chains of root functions. For example, such a system is naturally formed by root functions of the Laplace operator in the square $[0,1] \times [0,1]$ with the boundary conditions[38]

$$\begin{cases} u(0,x_2) = u(x_1,0) = 0, \\ \dfrac{\partial u}{\partial x_1}(0,x_2) = \dfrac{\partial u}{\partial x_1}(1,x_2),\ \dfrac{\partial u}{\partial x_2}(x_1,0) = \dfrac{\partial u}{\partial x_2}(x_1,1),\ 0 \leq x_1, x_2 \leq 1, \end{cases}$$

which are a "product" of nonlocal Samarskii–Ionkin problems with respect to each variable (see [131] for more details).

A specific feature of spectral problems for elliptic operators is a more complicated (as compared with the one-dimensional case) structure of the set of eigenvalues Λ. Even in the self-adjoint case, any sequence of real numbers can be a part of the set of eigenvalues of the Laplace operator [55, pp. 29–36]. As for the non-self-adjoint case, for any sequence of real numbers, for the Laplace operator in the square, we can find boundary conditions such that all numbers of this sequence are eigenvalues, and infinitely many eigenfunctions and infinitely many associated functions of the first order correspond to each such eigenvalue [59]. In this case, the system of all root functions of such a problem forms a Riesz basis in L_2 over the square. Thus, the characterization of the behavior of the "counting" function

$$n(r) = \sum_{\lambda \in \Lambda\colon |\lambda| \leq r} 1 \tag{4.9}$$

of the set of eigenvalues Λ simultaneously for all boundary-value problems connected with elliptic operators (4.1) is not possible.

Along with this, as follows from the example in [108], owing to the choice of the potential in the periodic boundary-value problem for the Schrödinger operator (4.4), we can attain the property that the orders l of associated functions in the system $\mathfrak{U}$ increase when $|\lambda|$ increases. Therefore, the maximum order of associated functions in chains is not uniformly bounded in general[39] (in contrast to Theorems 1.9 and 2.14 of the one-dimensional case).

The mentioned specific feature of spectral problems for operator (4.1) on the space of dimension greater than two stresses the complexity of studying biorthogonal expansions arising here simultaneously in the whole range of possible variants and explains in many respects the lesser obtained results as compared with the one-dimensional case.

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[38] That is, as a product of root functions of the one-dimensional problem (1.2).

[39] Other such examples are discussed in [133, 134].

4.2. Mean-value formula. One of the main tools in studying spectral properties of elliptic operator (4.1) considered here is the so-called mean-value formula for root functions.

In the case of the Laplace operator, the mean-value formula for eigenfunctions relates its mean value on any sphere in Ω with the value of the eigenfunction at the center of this sphere.

Let x be an arbitrary point of the domain $\Omega \subset \mathbb{R}^N$, and let the number r be less than or equal to $\operatorname{dist}(x, \partial\Omega)$. Denote by $\theta = (\theta_1, \dots, \theta_{N-1})$ angular hyperspherical coordinates, by $d\theta$ an area element in these coordinates, and by $\int_\theta F(x + r\theta)\, d\theta$ the integral taken over all angular coordinates on the surface of the N-dimensional sphere of radius r centered at the point x.

We set $\nu = (N-2)/2$ and denote by $\mu = \mu(\lambda)$ the square root of the eigenvalue λ whose real part is nonnegative. Then the mean-value formula for the eigenfunction of the Laplace operator has the form

$$\int\limits_\theta u_0(x + r\theta; \lambda)\, d\theta = (2\pi)^{N/2} (\mu r)^{-\nu} J_\nu(\mu r) u_0(x; \lambda). \tag{4.10}$$

For self-adjoint expansions of elliptic operators, the methodology elaborated on the basis of this formula [55] allows one to obtain very fine results on the convergence of the corresponding spectral expansions which are final in various classes of differentiable functions.

For an associated function of order $l \geq 1$, the mean-value formula connects its mean value on the sphere with the values of this function and all root functions of the corresponding chain of order less than l at the center of the sphere [66].

Here, we present the mean-value formula for root functions of the Schrödinger operator (4.4) obtained by A. S. Makin [132].

Let $n \in \mathbb{N}$, and let $Y_n(\theta)$ be any hyperspherical function of order n. We set

$$W_n(r, \rho; \lambda) = \frac{\pi}{2} \left(\frac{\rho}{r}\right)^\nu \rho \left[N_{\nu+n}(\mu r) J_{\nu+n}(\mu\rho) - N_{\nu+n}(\mu\rho) J_{\nu+n}(\mu r)\right], \tag{4.11}$$

where $J_{\nu+n}(z)$ and $N_{\nu+n}(z)$ are the cylindrical Bessel and Neumann functions of order $\nu + n$.

Theorem 4.1. *Let the potential $q(x)$ in (4.4) belong to the class $C^n(\Omega)$, $n = 0, 1, 2, \dots$, and let $N \geq 2$. Then for any eigenvalue $\lambda \neq 0$, the following relation holds:*

$$\begin{aligned}
&\alpha_n^{-1} \int\limits_\theta u_l(x + \theta r; \lambda) Y_n(\theta)\, d\theta \\
&= \sum_{k=0}^{l} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial\lambda^k} \left[\frac{J_{\nu+n}(\sqrt{\lambda} r) r^n}{(\sqrt{\lambda} r)^{\nu+n}}\right] \int\limits_\theta \frac{\partial^n u_{l-k}(x, \theta; \lambda)}{\partial r^n} Y_n(\theta)\, d\theta \\
&+ \sum_{k=0}^{l} \int\limits_0^r \frac{(-1)^k}{k!} \frac{\partial^k W_n(r, \rho; \lambda)}{\partial\lambda^k} \left[\alpha_n^{-1} \int\limits_\theta q(x + \theta\rho) u_{l-k}(x + \theta\rho; \lambda) Y_n(\theta)\, d\theta\right] d\rho,
\end{aligned} \tag{4.12}$$

where $\alpha_n = (n!)^{-1} \Gamma(\nu + n + 1) 2^{\nu+n}$ and the notation $\partial^n u_{l-k}(x, \theta; \lambda)/\partial r^n$ means the value of $\partial^n u_{l-k}(x + \theta r; \lambda)/\partial r^n$ for $r = 0$.

Following the approach suggested by E. I. Moiseev [146], the mean-value formula can also be written for the general elliptic operator (4.1) if the coefficients $a_{ij}(x)$ are sufficiently smooth[40] and $b_j(x) \equiv 0$, $j = \overline{1, N}$.

4.3. Estimates of root functions. As in the one-dimensional case, the principal role in obtaining the results on the properties of biorthogonal expansions is played by various estimates of root functions, and, in particular, the anti-a priori bound that connects the norm of an arbitrary root function with the norm of the root functions of order that is one less belonging to the same chain.

[40]The coefficients $a_{ij}(x)$ belong to the class $C^5(\Omega)$.

For the first time, the anti-a priori bound of root functions of an elliptic operator was obtained for the Schrödinger operator (4.4) with a bounded potential on Ω [65], and then it was generalized to an arbitrary second-order elliptic operator with coefficients satisfying the smoothness conditions (4.3) and (4.6) [47].

Theorem 4.2. *Let the eigenfunctions satisfy the Carleman condition*

$$|\operatorname{Im}\sqrt{\lambda}| \le C_0. \tag{4.13}$$

Then for any $l \in \mathbb{N}$ and any two compact sets K and K' of the domain Ω the first of which lies strictly inside the second, $K \subset \operatorname{int} K'$, there exists a constant $C_l = C_l(K, K') > 0$ such that the following estimate holds[41]:

$$\|u_{l-1}(\cdot;\lambda)\|_{2,K} \le C_l \left(1 + |\sqrt{\lambda}|\right) \|u_l(\cdot;\lambda)\|_{2,K'}. \tag{4.14}$$

The anti-a priori bound (4.14) is sharp with respect to the order in the sense that the exponent of power for the eigenvalue λ on the right-hand side cannot be decreased (as a corresponding example, one can consider the boundary-value problem for the Laplace operator in the N-dimensional cube $[0,1]^N$ with the condition $u|_{x1=0} = 0$, $u_{x_1}|_{x1=0} = u_{x_1}|_{x1=1}$, and $u = 0$ on the other faces of the cube [65]).

We note that the Carleman condition (4.3) in Theorem 4.2 can be omitted [79]. Also, we can make more clear the character of the dependence of the constant C_l on the order l of the associated function.[42] The final result in this direction was obtained in [136].[43]

Theorem 4.3. *Let operator* (4.1) *be uniformly elliptic on each compact set of the domain Ω, and let the smoothness conditions* (4.3) *hold. Then for any two compact sets K and K' of the domain Ω such that $K \subset \operatorname{int} K'$, there exists a constant $C = C(K, K') > 0$ such that for all $l \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$, the following estimate holds*:

$$\|u_{l-1}(\cdot;\lambda)\|_{2,K} \le C \left(l^2 + l \operatorname{Re}\sqrt{\lambda}\right) \|u_l(\cdot;\lambda)\|_{2,K'}. \tag{4.15}$$

Remark. If the eigenvalue λ lies in the domain $|\arg\sqrt{\lambda}| \ge \alpha > 0$, then the sum $l^2 + l \operatorname{Re}\sqrt{\lambda}$ on the right-hand side of (4.15) can be replaced by l^2, and, moreover, the resulting estimate is also sharp [136].

As a consequence of the anti-a priori bounds of root functions, we have the estimates of their derivatives in the metric of L_2.

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a multi-index, and let D^α denote the partial derivative $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}$ of order $|\alpha| = \sum_{i=1}^N \alpha_i$.

Theorem 4.4 ([139]). *Let operator* (4.1) *be uniformly elliptic on each compact set of the domain Ω, and let all its coefficients be infinitely differentiable in Ω. Then for any two compact sets K and K' of the domain Ω such that $K \subset \operatorname{int} K'$ and for any multi-index α, there exists a constant $C = C(K, K', \alpha) > 0$ such that for all $l = 0, 1, 2, \ldots$ and all $\lambda \in \mathbb{C}$, we have the* (*sharp with respect to λ and l*) *estimate*

$$\|D^\alpha u_l(\cdot;\lambda)\|_{2,K} \le C \left(l + \operatorname{Re}\sqrt{\lambda} + 1\right)^{|\alpha|} \|u_l(\cdot;\lambda)\|_{2,K'}. \tag{4.16}$$

For eigenvalues lying outside the Carleman parabola (4.13), the behavior of root functions can be made more clear.

Theorem 4.5 ([136]). *Let K' be an arbitrary compact set of the domain Ω. Then there exist constants $M > 0$ and $\delta > 0$ such that for any compact set $K \subset \operatorname{int} K'$ and any λ and l satisfying the inequality*

$$|\operatorname{Im}\sqrt{\lambda}| \ge Ml/\operatorname{dist}(K, \partial K'),$$

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[41] The constant C_l is independent of λ and depends only on the compact sets K and K'.

[42] In [47, 65, 79], this dependence was exponential.

[43] Also, an example showing the sharpness of the estimate with respect to λ and l was constructed.

we have the estimate

$$\|u_l(\cdot;\lambda)\|_{2,K} \le C\exp\left[-\delta\left|\operatorname{Im}\sqrt{\lambda}\right|\operatorname{dist}(K,\partial K')\right]\|u_l(\cdot;\lambda)\|_{2,K'}, \tag{4.17}$$

where $C = C(K,K') > 0$ *depends only on the compact sets* K *and* K' *considered.*

Remark. For the eigenfunctions ($l = 0$), the value δ is explicitly found:

$$\delta = \left[\max_{\|\xi\|=1, x\in K'} \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j\right]^{-1/2}$$

and cannot be increased [127].

Along with the anti-a priori bound, another important estimate is the estimate of the maximum of the module for the root function through its L_2 norm. For the first time, this estimate for the general second-order elliptic operator was obtained in [67].

Theorem 4.6. *Let the coefficients of the uniformly elliptic operator* (4.1) *satisfy conditions* (4.3), *and let* $a_{ij} \in C^5(\Omega)$. *Moreover, let the Carleman condition* (4.13) *hold. Then for any* $l = 0, 1, 2, \dots$ *and any compact sets* K *and* K' *of the domain* Ω *such that* $K \subset \operatorname{int} K'$, *there exists a constant* $C = C(K,K') > 0$ *such that the following estimate holds*:

$$\|u_l(\cdot;\lambda)\|_{\infty,K} \le C\left(1 + |\lambda|^{(N-1)/4}\right)\|u_l(\cdot;\lambda)\|_{2,K'}. \tag{4.18}$$

This estimate is sharp with respect to λ; this is justified by the example of the first boundary-value problem for the Laplace operator in the N-dimensional ball. The orthonormal eigenfunctions of this problem at the center of the ball are equal to $c_n\lambda_n^{(N-1)/4}$, where $\lim\limits_{n\to\infty}|c_n|$ depends only on N and the radius of the ball.

If condition (4.13) does not hold, then the constant $C(K,K')$ in (4.18) can be replaced by $\widetilde{C}(K,K')\exp[-\beta|\operatorname{Im}\sqrt{\lambda}|]$ with a certain $\beta > 0$ [80].

The final estimate, which is sharp with respect to λ and the order l, for the maximum of the module of the root function of the Schrödinger operator (4.4) was proved in [135].

Theorem 4.7. *Let the potential* $q(x)$ *in* (4.4) *be continuous on* Ω. *Then for any compact sets* K *and* K' *of the domain* Ω *such that* $K \subset \operatorname{int} K'$, *there exists a constant* $C = C(K,K') > 0$ *such that for all* $l \in \mathbb{N}$ *and* $\lambda \in \mathbb{C}$, *the following estimate holds*:

$$\|u_l(\cdot;\lambda)\|_{\infty,K} \le C\sqrt{l}\left(l^{(N-1)/2} + (\operatorname{Re}\sqrt{\lambda})^{(N-1)/2}\right)\|u_l(\cdot;\lambda)\|_{2,K'}. \tag{4.19}$$

5. Convergence of Spectral Expansions

We assume that the set Λ of eigenvalues of operator (4.1) has no finite accumulation points, and with each eigenvalue one associates finitely many root functions. We enumerate all eigenvalues from Λ in nondecreasing order with account for their multiplicity. Therefore, $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ and $|\lambda_i| \le |\lambda_j|$ for $i < j$.

As a result of such renumbering of the set Λ, each root function of the system $\mathfrak{U}$ obtains its own number: $\mathfrak{U} = \{u_k(x)\}_{k=1}^{\infty}$. We denote by l_k the order of the associated function $u_k(x)$; let $l_k = 0$ if $u_k(x)$ is an eigenfunction.

Taking into account (4.7) and (4.8), we see that root functions $u_k(x)$ satisfy the relations

$$\mathcal{L}\,u_k + \lambda_k u_k = \theta_k u_{k-1}, \tag{5.1}$$

where $\theta_k = 0$ if $u_k(x)$ is an eigenfunction and $\theta_k = 1$ if $u_k(x)$ is an associated function; moreover, in this case, $\lambda_{k-1} = \lambda_k$.

Let the system $\mathfrak{U}$ be closed and minimal in a certain space $L_p(\Omega)$, $p \ge 1$. Then for it, there exists a unique biorthogonally dual system $\mathfrak{V} = \{v_k(x)\}_{k=1}^{\infty}$ of functions of the space $L_{p/(p-1)}(\Omega)$. In a number of cases, we assume that the system $\mathfrak{V}$ consists of root functions of operator (4.5) formally adjoint to $\mathcal{L}$ and the following relations, which are analogous to (5.1), hold:

$$\mathcal{L}^* v_k + \overline{\lambda_k} v_k = \theta_{k+1} v_{k+1}. \tag{5.2}$$

Let us compose a partial sum of the biorthogonal expansion of an arbitrary function $f \in L_p(\Omega)$ in the system $\mathfrak{U}$:

$$\sigma_\lambda(x, f) = \sum_{|\lambda_k|<\lambda} (f, v_k) u_k(x), \tag{5.3}$$

and let us study the problem on the convergence of this sum to the function being expanded.

5.1. Uniform convergence. In [50], a method for proving the absolute and uniform convergence in the metric of L_2 of the biorthogonal expansion in the system $\mathfrak{U}$ of root functions of operator (4.1) was proposed by using only estimates of these functions in the L_2 norm and in the uniform metric, which are presented above in Subsection 4.3.

Theorem 5.1. *Let the coefficients of operator* (4.1) *uniformly elliptic in* Ω *be infinitely differentiable, and let* $\mathfrak{U} = \{u_k(x)\}_{k=1}^{\infty}$ *be the complete and minimal system of its root functions in* $L_2(\Omega)$ *such that the system biorthogonally dual to it consists of root functions of the operator* $\mathcal{L}^*$ *and relations* (5.2) *hold. Moreover, let the following conditions hold*:

(1) *the Carleman condition* (4.13) *holds*;

(2) *the orders* l_k *of root functions are uniformly bounded*:[44]

$$\sup_{k\in\mathbb{N}} l_k < \infty, \tag{5.4}$$

(3) *the "counting" function* (4.9) *satisfies the estimate*[45]

$$n(r) = O(r^{N/2}), \tag{5.5}$$

(4) *for each compact set* $K \subset \Omega$, *there exists a constant* $C = C(K) > 0$ *such that for all* $k \in \mathbb{N}$, *the following estimate holds*[46]:

$$\|u_k\|_2 \, \|v_k\|_{2,K} \le C(K), \tag{5.6}$$

(5) *the function* $f(x)$ *has a compact support in* Ω *and belongs to the Sobolev class* $W_2^{2s}(\Omega)$.

Then the partial sums $\sigma_\lambda(x, f)$ *of the biorthogonal expansion of the function* $f(x)$ *converge to it in the metric* $L_2(\Omega)$ *for* $s > N/2$ *and they converge absolutely and uniformly on any compact set for* $s > (3N-1)/4$.

The proof of this property is based not on application of the mean-value formula to root functions but on the fact that a function $f(x)$ from the class $W_2^{2s}(\Omega)$ admits the s-fold application of the operator $\mathcal{L}$ to itself.

Let us consider the functions $v_k^{[s]}(x)$ equal to $v_k(x)$ if $v_k(x)$ is an eigenfunction of the operator $\mathcal{L}^*$ and to

$$v_k(x) + \sum_{j=1}^{l} (\overline{\lambda_k})^{-j} C_{s+j-1}^{j} v_{k+j}(x)$$

if $v_k(x)$ are associated functions of the operator $\mathcal{L}^*$ of order $l \ge 1$. It follows from relation (5.2) that

$$\mathcal{L}^* v_k^{[s+1]} = -\lambda_k v_k^{[s]}, \quad s = 0, 1, 2, \ldots, \tag{5.7}$$

[44]Moreover, the multiplicity of eigenvalues can be not bounded.

[45]Such an estimate holds for a wide class of boundary-value problems related to operator (4.1). We will see from what will be done below that in this theorem, one can also consider functions $n(r)$ satisfying a more general estimate $n(r) = O(r^{\beta})$, $\beta > 0$.

[46]We can also consider the case where instead of $C(K)$, one has $C(K)(1 + |\lambda|)^{\gamma}$, $\gamma > 0$, in the right-hand side of (5.6).

where $v_k^{[0]}(x) = v_k(x)$. Relation (5.7) implies that for any function $f(x)$ satisfying the conditions of the theorem, the relation $(f, (\mathcal{L}^*)^s v_k^{[s]}) = (-\lambda_k)^s (f, v_k)$ holds, and, therefore,

$$(f, v_k) = (-\lambda_k)^{-s} (\mathcal{L}^s f, v_k^{[s]}). \tag{5.8}$$

To justify the convergence of the biorthogonal series to $f(x)$ in the metric $L_2(\Omega)$, it suffices to prove the boundedness of partial sums of the series $\sum_k \|(f, v_k) u_k(x)\|_2$, since, by the completeness of the system $\mathfrak{U}$ in $L_2(\Omega)$, only the function $f(x)$ itself can be the limit of the biorthogonal series converging in $L_2(\Omega)$.

By the compactness of the support $K \equiv \operatorname{supp} f$ in G, we have $|(f, v_k)| \le \|f\|_2 \, \|v_k\|_{2,K}$; taking into account (5.5) and (5.6), we obtain from this that

$$\sum_{|\lambda_k| \le 1} \|(f, v_k) u_k(x)\|_2 \le \|f\|_2 \cdot C(K) \sum_{|\lambda_k| \le 1} 1 = O(1) \|f\|_2. \tag{5.9}$$

Further, relation (5.8) implies the inequality

$$\sum_{|\lambda_k| > 1} \|(f, v_k) u_k(x)\|_2 \le \sum_{|\lambda_k| > 1} |\lambda_k|^{-s} \|\mathcal{L}^s f\|_2 \, \|v_k^{[s]}\|_{2,K} \, \|u_k\|_2.$$

By the anti-a priori bound (4.14) and condition (5.4), for any compact set $K' \subset \Omega$: $K \subset \operatorname{int} K'$, the estimate $\|v_k^{[s]}\|_{2,K} = O(1) \|v_k\|_{2,K'}$ holds. Thus, it follows from estimate (5.6) that to ensure the required boundedness of the partial sums, it suffices to ensure the convergence of the series $\sum\limits_{|\lambda_k| > 1} |\lambda_k|^{-s}$, which holds for $s > N/2$ by (5.5).

To study the absolute and uniform convergence of the biorthogonal expansion, as above, it suffices to prove the convergence of the series $\sum_k \bigl(|(f, v_k)| \cdot \sup_{x \in K} |u_k(x)|\bigr)$ for any compact set $K \subset \Omega$. But this reduces to the previous arguments via the application of estimate (4.18). Theorem 5.1 is proved.

We note that on the basis of estimates (4.15) and (4.17), in [136] A. S. Makin has succeeded in proving the result of Theorem 5.1 on the convergence of the biorthogonal expansion in $L_2(\Omega)$ under more general assumptions: without Carleman condition (4.13) and with the condition $l_k = o(|\lambda_k|^{1/2})$, $k \to \infty$, which is more general than (5.4).

With account for the refined estimate (4.19) of maxima of modules of root functions for the Schrödinger operator, the following result was proved in [139].

Theorem 5.2. *Let $\mathfrak{U} = \{u_k(x)\}_{k=1}^{\infty}$ be a complete and maximal system of root functions of the Schrödinger operator* (4.4) *in $L_2(\Omega)$ whose biorthogonally dual system consists of root functions of the corresponding formally adjoint operator $\mathcal{L}^*$, and let relations* (5.2) *hold. Moreover, let the following conditions hold*:

(1) *the orders l_k of root functions satisfy the estimate*

$$l_k = O(|\lambda_k|^{1/2 - \varepsilon} + 1), \quad k \to \infty, \tag{5.10}$$

where $0 < \varepsilon \le 1/2$;

(2) *the "counting" function $n(r)$ satisfies estimate* (5.5);

(3) *for each compact set $K \subset \Omega$, estimate* (5.6) *holds*;

(4) *the potential $q(x)$ of operator* (4.4) *belongs to the class $C^{[(3N/2) - \varepsilon] - 1}(\Omega)$;*

(5) *the function $f(x)$ has a compact support in Ω and belongs to the Sobolev class $W_2^{[(3N/2) - \varepsilon] + 1}(\Omega)$.*

Then the partial sums $\sigma_\lambda(x, f)$ of the biorthogonal expansion of the function $f(x)$ converge absolutely and monotonically to it on any compact set $K \subset \Omega$.

In [144], the convergence of the biorthogonal expansion in the system of root functions of the Schrödinger operator (4.4) with a discontinuous potential was studied. For a centrally symmetric potential, the following singularity was admitted[47]: $q(x) = |x - x_0|^{-1} q_0(|x - x_0|)$, where the function $q_0(t)$

[47]We note that these conditions ensure that the potential belongs to the class L_2 on any compact set in Ω.

is continuous for $t > 0$ and satisfies the estimate $|q_0(t)| \leq C_\varepsilon t^{\varepsilon-1}$; here, $\varepsilon > 1/2$ for $N = 3$ and $\varepsilon > 0$ for $N > 3$ (the case $N = 2$ was not considered). It was proved that the conditions of Theorem 5.1 guarantee the convergence of $\sigma_\lambda(x_0, f)$ for such an operator at the point of discontinuity of the potential for $f \in W_2^{2s}(\Omega)$, $s > N/2$, which becomes absolute for $s > (3N+1)/4$.

As the comparison with the self-adjoint case [55, Chap. 4] shows, the conditions for the uniform convergence of biorthogonal expansions obtained in Theorems 5.1 and 5.2 are far from being final even for the Laplace operator in the classes of functions being expanded. Therefore, in terms of Sobolev–Liouville classes $L_p^s(\Omega)$, the following inequalities are final conditions for the uniform convergence:

$$s \geq (N-1)/2, \quad sp > N, \quad p \geq 1. \tag{5.11}$$

On the basis of the mean-value formula for the Schrödinger operator and the Laplace operator, a number of results were obtained; in these results, the smoothness requirements on the function $f(x)$ are reduced (as compared with those in Theorems 5.1 and 5.2). We present one of them.

Theorem 5.3 ([131]). *Let $\mathfrak{U} = \{u_k(x)\}_{k=1}^\infty$ be a closed and minimal system of root functions of the Schrödinger operator* (4.4) *on $L_p(\Omega)$, $p \geq 1$, whose biorthogonally dual system consists of root functions of the corresponding formally adjoint operator $\mathcal{L}^*$, and let relations* (5.2) *hold. Moreover, let the following conditions hold*:

(1) *the Carleman condition* (4.13) *holds*;

(2) *the orders l_k of root functions are uniformly bounded*;

(3) *for each compact set $K \subset \Omega$, the following estimate uniform in $t \geq 0$ and $x \in K$ holds*[48]:

$$\sum_{t \leq |\sqrt{\lambda_k}| \leq t+1} \|v_k\|_{p/(p-1)} \sum_{\substack{0 \leq l \leq l_k \\ u_{k-l} \sim u_k}} \frac{|u_{k-l}(x)|}{1 + |\sqrt{\lambda_k}|^l} \leq C(K)(t^{N-1} + 1), \tag{5.12}$$

(4) *the potential $q(x)$ of operator* (4.4) *belongs to the class $C^N(\Omega)$*;

(5) *the function $f(x)$ has a compact support in the domain Ω and belongs to the class $W_p^{N-1}(\Omega)$, where $p > N/(N-1)$.*

Then $\sigma_\lambda(x, f)$ converges to $f(x)$ uniformly on any compact set of the domain Ω.

5.2. Riesz means of spectral expansions. As was already mentioned in the one-dimensional case, Riesz means of expansions in root functions of essentially non-self-adjoint problems should be defined in a specific way. For the first time, the correct definition of Riesz means was suggested by Ya. M. Salimov for biorthogonal expansions in root functions of the Laplace operator in [156–158]. It has the form[49]

$$\sigma_\lambda^\alpha(x, f) = \sum_{|\lambda_k| < \lambda} (f, v_k) \sum_{\substack{0 \leq l < \alpha+1 \\ u_{k-l} \sim u_k}} \frac{(-1)^l}{l!} \frac{\partial^l}{\partial \lambda_k^l} \left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha u_{k-l}(x). \tag{5.13}$$

We explain the natural character of such a definition for a natural α. Let us perform the differentiation in (5.13) and take into account that by (5.1), the expression $(\Delta + \lambda_k)^l u_k(x)$ is equal to zero if $u_k(x)$ is an eigenfunction and is equal to $u_{k-l}(x)$ if $u_k(x)$ is an associated function of order not exceeding l. Then

[48] It was Ya. M. Salimov who introduced such a condition on root functions of the Laplace operator in [156] for the first time; he used this condition in the theorem on the uniform convergence of the corresponding spectral expansion in [158].

[49] We will use a more compact form of writing them, which was suggested in [131].

the inner sum in (5.13) becomes

$$\sum_{\substack{0\le l<\alpha+1\\ u_{k-l}\sim u_k}} \frac{\Gamma(\alpha+1)}{l!\,\Gamma(\alpha+1-l)}\left(1-\frac{\lambda_k}{\lambda}\right)^{\alpha-l}\frac{u_{k-l}(x)}{\lambda^l}$$

$$= \sum_{\substack{0\le l<\alpha+1\\ u_{k-l}\sim u_k}} \frac{\Gamma(\alpha+1)}{l!\,\Gamma(\alpha+1-l)}\left(1-\frac{\lambda_k}{\lambda}\right)^{\alpha-l}\left(\frac{\Delta+\lambda_k}{\lambda}\right)^{l} u_k(x) = \left(1+\frac{\Delta}{\lambda}\right)^{\alpha} u_k(x).$$

We write the Riesz means (5.13) using the spectral function

$$\theta(x,y;\lambda) = \sum_{|\lambda_k|<\lambda} u_k(x)\overline{v_k(y)}.$$

If $\theta^\alpha(x,y;\lambda)$ denotes the Riesz means of this spectral function, then $\sigma_\lambda^\alpha(x,f) = \int_\Omega \theta^\alpha(x,y;\lambda)f(y)\,dy$; we obtain from this that

$$\theta^\alpha(x,y;\lambda) = \sum_{|\lambda_k|<\lambda} \overline{v_k(y)}\left(1+\frac{\Delta}{\lambda}\right)^{\alpha} u_k(x) = \left(1+\frac{\Delta_x}{\lambda}\right)^{\alpha}\theta(x,y;\lambda),$$

which is in correspondence with the classical definition of Riesz means.

Using definition (5.13) of Riesz means, Ya. M. Salimov proved the following result.

Let us introduce the modified partial sum of Riesz means of order α of the expansion of a function $f(x)$ into the N-fold Fourier integral by the relation

$$S_\lambda^\alpha(x,f) = \frac{2^\alpha\Gamma(\alpha+1)}{(2\pi)^{N/2}}(\sqrt{\lambda})^{(N/2)-\alpha}\int_{\mathbb{R}^N} |x-y|^{-\alpha-(N/2)}J_{\alpha+(N/2)}\left(\sqrt{\lambda}|x-y|\right)f(y)\,dy.$$

Theorem 5.4 ([156,157]). *Let $\mathfrak{U} = \{u_k(x)\}_{k=1}^\infty$ be a closed and minimal system of root functions of the Laplace operator in $L_p(\Omega)$, $p \ge 1$,*[50] *and let conditions* (1)–(3) *of Theorem* 5.3 *hold.*

Then for any function $f \in L_p(\Omega)$, the spectral expansion in the system $\mathfrak{U}$ and the expansion of the same function $f(x)$ into the N-fold trigonometric Fourier series are equisummable by using the Riesz method of order $\alpha > N-1$ on any compact set $K \subset \Omega$, and, moreover, the following estimate holds:

$$\sup_{x\in K}|\sigma_\lambda^\alpha(x,f) - S_\lambda^\alpha(x,f)| = O(1)\lambda^{N-1-\alpha}\|f\|_p. \tag{5.14}$$

Using the refined estimates (4.15) and (4.19) of root functions of the Schrödinger operator, A. S. Makin established the base conditions for Riesz means in L_2 on any compact set in Ω.

Theorem 5.5 ([133]). *Let $\mathfrak{U} = \{u_k(x)\}_{k=1}^\infty$ be a complete and minimal system of root functions of the Schrödinger operator* (4.4) *with an infinitely differential potential $q(x)$ in $L_2(\Omega)$. Moreover, let the following conditions hold*:

(1) *the Carleman condition* (4.13) *holds*;

(2) *the orders l_k of root functions satisfy estimate* (5.10) *with a certain $\varepsilon \in (0, 1/2]$*;

(3) *for each compact set $K \subset \Omega$, there exists a constant $C = C(K) > 0$ such that for all $t \ge 0$, the following estimate holds*[51]:

$$\sum_{t\le|\sqrt{\lambda_k}|\le t+1} \|u_k\|_{2,K}\,\|v_k\|_2 \le C(K)(t^{N-1}+1). \tag{5.15}$$

[50] In this case, it is not required that the biorthogonal system $\mathfrak{V}$ satisfy any differential equation.

[51] Estimate (5.15) holds, for example, in the case where the "counting" function (4.9) satisfies estimate (5.5) and the product of norms $\|u_k\|_{2,K}\,\|v_k\|_2$ is bounded uniformly in k.

Then for any α satisfying the conditions $\alpha > \alpha_0$ and $[\alpha] \geq \alpha_0$, where $\alpha_0 = (N - 2\varepsilon)/(2\varepsilon)$, and for any compact set $K \subset \Omega$, the following relation holds:

$$\lim_{\lambda\to\infty} \|\sigma_\lambda^\alpha(x, f) - f(x)\|_{2,K} = 0. \tag{5.16}$$

Remark. We note that in the self-adjoint case, Riesz means of order α greater than the critical value $(N - 1)/2$ converge to the function being expanded uniformly on any compact set (in Theorem 5.5, for $\varepsilon = 1/2$, i.e., when the orders of root functions are uniformly bounded, the order $\alpha_0 = N - 1$).

5.3. Abel–Poisson means of spectral expansions. Let us define the modified Abel–Poisson means of order 1/2 of the spectral expansion in root functions of operator (4.1). We set

$$H_k(x,t) = \sum_{\substack{l\geq 1\\ u_{k-l}\sim u_k}} \frac{1}{l!} \sum_{j=1}^{l} \frac{(l+j-2)!\, t^{l-j+1}}{(j-1)!\,(l-j)!\,(2\sqrt{\lambda_k})^{l+j-1}} u_{k-l}(x)$$

if $u_k(x)$ is an associated function and $H_k(x, t) = 0$ if $u_k(x)$ is an eigenfunction.

The *modified Abel–Poisson means of order* 1/2 are the series

$$A_t(x, f) = \sum_{k=1}^{\infty} \exp(-\sqrt{\lambda_k}t)(f, v_k)[u_k(x) + H_k(x, t)]. \tag{5.17}$$

Theorem 5.6 ([128]). *Let $\mathfrak{U} = \{u_k(x)\}_{k=1}^{\infty}$ be a complete and minimal system of root functions of operator* (4.1) *in $L_2(\Omega)$. Moreover, let the following conditions hold*:

(1) *the orders l_k of root functions in $\mathfrak{U}$ are bounded by one and the same constant $M > 0$*;

(2) *the "counting function" $n(r)$ satisfies estimate* (5.5);

(3) *for each compact set $K \subset \Omega$, there exists a constant $C = C(K) > 0$ such that the following inequality holds uniformly in $k \in \mathbb{N}$*:

$$\|u_k\|_{2,K}\, \|v_k\|_2 \leq C(K)(1 + |\lambda_k|)^\alpha \tag{5.18}$$

with the constant $\alpha \geq 0$ independent of the compact set K;

(4) *the coefficients $a_{ij}(x)$, $b_j(x)$, and $c(x)$ of operator* (4.1) *belong to the class $C^\beta(\Omega)$, where $\beta = 2[(N/2) + M + \alpha - 1] + 4$.*

Then for any function $f \in L_2(\Omega)$ and any compact set $K \subset \Omega$, the following relation holds:

$$\lim_{t\to 0+0} \|A_t(x, f) - f(x)\|_{2,K} = 0. \tag{5.19}$$

Relation (5.19) shows that the modified Abel–Poisson means (5.17) of order 1/2 have the basis property in L_2 for any compact set in Ω.

5.4. Some results for other operators. In [82], the following general elliptic operator of order $2m$ is considered:

$$\mathcal{L}\, u = \sum_{|\alpha|=m,|\beta|=m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u\right) + \sum_{|\gamma|\leq 2m-1} A_\gamma(x) D^\gamma u, \quad x \in \Omega \subset \mathbb{R}^N, \tag{5.20}$$

whose coefficients are infinitely differentiable, satisfy the uniform ellipticity condition on each compact set in Ω, i.e., the coefficients $A_{\alpha\beta}(x)$ are real, $A_{\alpha\beta}(x) = A_{\beta\alpha}(x)$, and for all $x \in K$ and any $\xi = (\xi_1, \dots, \xi_N)$, the inequality

$$(-1)^{m+1} \sum_{|\alpha|=m,|\beta|=m} A_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq C_0\|\xi\|^{2m}$$

holds with a certain positive constant C_0 depending on the compact set K; here, for the multi-index $\gamma = (\gamma_1, \dots, \gamma_N)$, we accept the notation $\xi^\gamma = \xi_1^{\gamma_1}\xi_2^{\gamma_2}\dots\xi_N^{\gamma_N}$.

For root functions of operator (5.20) understood in the generalized sense, the following anti-a priori bound was proved:

$$\|u_{l-1}(\cdot;\lambda)\|_{2,K} \le C(K,K')\left(1+|\operatorname{Re}\lambda|^{(2m-1)/(2m)}\right)\|u_l(\cdot;\lambda)\|_{2,K'}, \quad l \ge 1, \tag{5.21}$$

where K and K' are compact sets of the domain Ω satisfying the condition $K \subset \operatorname{int} K'$. As a consequence of estimate (5.21), the estimate of derivatives of root functions was also proved.

In [129], the anti-a priori bound was proved for root functions of a hypoelliptic operator with constant coefficients.

In a number of papers [77, 78, 81], the anti-a priori bound of root functions was proved and certain problems of convergence of spectral expansions corresponding to non-self-adjoint parabolic operators were considered.

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